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# Point-Free Foundation of Geometry and Multivalued Logic

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**Abstract** Whitehead, in two basic books, considers two different approaches to point-free geometry: the *inclusion-based approach*, whose primitive notions are regions and inclusion relation between regions, and the *connection-based approach*, where the connection relation is considered instead of the inclusion. We show that the latter cannot be reduced to the first one, although this can be done in the framework of multivalued logics.

## 1 Introduction

In recent times the interest about the research in the field of point-free geometry has been growing in different areas. As an example, we quote computability theory, lattice theory, computer science (for a comprehensive survey, see [9]). The basic ideas of point-free geometry were first formulated by Whitehead in *An Inquiry Concerning the Principles of Natural Knowledge* [12] and in *The Concept of Nature* [13], where he proposed as primitives the *events* and the *extension relation* between events. Instead, in order to define the points, the lines, and all the "abstract" geometrical entities, Whitehead proposed the notion of "abstractive class" representing the ability to imagine smaller and smaller regions. Now, as a matter of fact, these books are related to *mereology* (i.e., an investigation about the part-whole relation) rather than to point-free geometry. So, it is not surprising that later, in *Process and Reality* [14], Whitehead proposed a different approach, inspired by De Laguna [2], in which the topological notion of "contact between two regions" was assumed as a primitive and the inclusion was defined.

In this paper, we will give a mathematical reformulation of Whitehead's analysis (which is philosophical in nature) and this enables us to emphasize that there are technical reasons leading to the passage from the inclusion-based approach to the

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connection-based one. In fact, one proves that while it is possible to define the inclusion from the connection relation the converse fails. Moreover, the definition of point in an inclusion space is questionable. In spite of that, we show that the inclusion-based approach works well provided that we refer to multivalued logic and we consider a graded rather than a "crisp" inclusion relation. Indeed, in the resulting fuzzy structures we call *graded inclusion spaces of regions*, it is possible to define the contact relation. Moreover, we can give an adequate notion of point and this enables us to associate any graded inclusion space with a metric space. This suggests the possibility of finding a system of axioms in multivalued logic characterizing those inclusion spaces whose associated metric defines the Euclidean metric space (recall that there are very elegant approaches to Euclidean geometry metric in nature [1]). Some of the ideas in this paper were sketched in Miranda and Gerla [5]. A different metric approach to point-free geometry was proposed in [8].

## 2 Inclusion Spaces

In [12] Whitehead starts from a class of *events* and from a relation $K$ among events called *extension*. We adopt a different terminology which is related in a more strict way with the mathematical terminology and with the recent research in point-free geometry. So we use the word *region* instead of *event* and we call *inclusion relation* the converse of the extension relation. Also, we prefer to refer to the order relation $\leq$ rather than to the strict order. This enables us to reformulate the list of properties proposed by Whitehead in [12] by the following first-order theory with identity whose language $L_{\leq}$ contains only the binary relation symbol $\leq$.

**Definition 2.1** Consider the following list of axioms:

| | | |
|---|---|---|
| ***I1*** | $\forall x(x \leq x)$ | (reflexivity) |
| ***I2*** | $\forall x\forall y\forall z((x \leq z \wedge z \leq y) \Rightarrow x \leq y)$ | (transitivity) |
| ***I3*** | $\forall x\forall y(x \leq y \wedge y \leq x \Rightarrow x = y)$ | (anti-symmetry) |
| ***I4*** | $\forall z\exists x(x < z)$ | (there is no minimal region) |
| ***I5*** | $\forall x\forall y(x < y \Rightarrow \exists z(x < z < y))$ | (density) |
| ***I6*** | $\forall x\forall y(\forall x'(x' < x \Rightarrow x' < y) \Rightarrow x \leq y)$ | (below approximation) |
| ***I7*** | $\forall x\forall y\exists z(x \leq z \wedge y \leq z)$ | (upward-directed) |
| ***I8*** | $\forall z\exists x(z < x)$ | (there is no maximal region). |

We call *inclusion space* a model $(S, \leq)$ of ***I1–I7***, and *Whitehead inclusion space*, in brief *W-inclusion space*, a model $(S, \leq)$ of ***I1–I8***. Also, we call *regions* the elements of $S$ and *inclusion relation* the relation $\leq$.

Axiom *I6* is labeled "below approximation" since it is equivalent to the equality

$$x = \mathrm{Sup}\{x' \in S : x' < x\}. \tag{1}$$

In fact, $x$ is an upper bound of the class $\{x' \in S : x' < x\}$ and, given any upper bound $y$ of such a class, by *I6* we have $x \leq y$ and this proves (1). Conversely, it is evident that (1) entails *I6*.

Then an inclusion space is defined by a nonempty set $S$ and an order relation $\leq$ in $S$ with no minimal element, which is dense and upward-directed and in which every element can be approximate from below. If there is no maximal element for $\leq$, then we obtain a $W$-inclusion space. A trivial example of $W$-inclusion space is given by the set of real numbers with respect to the usual order. Another example, geometrical in nature, is given by the class of all the closed balls of the Euclidean plane ordered by the inclusion relation. A reasonable candidate to represent the idea of region is the notion of closed regular subsets.

**Definition 2.2** Given a topological space, denote by *cl* and *int* the closure and the interior operators, respectively, and put, for every set of points $x$,

$$\mathrm{creg}(x) = \mathrm{cl}(\mathrm{int}(x)).$$

Then we call *closed regular*, in brief *regular*, any fixed point of creg.

**Definition 2.3** We denote by $\mathbb{R}$ the real numbers set, by $\mathbb{R}^n$ the $n$-dimensional Euclidean space, and by $RC(\mathbb{R}^n)$ the class of all the closed regular subsets of $\mathbb{R}^n$.

There are several reasons suggesting the choice of the regular sets to represent the notion of region. As an example, in accordance with our intuition, all the subsets of $\mathbb{R}^n$ homeomorphic to a closed ball (with positive radius) are regular sets. Also, the points and the lines and all the geometrical entities whose dimension is less than $n$ are not regular and this reflects Whitehead's aim to define these geometrical notions by abstraction processes. A useful algebraic property is that the class $RC(\mathbb{R}^n)$ defines a complete atomic-free Boolean algebra. More precisely, we will consider suitable subclasses of $RC(\mathbb{R}^n)$. Indeed, we will consider

1. the class $\mathcal{R}_1$ of all the nonempty bounded and internally-connected elements of $RC(\mathbb{R}^n)$,
2. the class $\mathcal{R}_2$ of all the nonempty bounded elements of $RC(\mathbb{R}^n)$,
3. the class $\mathcal{R}_3$ of all nonempty internally-connected elements of $RC(\mathbb{R}^n)$,
4. the class $\mathcal{R}_4$ of all nonempty elements in $RC(\mathbb{R}^n)$,

where we say that a set $x$ is *internally-connected* if $\mathrm{int}(x)$ is connected.

In the following lemma we list some elementary topological facts.

**Lemma 2.4** *Let $x$ and $y$ be subsets of a locally connected topological space $S$. Then, while in general $\mathrm{cl}(x \cap y) \neq \mathrm{cl}(x) \cap \mathrm{cl}(y)$, in the case $x$ and $y$ are open subsets such that $x \cup y = S$,*

$$\mathrm{cl}(x \cap y) = \mathrm{cl}(x) \cap \mathrm{cl}(y). \tag{2}$$

*Equivalently, if $x$ is closed, $y$ is open, and $x \subseteq y$, then*

$$\mathrm{cl}(y - x) = \mathrm{cl}(y) - \mathrm{int}(x). \tag{3}$$

*Finally, if $x$ and $y$ are also regular and $\mathrm{cl}(y) \subseteq \mathrm{int}(x)$, then $x - y$ is a regular set.*

**Proof** To prove $\mathrm{cl}(x \cap y) \supseteq \mathrm{cl}(x) \cap \mathrm{cl}(y)$ let $P$ be an element in $\mathrm{cl}(x) \cap \mathrm{cl}(y)$. Then, for every open connected neighborhood $u$ of $P$ we have that $u \cap (x \cap y) \neq \varnothing$. Indeed otherwise, since $u \cap x \neq \varnothing$ and $u \cap y \neq \varnothing$ and $(u \cap x) \cup (u \cap y) = u$, the pair $u \cap x$ and $u \cap y$ should be an open partition of $u$. This proves that $\mathrm{cl}(x \cap y) \supseteq \mathrm{cl}(x) \cap \mathrm{cl}(y)$. Since it is evident that $\mathrm{cl}(x \cap y) \subseteq \mathrm{cl}(x) \cap \mathrm{cl}(y)$, (2) holds true. To prove (3), we apply the just proved equality to the open sets $y$ and $-x$. Finally, assume that $\mathrm{cl}(y) \subseteq \mathrm{int}(x)$, then

$$\mathrm{cl}(\mathrm{int}(x - y)) = \mathrm{cl}(\mathrm{int}(x) - \mathrm{cl}(y)) = \mathrm{cl}(\mathrm{int}(x)) - \mathrm{int}(\mathrm{cl}(y)) = x - y,$$

and this proves that $x - y$ is regular. □

**Lemma 2.5** *Let $c$ be a nonempty, closed, regular subset of $\mathbb{R}^n$ and let $b$ be an open ball such that $\mathrm{cl}(b) \subseteq \mathrm{int}(c)$. Then $c - b$ is a nonempty, closed, regular subset of $\mathbb{R}^n$. Moreover, if $c$ is internally-connected, then $c - b$ is internally-connected too. Finally, if $c \in \mathcal{R}_i$, then $c - b \in \mathcal{R}_i$, $i = 1, 2, 3, 4$.*

**Proof** By Lemma 2.4, $c - b$ is a regular closed set. Set, for every $x$, $\mathrm{fr}(x) = \mathrm{cl}(x) - \mathrm{int}(x) = \mathrm{cl}(x) \cap \mathrm{cl}(-x)$. Now, since $\mathrm{cl}(b) \subseteq \mathrm{int}(c)$, the distance $p$ between $\mathrm{fr}(c)$ and $b$ is different from 0. Indeed otherwise, since $b$ is bounded, there is a point $P \in \mathrm{fr}(b) \subseteq \mathrm{cl}(b)$ such that $P \in \mathrm{fr}(c)$ and therefore $P \notin \mathrm{int}(c)$. Let $r$ be the radius of $b$ and let $b'$ be the open ball concentric with $b$ and whose radius is $r + p/2$. Then the closure of $b'$ is contained in $c$ and $b' \supseteq \mathrm{cl}(b)$. We claim that $\mathrm{int}(c-b)$ is the union of the two overlapping connected sets $\mathrm{int}(c) - b'$ and $\mathrm{cl}(b') - \mathrm{cl}(b)$ and, therefore, that it is connected. In fact, it is evident that

$$\mathrm{int}(c - b) = \mathrm{int}(c) - \mathrm{cl}(b) = (\mathrm{int}(c) - b') \cup (\mathrm{cl}(b') - \mathrm{cl}(b))$$

and that, since $(\mathrm{int}(c)-b') \cap (\mathrm{cl}(b')-\mathrm{cl}(b))$ contains all the points in the frontier of $b'$, $(\mathrm{int}(c) - b') \cap (\mathrm{cl}(b') - \mathrm{cl}(b)) \neq \varnothing$. It is also evident that $\mathrm{cl}(b') - \mathrm{cl}(b)$ is connected. So we have only to prove that $\mathrm{int}(c) - b'$ is connected. Indeed otherwise, there are two nonempty disjoint open subsets, $u$ and $v$, in $\mathrm{int}(c) - b'$ such that $\mathrm{int}(c) - b' = u \cup v$. Then $\mathrm{fr}(b')$ is contained in $u \cup v$ and $G = u \cap \mathrm{fr}(b')$ and $H = v \cap \mathrm{fr}(b')$ are open sets in $\mathrm{fr}(b')$. Now, since $\mathrm{fr}(b')$ is connected, then $G = \varnothing$ or $H = \varnothing$. Let us suppose that $G = \varnothing$. Then $u \subseteq \mathrm{int}(c) - \mathrm{cl}(b')$ and, since $\mathrm{int}(c) - \mathrm{cl}(b')$ is open in $\mathrm{int}(c)$, $u$ is open in $\mathrm{int}(c)$. Therefore, $u$ and $b'$ are two nonempty disjoint open sets in $\mathrm{int}(c)$, so $u \cup b'$ is disconnected; this is a contradiction (see, for example, Exercise 6.1(c) in [4]). Thus $\mathrm{int}(c) - b'$ is connected. The remaining part of the proposition is evident. □

The following theorem extends a theorem given in [5].

**Theorem 2.6** *The structures $(\mathcal{R}_1, \subseteq)$ and $(\mathcal{R}_2, \subseteq)$ are W-inclusion spaces and $(\mathcal{R}_3, \subseteq)$ and $(\mathcal{R}_4, \subseteq)$ are inclusion spaces.*

**Proof** Trivially, all the considered structures satisfy *I1, I2, I3*, and *I4*. To prove *I5*, let $x$ and $y$ be nonempty regular sets such that $x \subset y$. Then, since $\mathrm{int}(y)$ is not contained in $x$, a point $P \in \mathrm{int}(y)$ exists such that $P \notin x$. Let $y'$ be a closed ball with center $P$ such that $y' \subset \mathrm{int}(y)$ and $y' \cap x = \varnothing$ . Now, by Lemma 2.5, $z = y - \mathrm{int}(y')$ is a nonempty closed regular internally-connected subset such that $x \subset z \subset y$, and this shows that all the considered structures satisfy *I5*. To prove *I6*, let us take two regions $x, y$ in $\mathcal{R}_i$, $(i = 1, \ldots, 4)$ and let us assume that all the subregions of $x$ are contained in $y$ and that $x$ is not contained in $y$. Then $\mathrm{int}(x)$ is not contained in $y$ too. Let $P$ be a point such that $P \in \mathrm{int}(x)$ and $P \notin y$. Then a real positive number $r$ exists such that the closure of the ball in $\mathbb{R}^n$ with center $P$ and radius $r$, which is an element of $\mathcal{R}_i$, for every $i = 1, \ldots, 4$, is contained in $\mathrm{int}(x)$ and disjoint from $y$, a contradiction.

To verify *I7*, we observe that, given two regions $x$ and $y$ in the structures $(\mathcal{R}_1, \subseteq)$ and $(\mathcal{R}_2, \subseteq)$ we can consider a closed ball $z$ containing both $x$ and $y$. Instead, in the structures $(\mathcal{R}_3, \subseteq)$ and $(\mathcal{R}_4, \subseteq)$ we can set $z$ equal to the whole space. Finally, it is evident that *I8* is satisfied by $(\mathcal{R}_1, \subseteq)$ and $(\mathcal{R}_2, \subseteq)$ and it is not satisfied by $(\mathcal{R}_3, \subseteq)$ and $(\mathcal{R}_4, \subseteq)$. □

In accordance with such a theorem, we give the following definition.

**Definition 2.7** Given $k = 1, \ldots, 4$, we call a *canonical k-inclusion space* in $\mathbb{R}^n$ the structure $(\mathcal{R}_k, \subseteq)$ defined in Theorem 2.6.

## 3 Contact Spaces

The inclusion relation is set-theoretical in nature and therefore rather unsatisfactory from a geometrical point of view (moreover, as we will see in Section 4, in the inclusion-based approach there are several technical difficulties). For this reason some years after the publication of [12] and [13], Whitehead, in [14], proposed a different idea based on the primitive notion of *connection relation*. This idea, topological in nature, was suggested by De Laguna in [2]. As in the inclusion-based approach, Whitehead was not interested in formulating the properties of this relation as a system of axioms and in reducing them at a logical minimum. So a very long list of "assumptions" was proposed. In this paper we refer to the following system which is equivalent to the first twelve assumptions (see [7]). We consider a language $L_C$ with a binary relation symbol $C$.

**Definition 3.1** Denote by $x \leq y$ the formula $\forall z(zCx \Rightarrow zCy)$ and by $x < y$ the formula $(x \leq y) \wedge (x \neq y)$. Then we call *contact theory* the first-order theory in $L_C$ whose axioms are

***C1*** $\forall x \forall y(xCy \Rightarrow yCx)$ (symmetry)
***C2*** $\forall z \exists x \exists y((x \leq z) \wedge (y \leq z) \wedge (\neg xCy))$
***C3*** $\forall x \forall y \exists z(zCx \wedge zCy)$
***C4*** $\forall x(xCx)$
***C5*** $\forall x \forall y((x \leq y \wedge y \leq x) \Rightarrow x = y)$
***C6*** $\forall x \exists y(x < y)$ (there is no maximal region).

We call *contact space* every model $(S, C)$ of ***C1–C5*** and *Whitehead contact space*, in brief *W-contact space*, every model of ***C1–C6***.

The intended interpretation is that the contact is either a surface contact or an overlap. As usual, we denote again by $C$ the interpretation of the relation symbol $C$. It is easy to prove that in any contact space the relation $\leq$ is an order relation. As in the case of inclusion structures, we can define four $n$-dimensional canonical contact structures. Indeed, it is possible to prove the following theorem, extending a result of Gerla and Tortora for the class $\mathcal{R}_2$ (see [6]).

**Theorem 3.2** *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$ be the classes defined in Section 2 and define the relation $C$ by setting*

$$xCy \Leftrightarrow x \cap y \neq \varnothing.$$

*Then $(\mathcal{R}_1, C)$, $(\mathcal{R}_2, C)$ are W-contact spaces and $(\mathcal{R}_3, C)$, $(\mathcal{R}_4, C)$ are contact spaces whose associated order relation coincides with the usual inclusion relation.*

**Proof** First, we will prove that in all the considered structures the order relation $\leq$ associated with $C$ coincides with the usual inclusion relation. Indeed, if $x \subseteq y$, then it is evident that for every region $z$ such that $z \cap x \neq \varnothing$, then $z \cap y \neq \varnothing$. This proves that $x \leq y$. Conversely, assume that $x \leq y$ and suppose that $x$ is not contained in $y$. Then, since $\mathrm{int}(x)$ is not contained in $y$, there exists a point $P \in \mathrm{int}(x)$ such that $P \notin y$. Let $b$ be a closed ball with center $P$ such that $b \subset \mathrm{int}(x)$ and $b \cap y = \varnothing$. So, $b$ is a region such that $bCx$ holds but $bCy$ is not true. This contradicts the hypothesis $x \leq y$. Thus $x \subseteq y$.

It is evident that in all the considered structures *C1, C2, C3*, and *C4* are satisfied. To prove *C5* it suffices to observe that $\leq$ is interpreted by the set theoretical inclusion. Finally, it is evident that *C8* is satisfied by $(\mathcal{R}_1, C)$ and $(\mathcal{R}_2, C)$ and it is not satisfied by $(\mathcal{R}_3, C)$ and $(\mathcal{R}_4, C)$. □

As in the case of the inclusion spaces, such a theorem enables us to give the following definition.

**Definition 3.3** Given $k = 1, 2, 3, 4$, we call *canonical contact k-space* in $\mathbb{R}^n$ the structure $(\mathcal{R}_k, C)$ defined in Theorem 3.2.

## 4 About the Definability of the Contact Relation

Let $I$ be an interpretation whose domain is $D$, $\alpha$ a first-order formula whose free variables are among $x_1, \ldots, x_n$ and $d_1, \ldots, d_n \in D$. Then we write $I \models \alpha\ [d_1, \ldots, d_n]$ to denote that the elements $d_1, \ldots, d_n$ satisfy $\alpha$. We call the *extension* of $\alpha$ in $I$ the relation $R_\alpha \subseteq D^n$ defined by

$$R_\alpha = \{(d_1, \ldots, d_n) : I \models \alpha[d_1, \ldots, d_n]\}$$

and in such a case we say that $R_\alpha$ is *definable by* $\alpha$. As an example, in the inclusion spaces and in the contact spaces the overlapping relation $O$ is defined by the formula $\exists z(z \leq x \wedge z \leq y)$. Also, Theorem 3.2 shows that in a canonical contact $k$-space the inclusion relation is definable by the formula $\forall z(zCx \Rightarrow zCy)$. Conversely, the question arises whether we can define the contact relation in a canonical inclusion $k$-space. A negative answer to this question should give theoretical support to Whitehead's passage from the inclusion-based approach to the contact-based one. We face this question by the following well-known property of the automorphisms.

**Proposition 4.1** *Let $I$ be an interpretation of a first-order language and $f : S \to S$ be an automorphism in $I$. Then*

$$I \models \alpha[d_1, \ldots, d_n] \Leftrightarrow I \models \alpha[f(d_1), \ldots, f(d_n)] \tag{4}$$

*for any formula $\alpha$ whose free variables are among $x_1, \ldots, x_n$ and for any $d_1, \ldots, d_n$ in $D$.*

The following theorem is an immediate extension of a theorem in [5].

**Theorem 4.2** *It is not possible to define the contact relation in a canonical inclusion k-space $(\mathcal{R}_k, \subseteq)$ for $k = 2, 3, 4$.*

**Proof** The language $L_{\leq}$ we are interested in has only a binary relation $\leq$ and therefore an automorphism in an interpretation $(S, \leq)$ is a one-to-one map $f : S \to S$ such that

$$d_1 \leq d_2 \Leftrightarrow f(d_1) \leq f(d_2).$$

In such a case, in accordance with Proposition 4.1, if a binary relation $C$ in $(S, \leq)$ is definable, then

$$d_1 C d_2 \Leftrightarrow f(d_1) C f(d_2). \tag{5}$$

Consider the case $k = 4$ and, by referring to the two dimensional case, set

$$r = \{(x, y) \in \mathbb{R}^2 | x = 0\};$$

$$p^< = \{(x, y) \in \mathbb{R}^2 | x < 0\};$$

$$p^> = \{(x, y) \in \mathbb{R}^2 | x > 0\}.$$

Also, define the map $g : \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$g((x, y)) = (x, y + 1) \qquad \text{if } (x, y) \in r \cup p^{>}$$
$$g((x, y)) = (x, y) \qquad \text{otherwise.}$$

This is a one-one map, which is continuous in $\mathbb{R}^2 - r$ and we can visualize as a *cut* of the Euclidean plane along the $y$-axis $r$ and a vertical translation of the half-plane $r \cup p^{>}$. If $x \in \mathcal{R}_4$, then $g(x)$ is not an element in $\mathcal{R}_4$, in general. Nevertheless, we have that $\text{int}(g(x)) \neq \varnothing$ and therefore that $\text{creg}(g(x))$ is a regular bounded nonempty subset of $\mathbb{R}^2$. In fact, since $\text{int}(x) \neq \varnothing$, either $\text{int}(x) \cap p^{>} \neq \varnothing$ or $\text{int}(x) \cap p^{<} \neq \varnothing$ and therefore either $g(\text{int}(x) \cap p^{>})$ or $g(\text{int}(x) \cap p^{>})$ is a nonempty open set contained in $g(x)$. We claim that the map $f : \mathcal{R}_4 \to \mathcal{R}_4$ defined by setting

$$f(x) = \text{creg}(g(x))$$

is an automorphism. In fact, it is evident that $x \subseteq y$ entails $f(x) \subseteq f(y)$. To prove the converse implication assume that $f(x) \subseteq f(y)$ and, by absurdity, that $x$ is not contained in $y$. Then $\text{int}(x)$ is not contained in $y$ and a closed ball $b$ exists such that $b \subseteq \text{int}(x)$ and $b \cap y = \varnothing$. Also, it is not restrictive to assume that $b$ is either completely contained in $p^{>}$ or completely contained in $p^{<}$ and therefore that $f(b) = g(b)$. Now, since $g$ is injective and since $b \cap y = \varnothing$, we have that $g(b) \cap g(y) = \varnothing$ and therefore $\text{int}(g(b)) \cap g(y) = \varnothing$.

On the other hand,

$$\text{int}(g(b)) \subseteq g(b) = f(b) \subseteq f(x) \subseteq f(y) \subseteq r \cup g(y).$$

Therefore, $\text{int}(g(b)) \subseteq r$, an absurdity. This proves that $f$ is an automorphism. Consider the closed balls $b_1 = \{(x, y) \in \mathbb{R}^2 | (x - 1)^2 + y^2 = 1\}$ and $b_2 = \{(x, y) \in \mathbb{R}^2 | (x + 1)^2 + y^2 = 1\}$. Then $b_1$ and $b_2$ are in contact in $(0, 0)$ but their images $f(b_1)$ and $f(b_2)$ are not in contact. Since $f$ transforms a bounded region into a bounded region, the same proof runs well in the case $k = 2$.

To examine the case $k = 3$, consider the *circle inversion*, that is, the map $g : \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}^2 - \{(0, 0)\}$ defined by setting

$$g(x, y) = (x/(x^2 + y^2), y/(x^2 + y^2))$$

and denote by $f$ the function defined by setting, for every nonempty set $x$,

$$f(x) = \text{cl}(g(\text{int}(x) - \{(0, 0)\})).$$

We claim that if $x$ is a nonempty internally-connected closed, regular subset, then $f(x)$ is a nonempty internally-connected regular set, too. Indeed, since the closure of any open set is a closed regular set, then $f(x)$ is a closed regular set. Moreover, observe that if $z$ is any open and connected set, then $\text{cl}(z)$ is internally-connected. In fact, assume that there are two nonempty disjoint open sets $a$ and $b$ such that $a \cup b = \text{int}(\text{cl}(z))$. Then, since $z \subseteq \text{int}(\text{cl}(z))$, $z \cap a$ and $z \cap b$ are disjoint open sets such that $(z \cap a) \cup (z \cap b) = z$. Since $z$ is connected, we have that either $z \cap a = \varnothing$ or $z \cap b = \varnothing$. As an example, assume that $z \cap a = \varnothing$. Then $a$ is a nonempty open set disjoint with $z$ and contained in $\text{cl}(z)$. This is an absurdum. Now, on account of the continuity of $g$, the set $g(\text{int}(x) - \{(0, 0)\})$ is connected and open. Thus $f(x) = \text{cl}(g(\text{int}(x) - \{(0, 0)\}))$ is internally-connected.

We claim that the map $f : \mathcal{R}_3 \to \mathcal{R}_3$ is an automorphism, with respect to the inclusion. In fact, trivially, if $x \subseteq y$, then $f(x) \subseteq f(y)$. Conversely, let us suppose that $x$ is not contained in $y$. Then $\text{int}(x)$ is not contained in $y$. Therefore, the closure of an open ball, $b$, is contained in $\mathrm{int}(x)$ and disjoint from $y \cup \{(0,0)\}$. It follows that $f(b)$ is contained in $f(x)$ but it is not contained in $f(y)$.

On the other hand, the contact relation is not preserved by $f$. In fact, two closed balls tangent in $(0,0)$ are in contact but their images under the map $f$ are not in contact. □

**Remark 4.3** In accordance with the example in the first part of the proof, we have that also the properties 'to be connected' and 'to be internally-connected' are not definable in the spaces $(\mathcal{R}_2, \subseteq)$ and $(\mathcal{R}_4, \subseteq)$.

It is still an open question whether or not we can define the contact relation in the space $(\mathcal{R}_1, \subseteq)$ of internally-connected regions. However, we are able to claim that if we refer to the connected regions the answer is positive.

**Theorem 4.4** *Denote*

1. *by $\mathcal{R}'_1$ the class of nonempty, closed regular and bounded connected subsets of $\mathbb{R}^n$,*
2. *by $\mathcal{R}'_3$ the class of nonempty, closed regular and connected subsets of $\mathbb{R}^n$.*

*Then $(\mathcal{R}'_1, \subseteq)$ is a W-inclusion space and $(\mathcal{R}'_3, \subseteq)$ is an inclusion space. Define the relation $C$ in $\mathcal{R}'_1$ and $\mathcal{R}'_3$ as in the cases $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, $\mathcal{R}_4$. Then $(\mathcal{R}'_1, C)$ is a W-contact space and $(\mathcal{R}'_3, C)$ is a contact space. Also, in both the structures $(\mathcal{R}'_1, \subseteq)$ and $(\mathcal{R}'_3, \subseteq)$ the contact relation $C$ is definable. Indeed, we have that $xCy$ if and only if the least upper bound $x \vee y$ exists; that is, $C$ is defined by the formula*

$$\exists z((x \leq z \wedge y \leq z) \wedge \forall m(x \leq m \wedge y \leq m \rightarrow z \leq m)).$$

**Proof** The first part of the proposition is immediate. Let $x, y$ be two elements in $\mathcal{R}'_1$. If $x \cap y \neq \emptyset$, then $x \cup y$ is connected and, trivially, $x \cup y = x \vee y$. Conversely, assume that $m = x \vee y$ exists. We claim that $m = x \cup y$. In fact, if $P \notin x \cup y$, then an open ball $b$ centered in $P$ exists such that $b \cap (x \cup y) = \emptyset$. Let $b'$ be a closed ball containing $x$ and $y$. Then $b' - b$ is an element in $\mathcal{R}'_1$ containing $x$ and $y$. As a consequence, $b' - b \supseteq m$ and therefore $P \notin m$. Thus, since by hypothesis $m$ is connected, $x \cap y \neq 0$ and therefore $xCy$. In the case of the structure $(\mathcal{R}'_3, \subseteq)$ the proof is similar. □

Observe that such a result is in accordance with the fact that the automorphism on $(\mathcal{R}_3, \subseteq)$, defined in Theorem 4.2, is not an automorphism in $(\mathcal{R}'_3, \subseteq)$ since it doesn't preserve the connection of a subset. Notice also that analogous results were proved in a series of basic papers of Pratt and Schoop (see, for example, [11]). In these papers Pratt refers to a different notion of canonical space.

## 5 Abstractive Classes and Geometrical Elements in the Inclusion Spaces

While in the point-based approaches to geometry a region is defined as a set of points, it is not surprising that in point-free geometry a point is defined by referring to a set of regions. Indeed, Whitehead in [12] defines the points by the following basic notion.

**Definition 5.1** Given an inclusion space $(S, \leq)$, we call an *abstractive class* any class $A$ of regions such that

(i) $A$ is totally ordered, that is, for every $x, y \in A$ either $x \leq y$ or $y \leq x$;
(ii) there is no region which is contained in all the regions in $A$.

We denote by AC the set of abstractive classes.

Whitehead's idea is that an abstractive class $A$ represents an "abstract object" which is obtained as a "limit" of the elements in $A$. On the other hand, it is possible that two different abstractive classes represent the same object. To face such a question, we define a preorder relation and the corresponding equivalence relation.

**Definition 5.2** The *covering* relation $\leq_c$ is defined by setting, for any $A_1$ and $A_2$ in AC,

$$A_1 \leq_c A_2 \Leftrightarrow \forall x \in A_2 \exists y \in A_1 y \leq x.$$

The covering relation $\leq_c$ is a preorder in AC, that is, it is reflexive and transitive. As is well known, we can obtain an order relation by a suitable quotient of such a preorder.

**Proposition 5.3** *Define the relation* $\equiv$ *by setting*

$$A_1 \equiv A_2 \Leftrightarrow A_1 \leq_c A_2 \text{ and } A_2 \leq_c A_1.$$

*Then* $\equiv$ *is an equivalence in* AC *and the related quotient* AC$/\equiv$ *is ordered by the relation* $\leq_c$ *defined by setting*

$$[A_1] \leq_c [A_2] \Leftrightarrow A_1 \leq_c A_2$$

*for every pair* $[A_1], [A_2]$ *of elements in* AC$/\equiv$.

Now we are able to give the definition of point remembering Euclid's definition "A *point* is that which has no part."

**Definition 5.4** We call a *geometrical element* any element of the quotient AC$/\equiv$, that is, any complete class of equivalence modulo $\equiv$. We call *point* any geometrical element which is minimal in the ordered set AC$/\equiv$ and we denote by Point$(S, \leq)$ the set of points of $(S, \leq)$.

In order to test the idea for which a geometrical element $[A]$ represents the "limit," that is, the "intersection" of an abstractive class $A$ representing $[A]$, we consider the following proposition.

**Proposition 5.5** *Consider the canonical structure* $(\mathcal{R}_i, \subseteq), i = 1, 2, 3, 4$ *and the related set* AC$_i$ *of abstractive classes. Also, consider the map* $h$ : AC$_i \to P(\mathbb{R}^n)$, *associating every abstractive class A with the related intersection.*

$$h(A) = \cap\{X | X \in A\}.$$

*Then*

$$A \leq_c B \Rightarrow h(A) \subseteq h(B).$$

*As a consequence, we can associate every geometrical element* $[A]$ *with a subset*

$$k([A]) = h(A)$$

*of* $\mathbb{R}^n$ *by obtaining an order-preserving correspondence.*

**Proof** Assume that $B$ covers $A$. Then for every region $X$ in $B$ there is $\underline{Y}$ in $A$ such that $X \supseteq \underline{Y}$, and therefore, since $\underline{Y} \supseteq \cap\{Y | Y \in A\} = h(A), X \supseteq h(A)$. Thus, $h(B) = \cap\{X | X \in B\} \supseteq h(A)$. □

In Whitehead, there is no hypothesis on the cardinality of the abstractive classes. Obviously, on account of condition (ii) of Definition 5.1, an abstractive class is necessarily infinite. Now, if we will express the effectiveness of the abstraction process, then it should be natural to assume the enumerability of the abstractive classes. Due to the fact that $\mathbb{R}^n$ is second countable and regular, the following proposition shows that such a choice is rather reasonable if we will refer to the canonical models. We say that an abstractive class is *sequential* if it is the set of elements of an injective, order-reversing sequence of regions.

**Proposition 5.6** *In a canonical model all the geometrical elements can be represented by a sequential abstractive class.*

**Proof** Consider an abstractive class $A$. Then, since $-h(A)$ is an open set, there is a sequence $B_m$ of balls in $\mathbb{R}^n$ such that $\cup_{m\in\mathbb{N}}\mathrm{cl}(B_m) = -h(A)$. Given a ball $B_m$, it is not possible that $\mathrm{cl}(B_m) \cap X \neq \varnothing$ for every $X \in A$ since in such a case the class $\{\mathrm{cl}(B_m) \cap X | X \in A\}$ of compact sets satisfies the finite intersection property and therefore

$$\cap_{X\in A}(\mathrm{cl}(B_m) \cap X) = \mathrm{cl}(B_m) \cap (\cap_{X\in A} X) = \mathrm{cl}(B_m) \cap h(A) \neq \varnothing.$$

Then, for every ball $B_m$, there is $X_m$ in $A$ such that $\mathrm{cl}(B_m) \cap X_m = \varnothing$. Then, since $\mathrm{cl}(B_m) \subseteq -X_m$, we have that $\cup_{m\in\mathbb{N}}\mathrm{cl}(B_m) \subseteq \cup_{m\in\mathbb{N}} - X_m$ and therefore that

$$h(A) = - \cup_{m\in\mathbb{N}} \mathrm{cl}(B_m) \supseteq \cap_{m\in\mathbb{N}} X_m \supseteq \cap_{X\in A} X = h(A).$$

So, $\cap_{n\in\mathbb{N}} X_n = h(A)$. If we set $C_m = \cap_{n\leq m} X_n$, we obtain an order-reversing sequence of elements in $A$ such that $\cap_{m\in\mathbb{N}} C_m = \cap_{n\in\mathbb{N}} X_n = h(A)$. The sequence $C_m$ is not injective, in general. Denote by $(A_n)_{n\in\mathbb{N}}$ an injective subsequence of $(C_m)_{m\in\mathbb{N}}$ such that $\cap_{n\in\mathbb{N}} A_n = \cap_{m\in\mathbb{N}} C_m$. We claim that $A$ is equivalent to $(A_n)_{n\in\mathbb{N}}$. In fact, trivially, $(A_n)_{n\in\mathbb{N}}$ dominates $A$. Conversely, assume that $(A_n)_{n\in\mathbb{N}}$ is not dominated by $A$. Then there is $X \in A$ such that no element $A_n$ is contained in $X$. Since $A$ is totally ordered with respect to the inclusion, this means for every $A_n$, $A_n \supseteq X$ and therefore $h(A) = \cap_{n\in\mathbb{N}} A_n \supseteq X$. This contradicts the fact that in $A$ there is no minimal element. □

Obviously, in the cases $(\mathscr{R}_3, \subseteq)$ and $(\mathscr{R}_4, \subseteq)$ it is possible that $h(A)$ is the empty set.

**Remark 5.7** The map $k$ is not injective, in general. In fact, let $P$ be a point in the Euclidean plane $\mathbb{R}^2$ and consider the sequence $B(P) = (B_n(P))_{n\in\mathbb{N}}$ of balls centered in $P$ and with radius $1/n$. Then $k([B(P)]) = \{P\}$. Assume, for example, that $P = (0, 0)$ and consider the sequences $B^-(P) = (B_n^-(P))_{n\in\mathbb{N}}$ and $B^+(P) = (B_n^+(P))_{n\in\mathbb{N}}$ of balls with radius $1/n$ and center in $(-1/n, 0)$ and $(1/n, 0)$, respectively. Then

$$k([B(P)]) = k([B^-(P)]) = k([B^+(P)]) = \{P\}.$$

On the other hand, $[B(P)]$, $[B^-(P)]$, and $[B^+(P)]$ are three different geometrical elements. More precisely, $[B^-(P)] <_c [B(P)]$, $[B^+(P)] <_c [B(P)]$, and $[B^-(P)]$ is not comparable with $[B^+(P)]$. This emphasizes also that the geometrical element $[(B(P)]$ is not minimal and therefore that $[(B(P)]$ is not a point with respect to Whitehead's definition. Obviously, even if it is intriguing to imagine a universe in which a Euclidean point $P = (0, 0)$ is split in three different "geometrical elements" $P_- = [B^-(P)]$, $P = [B(P)]$, $P_+ = [B^+(P)]$, this is surely far from Whitehead's aim and from the intuition. More generally, in spite of the fact that the main aim of Whitehead is to arrive at a good definition of point, the following theorem shows that Whitehead's project, as exposed in [12] and [13], fails since no point exists in the canonical models (we consider as the natural models).

**Theorem 5.8** *In any canonical inclusion space every geometrical element contains two noncomparable geometric elements. As a consequence no point exists.*

**Proof** Consider a geometrical element $[A]$ and, in accordance with Proposition 5.6, assume that $A$ is any sequential abstractive class $(A_n)_{n\in\mathbb{N}}$. Given $m \in \mathbb{N}$, since $A_m \neq A_{m+1}$ it is not possible that $\text{int}(A_m) \subseteq A_{m+1}$ since in such a case $A_m = \text{cl}(\text{int}(A_m)) \subseteq \text{cl}(A_{m+1}) = A_{m+1}$. Then $\text{int}(A_m) - A_{m+1}$ is a nonempty open set and we can consider two disjoint closed balls $D_m$ and $B_m$ contained in it. Set

$$\underline{D}_m = \text{creg}(\text{int}(A_m) - \cup_{n\geq m} D_n);$$
$$\underline{B}_m = \text{creg}(\text{int}(A_m) - \cup_{n\geq m} B_n).$$

Then, since $\text{int}(B_m)$ is contained in $\text{int}(A_m) - \cup_{n\geq m} D_n$, the interior of $\text{int}(A_m) - (\cup_{n\geq m} D_n)$ is nonempty and therefore $\underline{D}_m \neq \varnothing$. Obviously, $(\underline{D}_m)_{m\in\mathbb{N}}$ is order-reversing and, since $A_m \supseteq \underline{D}_m$, there is no region contained in all the set $\underline{D}_m$. This proves $(\underline{D}_m)_{m\in\mathbb{N}}$ is an abstractive class. In a similar way, we prove that $(\underline{B}_m)_{m\in\mathbb{N}}$ is an abstractive class. It is also evident that $(A_m)_{m\in\mathbb{N}}$ covers both $(\underline{D}_m)_{m\in\mathbb{N}}$ and $(\underline{B}_m)_{m\in\mathbb{N}}$, that $(\underline{D}_m)_{m\in\mathbb{N}}$ is not dominated by $(\underline{B}_m)_{m\in\mathbb{N}}$, and $(\underline{B}_m)_{m\in\mathbb{N}}$ is not dominated by $(\underline{D}_m)_{m\in\mathbb{N}}$. □

As we will see, these difficulties do not occur in the case of the contact spaces and this is a further reason in favor of such an approach.

## 6 Abstractive Classes and Geometrical Elements in the Contact Spaces

The notion of point in a contact space requires the one of nontangential inclusion. Observe that we prefer the expression "to have a tangential contact" instead of Whitehead's expression "externally connected."

**Definition 6.1** Given a contact space $(S, C)$, we say that two regions have a *tangential contact* when

(i) they are in contact,
(ii) they do not overlap.

We say that $x$ is *nontangentially included* in $y$ and we write $x \prec y$ provided that

(j) $x$ is included in $y$,
(jj) there is no region having a tangential contact with both $x$ and $y$.

The following is a simple characterization of the nontangential inclusion.

**Proposition 6.2** *The nontangential inclusion is the relation defined by the formula*

$$\forall z(zCx \Rightarrow zOy). \tag{6}$$

**Proof** We have to prove that the following claims are equivalent:

(a) $x \leq y$ and if $z$ has a tangential contact with $x$, then $z$ has not a tangential contact with $y$,
(b) if $zCx$, then $z$ overlaps $y$.

In fact, assume (a) and that $zCx$. Then, since $x \leq y$, in the case $z$ overlaps $x$ it is trivial that $z$ overlaps $y$. Otherwise, $z$ has a tangential contact with $x$ and therefore, by (a), $z$ overlaps $y$. This proves (b).

Assume (b), then trivially $x \leq y$. Let $z$ be a region with a tangential contact with $x$. Then, by (b), $z$ overlaps $y$ and therefore $z$ has not a tangential contact with $x$. $\square$

It is possible to prove that in a canonical space

$$x \prec y \Leftrightarrow x \subseteq \mathrm{int}(y).$$

**Definition 6.3** An *abstractive class* in a contact space is a set $A$ of regions such that

(j) $A$ is totally ordered by the nontangential inclusion,
(jj) there is no region which is contained in all the regions in $A$.

Observe that the sequences $B^-(P)$ and $B^+(P)$ defined in Remark 5.7 are not abstractive classes since they are not ordered with respect to the nontangential inclusion. The *geometrical elements* and the *points* are defined as in Definition 5.4.

**Proposition 6.4** *Define the maps h and k as in Proposition 5.5. Then, in the structures* $(\mathcal{R}_1, C)$ *and* $(\mathcal{R}_2, C)$*,*

$$A \leq_c B \Leftrightarrow h(A) \subseteq h(B).$$

*Consequently,*

$$A \equiv_c B \Leftrightarrow h(A) = h(B),$$

*and, therefore, k is an injective map.*

**Proof** Assume that $h(A) \subseteq h(B)$, let $X$ be a region in $B$, and $X' \in B$ such that $X' \prec X$. Then $\mathrm{int}(X) \supseteq X' \supseteq h(A)$ and, therefore,

$$-\mathrm{int}(X) \cap (\cap_{Y \in A} Y) = \cap_{Y \in A}(-\mathrm{int}(X) \cap Y) = \varnothing.$$

Since $(-\mathrm{int}(X) \cap Y)_{Y \in A}$ is an order-reversing family of compact sets, this entails that $Y_0 \in A$ exists such that $-\mathrm{int}(X) \cap Y_0 = \varnothing$. So $X \supseteq \mathrm{int}(X) \supseteq Y_0$. This proves that $A \leq_c B$. $\square$

We denote by $\mathrm{Point}(S, C)$ the set of points of $(S, C)$. Differently from the case of the inclusion spaces, we are able to prove that in the canonical spaces $(\mathcal{R}_1, C)$ and $(\mathcal{R}_2, C)$, Whitehead's definition of point works well.

**Theorem 6.5** *Consider the canonical spaces* $(\mathcal{R}_1, C)$ *and* $(\mathcal{R}_2, C)$ *in a Euclidean space* $\mathbb{R}^n$*. Then the points in* $(\mathcal{R}_1, C)$ *and* $(\mathcal{R}_2, C)$ *defined by the abstractive classes "coincide" with the usual points in* $\mathbb{R}^n$*(i.e., with the elements of* $\mathbb{R}^n$*). More precisely, the map associating every point P in* $\mathbb{R}^n$ *with the geometrical element* $[(B_n(P))_{n\in\mathbb{N}}]$ *is a one-to-one map from* $\mathbb{R}^n$ *and the set of points in* $(\mathcal{R}_i, C)$*,* $i = 1, 2$*.*

**Proof** Consider the canonical space defined by $\mathcal{R}_1$ and consider the map $f : \mathbb{R}^n \to \mathrm{Point}(\mathcal{R}_1, C)$ defined by setting, for every $P \in \mathbb{R}^n$, $f(P) = [(B_n(P))_{n\in\mathbb{N}}]$. To prove that $f(P)$ is a point, let $B$ be an abstractive class such that $B \leq_c (B_n(P))_{n\in\mathbb{N}}$. Then $h(B) \subseteq h((B_n(P))_{n\in\mathbb{N}}) = \{P\}$ and therefore, since $h(B) \neq \varnothing$, $h(B) = h((B_n(P))_{n\in\mathbb{N}})$. According to Proposition 6.4, this entails that $B \equiv_c (B_n(P))_{n\in\mathbb{N}}$.

It is evident that the map $f$ is injective. To prove that $f$ is surjective, let $[A] \in \mathrm{Point}(\mathcal{R}_1, C)$ and let $P$ be a point in $h(A)$. Then in accordance with Proposition 6.4, $(B_n(P))_{n\in\mathbb{N}}$ is dominated by $A$. So $f(P) = [(B_n(P))_{n\in\mathbb{N}}] = [A]$. In the case of the canonical space associated with $\mathcal{R}_2$, we proceed in the same way. □

Observe that we cannot extend these propositions to the canonical spaces $(\mathcal{R}_3, C)$ and $(\mathcal{R}_4, C)$. This is because in these cases it is possible that the intersection of all the regions in an abstractive class is empty. For example, consider the abstractive classes $A = (A_n)_{n\in\mathbb{N}}$ and $B = (B_n)_{n\in\mathbb{N}}$ defined by

$$\begin{aligned} A_n &= \{(x, y) | x \geq n, -1/n \leq y \leq 1/n\}, \\ B_n &= \{(x, y) | x \leq -n, -1/n \leq y \leq 1/n\}. \end{aligned} \tag{7}$$

Then both $h(A) = h(B) = \varnothing$ even if $A$ is not equivalent to $B$. On the other hand, our intuition says that the corresponding geometrical elements are two points which are "at infinity," in a sense. We can try to find more information on the points in these spaces by considering some compactification of the space $\mathbb{R}^n$. As an example, consider the open ball OB in the plane $\mathbb{R}^2$ defined by the inequality $x^2 + y^2 < 1$ and the homeomorphism $e : \mathbb{R}^2 \to \mathrm{OB}$ defined by the equations

$$\underline{x} = \frac{x}{\sqrt{x^2 + y^2 + 1}}; \qquad \underline{y} = \frac{y}{\sqrt{x^2 + y^2 + 1}} .$$

If we denote by CB the closure of OB, then CB is a compactification of $\mathbb{R}^2$. Observe that, given a closed regular subset in $\mathbb{R}^2$, its image under the embedding $e$ is a closed regular subset in OB but it is not, in general, a closed regular subset in CB. Denote by $g$ the map defined by setting $g(X) = \mathrm{cl}(e(X))$ where $X \in \mathcal{R}_4$ and consider an abstractive class $(X_n)_{n\in\mathbb{N}}$. Then, since $g$ is an order-preserving operator, $(g(X_n))_{n\in\mathbb{N}}$ is an order-reversing sequence of compact subsets of CB, and therefore we can consider the nonempty compact set $\cap g(X_n)$. Also, if $(Y_n)_{n\in\mathbb{N}}$ is an abstractive class covering $(X_n)_{n\in\mathbb{N}}$, then $(g(Y_n))_{n\in\mathbb{N}}$ is a sequence of subsets covering $(g(X_n))_{n\in\mathbb{N}}$ and therefore $\cap_{n\in\mathbb{N}} g(Y_n) \supseteq \cap_{n\in\mathbb{N}} g(X_n)$. Then two equivalent abstractive classes are associated with the same compact subset of CB. This means that it is possible to associate every geometrical element $[(X_n)_{n\in\mathbb{N}}]$ in the canonical space $(\mathcal{R}_4, C)$ with a nonempty compact subset

$$s([(X_n)_{n\in\mathbb{N}}]) = \cap_{n\in\mathbb{N}} g(X_n)$$

of CB. For example, if $X_n = \{(x, y) | (x - \underline{x})^2 + (y - \underline{y})^2 \leq 1/n\}$, then $s([(X_n)_{n\in\mathbb{N}}]) = \{e(\underline{x}, \underline{y})\}$. If $X_n = \{(x, y) | -1/n \leq y \leq 1/n\}$, then $s([(X_n)_{n\in\mathbb{N}}])$ is the diameter $\{(x, y) | -1 \leq x \leq 1, y = 0\}$ of CB. If we consider the abstractive classes $A$ and $B$ defined in (7), then $s([A]) = \{(1, 0)\}$ and $s([B]) = \{(-1, 0)\}$. Unfortunately, the map $s$ is not injective. In fact, for example, if we consider the classes

$$C_n = \{(x, y) | x \geq n, 0 \leq y \leq 1/n\}, \;\; D_n = \{(x, y) | x \geq n, -1/n \leq y \leq 0\}, \tag{8}$$

then these classes are not equivalent while

$$s([(C_n)_{n\in\mathbb{N}}]) = s([(D_n)_{n\in\mathbb{N}}]) = \{(1, 0)\}.$$

An open question is to find a geometrical interpretation of Whitehead's points in the structures $(\mathcal{R}_3, C)$ and $(\mathcal{R}_4, C)$.

## 7 Multivalued Logic for an Inclusion-Based Approach

As we have seen, there are some troubles in the inclusion-based approach to point-free geometry (see also [5]). Indeed, in natural models the topological notion of contact cannot be defined and there are difficulties in defining the notion of point. In the following we consider the inclusion-based approach moving to the framework of multivalued logic in order to go over these limits. We refer to first-order multivalued logics based on Archimedean triangular norms (see, for example, [10]). A *continuous triangular norm*, in brief a *t-norm*, is a continuous commutative and associative operation $\otimes$ in $[0, 1]$ which is isotone in both arguments and such that $x \otimes 1 = x$ for every $x$ in $[0, 1]$. Every continuous $t$-norm is associated with the implication operation defined by setting $x \to y = \mathrm{Sup}\{z \in [0, 1] | x \otimes z \leq y\}$. We say that a continuous norm $\otimes$ is *Archimedean* if, for any $x \neq 1$, $\lim_{n\to\infty} x^n = 0$ where, as usual, $x^n$ is defined by the equations $x^0 = 1$ and $x^{n+1} = x \otimes x^n$. These operations admit a very interesting characterization. We consider the extended interval $[0, \infty]$ and we set $x + \infty = \infty + x = \infty$ and $x \leq \infty$ for every $x \in [0, \infty]$.

**Definition 7.1** A map $f : [0, 1] \to [0, \infty]$ is an *additive generator* provided that $f$ is a continuous strictly decreasing function such that $f(1) = 0$. The *pseudoinverse* $f^{[-1]} : [0, \infty] \to [0, 1]$ of $f$ is defined by setting, for $y \in [0, \infty]$, $f^{[-1]}(y) = f^{-1}(y)$ if $y \in f([0, 1])$ and $f^{[-1]}(y) = 0$ otherwise.

The function $f^{[-1]}$ is continuous and order-reversing; moreover, for every $x \in S$,

$$f^{[-1]}(0) = 1;$$
$$f^{[-1]}(\infty) = 0;$$
$$f^{[-1]}(f(x)) = x;$$
$$f(f^{[-1]}(x)) = x \wedge f(0).$$

**Theorem 7.2** *An operation $\otimes : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous Archimedean t-norm if and only if there exists an additive generator $f : [0, 1] \to [0, \infty]$ such that*

$$x \otimes y = f^{[-1]}(f(x) + f(y)), \tag{9}$$

*for every $x, y$ in* $[0, 1]$.

In the case of the $t$-norm defined by (9), it is

$$x \to y = f^{[-1]}((f(y) - f(x)) \vee 0). \tag{10}$$

Given a continuous $t$-norm $\otimes$, we will consider a multivalued logic with logical connectives $\wedge, \to, \neg$, two logical constants $\underline{0}$ and $\underline{1}$ and with a modal operator Ct. The intended meaning of a formula as $\mathrm{Ct}(\alpha)$ is '$\alpha$ is completely true'. In such a logic, the set of truth values is $[0, 1]$, and

1. $\underline{0}$ and $\underline{1}$ are interpreted by 0 and 1, respectively,
2. the conjunction $\wedge$ is interpreted by $\otimes$,
3. the implication $\to$ is interpreted by the associated implication $\to$,
4. the negation $\neg$ is interpreted by the function $1 - x$,
5. Ct is interpreted by the map $\mathrm{ct} : [0, 1] \to [0, 1]$ such that $\mathrm{ct}(x) = 1$ if $x = 1$ and $\mathrm{ct}(x) = 0$ otherwise,
6. the universal and existential quantifiers are interpreted by the infimum and supremum operators, respectively.

Given a first-order language, an interpretation $I$ is defined by a domain $D$ and by associating every constant with an element in $D$, every $n$-ary operation name with an $n$-ary operation in $D$, and every $n$-ary relation name with an *$n$-ary fuzzy relation*, that is, a map $r : D^n \to [0, 1]$. As in the classical case, given an interpretation $I$, a formula $\alpha$ whose free variables are among $x_1, \ldots, x_n$ and $d_1, \ldots, d_n$ in $D$, we can define the valuation $\mathrm{Val}(\alpha, d_1, \ldots, d_n) \in [0, 1]$ of $\alpha$ in $d_1, \ldots, d_n$ in a truth functional way. We say that $d_1, \ldots, d_n$ *satisfy* $\alpha$ if $\mathrm{Val}(\alpha, d_1, \ldots, d_n) = 1$. Given a theory $T$, we say that $I$ is a *fuzzy model* of $T$ if $\mathrm{Val}(\alpha, d_1, \ldots, d_n) = 1$ for every $\alpha \in T$ and $d_1, \ldots, d_n$ in $D$. We call *crisp* a fuzzy relation assuming only the values 0 and 1 and we identify a classical relation $R \subseteq D^n$ with the crisp relation $c_R : D^n \to [0, 1]$ defined by setting $c_R(d_1, \ldots, d_n) = 1$ if $(d_1, \ldots, d_n) \in R$ and $c_R(d_1, \ldots, d_n) = 0$ otherwise. In other words, we can identify $R$ with its characteristic function $c_R$.

**Definition 7.3** Let $\alpha$ be a formula whose free variables are among $x_1, \ldots, x_n$. Then the *extension* of $\alpha$ in $I$ is the fuzzy relation $r_\alpha : D^n \to [0, 1]$ defined by setting $r_\alpha(d_1, \ldots, d_n) = \mathrm{Val}(\alpha, d_1, \ldots, d_n)$ for every $d_1, \ldots, d_n$ in $D$. In such a case we say that $r_\alpha$ *is defined by* $\alpha$. We call *crisp extension* of $\alpha$ the extension $r_{\mathrm{Cr}(\alpha)}$ of $\mathrm{Cr}(\alpha)$ and in such a case we say that $r_{\mathrm{Cr}(\alpha)}$ is the *crisp relation defined by* $\alpha$.

Then the crisp relation defined by $\alpha$ is the (characteristic function of the) relation

$$\{(d_1, \ldots, d_n) \in D^n | \alpha \text{ is satisfied by } d_1, \ldots, d_n\}.$$

In particular, we will consider a first-order language for the inclusion space theory. Such a language contains the predicate symbol Incl instead of $\leq$ and the prefix form is used to define the atomic formulas. Indeed, we will write $x \leq y$ to denote the formula $\mathrm{Ct}(\mathrm{Incl}(x, y))$. An interpretation of such a language is defined by a pair $(S, \mathrm{incl})$ where $S$ is a nonempty set and $\mathrm{incl} : S \times S \to [0, 1]$ a fuzzy binary relation. Then, the interpretation of $\leq$ (we call the *crisp inclusion* associated with incl) is the (characteristic function of the) relation defined by setting

$$x \leq y \Leftrightarrow \mathrm{incl}(x, y) = 1. \tag{11}$$

Let $E(x, y)$ denote the formula $\mathrm{Incl}(x, y) \wedge \mathrm{Incl}(y, x)$; then the interpretation of $E(x, y)$ (we call *the graded identity associated with* incl) is the fuzzy relation $\mathrm{eq} : S \times S \to [0, 1]$ defined by setting

$$\mathrm{eq}(x, y) = \mathrm{incl}(x, y) \otimes \mathrm{incl}(y, x). \tag{12}$$

In particular, we will consider the models of the following three formulas corresponding to the first three axioms in Definition 2.1.

**Definition 7.4** We call $\otimes$-*graded preordered set* a fuzzy model $(S, \mathrm{incl})$ of the following theory:

***A1*** $\forall x(\mathrm{Incl}(x, x))$,
***A2*** $\forall x \forall y \forall z((\mathrm{Incl}(x, z) \wedge \mathrm{Incl}(z, y)) \to \mathrm{Incl}(x, y))$.

Then a fuzzy relation incl is a $\otimes$-graded preorder if and only if

***a1*** $\mathrm{incl}(x, x) = 1$, (reflexivity)
***a2*** $\mathrm{incl}(x, y) \otimes \mathrm{incl}(y, z) \leq \mathrm{incl}(x, z)$, (transitivity)

for every $x, y, z \in S$.

In order to simulate Whitehead's definition of point, we will define the notion of *pointlikeness*, a property inspired to Euclid's definition of point as minimal element, that is, an element $x$ such that, for every $x'$, $x' \leq x$ entails $x' = x$.

**Definition 7.5** We call the *pointlikeness* property the property expressed by the formula which we denote by $\mathrm{Pl}(x)$,

$$\forall x'(x' \leq x \to E(x, x')).$$

The interpretation of Pl is the fuzzy subset of points pl defined by

$$\mathrm{pl}(x) = \mathrm{Inf}\{\mathrm{incl}(x, x') | x' \leq x\}. \tag{13}$$

Equivalently, we can obtain $\mathrm{pl}(x)$ by the formula

$$\mathrm{pl}(x) = \mathrm{Inf}\{\mathrm{incl}(x', x'') | x' \leq x, x'' \leq x\}. \tag{14}$$

The formula $\mathrm{Pl}(x)$ enables us to express the next two axioms. The former claims that if two regions $x$ and $y$ are points (approximately), then the graded inclusion is symmetric (approximately).

***A3*** $\mathrm{Pl}(x) \wedge \mathrm{Pl}(y) \to (\mathrm{Incl}(x, y) \to \mathrm{Incl}(y, x))$.

Such an axiom is satisfied if and only if, for every $x$ and $y$,

***a3*** $\mathrm{pl}(x) \otimes \mathrm{pl}(y) \leq (\mathrm{incl}(x, y) \to \mathrm{incl}(y, x))$.

The latter claims that every region $x$ contains a point:

***A4*** $\forall x \exists x'((x' \leq x) \wedge \mathrm{Pl}(x')$.

Such an axiom is satisfied if and only if for every $x$,

***a4*** $\mathrm{Sup}_{x' \leq x} \mathrm{pl}(x) = 1$;

that is, if and only if for every $x$,

$$\forall \epsilon > 0 \text{ there is } x' \leq x \text{ such that } \mathrm{pl}(x') \geq 1 - \epsilon. \tag{15}$$

**Definition 7.6** We call $\otimes$-*graded inclusion space of regions*, in brief, *graded inclusion space*, every model of ***A1–A4***.

## 8 Graded Inclusion Spaces and Hemimetrics

To obtain suitable examples of graded inclusion spaces it is useful to introduce the notion of hemimetric space.

**Definition 8.1** A *hemimetric space* is a structure $(S, d)$ such that $S$ is a nonempty set and $d : S \times S \to [0, \infty]$ is a mapping such that, for all $x, y, z \in S$,

***d1*** $d(x, x) = 0$;
***d2*** $d(x, y) \leq d(x, z) + d(z, y)$.

Then, a metric space is a hemimetric space which is symmetric, that is, such that $d(x, y) = d(y, x)$, for every $x, y \in S$, and such that $d(x, y) = 0$ only if $x = y$. An example, we call the *difference hemimetric*, is obtained by assuming that $S$ is a nonempty set, $f : S \to [0, \infty)$ is a map such that $\mathrm{Inf}(S) = 0$, and $d(x, y) = (f(x) - f(y)) \vee 0$. The hemimetric spaces are related with the preorders in the following way.

**Proposition 8.2** *Let $(S, d)$ be a hemimetric space. Then the relation $\leq$ defined by setting*

$$x \leq y \Leftrightarrow d(x, y) = 0$$

*for any $x, y \in S$ is a preorder such that $d$ is order-preserving with respect to the first variable and order-reversing with respect to the second variable. Conversely, let $\leq$ be any preorder in a set $S$ and define the mapping $d : S \times S \to [0, \infty]$ by setting*

*$d(x, y) = 0$ if $x \le y$ and $d(x, y) = 1$ otherwise. Then $(S, d)$ is a hemimetric space whose associated preorder is $\le$.*

For example, the preorder defined by a difference hemimetric is such that

$$x \le y \Leftrightarrow f(x) \le f(y).$$

This means that $\le$ is linear and, if there is $m \in S$ such that $f(m) = 0$, then $m$ is a minimum in $(S, \le)$.

**Definition 8.3** Given a hemimetric $d$ and $x \in S$, we call the *diameter* of $x$ the number

$$\delta(x) = \mathrm{Sup}\{d(x_1, x_2) | x_1 \le x, x_2 \le x\}. \tag{16}$$

Equivalently, since $d$ is order-preserving with respect to the first variable,

$$\delta(x) = \mathrm{Sup}\{d(x, y) | y \le x\}. \tag{17}$$

This means that all the atoms have diameter zero. Also, if a minimum $0 \in S$ exists, then

$$\delta(x) = d(x, 0). \tag{18}$$

Indeed, for every $y \le x$, $d(x, y) \le d(x, 0) + d(0, y) = d(x, 0)$.

In the case $(S, d)$ is a metric space, then the associated preorder is the identity relation and therefore all the regions are atoms and all the diameters are equal to zero. In the case of the difference hemimetric we have that $\delta(x) = f(x)$. When the hemimetric space is defined by a preorder with no minimum, we have that $\delta(x) = 0$ if $x$ is an atom and $\delta(x) = 1$ otherwise. The following proposition shows that the notion of hemimetric is "dual" of the one of graded preorder.

**Proposition 8.4** *Let $f : [0, 1] \to [0, +\infty]$ be an additive generator of a t-norm $\otimes$. Then for every hemimetric $d : S \times S \to [0, \infty]$ the fuzzy relation* incl *defined by setting*

$$\mathrm{incl}(x, y) = f^{[-1]}(d(x, y)) \tag{19}$$

*is a $\otimes$-graded preorder. Moreover,*

$$\mathrm{pl}(x) = f^{[-1]}(\delta(x)). \tag{20}$$

*Conversely, let* incl $: S \times S \to [0, 1]$ *be a $\otimes$-graded preorder and let $d$ be defined by setting*

$$d(x, y) = f(\mathrm{incl}(x, y)). \tag{21}$$

*Then $d$ is a hemimetric and*

$$\delta(x) = f(\mathrm{pl}(x)). \tag{22}$$

**Proof** Trivially, incl satisfies *a1*. To prove *a2* it is enough to take $x, y, z$ such that $d(x, y)$ and $d(y, z) \in f([0, 1])$. In such a case,

$$\begin{aligned}\mathrm{incl}(x, y) \otimes \mathrm{incl}(y, z) &= f^{-1}(d(x, y)) \otimes f^{-1}(d(y, z)) \\ &= f^{[-1]}(f(f^{-1}(d(x, y))) + f(f^{-1}(d(y, z)))) \\ &= f^{[-1]}(d(x, y) + d(y, z)) \le f^{[-1]}(d(x, z)) = \mathrm{incl}(x, z).\end{aligned}$$

Equation (20) is immediate since $f^{[-1]}$ is continuous and order-reversing.

Conversely, define $d$ by (21). Then it is immediate that $d(x, x) = 0$. Moreover, since

$$\mathrm{incl}(x, y) \otimes \mathrm{incl}(y, z) \le \mathrm{incl}(x, z),$$

we have that

$$f(\mathrm{incl}(x, y) \otimes \mathrm{incl}(y, z)) \geq f(\mathrm{incl}(x, z)),$$

and therefore, in accordance with the definition of $\otimes$,

$$f[f^{[-1]}(f(\mathrm{incl}(x, y)) + f(\mathrm{incl}(y, z)))] \geq f(\mathrm{incl}(x, z)).$$

Now, if $f(\mathrm{incl}(x, y)) + f(\mathrm{incl}(y, z))) \in f([0, 1]) = [0, f(0)]$, we obtain that

$$f(\mathrm{incl}(x, y)) + f(\mathrm{incl}(y, z)) \geq f(\mathrm{incl}(x, z)).$$

Otherwise, $f(\mathrm{incl}(x, y)) + f(\mathrm{incl}(y, z)) \geq f(0) \geq f(\mathrm{incl}(x, z))$. In both the cases this proves the triangular inequality.

Finally, (22) is immediate since $f$ is continuous and order-reversing. □

The following definition individuates the hemimetrics corresponding to the $\otimes$-graded inclusion spaces (see also [3]).

**Definition 8.5** A *hemimetric space of regions* is a hemimetric space $(S, d)$ such that for every $x$ and $y$,

***d3*** $|d(x, y) - d(y, x)| \leq \delta(x) + \delta(y)$,
***d4*** $\forall \epsilon > 0 \exists x' \leq x, \delta(x') \leq \epsilon$.

A difference hemimetric $d(x, y) = (f(x) - f(y)) \vee 0$ is an example of hemimetric space of regions. Indeed *d4* is trivial and

$$|d(x, y) - d(y, x)| = |f(x) - f(y)| \leq |f(x)| + |f(y)| = \delta(x) + \delta(y).$$

Let $(S, \leq)$ be a preordered set with no minimum and in which every element contains an atom. Then the associated hemimetric is a hemimetric space of regions. Indeed *d4* are immediate. To prove *d3* observe that in the case $|d(x, y) - d(y, x)| \neq 0$ the elements $x$ and $y$ are comparable and $x \neq y$. Assuming, for example, that $x < y$,

$$|d(x, y) - d(y, x)| = d(y, x) = 1 = \delta(y) \leq \delta(x) + \delta(y).$$

**Theorem 8.6** *Let $f : [0, 1] \to [0, +\infty]$ be an additive generator of a t-norm $\otimes$. Then, for every hemimetric space of regions $(S, d)$, the fuzzy relation* incl *defined by setting*

$$\mathrm{incl}(x, y) = f^{[-1]}(d(x, y)) \tag{23}$$

*defines a $\otimes$-graded inclusion space of regions. Conversely, let $(S, \mathrm{incl})$ be a $\otimes$-graded inclusion space of regions and let $d : S \times S \to [0, +\infty]$ be defined by setting*

$$d(x, y) = f(\mathrm{incl}(x, y)). \tag{24}$$

*Then $(S, d)$ is a hemimetric space of regions.*

**Proof** Let incl be defined by (23). Then it is immediate that $(S, \mathrm{incl})$ satisfies *A4*. To prove *A3*, at first we observe that, for $a, b, c$ positive real numbers,

$$|a \wedge c - b \wedge c| \leq |a - b| \wedge c;\ (a + b) \wedge c \leq a \wedge c + b \wedge c.$$

Also, it is not restrictive to assume that $\mathrm{incl}(x, y) > \mathrm{incl}(y, x)$ and therefore that $d(x, y) \leq d(y, x)$. Then

$$\begin{aligned}(\mathrm{incl}(x, y) \to \mathrm{incl}(y, x)) &= f^{[-1]}(f(\mathrm{incl}(y, x)) - f(\mathrm{incl}(x, y))) \\ &= f^{[-1]}(f(f^{[-1]}(d(y, x))) - f(f^{[-1]}(d(x, y)))) \\ &= f^{[-1]}(d(y, x) \wedge f(0) - d(x, y) \wedge f(0)).\end{aligned}$$

Moreover, because of the definition of $\otimes$,

$$\begin{aligned}\mathrm{pl}(x) \otimes \mathrm{pl}(y) &= f^{[-1]}(\delta(x)) \otimes f^{[-1]}(\delta(y)) \\ &= f^{[-1]}(f(f^{[-1]}(\delta(x))) + f(f^{[-1]}(\delta(y)))) \\ &= f^{[-1]}(\delta(x) \wedge f(0) + \delta(y) \wedge f(0)).\end{aligned}$$

On the other hand, by hypothesis,

$$d(y, x) - d(x, y) \leq \delta(x) + \delta(y)$$

and, therefore,

$$\begin{aligned}d(y, x) \wedge f(0) - d(x, y) \wedge f(0) &\leq (d(y, x) - d(x, y)) \wedge f(0) \\ &\leq (\delta(x) + \delta(y)) \wedge f(0) \leq \delta(x) \wedge f(0) + \delta(y) \wedge f(0).\end{aligned}$$

Since $f^{[-1]}$ is order-reversing,

$$\begin{aligned}(\mathrm{incl}(x, y) \to \mathrm{incl}(y, x)) &= f^{[-1]}(d(y, x) \wedge f(0) - d(x, y) \wedge f(0)) \\ &\geq f^{[-1]}(\delta(x) \wedge f(0) + \delta(y) \wedge f(0)) = \mathrm{pl}(x) \otimes \mathrm{pl}(y).\end{aligned}$$

To prove *A4*, by (15) we have to prove that $\forall \epsilon > 0$ there is $x' \leq x$ such that $f^{[-1]}(\delta(x)) \geq 1 - \epsilon$, that is, such that $\delta(x) \leq f(1 - \epsilon)$. This is an immediate consequence of *d4*.

Conversely, let $d$ be defined by (24). Then *d4* is immediate. To prove *d3* observe that by *a3*

$$f^{[-1]}(f(\mathrm{pl}(x)) + f(\mathrm{pl}(y))) \leq f^{[-1]}((f(\mathrm{incl}(y, x)) - f(\mathrm{incl}(x, y))) \vee 0).$$

Therefore,

$$f(\mathrm{pl}(x)) + f(\mathrm{pl}(y)) \geq (f(\mathrm{incl}(y, x)) - f(\mathrm{incl}(x, y))).$$

This entails *d3*. □

## 9 Defining the Points in a Graded Inclusion Space of Regions

We obtain the notion of point in a graded inclusion space by extending the pointlikeness property to the abstraction processes.

**Definition 9.1** Given a graded inclusion space, we call *abstraction process* any sequence $\langle p_n \rangle_{n \in \mathbb{N}}$ of regions which is order-reversing with respect to the order associated with the graded inclusion. We extend the pointlikeness property to the abstraction processes by setting

$$\mathrm{pl}(\langle p_n \rangle_{n \in \mathbb{N}}) = \mathrm{Sup}_n \mathrm{pl}(p_n) \tag{25}$$

and we say that $\langle p_n \rangle_{n \in \mathbb{N}}$ *represents a point* if $\mathrm{pl}(\langle p_n \rangle_{n \in \mathbb{N}}) = 1$. We denote by Pr the class of abstraction processes representing a point.

Observe that *A4* enables us to prove that every region "contains" an abstraction process representing a point and therefore that $\mathrm{Pr} \neq \varnothing$. The following theorem shows that the class of abstraction processes representing points is a pseudometric space.

**Theorem 9.2** *Let* $(S, \mathrm{incl})$ *be a* $\otimes$*-graded inclusion space and* $d'$ *the associated hemimetric. Then the map* $d : \mathrm{Pr} \times \mathrm{Pr} \to [0, \infty]$ *obtained by setting*

$$d(\langle p_n\rangle_{n\in\mathbb{N}}, \langle q_n\rangle_{n\in\mathbb{N}}) = \lim_{n\to\infty} d'(p_n, q_n), \tag{26}$$

*defines a pseudometric space* $(\mathrm{Pr}, d)$.

**Proof** To prove the convergence of the sequence $\langle d'(p_n, q_n)\rangle_{n\in\mathbb{N}}$, let $n$ and $k$ be natural numbers and assume that $n \geq k$. Then, since $d'(q_k, q_n) \leq \delta(q_k)$ and $d'(p_n, p_k) = 0$,

$$d'(p_n, q_n) \leq d'(p_n, p_k) + d'(p_k, q_k) + d'(q_k, q_n) \leq \delta(q_k) + d'(p_k, q_k),$$

and, therefore,

$$d'(p_n, q_n) - d'(p_k, q_k) \leq \delta(q_k).$$

Likewise, since $d'(p_k, p_n) \leq \delta(p_k)$ and $d'(q_n, q_k) = 0$,

$$d'(p_k, q_k) \leq d'(p_k, p_n) + d'(p_n, q_n) + d'(q_n, q_k) \leq d'(p_n, q_n) + \delta(p_k),$$

and, therefore,

$$d'(p_k, q_k) - d'(p_n, q_n) \leq \delta(p_k).$$

This entails

$$|d'(p_n, q_n) - d'(p_k, q_k)| \leq \max\{\delta(q_k), \delta(p_k)\}.$$

Obviously, in the case $n \leq k$,

$$|d'(p_n, q_n) - d'(p_k, q_k)| \leq \max\{\delta(q_n), \delta(p_n)\}.$$

Thus,

$$|d'(p_n, q_n) - d'(p_k, q_k)| \leq \max\{\delta(q_n), \delta(p_n), \delta(q_k), \delta(p_k)\}.$$

The convergence of the diameters entails that $\langle d'(p_n, q_n)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence.

It is evident that $d(\langle p_n\rangle_{n\in\mathbb{N}}, \langle p_n\rangle_{n\in\mathbb{N}}) = 0$ and that $d$ satisfies the triangular inequality. To prove the symmetry, observe that, by *d3*, $|d(p_n, q_n) - d(q_n, p_n)| \leq \delta(p_n) + \delta(q_n)$ and that $\lim_{n\to\infty} \delta(p_n) + \delta(q_n) = 0$. □

Such a proposition enables us to associate any $\otimes$-graded inclusion space with a metric space. Indeed, recall that the *quotient* of a pseudometric space $(X, d)$ is the metric space $(\underline{X}, \underline{d})$ defined by assuming that

1. $\underline{X}$ is the quotient of $X$ modulo the relation $\equiv$ defined by setting $x \equiv x'$ if and only if $d(x, x') = 0$,
2. $\underline{d}([x], [y]) = d(x, y)$ for every $[x], [y] \in X'$.

**Definition 9.3** We call *metric space associated with a graded inclusion space* $(S, \mathrm{incl})$ the quotient $(\underline{\mathrm{Pr}}, \underline{d})$ of the pseudometric space $(\mathrm{Pr}, d)$. We call *point* any element in $\underline{\mathrm{Pr}}$.

Then, the metric space $(\underline{\mathrm{Pr}}, \underline{d})$ associated with a graded inclusion space $(S, \mathrm{incl})$ is obtained

1. by starting from the class Pr of abstraction processes;
2. by setting $\underline{\mathrm{Pr}}$ equal to the quotient of Pr modulo the equivalence relation $\equiv$ defined by

$$\langle p_n\rangle_{n\in\mathbb{N}} \equiv \langle q_n\rangle_{n\in\mathbb{N}} \Leftrightarrow \lim_{n\to\infty} \mathrm{incl}(p_n, q_n) = 1;$$

3. by defining $\underline{d} : \underline{\mathrm{Pr}} \times \underline{\mathrm{Pr}} \to [0, \infty]$ by the equation,
$$\underline{d}(P, Q) = \lim_{n\to\infty} f(\mathrm{incl}(p_n, q_n)) \tag{27}$$
where $P = [\langle p_n \rangle_{n\in\mathbb{N}}]$ and $Q = [\langle q_n \rangle_{n\in\mathbb{N}}]$ are elements in $\underline{\mathrm{Pr}}$.

## 10 In a Canonical Graded Inclusion Space the Connection is Definable

The more famous hemimetric is the excess measure usually considered in literature to define the Hausdorff distance.

**Definition 10.1** Given a metric space $(M, d)$ the *excess measure* is the map $e : P(M) \times P(M) \to [0, \infty]$ defined, for every pair $x$ and $y$ of subsets of $M$, by setting
$$e(x, y) = \mathrm{Sup}_{P\in x}\mathrm{Inf}_{Q\in y} d(P, Q). \tag{28}$$

In [3] the following proposition is proved.

**Proposition 10.2** *The excess measure defines in each class $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ a hemimetric space of regions. Consequently, if $f : [0, 1] \to [0, +\infty]$ is an additive generator of $\otimes$, the function*
$$\mathrm{incl}(x, y) = f^{[-1]}(e(x, y))$$
*is a $\otimes$-graded inclusion space. The induced order is the usual set theoretical inclusion and the pointlikeness property is defined by*
$$\mathrm{pl}(x) = f^{[-1]}(|x|)$$
*where $|x|$ is the usual diameter in a metric space.*

As an example, by setting $f(x) = \mathrm{Log}(x)$, we have that $\otimes$ is the usual product and the equation
$$\mathrm{incl}(x, y) = 10^{-e(x,y)}$$
defines a $\otimes$-graded inclusion space in each class $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$.

**Definition 10.3** Given $i \in \{1, 2, 3, 4\}$, the $\otimes$-graded inclusion space $(\mathcal{R}_i, \mathrm{incl})$ is called *canonical i-space*.

We will show that, differently from Whitehead's inclusion spaces, in a $\otimes$-graded inclusion space we can define the contact relation by a formula expressing, in a sense, the overlapping relation. First, we have to prove the following two lemmas.

**Lemma 10.4** *Consider the $\otimes$-graded inclusion spaces $(\mathcal{R}_i, \mathrm{incl})$ associated with the excess and define $C$ by setting $xCy$ if and only if $x \cap y \neq \emptyset$. Then for every pair of bounded regions $x, y \in \mathcal{R}_i$, the following are equivalent:*

(j) *$xCy$,*
(jj) *for every $0 < \epsilon < 1$, a region $z$ in $\mathcal{R}_i$ exists such that $\mathrm{incl}(z, x) \geq \epsilon$ and $\mathrm{incl}(z, y) \geq \epsilon$.*

**Proof** (j) $\Rightarrow$ (jj) Let $P$ be a point in $x \cap y$ and $\epsilon$ such that $0 < \epsilon < 1$. Then, since $x$ is regular, the open ball $b$ centered in $P$ and with diameter $f(\epsilon)$ has a nonempty intersection with $\mathrm{int}(x)$. Consequently, the set $z = \mathrm{cl}(\mathrm{int}(x) \cap b)$ is a nonempty regular, closed, bounded subset of $\mathbb{R}^n$ and we have $e(z, y) \leq e(\mathrm{cl}(b), y) \leq f(\epsilon)$. So, $\mathrm{incl}(z, x) = 1$ and $\mathrm{incl}(z, y) = f^{[-1]}(e(z, y)) \geq f^{[-1]}(f(\epsilon)) = \epsilon$. Notice that if $x$ is internally connected then $z$ is internally connected.

(jj) $\Rightarrow$ (j) Since both the regions $x$ and $y$ are bounded, to prove that $x \cap y \neq \varnothing$ it is sufficient to prove that for every natural number $k$ there are two points $P \in x$ and $Q \in y$ such that $d(P, Q) < 1/k$. Now, set $\epsilon = f^{[-1]}(1/2k)$ and let $z$ be a region such that

$$\mathrm{incl}(z, x) = f^{[-1]}(e(z, x)) \geq \epsilon = f^{[-1]}(1/2k), \text{ and}$$
$$\mathrm{incl}(z, y) = f^{[-1]}(e(z, y)) \geq \epsilon = f^{[-1]}(1/2k).$$

Now, if $Z$ is a point in $z$, then

$$f^{[-1]}(e(Z, x)) \geq f^{[-1]}(e(z, x)) \geq f^{[-1]}(1/2k), \text{ and}$$
$$f^{[-1]}(e(Z, y)) \geq f^{[-1]}(e(z, y)) \geq f^{[-1]}(1/2k),$$

and, therefore,

$$e(Z, x) \leq 1/2k \text{ and } e(Z, y) \leq 1/2k.$$

Let $P \in x$ and $Q \in y$ such that $e(Z, x) = d(Z, P)$ and $e(Z, y) = d(Z, Q)$. Then

$$d(P, Q) \leq d(P, Z) + d(Z, Q) = e(Z, x) + e(Z, y) \leq 1/k.$$

□

**Lemma 10.5** *Denote by* $\mathrm{Bounded}(x)$ *the formula* $\neg\mathrm{Ct}(\neg\mathrm{Pl}(x))$*. Then in any* $\otimes$*-graded inclusion spaces* $(\mathcal{R}_i, \mathrm{incl})$*,* $\mathrm{Bounded}(x)$ *is satisfied by a region* $r$ *at degree* 1 *if and only if* $|r| < f(0)$.

**Proof** Observe that the formula $\neg\mathrm{Ct}(\neg\mathrm{Pl}(x))$ is interpreted by the fuzzy set $1-\mathrm{ct}(1-\mathrm{pl}(x))$ and that $1-\mathrm{ct}(1-\mathrm{pl}(r)) = 1 \Leftrightarrow \mathrm{ct}(1-\mathrm{pl}(r)) = 0 \Leftrightarrow 1-\mathrm{pl}(r) \neq 1$ $\Leftrightarrow \mathrm{pl}(r) \neq 0 \Leftrightarrow |r| < f(0)$. □

We denote by *bounded* the fuzzy subset interpreting the formula $\mathrm{Bounded}(x)$.

**Theorem 10.6** *Denote by* $O(x, y)$ *the formula* $\exists z(\mathrm{Incl}(z, x)) \wedge \mathrm{Incl}(z, y))$ *and by* $C(x, y)$ *the formula,*

$$\exists x' \exists y' \mathrm{Ct}((\mathrm{Bounded}(x') \wedge \mathrm{Bounded}(y') \wedge (x' \leq x) \wedge (y' \leq y) \wedge O(x', y')).$$

*Then in all the graded inclusion spaces* $(\mathcal{R}_i, \mathrm{incl})$ *the contact relation is definable by* $C(x, y)$*. In* $(\mathcal{R}_1, \mathrm{incl})$ *and* $(\mathcal{R}_2, \mathrm{incl})$ *the contact relation is definable by the formula* $\mathrm{Ct}(O(x, y))$.

**Proof** Assume that the two regions $r$ and $r'$ satisfy $C(x, y)$. Then there are $\underline{r} \leq r$ and $\underline{r}' \leq r'$ such that $\mathrm{bounded}(\underline{r}) = 1, \mathrm{bounded}(\underline{r}') = 1$ and $\mathrm{Sup}\{\mathrm{incl}(z, \underline{r}) \otimes \mathrm{incl}(z, \underline{r}')\} = 1$. In accordance with Lemma 10.4, this is equivalent to say that $\underline{r}$ is connected with $\underline{r}'$ and therefore that $r$ is connected with $r'$.

Conversely, assume that $rCr'$. Then a point $P$ exists in $r \cap r'$. Let $b$ be an open ball centered in $P$ and with diameter less than $f(0)$. Then, since $x$ and $y$ are closed and regular, $b \cap \mathrm{int}(r) \neq \varnothing$ and $b \cap \mathrm{int}(r') \neq \varnothing$. This entails that $\underline{r} = \mathrm{cl}(b \cap \mathrm{int}(r))$ and $\underline{r}' = \mathrm{cl}(b \cap \mathrm{int}(r'))$ are nonempty elements in $\mathcal{R}_i$ whose diameter is less than $f(0)$. Since $P \in \underline{r} \cap \underline{r}'$, by Lemma 10.4, $\mathrm{Sup}\{\mathrm{incl}(z, \underline{r}) \otimes \mathrm{incl}(z, \underline{r}')\} = 1$. Then the formula $C(x, y)$ is satisfied by $r$ and $r'$. The remaining part of the theorem is evident. □

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