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Polyadic Quantification via Denoting Concepts

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Abstract The question of the origin of polyadic expressivity is explored and the results are brought to bear on Bertrand Russell's 1903 theory of denoting concepts, which is the main object of criticism in his 1905 "On Denoting." It is shown that, appearances to the contrary notwithstanding, the background ontology of the earlier theory of denoting enables the full-blown expressive power of first-order polyadic quantification theory without any syntactic accommodation of scopal differences among denoting phrases such as 'all φ', 'every φ', and 'any φ' on the one hand, and 'some φ' and 'a φ' on the other. The case provides an especially vivid illustration of the general point that structural (or ideological) austerity can be paid for in the coin of ontological extravagance.

What accounts for the expressive power of first-order polyadic quantification theory compared to its relatively impoverished monadic kin? The standard answer is that the former permits interaction among distinct quantifiers binding distinct variables within one and the same formula. This answer, while surely correct, does not pinpoint precisely where expressive power lies. I propose to delve deeper into the issue. Doing so will also lead to a surprising discovery about an important but often-neglected episode in the history of logic.

We begin by considering a simple formalization exercise. Take the argument from 'Everyone loves everyone' to 'Everyone is loved by someone or other'. In weighing the polyadic vs. monadic formalization alternatives, two connected features of the argument present themselves. First, the argument, while clearly valid, will not be rendered so by the resources made available by monadic quantification theory, whereby the premise would be formalized along the lines of $\forall x L'x$ and the conclusion along the lines of $\forall x L''x$.[1] This is a familiar expressive shortcoming of the monadic framework relative to the polyadic one. Second, but no less familiar, is the fact that in a monadic setting the argument cannot be distinguished formally from arguments that are obviously distinct from it, such as the valid one from 'Everyone

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loves everyone' to 'Everyone loves someone or other'. This latter claim would be captured by something along the lines of $\forall x L'''x$. Polyadic quantification theory offers an account of both the validity of the argument and its distinctness from the second argument. First, the argument can be formalized as a valid inference from $\forall x \forall y Lxy$ to $\forall y \exists x Lxy$. Second, the inference in question can be distinguished from the equally valid but distinct inference from $\forall x \forall y Lxy$ to $\forall x \exists y Lxy$. Moreover, we can further distinguish these arguments from other valid arguments in the vicinity, such as the one from 'Everyone loves everyone' to 'Someone is loved by everyone', captured by the valid inference from $\forall x \forall y Lxy$ to $\exists y \forall x Lxy$.

What accounts for such expressive variety? The answer will emerge clearly when we compare the above three consequences of 'Everyone loves everyone' and their nonequivalent formalizations.

(a)	'Everyone is loved by someone or other.'	$\forall y \exists x Lxy$
(b)	'Everyone loves someone or other.'	$\forall x \exists y Lxy$
(c)	'Someone is loved by everyone.'	$\exists y \forall x Lxy$

Let us register a few useful observations. First, while the order of the quantifiers in (a) and (b) is the same, the order of the binding occurrences of the variables is different.[2] Second, while the order of the quantifiers in (a) and (c) is different, the order of the binding occurrences of the variables is the same. Finally, the order of the bound occurrences of the variables is fixed throughout. We can stipulate, as is common among some practitioners of formalization into polyadic quantification theory, that the order of bound occurrences of variables within a given occurrence of a predicate letter is to remain fixed as in (a)–(c) above.[3] Then under such a stipulation the logical variety exhibited by (a)–(c) arises from interaction between two distinct 'orders': the order of the quantifiers themselves and the order of the binding occurrences of the variables.

Let us change our stipulation, however. Let us stipulate, as is also common among some formalizers, that the order of binding occurrences of variables in formulas is to remain fixed. Under this stipulation we retain the above logical variety by letting the order of the bound occurrences of the variables vary. So according to this option we replace Lxy representing x loving y with Lyx representing y loving x as needed. Specifically, the above sentences will be relettered as follows.

(a′)	'Everyone is loved by someone or other.'	$\forall x \exists y Lyx$
(b′)	'Everyone loves someone or other.'	$\forall x \exists y Lxy$
(c′)	'Someone is loved by everyone.'	$\exists x \forall y Lyx$

Under the stipulation that the order of the binding occurrences of the variables is fixed, logical variety derives from the interaction between the order of the quantifiers on the one hand and the order of the bound occurrences of the variables on the other. In sum, on either alternative the match between the binding and the bound occurrences of variables determines for each quantifier which position(s) in the sentence it is to govern. All that matters is that the quantifiers have the right 'addresses'. So if we compare, for example, $\forall y \exists x Lxy$ and $\forall x \exists y Lyx$, the two sentences are expressively equivalent to the extent that the first quantifier universally quantifies into the second position in the predicate letter whereas the second quantifier existentially quantifies into the first position.

But there is yet another alternative for observing the stipulation that binding occurrences of variables have a fixed order without loss of logical variety. This third alternative involves enriching the nonlogical vocabulary. Take (c), for example. Instead of relettering $\exists y \forall x Lxy$ to get the (c′) alternative $\exists x \forall y Lyx$ we can observe the stipulation of alphabetical order for the binding occurrences of the variables and observe the further initial stipulation of alphabetical order for the bound occurrences of the variables by resorting to a distinct predicate letter L^{-1} expressing the converse relation of being loved. On this third option our variety is as follows.

(a″)	‘Everyone is loved by someone or other.’	$\forall x \exists y L^{-1} xy$
(b″)	‘Everyone loves someone or other.’	$\forall x \exists y Lxy$
(c″)	‘Someone is loved by everyone.’	$\exists x \forall y L^{-1} xy$

Call the difference between L and L^{-1} a difference in a predicate letter's ‘direction’. What we have done is stipulate both that the order of the binding occurrences of variables be fixed and that the order of the bound occurrences of the variables be fixed. What accounts for logical variety now is not the interaction between the order of the quantifiers and the order of binding occurrences of variables, as in (a)–(c). Nor is it the interaction between the order of the quantifiers and the order of the bound occurrences of the variables as in (a′)–(c′). Rather, what accounts for logical variety now is the interaction between the order of the quantifiers and the directions of the predicate letters. In such cases as the ones above, where there is a perfect match between the number of argument places in a lexically simple matrix and the number of quantifiers that govern it, we might as well dispense with the variables and represent our variety as follows.

(a‴)	‘Everyone is loved by someone or other.’	$L^{-1}\forall\exists$
(b‴)	‘Everyone loves someone or other.’	$L\forall\exists$
(c‴)	‘Someone is loved by everyone.’	$L^{-1}\exists\forall$

Of course, this will not work in general because it will not work, for example, for cases where multiple argument places are governed by a single quantifier. It is only offered here as a heuristic aid for things to come. But, in general, if we enrich the language to include for every (n-place) predicate letter the rest of its $(n! - 1)$ directional variants, we can observe the stipulation that the order of the binding occurrences of variables is alphabetical while observing the further stipulation that the order of initial (among possibly multiple) bound occurrences of distinct variables within each predicate letter is alphabetical as well.[4] Of course, if we were to draw inferences under such a formalization protocol we would need special provisions linking the various predicate letters of our enriched language. Thus, for example, in the absence of such provisions we could not infer $\exists x \forall y L^{-1} xy$ from $\forall x \forall y Lxy$. But this limitation on the formal *system* does not spoil the point that our enriched formal *language* can capture the full expressivity of first-order polyadic quantificational theory with the above stipulations in place.

The point bears a brief illustration. Let ψ be a triadic predicate in the original language and let u, v, w be variables such that the first precedes the second alphabetically while the second precedes the third. We add the $3! - 1$ directional variants of ψ under the following mapping ξ:

$$\xi(\psi uvw) = \psi uvw$$
$$\xi(\psi uwv) = \psi^{-1}uvw$$
$$\xi(\psi vuw) = \psi^{-2}uvw$$
$$\xi(\psi vwu) = \psi^{-3}uvw$$
$$\xi(\psi wuv) = \psi^{-4}uvw$$
$$\xi(\psi wvu) = \psi^{-5}uvw.$$

Consider now the following validity:

$$\forall w(\exists u \psi uww \rightarrow \exists v \psi vww).$$

To obtain the result that the order of binding occurrences of variables is alphabetical we reletter as follows:

$$\forall u(\exists v \psi vuu \rightarrow \exists w \psi wuu).$$

Next, to obtain the desired result of alphabetical order for (initial) bound occurrences of variables as well, we translate into the new vocabulary, scanning bound occurrences from left to right and scanning the list specifying ξ from top to bottom:

$$\forall u(\exists v \psi^{-2}uvu \rightarrow \exists w \psi^{-4}uuw).$$

Finally, to restore the validity we would need to include in the transformation rules of the formal system provisions for 'backward' translation, reversing the last step via ξ^{-1}.

Let us now go back and consider our original language and some arbitrary polyadic sentence S. S has an equivalent in Prenex Normal Form, $Q_1u_1 \dots Q_nu_nM$, where for each i, $1 \leq i \leq n$, Q_i is either $\exists$ or $\forall$ and M is the quantifier-free matrix with the u_i as its free variables and where the order of the binding occurrences of the variables in the prefix is fixed alphabetically. Now, under the intended interpretation of the language the matrix M defines an n-place relation R. We can introduce into the language a new n-place predicate letter whose extension is R. If for every polyadic sentence S in the original language we introduce such an n-place predicate letter together with the rest of its directional variants φ_j^S, $1 \leq j \leq n!$, then for each polyadic sentence of the original language there will be a PNF equivalent with the following four features: (1) its matrix is a single predicate letter, (2) the number of quantifiers in its prefix is identical to the number of argument places in that predicate letter, (3) the order of the binding occurrences of its variables is fixed alphabetically, and (4) the order of the bound occurrences of its variables is fixed alphabetically as well. In effect, this last option makes the order of the quantifiers the sole determinant of polyadic quantificational variety, provided, of course, that the language is sufficiently rich to include for each sentence S of the old language the new predicate letters φ_j^S. Such a PNF equivalent, using one of those newly introduced predicate letters, might just as well be written as $\varphi_j^S Q_1, \dots, Q_n$.[5]

Let us now bring this general lesson to bear on an all-but-forgotten episode in the history of logic. Russell's theory of denoting concepts in [5] is far less familiar than its famous later rival, the theory of Russell [6]. The theory of denoting concepts encapsulates Russell's early attempt to account for generality within his overall metaphysics of discreteness, the view Hylton called 'Platonic Atomism'.[6] Denoting concepts are an exception to the rule that Russellian propositions contain what they are about. These denoting concepts are expressed by the so-called denoting phrases: phrases of the form 'a φ', 'some φ', 'any φ', 'every φ', 'all φ', and 'the φ', and their job is to denote certain combinations of entities.[7] Thus, for example, in the proposition expressed by 'Mary is out', Mary herself occurs as a constituent, but in the proposition expressed by 'Every girl is out', the *every girl* denoting concept occurs as a constituent and denotes a certain complex of girls. Similarly, in the proposition expressed by 'Some boy is in', the *some boy* denoting concept occurs as a constituent and denotes a certain complex of boys. And yet it is not the case that every girl occurs as a constituent in the second proposition, nor is it the case that some boy occurs as a constituent in the third. Rather, the propositions in question contain denoting concepts, items whose job it is to further denote what the propositions are about:

> A concept *denotes* when, if it occurs in a proposition, the proposition is not *about* the concept, but about a term connected in a certain peculiar way with the concept. If I say, 'I met a man', the proposition is not about *a man*: this is a concept which does not walk the streets, but lives in the shadowy limbo of the logic-books. What I met was a thing, not a concept, an actual man with a tailor and a bank-account. [5, p. 53]

Russell [6] poses various difficulties for this theory, which Russell wrongly identifies with Frege's theory of sense. Chief among them is the problem posed in the difficult Grey's Elegy passage, which will not be rehearsed here, but which seems to bear a certain resemblance to Frege's concept *horse* problem.

One issue not raised in [6], however, is the capacity of the earlier theory of denoting concepts to capture first-order polyadic quantificational expressivity, in particular, the theory's ability to capture scope distinctions. And yet it is not uncommon to suppose that the theory is deficient on this score.[8] This raises an interesting interpretive problem: given the close scrutiny to which the theory is put in [6], and given Russell's obvious preoccupation with matters of scope, how is it that this most glaring flaw of the theory is never discussed? The correct answer is that no such flaw exists. Within Russell's early metaphysical framework the theory of denoting concepts is perfectly suitable to account for the full expressivity of first-order polyadic quantification theory.

The apparent problem with the theory is this. It can easily seem as though denoting concepts cannot track scope distinctions. This is so to the extent that denoting concepts do not admit of nesting, each doing its denoting work independently of the others. Take, for example, the scopally ambiguous 'Every boy loves some girl'. In polyadic quantification theory we render such ambiguity as follows,

(i) $\forall x(Bx \rightarrow \exists y(Gy \wedge Lxy))$
(ii) $\exists y(Gy \wedge \forall x(Bx \rightarrow Lxy))$,

but the contrast is perhaps better discerned via the restricted quantification notation,

(i′) [every $x : Bx$][some $y : Gy$](Lxy)
(ii′) [some $y : Gy$][every $x : Bx$](Lxy).

The theory of denoting concepts, on the other hand, can only render the contrast as a contrast between these two propositions:

(i″) [LOVE (*every boy* denoting concept)(*some girl* denoting concept)]
(ii″) [LOVE^{-1}(*some girl* denoting concept)(*every boy* denoting concept)].[9]

But now it can easily seem that (ii$''$) would be rendered more simply as (i$''$) by reversing the order of the relata and replacing the relation LOVE^{-1} with its converse LOVE, in which case the supposed contrast disappears. It can thus easily seem that the theory of denoting concepts is expressively inadequate.

Not so, however, on Russell's own metaphysical terms.[10] For Russell a proposition of the form Rab is distinct from one of the form $R^{-1}ba$, for the relation R and its converse R^{-1} are themselves distinct entities. Relations, for Russell, have directions by their very nature.[11] So insofar as these distinct relations are constituent entities of their respective propositions, the propositions in question are themselves distinct. And so, (i$''$) $\neq$ (ii$''$). In general, for each relation R of adicity n, Russell will distinguish $n!$ directionally distinct R-relations.[12] What accounts for logical variety in this rich Russellian realm of propositions is the interaction between the order of the relata and the directions of the relations that relate them. Let $\varphi(u_1, \ldots, u_n)$ be a quantifier-free formula that captures one of the directionally distinct n-place R-relations. Then there will be an intended one-to-one mapping f^{φ} from the $n!$ possible reletterings among the free u_is of the formula onto the $n!$ directionally distinct R-relations.[13]

With such resources in hand, the expressive adequacy of the theory of denoting concepts is easily demonstrated. Let S be a polyadic sentence.[14] Then it has a PNF equivalent $Q_1 u_1 \ldots Q_n u_n M$ where the order of the binding occurrences of the variables in the prefix is fixed alphabetically. The Russellian interpretation of this PNF sentence will be

$$[f^M(M)(Q_1 \text{ denoting concept}), \ldots, (Q_n \text{ denoting concept})].$$ [15]

And this latter proposition will be true just in case S is true under the intended interpretation. The theory of denoting concepts is thus vindicated, offering the full expressivity of first-order polyadic quantification theory without variable-binding.

What about logical relations among these Russellian propositions? Earlier we noted that one way to secure polyadic expressivity is to enrich the language to include for each predicate letter all of its directional variants but that the price of such enrichment is the need for special provisions linking these distinct directional variants in order to facilitate intuitively valid inferences. With the Russellian accommodation of polyadic expressivity the situation is somewhat different. Russellian propositions are possible facts and their constituents are worldly items. As we saw, the ontology is rich enough to include for each n-place relation its $n! - 1$ directional variants. But Russell also held that implication among propositions is built into this ontology and is determined by the propositions' constitutional makeup.[16] And this makeup is determined, once again, by the interaction between the direction of the relation and the order of the relata. It is then by appeal to general features of reality—such as that if one of the girls is loved by each and every one of the boys then each of the boys loves some girl or other but not the other way around—that the Russellian logician can maintain that while (ii$''$) above implies (i$''$), (i$''$) fails to imply (ii$''$). Admittedly, in [5] Russell worries about the truth conditions for nested polyadic quantification and what he offers in this regard does seem inadequate.[17] The present point is that if we abstract from Russell's own failed attempts to specify those truth conditions and focus on the metaphysics of the propositions themselves, his early theory of denoting concepts has the resources to capture full polyadic expressivity.

Finally, it is often supposed that the later theory of [6] achieved a certain ontological economy over its predecessor.[18] On the earlier view, given that 'every boy loves some girl' is meaningful, the denoting phrases 'every boy' and 'some girl' had each to make its own contribution of a distinctive constituent to the proposition expressed by the entire sentence. But with the later theory it is no longer assumed that it is a condition on the meaningfulness of denoting phrases that they contribute their distinctive constituents to the propositions expressed. With the advent of contextual definitions, denoting concepts could be eliminated from Russell's ontology in one fell swoop. And if I am right, with the later theory in place there is no longer any logical need for a multiplicity of directionally distinct variants of a given relation either, which constitutes yet another easily overlooked pruning of Russell's ontology.

Notes

1. Or $\forall x(Px \rightarrow L'x)$ and $\forall x(Px \rightarrow L''x)$, respectively, but we shall assume throughout that quantification is restricted to persons. In all that follows we let use-mention ambiguities be settled by the context.

2. By a 'binding' occurrence of a variable we mean an occurrence of a variable immediately following a quantifier. By a 'bound' occurrence of a variable we mean an occurrence of a variable in a position that is quantified into. Thus, in $\exists u \varphi(u)$ the first occurrence of the variable is a binding occurrence while the second occurrence is a bound occurrence. Writers on logic often regard occurrences of quantifiers as already including the binding occurrences of the variables, but for present purposes we refrain from doing so.

3. More precisely, in any polyadic formula we can stipulate that within a *given* occurrence of a predicate letter the order of initial (among possibly multiple) bound occurrences of distinct variables be ordered alphabetically. Thus, in any polyadic formula of the form $\ldots \psi(\ldots u \ldots v \ldots) \ldots$, where u and v are initial (among possibly multiple) bound occurrences of distinct variables, we can stipulate that u precede v alphabetically and reletter the rest of the formula to accommodate the stipulation. For the sake of readability we proceed as though our variables are just x, y, and z. Assuming infinitely many variables at our disposal, alphabetical order will be $\langle x_1, y_1, z_1, x_2, y_2, z_2, x_3, \ldots \rangle$.

4. More precisely, if ψ is a predicate letter with u and v among its variables and u is earlier in the alphabet than v, we can stipulate both that the binding occurrence of u occur before the binding occurrence of v in the overall formula and that the initial bound occurrence of u in ψ occur before the initial bound occurrence of v in ψ.

5. Compared with Quine's familiar syntactic Schönfinkel-inspired elimination of variables in [4] the present method is semantic, proceeding as it does via a model.

6. See Hylton [3], especially Chapter 4.

7. In all that follows we focus on the first five cases and ignore the sixth.

8. Even those who are most sympathetic to the theory assume that it requires some constructive reinterpretation to accommodate logical variety. Thus, for example, in Dau [1] it is claimed that Russell's theory can capture scope distinctions only if we understand it as requiring that the various denoting concepts (say, the *some girl* denoting concept and the

a girl denoting concept) are to take different scopes (wide and narrow, respectively). We, on the other hand, refrain from making any such assumption about denoting concepts. Indeed, for present purposes we follow the logician's habit of treating all denoting phrases (with the exception of 'the'-phrases) as being either existential or universal.

9. The brackets indicate Russellian propositions, which are also possible facts. The verb in all caps indicates the relation in question.

10. From this point on, by 'Russell' I shall mean the Russell of [5].

11. "It is characteristic of a relation of two terms that it proceeds, so to speak, *from* one *to* the other" ([5, p. 95], italics in the original). As Russell makes clear in the surrounding discussion, *all* relations have directions, including symmetric ones. Thus, for example, the relation of meeting (or MEET) will have the relation of being met by (or MEET^{-1}) as its converse. And while [MEETab] will imply and be implied by [$\text{MEET}^{-1}ba$], the two propositions will be distinct, differing both in the order of the relata and in the relation itself.

12. "The relation which holds between b and a whenever R holds between a and b will be called the *converse* of R, and will be denoted $\breve{R}$. The relation of R to $\breve{R}$ is the relation of oppositeness, or difference of sense" [5, p. 96].

13. Williamson [7] criticizes the view that relations are distinct from their converses on the grounds that such distinctness would render relational expressions semantically indeterminate and that polyadic expressivity has no use for it. Indeed, as we saw above, by the standards of contemporary polyadic quantification theory the difference between Rab and Rba resides only in the order in which the univocal R relates a and b. As for the charge of semantic indeterminacy, Russell would have little interest in such language-oriented matters, falling as they do outside the purview of logic proper. When it comes to denoting concepts, on the other hand, he would distinguish the denoting concept *the relation* R from the denoting concept *the relation* $\breve{R}$ and would say that the first inherently denotes R while the second inherently denotes $\breve{R}$.

14. If S is quantifier-free, then it has a straightforward Russellian capture as the bearing of an n-place relation, $n \geq 1$, among n individuals.

15. Assuming, contra Russell, that both 'all' and 'any' are equivalent to 'every' and that 'a' is equivalent to 'some', each of these Q_i denoting concepts, $1 \leq i \leq n$, will be either the *everything* denoting concept or else the *something* denoting concept.

16. Thus Russell says of two-place relations R: "they all have a converse, that is, a relation $\breve{R}$ such that aRb implies and is implied by $b\breve{R}a$, whatever a and b may be" [5, p. 97].

17. See Chapter VIII of [5]. See also Chapter 3 of [2].

18. Thus Hylton: "According to the theory of denoting concepts, the denoting concept *every person*, say, was such an entity [an entity in the proposition corresponding to a denoting phrase in the sentence —OS]; according to the *OD* theory, however, there is no such entity" [3, p. 239].

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