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Abstract Elementary Classes with Löwenheim-Skolem Number Cofinal with ω

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Abstract In this paper we study abstract elementary classes with finite character and Lowenheim-Skolem number κ, where κ is cofinal with ω. We generalize results obtained by Kueker for $\kappa = \omega$. In particular, we show that $\mathbb{K}$ is closed under $L_{\infty,\kappa}$-elementary equivalence and obtain sufficient conditions for $\mathbb{K}$ to be $L_{\infty,\kappa}$-axiomatizable. In addition, we provide an example to illustrate that if κ is uncountable regular then $\mathbb{K}$ is not closed under $L_{\infty,\kappa}$-elementary equivalence.

1 Introduction

Kueker [7] recently showed that an abstract elementary class with Löwenheim-Skolem number κ implies closure under L_{∞,κ^+}-elementary equivalence. In addition, Kueker proved that the assumption of finite character along with Löwenheim-Skolem number ω implies closure under $L_{\infty,\omega}$-elementary equivalence and noted the necessity of finite character. In this paper we investigate finite character for abstract elementary classes $(\mathbb{K}, \prec_{\mathbb{K}})$ of uncountable Löwenheim-Skolem number κ. We show that if the cofinality of κ is ω then $\mathbb{K}$ is closed under $L_{\infty,\kappa}$-elementary equivalence, and we obtain versions of some of Kueker's other results on categoricity and axiomatizability. On the other hand, if κ is a regular uncountable cardinal, we show that an example due to Morley has finite character but is not closed under $L_{\infty,\kappa}$-elementary equivalence.

Abstract elementary classes were introduced in the 1980s by Shelah [9] as generalizations of elementary classes. They consist of a class of models along with a notion of strong substructure and were proposed as the broadest possible class of structures to potentially have a feasible model theory.

Definition 1.1 For a given vocabulary L, an *abstract elementary class (AEC)*, $(\mathbb{K}, \prec_{\mathbb{K}})$, is a family of L-structures $\mathbb{K}$, together with a binary relation $\prec_{\mathbb{K}}$ satisfying the following axioms:

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(1) *Closure under isomorphism*	If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{N} \cong \mathcal{M}$, then $\mathcal{N} \in \mathbb{K}$; if $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ and $(\mathcal{N}, \mathcal{M}) \cong (\mathcal{N}', \mathcal{M}')$, then $\mathcal{M}' \prec_{\mathbb{K}} \mathcal{N}'$.
(2) $\prec_{\mathbb{K}}$ *is a strong substructure*	If $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$, then $\mathcal{M} \subseteq \mathcal{N}$; if $\mathcal{M} \in \mathbb{K}$, then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{M}$; if $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$ and $\mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{M}_2$, then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_2$.
(3) *Löwenheim-Skolem axiom*	There is an infinite cardinal number $LS(\mathbb{K})$ such that for every $\mathcal{M} \in \mathbb{K}$ and for every subset $A \subseteq \mathcal{M}$ there is some $\mathcal{M}' \prec_{\mathbb{K}} \mathcal{M}$ such that $A \subseteq \mathcal{M}'$ and $\lvert\mathcal{M}'\rvert \leq \max\{\lvert A\rvert, LS(\mathbb{K})\}$.
(4) *Union axiom*	Let $\{\mathcal{M}_i\}_{i<\delta}$ be a continuous $\prec_{\mathbb{K}}$-chain. Then (i) $\bigcup_{i<\delta} \mathcal{M}_i \in \mathbb{K}$, (ii) for each $j < \delta$, $\mathcal{M}_j \prec_{\mathbb{K}} \bigcup_{i<\delta} \mathcal{M}_i$, (iii) if $\mathcal{M}_i \prec_{\mathbb{K}} \mathcal{N}$ for all $i < \delta$, then $\bigcup_{i<\delta} \mathcal{M}_i \prec_{\mathbb{K}} \mathcal{N}$.
(5) *Coherence axiom*	If $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2 \in \mathbb{K}$, $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_2$, $\mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{M}_2$, and $\mathcal{M}_0 \subseteq \mathcal{M}_1$, then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$.

Shelah proved in [9] that if a class of L-structures satisfies the axioms of an AEC then the union axiom can be generalized to unions of $\prec_{\mathbb{K}}$-directed families. We refer to a set of models S as a $\prec_{\mathbb{K}}$-directed family if for any $\mathcal{M}_0, \mathcal{M}_1 \in S$ there exists $\mathcal{M}_2 \in S$ such that $\mathcal{M}_0, \mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{M}_2$.

Lemma 1.2 ([9]) *Let* $(\mathbb{K}, \prec_{\mathbb{K}})$ *be an* AEC *and let* S *be a* $\prec_{\mathbb{K}}$*-directed family of models from* $\mathbb{K}$*. Further, let* $\mathcal{N} = \bigcup S$*. Then the following hold:*

(1) $\mathcal{N} \in \mathbb{K}$.
(2) $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ *for all* $\mathcal{M} \in S$.
(3) *Given a model* $\mathcal{A} \in \mathbb{K}$*, if* $\mathcal{M} \prec_{\mathbb{K}} \mathcal{A}$ *for all* $\mathcal{M} \in S$*, then* $\mathcal{N} \prec_{\mathbb{K}} \mathcal{A}$.

In the study of AECs, we frequently restrict ourselves to AECs with two additional "nice" properties.

Definition 1.3 Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an abstract elementary class.

(1) $(\mathbb{K}, \prec_{\mathbb{K}})$ has the *amalgamation property* if and only if for all models $\mathcal{M}$, $\mathcal{N}_1$, $\mathcal{N}_2 \in \mathbb{K}$ such that $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}_1$ and $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}_2$ there is a model $\mathcal{N} \in \mathbb{K}$ and $\mathbb{K}$-embeddings f_1 and f_2 such that f_i maps $\mathcal{N}_i$ into $\mathcal{N}$ and $f_1(\mathcal{M}) = f_2(\mathcal{M})$.
(2) $(\mathbb{K}, \prec_{\mathbb{K}})$ has the *joint embedding property* if and only if for all models $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathbb{K}$ there is a model $\mathcal{N} \in \mathbb{K}$ and $\mathbb{K}$-embeddings f_i of $\mathcal{M}_i$ into $\mathcal{N}$.

For results depending on the assumptions of amalgamation, joint embedding, and arbitrarily large models, we use the notation (AP, etc.). In addition to these properties, most of the AEC results that we will establish in this paper rely on the assumption of *finite character*. Finite character was introduced by Hyttinen and Kesälä [2] in order to indicate that the definition of strong substructure in the AEC is a local property. The following definition formulated by Kueker [7] is not the same as the notion introduced by Hyttinen and Kesälä, but it is equivalent under the assumption of amalgamation.

Definition 1.4 An AEC $(\mathbb{K}, \prec_{\mathbb{K}})$ has *finite character* if and only if for all models $\mathcal{M}, \mathcal{N} \in \mathbb{K}$, $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ and for every finite tuple $a_0, \ldots, a_n \in \mathcal{M}$ there is a $\mathbb{K}$-embedding of $\mathcal{M}$ into $\mathcal{N}$ fixing $a_0, \ldots, a_n$ pointwise.

One tool used throughout this paper to analyze abstract elementary classes is infinitary logic. We will heavily apply concepts of first-order infinitary logic allowing either infinitely many conjunctions and disjunctions or infinitely many variables (or both). For those unfamiliar with infinitary logics, the essential definitions and results can be found in [5] and [4].

In $L_{\infty,\mu}$ and $L_{\chi,\mu}$ there are two schools of thought on how to define elementary equivalence. We will use the more restrictive definition of elementary equivalence in which $(\mathcal{M}, \bar{a})$ does not add new constants to the language for $\bar{a}$ but merely refers to formulas from $L_{\infty,\mu}$ (or $L_{\chi,\mu}$) applied to elements of the sequence $\bar{a}$. We state below the definitions of $L_{\infty,\mu}$ and $L_{\chi,\mu}$-elementary equivalence that will be used throughout this paper.

Definition 1.5 Given L-structures $\mathcal{M}$ and $\mathcal{N}$, let $\bar{a} \subseteq \mathcal{M}$ and $\bar{b} \subseteq \mathcal{N}$ be sequences of the same length. Then $(\mathcal{M}, \bar{a}) \equiv_{\infty,\mu} (\mathcal{N}, \bar{b})$ *if and only if* for every $\varphi(\bar{x}) \in L_{\infty,\mu}$ with $lh(\bar{x}) = \delta$, $\mathcal{M} \models \varphi(\langle a_{i(j)}\rangle_{j\in\delta}) \leftrightarrow \mathcal{N} \models \varphi(\langle b_{i(j)}\rangle)$ for every $i \in {}^{\delta}lh(\bar{a})$. Note that $\delta < \mu$ necessarily, since $\varphi(\bar{x}) \in L_{\infty,\mu}$. $L_{\chi,\mu}$-elementary equivalence is defined analogously.

We conclude the background section by citing several of Kueker's recent results that motivated this paper. Recall these results require the assumption of a countable Löwenheim-Skolem number.

Theorem 1.6 ([7]) *If* $(\mathbb{K}, \prec_{\mathbb{K}})$ *has finite character, then*

(1) *if* $\mathcal{M} \in \mathbb{K}$ *and* $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$, *then* $\mathcal{N} \in \mathbb{K}$,
(2) *if* $\mathcal{M} \in \mathbb{K}$ *and* $\mathcal{M} \prec_{\infty,\omega} \mathcal{N}$, *then* $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

Theorem 1.7 ([7] AP, etc.) *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *has finite character. If* $(\mathbb{K}, \prec_{\mathbb{K}})$ *is* λ-*categorical for some* $\lambda \geq \omega$, *then there is a complete sentence* $\sigma \in L_{\omega_1,\omega}$ *such that for all* L-*structures* $\mathcal{M}$ *with* $|\mathcal{M}| \geq \lambda$, $\mathcal{M} \in \mathbb{K}$ *if and only if* $\mathcal{M} \models \sigma$.

Theorem 1.8 ([7]) *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *has finite character. Assume that* $\mathbb{K}$ *contains at most* λ-*many models of cardinality* λ *for some infinite* λ. *Then* $\mathbb{K} = \mathrm{Mod}(\theta)$ *for some* $\theta \in L_{\infty,\omega}$. *If* $\mathbb{K}$ *also contains at most* λ-*many models of cardinality* $< \lambda$, *then we can find* $\theta \in L_{\lambda^+,\omega}$.

2 The Filter

Assume κ is an infinite cardinal with cofinality ω. We will choose (and fix) a countable, increasing sequence of infinite cardinals $\langle\kappa_i\rangle_{i\in\omega}$ such that $\kappa = \bigcup_{i\in\omega} \kappa_i$. Any exceptions to this assumption will be explicitly noted.

For the case of Löwenheim-Skolem number ω, Kueker defined the concept of a countable approximation and what is meant by a property of a model to occur in *almost all* countable approximations. For any set s, any countable vocabulary L, and any L-structure $\mathcal{M}$, we use the notation $\mathcal{M}^s$ to denote the substructure of $\mathcal{M}$ generated by $(\mathcal{M} \cap s)$. If s is countable, then we call $\mathcal{M}^s$ a *countable approximation* and if s is of size λ, then we call $\mathcal{M}^s$ a λ-approximation. Additionally, for any set C, we construct a filter on $\mathcal{P}_{\omega_1}(C)$ in order to define the notion of *almost all* $s \subseteq C$.

Definition 2.1 Fix a set C and let $X \subseteq \mathcal{P}_{\omega_1}(C)$.

(1) X is *ω-closed* if and only if X is closed under unions of countable chains.
(2) X is *ω-unbounded* if and only if for every $s_0 \in \mathcal{P}_{\omega_1}(C)$ there is an $s \in X$ such that $s_0 \subseteq s$.

Definition 2.2 $D_{\omega_1}(C)$ is the set of all $X \subseteq \mathcal{P}_{\omega_1}(C)$ such that X contains an ω-closed and ω-unbounded subset.

It is a straightforward proof to show that $D_{\omega_1}(C)$ satisfies the definition of a filter on $\mathcal{P}_{\omega_1}(C)$. Additionally, we note that $D_{\omega_1}(C)$ is defined in such a way to guarantee ω_1-completeness and closure under diagonalization for sets indexed by finite sequences. These properties are crucial to most of the results obtained using the filter, and analogues of them will need to hold when defining filters in higher cardinalities.

The filter $D_{\omega_1}(C)$ has a game theoretic characterization that is useful in proving many results regarding countable approximations and is integral to the generalization of the filter to higher cardinalities. Given a set C and a collection of subsets $X \subseteq \mathcal{P}_{\omega_1}(C)$, we define the ω-length game $G_\omega(X)$ by having player I_X and II_X alternately choose single elements $a_i \in C$. We say player II_X *wins* the game if $\{a_i\}_{i\in\omega} \in X$.

Theorem 2.3 ([6]) *Fix a set C and let $X \subseteq \mathcal{P}_{\omega_1}(C)$. $X \in D_{\omega_1}(C)$ if and only if player II has a winning strategy in the game $G_\omega(X)$.*

A set C is large enough to approximate $\mathcal{M}$ if and only if $\mathcal{M} \subseteq C$. A property of one or more models and/or formulas is said to hold *almost everywhere* (a.e.) if and only if it holds for all $s \in X$ for some $X \in D_{\omega_1}(C)$.

The crux of generalizing Kueker's results to Löwenheim-Skolem number κ is defining an appropriate filter and demonstrating that it upholds the appropriate properties. To do this, we must first define the particular generalizations of the game $G_\omega(X)$ to cardinality κ that will be used.

Definition 2.4 Let C be a set and $X \subseteq \mathcal{P}_{\kappa^+}(C)$. We define

(1) $G_\kappa(X)$ as the ω-length game in which players I_X and II_X alternately choose $s_i \in \mathcal{P}_\kappa(C)$; we say that player II_X *wins* the game $G_\kappa(X)$ if and only if $\bigcup_{i\in\omega} s_i \in X$;

(2) $G^*_\kappa(X)$ as the ω-length game in which players I^*_X and II^*_X alternately choose $s_i \in \mathcal{P}_\kappa(C)$ such that $|s_{2n}|, |s_{2n+1}| \le \kappa_n$; we say that player II^*_X *wins* if and only if $\bigcup_{i\in\omega} s_i \in X$.

Theorem 2.5 *Let C be a set and $X \subseteq \mathcal{P}_{\kappa^+}(C)$. Player II_X has a winning strategy in the game $G_\kappa(X)$ if and only if player II^*_X has a winning strategy in the game $G^*_\kappa(X)$.*

Proof First assume player II_X has a winning strategy in $G_\kappa(X)$. We define player II^*_X's winning strategy by playing two parallel games. For each $n \in \omega$, at stage n suppose player I^*_X has chosen $s^*_{2n} \in \mathcal{P}_\kappa(C)$ such that $|s^*_{2n}| \le \kappa_n$ in $G^*_\kappa(X)$. Let player I_X choose $s_{2n} = s^*_{2n}$ at stage n in $G_\kappa(X)$. Player II_X uses his winning strategy to choose $s_{2n+1} \in \mathcal{P}_\kappa(C)$. Finally, let player II^*_X choose $s^*_{2n+1} = \bigcup \{s_{2i+1} : i \le n, |s_{2i+1}| \le \kappa_n\}$. Since player II_X used his winning strategy in $G_\kappa(X)$, $\bigcup_{i\in\omega} s_i \in X$. By construction, $\bigcup_{i\in\omega} s_i = \bigcup_{i\in\omega} s^*_i$. Hence, player II^*_X has a winning strategy in $G^*_\kappa(X)$.

Conversely, assume player II^*_X has a winning strategy in $G^*_\kappa(X)$. We again define player II_X's winning strategy by playing parallel games. At stage n, suppose player I_X has chosen $s_{2n} \in \mathcal{P}_\kappa(C)$. Let player I^*_X choose $s^*_{2n} = \{s_{2i} : i \le n, |s_{2i}| \le \kappa_n\}$. Player II^*_X uses his winning strategy to choose $s^*_{2n+1} \in \mathcal{P}_\kappa(C)$ such that $|s^*_{2n+1}| \le \kappa_n$. Player II_X then chooses $s_{2n+1} = s^*_{2n+1}$. Again by construction, $\bigcup_{i\in\omega} s_i = \bigcup_{i\in\omega} s^*_i$.

Since player II_X^* used his winning strategy, $\bigcup_{i\in\omega} s_i \in X$. Thus, player II_X has a winning strategy in $G_\kappa(X)$. □

Remark 2.6 It can also be shown that player II_X having a winning strategy in the game $G_\kappa(X)$ is equivalent to player II_X having a winning strategy in the κ-length game where players I and II choose a single element at a time. It can additionally be shown that these are equivalent to player II_X having a winning strategy in the game $G_{\kappa^+}(X)$ defined analogously to the game $G_\kappa(X)$. The proofs of these equivalences can be found in [3].

We can now define the set $D_{\kappa^+}(C)$, which will be our filter, based on the game theoretic characterization of the filter $D_{\omega_1}(C)$ from Theorem 2.3. A further discussion on the properties of this filter can be found in [1].

Definition 2.7 Given a set C, define the set $D_{\kappa^+}(C)$ such that

$$D_{\kappa^+}(C) = \{X \subseteq \mathcal{P}_{\kappa^+}(C) : \mathrm{II}_X \text{ has a winning strategy in } G_\kappa(X)\}.$$

Remark 2.8 By Theorem 2.5, if a set X is in $D_{\kappa^+}(C)$ then player II_X has a winning strategy in both the game $G_\kappa(X)$ as well as the game $G_\kappa^*(X)$.

Note that unlike Kueker's filter on $\mathcal{P}_{\omega_1}(C)$, it is not true that each $X \in D_{\kappa^+}(C)$ contains a κ-closed and κ-unbounded subset. However, the converse is true. In fact, if X contains merely an ω-closed and κ-unbounded subset then $X \in D_{\kappa^+}(C)$. The notion of κ-unbounded is the obvious analogue of ω-unbounded defined before.

Theorem 2.9 *Let C be a set and $X \subseteq \mathcal{P}_{\kappa^+}(C)$. If X contains an ω-closed and κ-unbounded subset, then $X \in D_{\kappa^+}(C)$.*

Proof Using κ-unboundedness, ω-closure, and the cofinality of κ, it can easily be demonstrated that player II_X always has a winning strategy in the game $G_\kappa(X)$. □

We proceed to show some other desirable properties that $D_{\kappa^+}(C)$ exhibits. First we show that it is closed under κ-many intersections.

Lemma 2.10 *$D_{\kappa^+}(C)$ is κ^+-complete.*

Proof Let $X_\alpha \in D_{\kappa^+}(C)$ for each $\alpha \in \kappa$ and let $Y = \bigcap_{\alpha\in\kappa} X_\alpha$. In order to show that player II_Y^* has a winning strategy in the game $G_\kappa^*(Y)$, we play κ-many concurrent games and employ the winning strategies of players $\mathrm{II}_{X_\alpha}^*$. It is important to note how the gameplay proceeds. At the time player I_Y^* plays his first move, we start the first κ_0-many games, $G_\kappa^*(X_\alpha)$ for $\alpha < \kappa_0$. When player I_Y^* plays his second move, the first κ_0-many games continue and the games $G_\kappa^*(X_\alpha)$ start for $\kappa_0 \leq \alpha < \kappa_1$. We continue to stagger the beginning of each game $G_\kappa^*(X_\alpha)$ in this manner.

At stage n, if $\alpha < \kappa_n$ then we assume player $\mathrm{I}_{X_\alpha}^*$ has chosen $\bigcup_{i\leq 2n} s_i$ for his move and player $\mathrm{II}_{X_\alpha}^*$ responds with his winning strategy. For simplicity sake, we denote player $\mathrm{II}_{X_\alpha}^*$'s response as s_{2n+1}^α. Player II_Y^* then responds to player I_Y^* with $\bigcup_{\alpha<\kappa_n} s_{2n+1}^\alpha$.

By construction, $s = \bigcup_{i\in\omega} s_i = \bigcup_{i\in\omega} s_i^\alpha$ for all $\alpha \in \kappa$. Since players $\mathrm{II}_{X_\alpha}^*$ used their winning strategies once the game started, $s \in X_\alpha$ for all $\alpha \in \kappa$. Thus $s \in Y$ as desired. □

Remark 2.11 From Lemma 2.10 and the observation that $D_{\kappa^+}(C)$ is upward closed, it follows that $D_{\kappa^+}(C)$ is a filter on $\mathcal{P}_{\kappa^+}(C)$.

Finally we state that our filter is closed under the diagonalization of sets indexed by finite sequences. The proof is omitted but follows easily from Lemma 2.10. This result is potentially of future use when considering AECs with finite character.

Lemma 2.12 *$D_{\kappa^+}(C)$ is closed under diagonalization for sets indexed by finite sequences. That is, if $X_{\langle i_0,\ldots,i_n\rangle} \in D_{\kappa^+}(C)$ for all $n \in \omega$ and for every $i_0,\ldots,i_n \in I$, where $I \subseteq C$, then $\bar{X} \in D_{\kappa^+}(C)$ where $\bar{X} = \{s \in \mathcal{P}_{\kappa^+}(C) : s \in X_{\langle i_0,\ldots,i_n\rangle}$ for all $n \in \omega$ and for all $i_0,\ldots,i_n \in (I \cap s)\}$.*

3 Main Results

For the entirety of this section we assume $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC with $LS(\mathbb{K}) = \kappa$. Our notation and terminology for a property of a model to occur *almost everywhere* is analogous to those used for countable approximations. A property of κ-approximations to one or more models is said to hold *κ-almost everywhere* (or κ-a.e.) if and only if it holds for all $s \in X$ for some $X \in D_{\kappa^+}(C)$, where C is large enough to approximate all the structures involved.

Note that the set $\{s : (\mathcal{M} \cap s) = \mathcal{M}^s\}$ is ω-closed and κ-unbounded and thus is in $D_{\kappa^+}(C)$ for any $C \supseteq \mathcal{M}$ by Theorem 2.9. Since $D_{\kappa^+}(C)$ is closed under intersections, this observation enables us to assume $\mathcal{M}^s = (\mathcal{M} \cap s)$ in our results.

The following application of κ-approximations is implied by the Löwenheim-Skolem axiom but will be more helpful to us stated in this form.

Lemma 3.1 *Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an* AEC *with $LS(\mathbb{K}) \leq \kappa$.*

(1) *If $\mathcal{M} \in \mathbb{K}$, then $\mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$ κ-a.e.*
(2) *If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ such that $|\mathcal{M}_0| = \kappa$, then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s$ κ-a.e.*

Proof

(1) Let $X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{M}) : \mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}, \mathcal{M}^s = s\}$. It follows from the coherence and union axioms that X is ω-closed. In addition, it follows from the Löwenheim-Skolem axiom that X is κ-unbounded. Hence, $X \in D_{\kappa^+}(\mathcal{M})$ by Theorem 2.9 and thus $\mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$ κ-a.e.
(2) Note that $\mathcal{M}_0 \subseteq \mathcal{M}^s$ κ-a.e. From part (1) and the coherence axiom it follows that $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s$ κ-a.e. □

We recall the game theoretic characterization of $L_{\infty,\lambda}$-elementary equivalence since it will be a key tool in proving Lemma 3.4.

Definition 3.2 Let $\mathcal{M}$ and $\mathcal{N}$ be two L-structures for some vocabulary L. Define the game $G_\lambda(\mathcal{M}, \mathcal{N})$ as the 2-person, ω-length game such that players I and II alternately choose sequences $\bar{a}^n \subseteq \mathcal{M}$ and $\bar{b}^n \subseteq \mathcal{N}$ of length less than λ. We say that player II *wins the game if the map h defined as $h(\bar{a}^i) = \bar{b}^i$ for all $i \in \omega$ is a partial isomorphism.*

Theorem 3.3 ([5]) *Let λ be an infinite cardinal. For L-structures $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M} \equiv_{\infty,\lambda} \mathcal{N}$ if and only if player II has a winning strategy in the game $G_\lambda(\mathcal{M}, \mathcal{N})$.*

The following are essential technical lemmas needed for our major results. We require cofinality ω for the following proofs so that back-and-forth arguments can be completed in ω-many steps.

Lemma 3.4 *Assume $\mathcal{M} \in \mathbb{K}$, $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ of cardinality κ, $n \in \omega$ and $\bar{a}_0, \ldots, \bar{a}_{n-1} \subseteq \mathcal{M}_0$ are sequences of length $< \kappa$. Let $\mathcal{N}$ be an arbitrary L-structure and $\bar{b}_0, \ldots, \bar{b}_{n-1} \subseteq \mathcal{N}$ be such that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}) \equiv_{\infty,\kappa} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n})$. Then [there is a $\mathbb{K}$-embedding h of $\mathcal{M}_0$ into $\mathcal{N}^s$ such that $h(\bar{a}_i) = \bar{b}_i$ for all $i < n$] κ-a.e.*

Proof Let $Y = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{N})$: there is a $\mathbb{K}$-embedding h of $\mathcal{M}_0$ into $\mathcal{N}^s$ such that $h(\bar{a}_i) = \bar{b}_i\ \forall i < n\}$. We will show player II_Y has a winning strategy in $G_\kappa(Y)$.

Let $X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{M}) : \mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s, \mathcal{M}^s = s\}$ which is an element of $D_{\kappa^+}(\mathcal{M})$ by Lemma 3.1. Thus, player II_X has a winning strategy in $G_\kappa(X)$. Using this strategy and the game theoretic characterization of $L_{\infty,\kappa}$-elementary equivalence we can construct a winning strategy for player II_Y.

Assume player I_Y has chosen $t_0 \in \mathcal{P}_\kappa(\mathcal{N})$. Let $\bar{d}_0 \subseteq \mathcal{N}$ be a sequence of length $< \kappa$ such that $\mathrm{ran}(\bar{d}_0) = t_0$. There is $\bar{c}_0 \subseteq \mathcal{M}$ such that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}, \bar{c}_0) \equiv_{\infty,\kappa} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n}, \bar{d}_0)$. Assume player I_X has chosen $s_0 = \mathrm{ran}(\bar{c}_0)$ in $G_\kappa(X)$. Player II_X then uses his winning strategy to choose $s_1 \in \mathcal{P}_\kappa(\mathcal{M})$. Let $\bar{c}_1 \subseteq \mathcal{M}$ be a sequence of length $< \kappa$ such that $\mathrm{ran}(\bar{c}_1) = s_1$. There exists $\bar{d}_1 \subseteq \mathcal{N}$ such that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}, \bar{c}_0, \bar{c}_1) \equiv_{\infty,\kappa} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n}, \bar{d}_0, \bar{d}_1)$. Finally, let player II_Y choose $t_1 = \mathrm{ran}(\bar{d}_1)$ in response to player I_Y's choice of t_0. Continue this process for all t_i for all $i \in \omega$.

Let $\bar{c} = \bigcup_{i\in\omega} \bar{c}_i \subseteq \mathcal{M}$ and let $\bar{d} = \bigcup_{i\in\omega} \bar{d}_i \subseteq \mathcal{N}$. Since $\mathrm{cof}(\kappa) = \omega$, it is not necessarily true that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}, \bar{c}) \equiv_{\infty,\kappa} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n}, \bar{d})$. However, we can say that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}, \bar{c}) \equiv_{\infty,\omega} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n}, \bar{d})$ since $L_{\infty,\omega}$-formulas only have finitely many free variables. Since player II_X used his winning strategy, $\mathrm{ran}(\bar{c}) = s \in X$. Thus $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s$ and $\mathcal{M}^s = s = \mathrm{ran}(\bar{c})$.

Let $t = \mathrm{ran}(\bar{d})$. Define $g : \mathcal{M}^s \to \mathcal{N}$ by $g(\bar{c}_i) = \bar{d}_i$ for all $i \in \omega$. Then g is an isomorphism of $\mathcal{M}^s$ onto a substructure $\mathcal{N}^t$ of $\mathcal{N}$ such that $\mathcal{N}^t = t$. If we let $\mathcal{N}_0 = g(\mathcal{M}_0)$ then $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^t$ because $\prec_{\mathbb{K}}$ is preserved under isomorphism. In addition, $g(\bar{a}_i) = \bar{b}_i$ for all $i < n$. If we let $h = g \restriction \mathcal{M}_0$ then h is a $\mathbb{K}$-embedding of $\mathcal{M}_0$ into $\mathcal{N}^t$ such that $h(\bar{a}_i) = \bar{b}_i$ for all $i < n$. Thus $t \in Y$ and player II_Y has a winning strategy. □

Lemma 3.5 *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character. Let $\mathcal{M} \in \mathbb{K}$, $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ where $|\mathcal{M}_0| \leq \kappa$ and $\bar{a} \subseteq \mathcal{M}$ such that $\mathrm{ran}(\bar{a}) = \mathcal{M}_0$. Let $\mathcal{N}$ be an arbitrary L-structure and let $\bar{b} \subseteq \mathcal{N}$ be a sequence of the same length as $\bar{a}$. If $(\mathcal{M}, a_{i_0}, \ldots, a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \ldots, b_{i_n})$ for all $i_0, \ldots, i_n \in |\bar{a}|$ and for all $n \in \omega$ then $\mathrm{ran}(\bar{b}) = \mathcal{N}_0$ where $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. and $\mathcal{M}_0 \cong \mathcal{N}_0$.*

Proof Let $Y^{b_{i_0},\ldots,b_{i_n}} = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{N})$: there exists a $\mathbb{K}$-embedding $h : \mathcal{M}_0 \to \mathcal{N}^s$ such that $h(a_{i_k}) = b_{i_k}\ \forall k \leq n\}$. Lemma 3.4 implies that $Y^{b_{i_0},\ldots,b_{i_n}} \in D_{\kappa^+}(\mathcal{N})$ for all finite sequences $\langle b_{i_0}, \ldots, b_{i_n} \rangle \subseteq \bar{b}$. Thus $Z = \bigcap Y^{b_{i_0},\ldots,b_{i_n}} \in D_{\kappa^+}(\mathcal{N})$ by κ^+-completeness.

Define the map $g : \mathcal{M}_0 \to \mathcal{N}$ as $g(a_i) = b_i$ for all $i \in \kappa$. As in the previous proof we can state that $(\mathcal{M}, \bar{a}) \equiv_{\infty,\omega} (\mathcal{N}, \bar{b})$ and thus g is an isomorphism of $\mathcal{M}_0$ onto some substructure $\mathcal{N}_0 \subseteq \mathcal{N}$ where $\mathrm{ran}(\bar{b}) = \mathcal{N}_0$. Fix $s \in Z$. For any finite sequence $\langle b_{i_0}, \ldots, b_{i_n} \rangle \subseteq \mathcal{N}_0$ the map $h \circ g^{-1}$ is a $\mathbb{K}$-embedding of $\mathcal{N}_0$ into $\mathcal{N}^s$ fixing $b_{i_0}, \ldots, b_{i_n}$. Hence, by finite character, $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$. Therefore, $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. as desired. □

Lemma 3.6 *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character. Let $\mathcal{M} \in \mathbb{K}$ and assume $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ for some L-structure $\mathcal{N}$. Then for every subset $B_0 \subseteq \mathcal{N}$ of cardinality $\leq \kappa$ there is a substructure $\mathcal{N}_0 \subseteq \mathcal{N}$ of cardinality κ such that $B_0 \subseteq \mathcal{N}_0$ and $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e.*

Proof Let $X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{M}) : \mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$ and $\mathcal{M}^s = s\}$. By Lemma 3.1 $X \in D_{\kappa^+}(\mathcal{M})$ and thus player II_X has a winning strategy in $G_\kappa(X)$.

Enumerate B_0 as $\langle \bar{b}_{2i} \rangle_{i \in \omega}$ such that $\bar{b}_{2i} \subseteq \bar{b}_{2(i+1)}$ and $|\bar{b}_{2i}| < \kappa$ for all $i \in \omega$. This is possible since the cofinality of κ is ω.

Let $\bar{a}_0 \subseteq \mathcal{M}$ be such that $(\mathcal{M}, \bar{a}_0) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b}_0)$. Let player I_X choose $s_0 = \mathrm{ran}(\bar{a}_0)$ in $G_\kappa(X)$. Player II_X will then use his winning strategy to choose $s_1 \in \mathcal{P}_\kappa(\mathcal{M})$. Let $\bar{a}_1 \subseteq \mathcal{M}$ be such that $\mathrm{ran}(\bar{a}_1) = s_1$. Let $\bar{b}_1 \subseteq \mathcal{N}$ be such that $(\mathcal{M}, \bar{a}_0, \bar{a}_1) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b}_0, \bar{b}_1)$. Continue in this manner for all $n \in \omega$.

Since player II_X used his winning strategy, we know $s = \bigcup_{i \in \omega} \bar{a}_i \in X$. Thus $\mathrm{ran}(\bar{a}) = \mathcal{M}_0$ where $\mathcal{M}_0 = \mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$.

By construction, $B_0 \subseteq \mathrm{ran}(\bar{b})$ and for any $i_0, \ldots, i_n \in \kappa$ and any $n \in \omega$ we know that $(\mathcal{M}, a_{i_0}, \ldots, a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \ldots, b_{i_n})$. By Lemma 3.5 we can conclude that $\mathrm{ran}(\bar{b}) = \mathcal{N}_0$ where $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. and $B_0 \subseteq \mathcal{N}_0$ as desired. □

We are now able to use κ-approximations to prove that AECs with finite character and a Löwenheim-Skolem number of κ are closed under $L_{\infty,\kappa}$-elementary equivalence.

Theorem 3.7 *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character. Let $\mathcal{M} \in \mathbb{K}$ and $\mathcal{N}$ be an arbitrary L-structure. If $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$, then $\mathcal{N} \in \mathbb{K}$.*

Proof Let $S = \{\mathcal{N}_0 \subseteq \mathcal{N} : |\mathcal{N}_0| = \kappa, \mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s \ \kappa\text{-a.e.}\}$. By Lemma 1.2 it suffices to show that S is a family of $\mathbb{K}$-structures directed under $\prec_{\mathbb{K}}$ and that $\bigcup S = \mathcal{N}$.

Assume $\mathcal{N}_0, \mathcal{N}_1 \in S$. By Lemma 3.6 there is a $\mathbb{K}$-structure $\mathcal{N}_2$ such that $\mathcal{N}_2 \subseteq \mathcal{N}$, $\mathcal{N}_0 \cup \mathcal{N}_1 \subseteq \mathcal{N}_2$, $|\mathcal{N}_2| = \kappa$, and $\mathcal{N}_2 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. Thus $\mathcal{N}_2 \in S$ and it follows that S is a family of κ-size $\mathbb{K}$-structures directed under $\subseteq$. Furthermore, if $\mathcal{N}_0, \mathcal{N}_1 \in S$ and $\mathcal{N}_0 \subseteq \mathcal{N}_1$ then there will be some $\mathcal{N}^s \subseteq \mathcal{N}$ such that both $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ and $\mathcal{N}_1 \prec_{\mathbb{K}} \mathcal{N}^s$. Therefore, $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}_1$ by the coherence axiom. Hence, S is directed under $\prec_{\mathbb{K}}$. In addition, it follows from Lemma 3.6 that $\bigcup S = \mathcal{N}$ and thus S is as desired. □

Under the assumption of finite character we obtain two noteworthy corollaries. We state them here without proof. The first corollary states that $\prec_{\mathbb{K}}$ is preserved by $L_{\infty,\kappa}$-elementary equivalence.

Corollary 3.8 *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character. Further assume $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ and $\bar{a} \subseteq \mathcal{M}$ such that $\mathrm{ran}(\bar{a}) = \mathcal{M}_0$. If $\bar{b}$ is a sequence of length $|\bar{a}|$ from a model $\mathcal{N}$ and $(\mathcal{M}, a_{i_0}, \ldots, a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \ldots, b_{i_n})$ for all $i_0, \ldots, i_n \in |\bar{a}|$ and for all $n \in \omega$, then $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}$ where $\mathrm{ran}(\bar{b}) = \mathcal{N}_0$ and $\mathcal{M}_0 \cong \mathcal{N}_0$.*

The final corollary to Theorem 3.7 states that $L_{\infty,\kappa}$-substructures are also $\mathbb{K}$-substructures.

Corollary 3.9 *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character. If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$, then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.*

As a consequence of closure under $L_{\infty,\kappa}$-elementary equivalence, we can axiomatize $(\mathbb{K}, \prec_{\mathbb{K}})$ by a sentence of $L_{\infty,\kappa}$ if there are few models of sufficiently high cardinality. We found many axiomatizability results analogous to Kueker's recent results with $\kappa = \omega$.

Assuming few models of some cardinality (with a condition on this cardinality) we are able to obtain our first axiomatizability result. To prove this, we utilized a generalization of Scott's Theorem and a result of Kueker's on λ-approximations [6] to disjunct a large number of sentences incorporating the Scott sentence of each model.

Theorem 3.10 ([3]) *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character. If $\mathbb{K}$ has at most λ-many models of cardinality λ for some λ such that $\lambda^{<\kappa} = \lambda$, then $\mathbb{K} = \mathrm{Mod}(\sigma)$ for some $\sigma \in L_{\infty,\kappa}$. If there are at most λ-many models of cardinality $< \lambda$, then we can find $\sigma \in L_{\lambda^+,\kappa}$.*

We proceeded to investigate axiomatizability by exploring a.e.c.'s that were categorical in some cardinality. In order to work with these a.e.c.'s, we needed to define a notion of a λ-galois saturated model over sets that, under the assumption of a monster model, was consistent with the traditional definition λ-galois saturated model when $\lambda > LS(\mathbb{K})$ and extends to $\lambda = LS(\mathbb{K})$. We use this definition to construct a sentence of $L_{\infty,\kappa}$ describing κ-galois saturation. Thus, we get the following theorem stating that there is a complete $L_{\infty,\kappa}$-sentence closely approximating the a.e.c.

Theorem 3.11 (AP, etc. [3]) *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character. Let $\mathbb{K}$ be λ-categorical for $\lambda > \kappa$ and $\mathrm{cof}(\lambda) > \kappa$. Then there is a complete sentence $\sigma \in L_{\infty,\kappa}$ such that*

1. *$\mathrm{Mod}(\sigma) \subseteq \mathbb{K}$ and σ has a model of cardinality κ^+,*
2. *$\mathbb{K}$ and $\mathrm{Mod}(\sigma)$ contain precisely the same models of cardinality $\geq \lambda$,*
3. *if $\mathcal{M}, \mathcal{N} \models \sigma$, then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ if and only if $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$.*

Remark 3.12 It is still an open question as to whether or not σ must have a model of cardinality κ.

Taking the sentence from Theorem 3.11 and disjuncting it with each sentence describing the models below the categoricity cardinal, we get the following axiomatizability result.

Corollary 3.13 (AP, etc. [3]) *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character. Let $\mathbb{K}$ be λ-categorical for $\lambda > \kappa$ and $\mathrm{cof}(\lambda) > \kappa$. Then there is a sentence $\theta \in L_{\infty,\kappa}$ such that $\mathbb{K} = \mathrm{Mod}(\theta)$.*

4 Examples

In this section we will provide several examples to show that the assumptions made in the previous section are necessary and that closure under $L_{\infty,\kappa}$-equivalence is the best possible result.

First, we will show that if we remove the assumption of finite character, we cannot assure closure under $L_{\infty,\kappa}$-equivalence. The following example, due to Kueker, illustrates a very simple case of an AEC without finite character.

Example 4.1 Define the vocabulary $L = \{P\}$ where P is a unary predicate symbol. Let $\mathbb{K} = \{\mathcal{M} : \mathcal{M}$ is an L-structure, $|P^{\mathcal{M}}| = \kappa$, $|\neg P^{\mathcal{M}}| \geq \kappa\}$. In addition, define $\prec_{\mathbb{K}}$ as $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ if and only if $\mathcal{M} \subseteq \mathcal{N}$ and $P^{\mathcal{M}} = P^{\mathcal{N}}$.

It is a very simple exercise to verify that $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC satisfying (AP, etc.). It also follows easily that $(\mathbb{K}, \prec_{\mathbb{K}})$ fails to satisfy finite character. To see this, let $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ be such that $\mathcal{M} \subseteq \mathcal{N}$ and there is just a single element $b \in P^{\mathcal{N}} \setminus P^{\mathcal{M}}$. For any $n \in \omega$ and $a_0, \ldots, a_n \in \mathcal{M}$ there is a $\mathbb{K}$-embedding $f : \mathcal{M} \to \mathcal{N}$ fixing $a_0, \ldots, a_n$ (since $|P^{\mathcal{M}} \setminus \{a_0, \ldots, a_n\}| = |P^{\mathcal{N}} \setminus \{a_0, \ldots, a_n\}| = \kappa$). However, $P^{\mathcal{M}} \neq P^{\mathcal{N}}$ and thus $\mathcal{M} \not\prec_{\mathbb{K}} \mathcal{N}$. Therefore, $\mathbb{K}$ fails to have finite character.

To demonstrate $(\mathbb{K}, \prec_{\mathbb{K}})$ is not closed under $L_{\infty,\kappa}$-elementary equivalence, let $\mathcal{M}, \mathcal{N}$ be L-structures such that $|P^{\mathcal{M}}| = \kappa$ and $|\neg P^{\mathcal{M}}| = \kappa^+$ but $|P^{\mathcal{N}}| = \kappa^+$ and $|\neg P^{\mathcal{N}}| = \kappa^+$. Thus, $\mathcal{M} \in \mathbb{K}$ and $\mathcal{N} \notin \mathbb{K}$. However, $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$.

Using the template of the previous example, we can construct an AEC that satisfies finite character in order to demonstrate that $L_{\infty,\kappa}$ is the best possible result.

Example 4.2 Define the vocabulary L as in Example 4.1 but this time let $\mathbb{K} = \{\mathcal{M} : \mathcal{M}$ is an L-structure, $|P^{\mathcal{M}}| \geq \kappa, |\neg P^{\mathcal{M}}| \geq \kappa\}$ and define $\prec_{\mathbb{K}}$ as ordinary substructure. It again follows easily that $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC satisfying (AP, etc.). In addition, it satisfies finite character since $\prec_{\mathbb{K}} = \subseteq$. By Theorem 3.7 $(\mathbb{K}, \prec_{\mathbb{K}})$ is closed under $L_{\infty,\kappa}$-elementary equivalence; however, it is not closed under $L_{\infty,\tau}$-elementary equivalence for any $\tau < \kappa$. Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures such that $|P^{\mathcal{M}}| = \kappa$, $|\neg P^{\mathcal{M}}| = \kappa$, $|P^{\mathcal{N}}| = \tau$, and $|\neg P^{\mathcal{M}}| = \kappa$. Therefore, $\mathcal{M} \equiv_{\infty,\tau} \mathcal{N}$ but $\mathcal{M} \in \mathbb{K}$ and $\mathcal{N} \notin \mathbb{K}$.

Finally, we will demonstrate that if κ is uncountable and regular then closure under $L_{\infty,\kappa}$-elementary equivalence fails. Morley [8] provided the following example of two models of size $\aleph_1$ that are L_{∞,ω_1}-elementary equivalent but are not isomorphic. We will use this example to construct an AEC with $LS(\mathbb{K}) = \aleph_1$ that has finite character but is not closed under L_{∞,ω_1}-elementary equivalence. Similar examples will work for any regular cardinal.

Example 4.3 ([8]) There exists a well-founded tree of cardinality $\aleph_1$, $\mathcal{M}$ such that

(1) every element has exactly ω_1 immediate successors,
(2) for every $a_0 \in \mathcal{M}$, $\mathcal{M} \cong \mathcal{M} \upharpoonright \{a : a_0 \leq a\}$,
(3) every branch is countable but there are arbitrarily long countable branches.

Define $\mathcal{N}$ by starting with $(\omega_1, <)$ and putting a copy of $\mathcal{M}$ above every element of ω_1. Thus, $|\mathcal{N}| = \aleph_1$, $\mathcal{M} \equiv_{\infty,\omega_1} \mathcal{N}$ by a simple back-and-forth argument, but $\mathcal{M} \ncong \mathcal{N}$.

Let $\mathbb{K} = \{\mathcal{A} : \mathcal{N}$ is isomorphically embeddable in $\mathcal{A}\}$ and define $\prec_{\mathbb{K}}$ as ordinary substructure. It is clear to see that $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC satisfying finite character and (AP, etc.) with $LS(\mathbb{K}) = \aleph_1$. Observe that $\mathcal{N} \in \mathbb{K}$ and $\mathcal{M} \notin \mathbb{K}$. Thus $(\mathbb{K}, \prec_{\mathbb{K}})$ fails to be closed under L_{∞,ω_1}-elementary equivalence.

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