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On Generalizing Kolmogorov

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Abstract In his "From classical to constructive probability," Weatherson offers a generalization of Kolmogorov's axioms of classical probability that is neutral regarding the logic for the object-language. Weatherson's generalized notion of probability can hardly be regarded as adequate, as the example of supervaluationist logic shows. At least, if we model credences as betting rates, the Dutch-Book argument strategy does not support Weatherson's notion of supervaluationist probability, but various alternatives. Depending on whether supervaluationist bets are specified as (a) conditional bets (Cantwell), (b) unconditional bets with graded payoffs (Milne), or (c) unconditional bets with ungraded payoffs (Dietz), supervaluationist probability amounts to (a) conditional probability of truth given a truth-value, (b) the expected truth-value, or (c) the probability of truth, respectively. It is suggested that for supervaluationist logic, the third option is the most attractive one, for (unlike the other options) it preserves respect for single-premise entailment.

1 Weatherson's Generalized Calculus of Probability

According to Kolmogorov's classical axioms of probability, any function defined on sentences of a given language $\mathcal{L}$ of propositional logic is a *probability function* just in case it (i) takes nonnegative real numbers as values, (ii) it takes for tautologies the value 1, and (iii) it meets the finite additivity constraint, which says that for any pair of sentences A and B, if they are jointly classically inconsistent, their disjunction takes as value the sum of the values of A and of B, respectively.[1] Classical probability fixes the logic as classical logic. In his "From classical to constructive probability" [12, p. 12], Weatherson generalizes Kolmogorov's axioms to obtain conditions defining, for *any* given logic, a class of probability functions. According to this, if $\models$ is the entailment relation on our language $\mathcal{L}$, any function P that takes real numbers as values for sentences of $\mathcal{L}$ is a probability function just in case it satisfies for every pair of sentences of $\mathcal{L}$, A and B, the following conditions:

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(GP1) If $\{A\} \models$, then $\mathrm{P}(A) = 0$.
(Value zero for inconsistent sentences)

(GP2) If $\{\,\} \models A$, then $\mathrm{P}(A) = 1$.
(Value one for logically true sentences)

(GP3) If $\{A\} \models B$, then $\mathrm{P}(A) \leq P(B)$.
(Respect for single premise entailment)

(GP4) $\mathrm{P}(A \vee B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \,\&\, B)$.
(General additivity)

Call probability in Weatherson's generalized sense *generalized probability*. In the special case of classical logic, the respective class of probability functions will be just the class of classical probability functions. But as far as nonclassical logics are concerned, the generalized probability calculus may yield nonclassical value distributions—for example, for intuitionist logic, instances of some classical logical laws like the law of excluded middle may receive a value lower than one.

Weatherson does not offer any argument in favor of generalized probability. He introduces the calculus as characterizing the class of "essentially Kolmogorovian" probability functions (p. 111). If one takes respect for logic, and general additivity as 'essential' features of any notion of probability, trivially, one ends up with generalized probability. But insofar as generalized probability is meant to have a normative bearing on the structure of subjective probability—as Weatherson suggests—the question arises whether these "essentially Kolmogorovian" features are adequate. In particular, the question arises whether they are adequate for *every* logical framework. I contend that they are not. For a case in point, it is shown that generalized probability runs into serious trouble if it is applied to *supervaluationist* logic.

Consider a standard **S5** semantics for a language of propositional logic with a sentence operator 'D', where the latter is treated as a necessity operator. By reinterpreting 'D' as an operator of truth and defining entailment in terms of preservation of truth, we obtain a supervaluationist semantics. Specifically, supervaluationist entailment ($\models_{\mathrm{SV}}$) can be characterized in terms of **S5** entailment ($\models_{\mathbf{S5}}$) as follows: $\Sigma \models_{\mathrm{SV}} \alpha \Leftrightarrow \Sigma^* \models_{\mathbf{S5}} \mathrm{D}\alpha$, where Σ^* is obtained from Σ by attaching the D-operator to every member of Σ.[2] For our purposes, it does to point out the following features of supervaluationism:

(a) Classical entailment implies supervaluationist entailment.
(b) For any sentence A, $\ulcorner \mathrm{D}A \urcorner$ and A are logically equivalent.
(c) Sentences of the form $\ulcorner \mathrm{D}A \,\&\, \mathrm{D} \sim A \urcorner$ are inconsistent ($\mathrm{D}A \,\&\, \mathrm{D} \sim A \models_{\mathrm{SV}}$).
(d) Finally, for some sentences A, we can consistently make an assumption of the form $\ulcorner {\sim}\mathrm{D}A \,\&\, {\sim}\mathrm{D} \sim A \urcorner$; that is, assume A to be gappy (neither true nor false).[3]

With these results in place, consider the notion of probability that we obtain from generalized probability for supervaluationist logic—call it supervaluationist *generalized probability*, henceforth abbreviated as *SGP*. Call any SGP-function P *dogmatic* with respect to a logically contingent sentence A just in case $\mathrm{P}(A) = 0$ or $\mathrm{P}(A) = 1$.[4] It turns out that SGP is *strongly dogmatic* with respect to every logically contingent sentential expression of gappiness of the form $\ulcorner {\sim}\mathrm{D}A \,\&\, {\sim}\mathrm{D} \sim A \urcorner$ in the sense that every SGP-function is dogmatic with respect to it. Specifically, any sentence of this form is to receive the value zero, as can be seen by the following reasoning. For any SGP-function P, we have

By (a) and (GP1) (i) $\mathrm{P}(A \ \& \ \sim A) = 0$.
By (a) and (GP2) (ii) $\mathrm{P}(A \vee \sim A) = 1$.
By (b) and (GP3) (iii) $\mathrm{P}(A) = \mathrm{P}(\mathrm{D}A)$ and $\mathrm{P}(\sim A) = \mathrm{P}(\mathrm{D} \sim A)$.
By (c) and (GP1) (v) $\mathrm{P}(\mathrm{D}A \ \& \ \mathrm{D} \sim A) = 0$.
From (i)–(v) and (GP4), it follows that $\mathrm{P}(\mathrm{D}A \vee \mathrm{D} \sim A) = 1$.
From this, by (a) and (GP1)–(GP4), $\mathrm{P}(\sim \mathrm{D}A \ \& \ \sim \mathrm{D} \sim A) = 0$.

That is, it follows that SGP is strongly dogmatic with respect to *any* sentential expression of gappiness of the form $\ulcorner \sim \mathrm{D}A \ \& \ \sim \mathrm{D} \sim A \urcorner$. But, by (d), there are logically contingent sentences of this form.

This result highlights a serious problem for SGP. To clarify what the problem is about, let me begin with two arguable objections that I do not subscribe to.

Objection from credence as a measure of certainty Relative to any given epistemic state, a maximal (minimal) credence is to be awarded only to sentences which are certainly true (false). Thus any notion of probability that is strongly dogmatic with respect to some logically contingent sentence fails to be adequate as a model of credence, as it does not supply means of measuring certainty for the everyday case where there is uncertainty with regard to some of those sentences.

This objection draws on an assumption that is even rejectable for classical probability. For one, the assumption that certainty amounts to a subjective probability one is false for classical distributions over possibility-spaces with uncountably many elements. Specifically, such distributions cannot satisfy the regularity constraint: a probability distribution over a set of propositions (that is, sets of worlds) is *regular* just in case it assigns probability zero only to the empty proposition and probability one only to the universal proposition. As a consequence, some propositions would receive a value one or a value zero, even though they are contingent.

> The finite additivity principle implies that for any possibility-space, there are at most n elements with a probability at least $1/n$. From this, it follows that at most countably many elements can have a positive probability. (Compare [13, p. 173] and [5, p. 75])

For another, even for classical distributions on possibility spaces with countably many elements, regularity cannot be taken for granted.

> See the argument in [13, p. 174], which relies on a uniformity constraint on assignments for elements in possibility-spaces.

The requirement that subjective probability be a measure of certainty sets a standard that is not even met by classical probability. Thus, if the objection from credence as a measure of certainty applies to Weatherson's generalization, it also applies to classical probability. And insofar as the objection is only aimed at Weatherson's way of generalizing classical probability, it misfires. Here is another objection.

Objection from coherent partial believability for logically contingent sentences Whatever is not logically impossible may be also coherently believed to a positive degree. Any notion of probability that is strongly dogmatic with respect to some logically contingent sentence is therefore inadequate, because for some logically contingent sentence A, it does not allow us to believe partially that A to some positive degree.

The idea that whatever is logically possible may be also coherently believed to a positive degree is not indisputable. For a case in point against it, consider the Dutch-Book strategy of vindicating classical probability. The default assumption in standard Dutch-Book arguments is that the modal space of possible outcomes of a bet is not affected by the placement of the bet. That this assumption is not always safe is shown by bets regarding hypotheses such as

(#) The winner of the next horse race is Golden Arrow, and a zero betting quotient regarding "The winner of the next horse race is Golden Arrow" is accepted as fair.[5]

If we accept a positive betting-quotient for (#) as fair, then it is to be ruled out that (#) may turn out true—for the second conjunct is then in any case false. Any bet regarding (#) with a positive quotient is hence susceptible to a Dutch Book.

> The argument runs analogously to the Dutch-Book argument to the effect that value zero is to be assigned to logical falsities.

(#) however is clearly logically contingent. Hence, we have an argument that our degrees of belief are to be dogmatic with respect to some particular logically contingent sentences. This argument strategy seems to be of no avail though for motivating Weatherson's suggested kind of dogmatism. For example, if we award a positive chance to the possibility that a given patch is a borderline case of redness, then this possibility is not *thereby* ruled out. This leads me to the objection I would like to raise.

Objection from Dutch-Book arguments The strong dogmatism of SGP with respect to any logically contingent sentential expression of gappiness is ill-motivated in that it lacks any underpinning in the form of an interpretation of probability and an associated argument for the conclusion that probabilities are to be dogmatic in the suggested way. At least insofar as we can model credences as betting quotients, the standard Dutch-Book argument strategy does not supply means of motivating SGP. Rather any plausible version of supervaluationist bets suggests that we need to abandon either respect for logic or general additivity. Importantly, no plausible version of supervaluationist bets suggests that we are to be strongly dogmatic with respect to any logically contingent sentential expression of gappiness. Or so I am going to argue.

2 From Classical to Supervaluationist Probability

Weatherson gives a motivation for the intuitionist version of generalized probability by way of a Dutch-Book argument. Thus, it cannot be considered as unfair also to assess the supervaluationist version of generalized probability by the same standard and to look for support in the form of a Dutch-Book argument. In what follows, I discuss what I take to be the most natural options of generalizing classical bets for a supervaluationist framework. All of the associated notions of probability that can be vindicated by way of Dutch-Book arguments are distinct from SGP.

Recall how the Dutch-Book argument strategy runs for a classical framework. A Book consists of a function P of 'betting-quotients' and a function S of 'stakes', where both functions map the language into reals—with S taking nonzero values only for finitely many sentences. The betting-quotient distribution is chosen by the bettor; the stake distribution is chosen by the bookie. If the stake for a hypothesis is positive, the bettor places a bet *on* the hypothesis; if the stake for a hypothesis is negative, she places a bet *against* the hypothesis. Possible outcomes are representable as classical interpretations of the language, taking either value 1 ('truth') or 0 ('false') for sentences. The payoff for a bet regarding a hypothesis A is $(1 - \mathrm{P}(A)) \times \mathrm{S}(A)$ if and only if A is true; otherwise, the payoff is $-\mathrm{P}(A) \times \mathrm{S}(A)$. The payoff for a Book for any classical interpretation I_{CL} is thus given by

$$(\text{Payoff}_{\text{classical}}) \qquad \Sigma_{A\in\mathcal{L}}((\mathrm{I}_{\mathrm{CL}}(A) - \mathrm{P}(A)) \times \mathrm{S}(A)).$$

The idea of Dutch-Book arguments is to characterize the class of coherent betting-quotient distributions as the class of distributions that are safe from a sure loss contract (a 'Dutch-Book'), that is, a Book where the payoff is in any case negative.

Supervaluationist interpretations are not bivalent. They take either value 1 ('truth'), 0 ('false'), or a third value ('neither true nor false'). So how to generalize the Dutch-Book argument strategy for supervaluationist interpretations? The question boils down to the question of how to generalize the payoff conditions for nonbivalent frameworks where sentences may be either true, false, or gappy. I discuss three natural options: Conditional bets (Cantwell [1]), unconditional bets with graded payoffs (Milne [11]), and unconditional bets with ungraded payoffs (Dietz [2]).

2.1 Conditional bets (a) The idea is to let any bet regarding a hypothesis which is neither true nor false be canceled. Otherwise, the payoffs are as for unconditional classical bets. The payoff for any nonbivalent interpretation I of the language is then given by

$$(\text{Payoff}_{\text{conditional}}) \qquad \Sigma_{\{A\in\mathcal{L}:\mathrm{I}(A)\in\{0,1\}\}}((\mathrm{I}(A) - \mathrm{P}(A)) \times \mathrm{S}(A)).$$

For associated Dutch-Book theorems in both directions, I can refer to a general result in Cantwell's paper [1] for a language $\mathcal{L}$ of propositional logic including an operator for truth (D). Cantwell's Dutch-Book argument strategy applies to any formal semantics for $\mathcal{L}$ that meets the following minimal provisos:

(i) Standard connectives of negation, disjunction, and conjunction obey the classical truth-tables if the immediate components are all truth-valued;
(ii) 'D' as attached to a sentence A forms a sentence that is true just in case A is true; otherwise it is false;
(iii) entailment ($\models$) satisfies both

(Equivalence) A is logically equivalent to B just in case A and B agree in truth-value, for any interpretation where A or B is truth-valued, and

(Mutual incompatibility) $\{A, B\} \models$ just in case A and B are not both true, for any interpretation where A or B is truth-valued.

All these provisos are met in a standard supervaluationist framework. For our purposes, it does to focus on the supervaluationist instance of Cantwell's probability calculus: Any function P that takes reals as values for sentences of $\mathcal{L}$ is a *supervaluationist conditional probability function* just in case for every pair of *truth-determinate* sentences, A and B, that is, sentences that are truth-valued for every supervaluationist interpretation,

(1) $0 \leq \mathrm{P}(A) \leq 1$,
(2) If A is (relative to $\models_{\mathrm{SV}}$) logically equivalent to B, then $\mathrm{P}(A) = \mathrm{P}(B)$,

(3) $\mathrm{P}({\sim}A) = 1 - \mathrm{P}(A)$,
(4) If $\{A, B\} \models_{\mathrm{SV}}$, then $\mathrm{P}(A \vee B) = \mathrm{P}(A) + \mathrm{P}(B)$,
and for every sentence A,
(5) $\mathrm{P}(A) = \mathrm{P}(\mathrm{D}A) \div \mathrm{P}(\mathrm{D}A \vee \mathrm{D} {\sim}A)$, if $\mathrm{P}(\mathrm{D}A \vee \mathrm{D} {\sim}A) > 0$.[6]

For the truth-determinate fragment of $\mathcal{L}$, probability functions of this type are classical probability functions. For sentences A of $\mathcal{L}$ in general, supervaluationist conditional probability amounts to classical conditional probability that A is true, given it has a truth-value. As a first result, we are safe from Weatherson's strong dogmatism with respect to every expression of gappiness: For every statement of gappiness, the classical probability of truth may be positive just in case the statement in question is logically contingent. Supervaluationist conditional probability thus is not as strong as Weatherson's SGP. Let us have a closer look at the individual axioms. The classical constraints (1) and (3) are generally valid.[7] Supervaluationist logical truths and supervaluationist inconsistent sentences receive the value one and zero, respectively. The constraint (2), however, is not generally valid. For any supervaluationist conditional probability function P where $0 < \mathrm{P}(\mathrm{D}A \vee \mathrm{D} {\sim}A) < 1$, we have $\mathrm{P}(A) > \mathrm{P}(\mathrm{D}A)$. However, for any sentence A, A and $\ulcorner \mathrm{D}A \urcorner$ are equivalent in supervaluationist logic. Hence, a fortiori, supervaluationist single-premise entailment is not generally respected. Also the finite additivity constraint (4) is not generally valid, which implies that general additivity also does not generally hold.[8] Consequently, two principles of generalized probability, (GP3) and (GP4), are invalid for the general case where also truth-indeterminate sentences may be considered.

2.2 Unconditional bets with graded payoffs (b) We may think of supervaluationist interpretations in analogy to many-valued interpretations, as ranging over *degrees of truth*, with the maximal value 1 for 'truth', the minimal value 0 for 'falsity', and an intermediate value c for 'gappiness'—plausibly c is 0.5, but it may be also any other real number greater than 0 and smaller than 1. The natural idea is then to let the payoff for a bet on (against) a hypothesis be the higher (the lower) the truer it comes out. This comes down to classical payoffs, just with nonbivalent interpretations I, which may take intermediate degrees and which allow for graded payoffs:

$$(\mathrm{Payoff}_{\text{unconditional-graded}}) \qquad \Sigma_{A \in \mathcal{L}}((\mathrm{I}(A) - \mathrm{P}(A)) \times \mathrm{S}(A)).$$

Milne [11] gives Dutch-Book arguments for unconditional bets of this type for fuzzy and many-valued logics where valuations are *additive*; that is, for any valuation v and any sentences A and B, $v(A \,\&\, B) + v(A \vee B) = v(A) + v(B)$. Supervaluationist logic is not additive. For example, it allows pairs of sentences A and B both to be gappy, with their disjunction being gappy as well and their conjunction being false: for instance, the sentences 'This patch is red', 'This patch is green', and 'This patch is red or green' may all receive the intermediate value c ('is gappy'), with 'This patch is both red and green' receiving the value 0 ('is false'). As $c \neq 0$, additivity thus fails. If we apply the Dutch-Book argument strategy for unconditional bets with graded payoffs to supervaluationist logic, we can vindicate the following notion of probability: Any function P that takes reals as values for sentences of $\mathcal{L}$ is a *supervaluationist unconditional probability function for graded payoffs* just in case for every pair of *truth-determinate* sentences, A and B, that is, sentences that are truth-valued for every supervaluationist interpretation, P satisfies (1)–(4), and for every sentence A, P satisfies

$$\mathrm{P}(A) = \mathrm{P}(\mathrm{D}A) + (\mathrm{P}({\sim}\mathrm{D}A \ \& \ {\sim}\mathrm{D} {\sim} A) \times c).^{9} \qquad (5^*)$$

For the truth-determinate fragment, probability functions of this type are classical probability functions. (5*) rewrites

$$\mathrm{P}(A) = (\mathrm{P}(\mathrm{D}A) \times 1) + (\mathrm{P}({\sim}\mathrm{D}A \ \& \ {\sim}\mathrm{D} {\sim} A) \times c) + (\mathrm{P}(\mathrm{D} {\sim} A) \times 0). \qquad (5^{*\prime})$$

As $\ulcorner \mathrm{D}A \urcorner$, $\ulcorner {\sim}\mathrm{D}A \ \& \ \mathrm{D} {\sim} A \urcorner$, and $\ulcorner \mathrm{D} {\sim} A \urcorner$ are all truth-determinate sentences, their values meet finite additivity. Consequently, in the general case, supervaluationist unconditional probability for graded payoffs is given by the expected truth-value. To begin with, expected truth-values for supervaluationist logic allow any statement of gappiness to receive a positive value—we are hence again safe from Weatherson's suggested type of strong dogmatism. Expected truth-values for supervaluationist logic satisfy (1).[10] They also satisfy the constraint (3) just in case the intermediate semantic value c is 0.5.[11] But however we fix the intermediate value c (with $0 < c < 1$), the finite additivity constraint (4) is not satisfied, by failure of the additivity constraint for semantic values.[12] The expected truth-value for logical truths is one, and the expected truth-value of logical contradictions is zero. Thus as the expected truth-values of logical contradictions is zero, general additivity also fails. Even worse, not even (2) is satisfied, whatever real c (with $0 < c < 1$) we chose as intermediate value. Thus, a fortiori, single premise entailment is not respected.[13] What we end up with is hence a notion of supervaluationist probability that violates the same principles of generalized probability as conditional supervaluationist probability: neither (GP3) nor (GP4) is valid.

The foregoing two notions of supervaluationist probability are neither additive nor respecting single-premise entailment. The following option seems in this respect the most attractive one in that it validates all principles of generalized probability that make sure that logic is respected, albeit general additivity turns out again invalid.

2.3 Unconditional bets with ungraded payoffs (c) In [2], I suggest keeping to ungraded payoffs and treating untrue hypotheses like false hypotheses. This comes down to

$$(\mathrm{Payoff}_{\text{unconditional-ungraded}}) \qquad \Sigma_{A\in\mathcal{L}}((\mathrm{V}(\mathrm{I}, A) - \mathrm{P}(A)) \times \mathrm{S}(A))$$

where V is a two-place function that maps pairs of nonbivalent interpretations I and sentences of $\mathcal{L}$ into $\{0, 1\}$ as follows:

$$\mathrm{V}(\mathrm{I}, A) = 1 \text{ if } \mathrm{I}(A) = 1;$$
$$\mathrm{V}(\mathrm{I}, A) = 0 \text{ otherwise.}$$

Starting from payoff conditions of this type, we obtain for supervaluationist logic by way of Dutch-Book arguments the following notion of probability: Any function P that takes reals as values for sentences of $\mathcal{L}$ is a *supervaluationist unconditional probability function for ungraded payoffs* just in case for every pair of *truth-determinate* sentences, A and B, that is, sentences that are truth-valued for every supervaluationist interpretation, P satisfies (1)–(4), and for every sentence A, P satisfies

$$\mathrm{P}(A) = \mathrm{P}(\mathrm{D}A).^{14} \qquad (5^{**})$$

For the truth-determinate fragment, probability functions of this type are classical probability functions. In the general case, supervaluationist unconditional probability for ungraded payoffs are classical probabilities of truth. Probabilities of truth satisfy (1) and also (2)—importantly, they also respect single premise-entailment. However, they neither satisfy (3) nor the finite additivity constraint (4). Importantly, for supervaluationist logic, classical probabilities of truth allow positive values for any logically contingent statement of gappiness—again we are safe from Weatherson's suggested type of strong dogmatism. That supervaluationist unconditional probability for ungraded payoffs is a weakening of SGP is clearer to see on the following reformulation. Any function P that takes reals as values for sentences of $\mathcal{L}$ is a supervaluationist unconditional probability function for ungraded payoffs just in case it satisfies for every pair of sentences of $\mathcal{L}$, A and B, the following conditions:[15]

(GP1_{SV}) If $\{A\} \models_{\text{SV}}$, then $\text{P}(A) = 0$.
(GP2_{SV}) If $\{\ \} \models_{\text{SV}} A$, then $\text{P}(A) = 1$.
(GP3_{SV}) If $\{A\} \models_{\text{SV}} B$, then $\text{P}(A) \leq \text{P}(B)$.
($\text{GP4}_{\text{SV}}*$) $\text{P}(\text{D}A \vee \text{D}B) = \text{P}(A) + \text{P}(B) - \text{P}(A \,\&\, B)$.

As a result, the general additivity constraint (GP4) is not valid. But unlike in the foregoing alternatives, probability respects logic in all relevant regards. In view of these features, it seems fair to view probability of truth as the most attractive candidate for modeling degree of belief in a supervaluationist framework.[16]

3 Conclusion

Weatherson suggests that our degrees of belief are to respect logic and to be additive, whatever logic we may adopt. I argued that this general requirement runs into serious trouble for supervaluationist logic in that it suggests an ill-motivated strong dogmatism with respect to every logically contingent statement of gappiness. If we model degrees of beliefs as betting quotients considered as fair, all natural generalizations of classical bets for supervaluationist frameworks end up in notions of probability that are safe from Weatherson's suggested type of strong dogmatism. None of these notions is as strong as SGP: either they fail to respect single-premise entailment and/or are nonadditive or both. It turns out that in contrast to (a) conditional probabilities of truth given a truth-value and (b) expected truth-values, (c) probabilities of truth do respect logic, albeit they are not additive. In view of this result, it seems fair to award probability of truth the most credit for being a candidate of supervaluationist probability.

Appendix

Theorem 3.1 *Any set of betting quotients that violates* (5*) *for some sentence or some law among* (1)–(4) *for some truth-determinate sentence can be Dutch-booked.*

Proof For truth-determinate sentences A and B, supervaluationist entailment satisfies the constraints (Equivalence) and (Mutual Incompatibility). By the standard Dutch-Book arguments for bivalent probability, we are hence free to assume that (1)–(4) hold with respect to truth-determinate sentences. We only need to show then that a violation of (5*) can be Dutch-booked. Assume $\text{P}(A) \neq \text{P}(\text{D}A) + (\text{P}({\sim}\text{D}A \,\&\, {\sim}\text{D} {\sim}A) \times c)$, where $0 < c < 1$.

Case (a) $\mathrm{P}(A) < (\mathrm{P}(\mathrm{D}A) + \mathrm{P}(\sim\mathrm{D}A\ \&\ \sim\mathrm{D}\sim A) \times c)$. Take a stake distribution S where $\mathrm{S}(A) = -1$, $\mathrm{S}(\mathrm{D}A) = 1$, $\mathrm{S}(\sim\mathrm{D}A\ \&\ \sim\mathrm{D}\sim A) = c$ and where for the remaining sentences, S takes value zero. The payoffs for this Book amount then for any interpretation to $\mathrm{P}(A) - \mathrm{P}(\mathrm{D}A) - (\mathrm{P}(\sim\mathrm{D}A\ \&\ \sim\mathrm{D}\sim A) \times c)$, which by assumption (a) is negative.

Case (b) $\mathrm{P}(A) > (\mathrm{P}(\mathrm{D}A) + \mathrm{P}(\sim\mathrm{D}A\ \&\ \sim\mathrm{D}\sim A) \times c)$. Take a stake distribution S where $\mathrm{S}(A) = 1$, $\mathrm{S}(\mathrm{D}A) = -1$, $\mathrm{S}(\sim\mathrm{D}A\ \&\ \sim\mathrm{D}\sim A) = -c$ and where the remaining sentences take value zero. The payoffs for this Book amount then for any interpretation to $\mathrm{P}(\mathrm{D}A) + (\mathrm{P}(\sim\mathrm{D}A\ \&\ \sim\mathrm{D}\sim A) \times c) - \mathrm{P}(A)$, which by assumption (b) is negative. Thus if P violates (5*), it can be Dutch-booked in any case. □

Theorem 3.2 *No set of betting quotients that satisfies both* (1)–(4) *for every truth-determinate sentence and* (5*) *for every sentence can be Dutch-booked.*

Proof Suppose that P is a betting-quotient distribution that satisfies both (1)–(4) for every truth-determinate sentence and (5*) for every sentence, and that there is a Dutch Book B_0 for P. Then the stakes for finitely many truth-indeterminate sentences must be distinct from zero; for by the converse Dutch-Book argument for classical probability for bivalent languages (Kemeny [7]), there is no Dutch Book for betting quotients for truth-determinate sentences meeting (1)–(4). Let $\{A_1, \ldots, A_n\}$ be the set of truth-indeterminate sentences for which the stake distribution S_0 in B_0 takes a nonzero value. Define a new stake function S_1 as follows:

$$\mathrm{S}_1(A) = 0, \mathrm{S}_1(\mathrm{D}A) = S_0(\mathrm{D}A) + \mathrm{S}_0(A), \mathrm{S}_1(\sim\mathrm{D}A\ \&\ \sim\mathrm{D}\sim A) = \mathrm{S}_0(\sim\mathrm{D}A\ \&\ \sim\mathrm{D}\sim A) + (\mathrm{S}_0(A) \times c).$$

The resulting Book B_1 has the same net gain for any outcome as B_0. By $(n-1)$-many reiterations of this procedure, we obtain a Book B_n that agrees with B_0 in the net gains for all possible outcomes, and where only truth-determinate sentences have a nonzero stake value. As, by (1)–(4) for truth-determinate sentences, the betting-quotients for these sentences have a classical structure, it follows by the converse Dutch-Book argument for classical probability for bivalent languages that B_n cannot be a Dutch Book. By reductio, P cannot be Dutch-booked. □

Theorem 3.3 *Any set of betting quotients that violates* (5**) *for some sentence or some law among* (1)–(4) *for some truth-determinate sentence can be Dutch-booked.*

Proof For truth-determinate sentences A and B, supervaluationist entailment satisfies the constraint (Equivalence) and (Mutual Incompatibility). By the standard Dutch Book arguments for bivalent probability, we are hence free to assume that (1)–(4) hold for truth-determinate sentences. We only need to show then that a violation of (5**) can be Dutch-booked. Assume $\mathrm{P}(A) \neq \mathrm{P}(\mathrm{D}A)$.

Case (a) $\mathrm{P}(A) < \mathrm{P}(\mathrm{D}A)$. Set $\mathrm{S}(A) = -1, \mathrm{S}(\mathrm{D}A) = 1$, and for the remaining sentences, give zero stakes. The payoffs for this book for any interpretation I are then given as

I(A)	Payoff for I
1	$-(1 - \mathrm{P}(A)) + (1 - \mathrm{P}(\mathrm{D}A))$
0	$\mathrm{P}(A) - \mathrm{P}(\mathrm{D}A)$
–	$\mathrm{P}(A) - \mathrm{P}(\mathrm{D}A)$.

For every possible assignment, the net gains are thus negative.

Case (b) For the converse case (b) that $\mathrm{P}(A) > \mathrm{P}(\mathrm{D}A)$, set $\mathrm{S}(A) = 1$ and $\mathrm{S}(\mathrm{D}A) = -1$, and for the remaining sentences, give zero stakes. Again (by parity of reasoning) the net gains are for every possible assignment negative. Consequently, if $\mathrm{P}(A) \neq \mathrm{P}(\mathrm{D}A)$, P can be Dutch-booked. □

Theorem 3.4 *No set of betting quotients that satisfies both* (1)–(4) *for every truth-determinate sentence and* (5**) *for every sentence can be Dutch-booked.*

Proof Suppose there is a set of betting quotients P that both satisfies the constraints (1)–(4) for every truth-determinate sentence and (5**) for every sentence, and is susceptible to a Dutch Book B_0. Then the stakes for some truth-indeterminate sentences must be distinct from zero; for by the converse Dutch-Book argument for classical probability for bivalent languages (Kemeny [7]), there is no Dutch Book for betting quotients for truth-determinate sentences that meet (1)–(4). Let $\{A_1, \ldots, A_n\}$ be the set of truth-indeterminate sentences for which the stake distribution S_0 in B_0 takes values distinct from zero. Define a new stake function S_1 on $\mathcal{L}$ as follows: $\mathrm{S}_1(A_1) = 0$, $\mathrm{S}_1(\mathrm{D}A_1) = \mathrm{S}_0(\mathrm{D}A_1) + \mathrm{S}_0(A_1)$; for all other sentences A, $\mathrm{S}_1(A) = \mathrm{S}_0(A)$. The net gains for the resulting book B_1 are then for every assignment the same as on B_0. By $(n-1)$-many reiterations of this procedure, we obtain from B_0 a book B_n that agrees with in the net gains for all possible outcomes, and where only truth-determinate sentences take a nonzero stake value. As, by (1)–(4) for truth-determinate sentences, the betting-quotient distribution for these sentences is classical, by the converse Dutch-Book theorem for classical probability for bivalent languages, B_n cannot be a Dutch Book. Thus by reductio, P cannot be Dutch-booked. □

Theorem 3.5 *Any function* P *that satisfies both* (1)–(4) *for every truth-determinate sentence and* (5**) *for every sentence satisfies* ($\mathrm{GP1}_{\mathrm{SV}}$)–($\mathrm{GP4}_{\mathrm{SV}}$*) *for every sentence, and vice versa.*

Proof ($\Rightarrow$) To begin with ($\mathrm{GP2}_{\mathrm{SV}}$), if $\{\,\} \models_{\mathrm{SV}} A$, then A is truth-determinate and SV-equivalent to $\ulcorner A \vee \sim A \urcorner$, which is truth-determinate as well. Hence, by (2), $\mathrm{P}(A) = \mathrm{P}(A \vee \sim A)$. As the negation of A is truth-determinate, if A is truth-determinate, by (3) and (4), $\mathrm{P}(A \vee \sim A) = 1$. Hence $\mathrm{P}(A) = 1$. On ($\mathrm{GP1}_{\mathrm{SV}}$), if $A \models_{\mathrm{SV}}$, then also $\{\mathrm{D}A\} \models_{\mathrm{SV}}$. $\mathrm{D}A$ is truth-determinate and SV-equivalent to $A \,\&\, \sim A$, which is truth-determinate as well. Hence, by (2), $\mathrm{P}(\mathrm{D}A) = \mathrm{P}(A \,\&\, \sim A)$. By ($\mathrm{GP2}_{\mathrm{SV}}$), (2) and (3), then $\mathrm{P}(\mathrm{D}A) = 0$. Hence, by (5**), $\mathrm{P}(A) = 0$. On ($\mathrm{GP3}_{\mathrm{SV}}$), assume $\{A\} \models_{\mathrm{SV}} B$. Then $\{\mathrm{D}A\} \models_{\mathrm{SV}} \mathrm{D}B$. Now we have $\{\mathrm{D}A\} \models_{\mathrm{SV}} \mathrm{D}B$ just in case $\ulcorner \mathrm{D}A \vee (\sim \mathrm{D}A \,\&\, \mathrm{D}B) \urcorner$ is SV-equivalent to $\ulcorner \mathrm{D}B \urcorner$, either of which is truth-determinate. Hence, by (2), $\mathrm{P}(\mathrm{D}A \vee (\sim \mathrm{D}A \,\&\, \mathrm{D}B)) = \mathrm{P}(\mathrm{D}B)$. As $\ulcorner \mathrm{D}A \urcorner$ and $\ulcorner \sim \mathrm{D}A \,\&\, \mathrm{D}B \urcorner$ are mutually incompatible and both are truth-determinate, from (4), as an instance, $\mathrm{P}(\mathrm{D}A \vee (\sim \mathrm{D}A \,\&\, \mathrm{D}B)) = \mathrm{P}(\mathrm{D}A) + \mathrm{P}(\sim \mathrm{D}A \,\&\, \mathrm{D}B)$. Hence, by (1), $\mathrm{P}(\mathrm{D}A) \leq \mathrm{P}(\mathrm{D}B)$. From this, by (5**), it follows that $\mathrm{P}(A) \leq \mathrm{P}(B)$. On ($\mathrm{GP4}_{\mathrm{SV}}$*), for any A and B, $\ulcorner \mathrm{D}A \vee \mathrm{D}B \urcorner$ is SV-equivalent to $\ulcorner (\mathrm{D}A \,\&\, \mathrm{D}B) \vee (\mathrm{D}A \,\&\, \sim \mathrm{D}B) \vee (\sim \mathrm{D}A \,\&\, \mathrm{D}B)) \urcorner$. As the latter two sentences are truth-determinate, by (2), $\mathrm{P}(\mathrm{D}A \vee \mathrm{D}B) = \mathrm{P}((\mathrm{D}A \,\&\, \mathrm{D}B) \vee (\mathrm{D}A \,\&\, \sim \mathrm{D}B) \vee (\sim \mathrm{D}A \,\&\, \mathrm{D}B))$. As $\ulcorner \mathrm{D}A \,\&\, \mathrm{D}B \urcorner$, $\ulcorner \mathrm{D}A \,\&\, \sim \mathrm{D}B \urcorner$, and $\ulcorner \sim \mathrm{D}A \,\&\, \mathrm{D}B \urcorner$ are all truth-determinate and pairwise incompatible, from iterated application of (4), we get $\mathrm{P}(\mathrm{D}A \vee \mathrm{D}B) =$

$P(DA \ \& \ DB) + P(DA \ \& \ {\sim}DB) + P({\sim}DA \ \& \ DB)$. By parity of reasoning, $P(DA) = P(DA \ \& \ DB) + P(DA \ \& \ {\sim}DB)$ and $P(B) = P(DA \ \& \ DB) + P({\sim}DA \ \& \ DB)$. Furthermore, as $\ulcorner DA \ \& \ DB \urcorner$ is SV-equivalent to $\ulcorner D(A \ \& \ B) \urcorner$, which are both truth-determinate, by (2), $P(DA \ \& \ DB) = P(D(A \ \& \ B))$. By (5**), $P(D(A \ \& \ B)) = P(A \ \& \ B)$. Hence, $P(DA \vee DB) = P(A) + P(B) - P(A \ \& \ B)$.

($\Leftarrow$) On (1), it follows from ($GP1_{SV}$)–($GP3_{SV}$). On (2), it follows from ($GP3_{SV}$). On (3), if A is truth-determinate, so is its negation. In this case, $\ulcorner A \vee {\sim}A \urcorner$ is SV-equivalent to $\ulcorner DA \vee D{\sim}A \urcorner$. Hence, by ($GP3_{SV}$), $P(A \vee {\sim}A) = P(DA \vee D{\sim}A)$. By ($GP4_{SV}$*), $P(A \vee {\sim}A) = P(A) + P({\sim}A) - P(A \ \& \ {\sim}A)$. As $\models_{SV} A \vee {\sim}A$, by ($GP1_{SV}$), $P(A \vee {\sim}A) = 1$. As $\{A \ \& \ {\sim}A\} \models_{SV}$, by ($GP2_{SV}$), $P(A \ \& \ {\sim}A) = 0$. Thus $P({\sim}A) = 1 - P(A)$. On (4), on the assumption that A and B are truth-determinate, $\ulcorner A \vee B \urcorner$ is logically equivalent with $\ulcorner (DA \vee DB) \urcorner$. By ($GP4_{SV}$*), from this, it follows that $P(A \vee B) = P(A) + P(B) - P(A \ \& \ {\sim}B)$. On the further assumption that A and B are mutually incompatible, by ($GP2_{SV}$), we have $P(A \ \& \ {\sim}B) = 0$. Thus $P(A \vee B) = P(A) + P(B)$. On (5**), as A is SV-equivalent to $\ulcorner DA \urcorner$, by ($GP3_{SV}$), $P(A) = P(DA)$. □

Notes

1. Kolmogorov's calculus is formulated for probability functions on algebras of events; see his [8, p. 2]. Since Weatherson discusses probability assignments to sentences in languages, I refer here to the translation of Kolmogorov's axioms for languages of propositional logic; compare Howson and Urbach [6, p. 21].

2. Compare Kremer & Kremer [9]. For the standard system of supervaluationist logic, see Fine's seminal paper [4].

3. On supervaluationist logic, for instance, for all logically contingent sentences A that do not involve any occurrence of 'D', $\ulcorner {\sim}DA \ \& \ {\sim}D{\sim}A \urcorner$ in turn is logically contingent.

4. I adopt this terminology from Howson [5, p. 70].

5. For other arguable cases in point, see Milne's Dutch-Book arguments in [10] suggesting that rational agents are to be perfectly accurate about their degree of beliefs.

6. To obtain Cantwell's calculus for the general case where we consider any semantics where interpretations and the associated entailment relation $\models$ meet the constraints (i)–(iii) (as given on p. 10), we just need to replace reference to $\models_{SV}$ in (2) and (4) by reference to $\models$.

7. On (1), it is valid, by (5) and (1), for truth-determinate sentences. On (3), it is valid, by (5) and the fact that by classical probability for truth-determinate sentences, $P(DA \vee D{\sim}A) = P(DA) + P(D{\sim}A)$.

8. For a counterexample, consider any truth-indeterminate sentence A and any SCP-function P where $P(DA) = P(D{\sim}A) = 0$. We have then $P(A \vee {\sim}A) = 1$, but $P(A) = P({\sim}A) = 0$.

9. See Appendix, Theorems 3.1 and 3.2.

10. By (5*), (1) for truth-determinate sentences, and arithmetic.

11. By classical probability for truth-determinate sentences, $\mathrm{P}(\mathrm{D}A)+\mathrm{P}({\sim}\mathrm{D}A \ \& \ {\sim}\mathrm{D} {\sim}A)+\mathrm{P}(\mathrm{D} {\sim}A)=1$. Thus, if $\mathrm{P}({\sim}\mathrm{D}A \ \& \ {\sim}\mathrm{D} {\sim}A)>0$, $\mathrm{P}(\mathrm{D}A)+(\mathrm{P}({\sim}\mathrm{D}A \ \& \ {\sim}\mathrm{D} {\sim}A)\times c)+(\mathrm{P}({\sim}\mathrm{D}A \ \& \ {\sim}\mathrm{D} {\sim}A)\times c)+\mathrm{P}(\mathrm{D} {\sim}A)=1$ if and only if $c=0.5$.

12. For example, if the space of possible interpretations narrows down to one interpretation where (1) 'The patch is red or green', (2) 'The patch is red', (3) 'The patch is green' are all neither true nor false, and where (4) 'The patch is red and green' is false, the expected truth-value for (1)–(3) each is c, and the expected truth-value of (4) is 0. But as $0<c<1$, $c \neq c+c-0$.

13. For any A, A is SV-equivalent to $\ulcorner \mathrm{D}A \urcorner$. But the expected truth-values for A and $\ulcorner \mathrm{D}A \urcorner$ are c and 0, respectively, if the space of possible interpretations shrinks down to interpretations where A is gappy and $\ulcorner \mathrm{D}A \urcorner$ hence false, and c is constrained to be in any case distinct from zero.

14. See Appendix, Theorems 3.3 and 3.4.

15. See Appendix, Theorem 3.5.

16. In [2], I am primarily concerned with a nonclassical calculus of probability suggested in Field's [3], of which I show that it is equivalent to supervaluationist unconditional probability for ungraded payoffs. The discussion in this paper goes essentially beyond my [2] in that it gives an appraisal of the pros and cons of alternative ways of modeling supervaluationist probability.

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