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On the Degrees of Diagonal Sets and the Failure of the Analogue of a Theorem of Martin

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Abstract Semi-hyperhypersimple c.e. sets, also known as diagonals, were introduced by Kummer. He showed that by considering an analogue of hyperhypersimplicity, one could characterize the sets which are the Halting problem relative to arbitrary computable numberings. One could also consider half of splittings of maximal or hyperhypersimple sets and get another variant of maximality and hyperhypersimplicity, which are closely related to the study of automorphisms of the c.e. sets. We investigate the Turing degrees of these classes of c.e. sets. In particular, we show that the analogue of a theorem of Martin fails for these classes.

1 Introduction

The study of computably enumerable (c.e.) sets and their Turing degrees has a long and rich history. In this paper, we are concerned with several classes of c.e. sets, which show up naturally in several contexts, and in their Turing degrees. In particular, these classes arose from the study of automorphisms of the lattice of c.e. sets, as well as in the classification of the sets which are the analogues of the Halting problem. We will first discuss the origins of these classes of sets and then state the motivations behind our interest in these sets.

The c.e. sets we are concerned with in this paper are the analogues of maximal(max) and hyperhypersimple(hhs) sets, when we replace the congruence $=^*$ (finite symmetric difference) by a more general congruence relation $=^{\text{c.e.}}$ where $A =^{\text{c.e.}} B$ if and only if the summetric difference of A and B is computably enumerable. Formally, we have the following definition.

Definition 1.1 A weak array $\{V_x\}_{x\in\mathbb{N}}$ is a sequence of uniformly c.e. sets; that is, there is some computable f such that $V_x = W_{f(x)}$ for all x. A sequence of

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sets is disjoint if the sets are mutually disjoint. We say that a c.e. set A is *semi-hyperhypersimple* if for every weak array of disjoint sets $\{V_x\}_{x\in\mathbb{N}}$, there is some x such that $V_x - A$ is c.e. A set A is *semi-maximal* if for every pair of disjoint c.e. sets W_i, W_j, we have either $W_i - A$ is c.e. or $W_j - A$ is c.e.

The sets in Definition 1.1 were introduced by Kummer in [11]. Kummer showed that the semi-hhs sets arose naturally in degree theory, more specifically, in the study of different versions of the Halting problem. Recall that $F(i, x)$ is a computable numbering of $P :=$ {all partial computable functions of a single variable} if F is partial computable (in two variables) and $\{\lambda x F(i, x)\}_{i\in\mathbb{N}} = P$. A Gödel numbering is a computable numbering F where there is an effective way of getting between F and the standard numbering of P. That is, there are computable functions a and b such that for every i, $\lambda x F(i, x) = \varphi_{a(i)}$ and $\varphi_i = \lambda x F(b(i), x)$.

It is not hard to see that not every computable numbering is a Gödel numbering; for instance, Friedberg [9] gave a numbering of P (in fact, of the c.e. sets) without repetition, generally known as Friedberg numberings. The diagonal (i.e., a version of the Halting problem) induced by Gödel numberings corresponds to the creative sets. Diagonals induced by arbitrary computable numberings are therefore a natural thing to study. Kummer then proved that the diagonal sets which are Halting problems relative to arbitrary computable numberings can be characterized in terms of a class of c.e. sets defined in a seemingly unrelated way.

Theorem 1.2 *A c.e. set is a diagonal of some computable numbering if and only if it is not semi-hyperhypersimple.*

It was also shown in [11] that the semi- and plain versions of both maximality and hyperhypersimplicity coincided for simple sets further demonstrating that these sets are natural extensions of maximality and hyperhypersimplicity.

In [10], Hermann and Kummer explored the lattice theoretic properties of semi-max and semi-hhs sets. Let $\mathcal{L}^*(A)$ and $\mathcal{L}^{\text{c.e.}}(A)$ denote the lattice of c.e. supersets of A modulo the ideal of supersets B of A such that $B - A$ is finite (and c.e., respectively). They showed that A is semi-hhs if and only if $\mathcal{L}^{\text{c.e.}}(A)$ is a Boolean algebra, generalizing the well-known result of Lachlan [12] where A is hyperhypersimple if and only if $\mathcal{L}^*(A)$ is a Boolean algebra. They obtained the corollary.

Corollary 1.3 *The property of being a diagonal is elementarily lattice theoretic.*

This means that being the Halting problem relative to some arbitrary computable numbering is elementarily lattice theoretic, generalizing a well-known theorem of Harrington that being the Halting problem relative to the standard numbering is elementary lattice theoretic. They also showed that a coinfinite set A is semi-max if and only if $|\mathcal{L}^{\text{c.e.}}(A)| = 2$; again this result is analogous to the relationship between maximality and the cardinality of $\mathcal{L}^*(A)$. For these reasons, the noncomputable semi-max sets are sometimes called $\mathcal{D}$-maximal, and the noncomputable semi-hhs sets are sometimes referred to as $\mathcal{D}$-hhs in literature. These results further reinforce the idea that the semi-maximal and semi-hyperhypersimple c.e. sets are in many ways analogues of maximal and hyperhypersimple c.e. sets.

Besides being connected to different versions of the Halting problem, these c.e. sets also show up in relation to the study of automorphisms of c.e. sets, as well as their connection with the hemi-maximal and Hermann sets. A splitting of a c.e. set

A is a pair A_0, A_1 of disjoint c.e. sets such that $A_0 \sqcup A_1 = A$. The splitting is nontrivial if both A_0, A_1 are noncomputable. A *hemi-maximal* (*hemi-hyperhypersimple*) set is half of a nontrivial splitting of a maximal (hyperhypersimple) set. Let **SM**, **SHHS**, **HM**, and **HHHM** denote the classes of c.e. Turing degrees which contain, respectively, a semi-max, semi-hhs, hemi-max, and hemi-hhs set. The following shows the relationship between the various degree classes in which we have an interest; these can be shown easily. The upward arrows follow from the proof that every maximal set is hhs, and the other implications are in Kummer [11].

$$\begin{tikzcd}
\mathbf{HHHS} \arrow[r] & \mathbf{SHHS} \\
\mathbf{HM} \arrow[u] \arrow[r] & \mathbf{SM} \arrow[u]
\end{tikzcd}$$

Harrington proved that the creative sets form a definable orbit which realizes only sets of complete m-degrees. He proved that any two creative sets were effectively automorphic. It was realized that simply considering effective automorphisms alone was not enough to discover more orbits. In an influential paper [14], Soare developed what is known today as the automorphism machinery, which gave a general method of constructing Δ_3^0 automorphisms via the extension lemma. He used it to show that the maximal sets form an orbit, which contains only sets of high degree. Despite Soare's discovery, there are still very few known definable orbits to date, and many known orbits are generated by the splittings of sets which are themselves known to produce orbits. Downey and Stob [5] showed that by considering half of nontrivial splittings of maximal sets (the hemi-maximal sets), one could get a new definable orbit. They also showed in [8] that these sets occur in every jump class. These results on the orbit of the hemi-maximal sets lead to an interesting question about the relationship between orbits in $\mathrm{Aut}(\mathcal{E})$ and the Turing degree of sets realizing these orbits. In particular, the hemi-maximal sets give us an example of a definable orbit which is relatively large in the sense that it contains representatives in every possible jump class. Is it possible for there to be an orbit which contains a member of every noncomputable c.e. Turing degree? In [6] Downey and Harrington showed that there can be no such fat orbit.

The semi-max and semi-hhs sets show up again in the search for orbits. Interestingly enough, semi-maximality together with strong r-separability gives us a class of sets which forms a new orbit. A *Hermann* set is a noncomputable semi-maximal set which is strongly r-separable in the sense that every c.e. set disjoint from it can be separated by a computable set in an infinite way. Clearly every maximal, hemi-max, and Hermann set is semi-maximal. Cholak, Downey, and Hermann [2] showed that the Herman sets form an orbit, again with representatives from every jump class. These sets play a central role in a recent paper of Cholak, Downey, and Harrington [3], where they showed that not every orbit is elementarily definable. In particular, they constructed an orbit which is as complicated as can be.

Various degree theoretic results about the hemi-maximal sets were obtained by Downey and Stob in [8; 5; 7]. They showed that every high c.e. degree contains a hemi-maximal set and that **HM** was downward dense. The latter proof is a straightforward modification of the Friedberg maximal set construction. The reason why we cannot combine permitting with a maximal set construction M is due to the following: when we are required to move a marker, say m_i with current value $m_i[s] = p$, for the sake of improving its e-state, we might not have permission from the given set. On the other hand, if we merely wanted to make M hemi-maximal, we could move the marker m_i by dumping the value p into the other half of the splitting. The only trouble we get by doing this is that we are not able to enumerate p into A anymore. This is not an issue if we only needed to make A noncomputable. This is characteristic of proofs involving sets with hemi- (and even semi-) properties and provides a way of exploiting the hemi- (or semi-) properties. Downey and Stob showed further that not every c.e. degree is in **HM**, that these degrees can be found in every jump class, and that they are nowhere dense in the low c.e. degrees. These results refute a number of conjectures. As a corollary it follows that there is an orbit containing sets of every high degree and yet does not only contain sets of high degree (unlike the orbit of the maximal sets). It also shows that the degrees of sets in an orbit are not necessarily closed upward.

Our interest in these sets is generated by Martin's results. Martin, in a classic paper [13], showed the coincidence of several classes of degrees. He showed that the c.e. degrees containing a maximal set were exactly the high degrees, that is, those degrees $\boldsymbol{a}$ such that $\boldsymbol{a}' = \boldsymbol{0}''$. These were also exactly the c.e. degrees containing a hyperhypersimple set, as well as the degrees containing a dense simple set. Our work here is to contribute to the understanding of the Turing degrees of such sets. The combined work of Kummer [11] and Cholak, Downey, and Hermann [2] showed that the degrees in **SM** and **SHHS** also satisfy the same degree theoretic facts listed above for **HM**. The similarity in the Turing degree structures led Kummer [11] to ask if his result was a strict improvement, that is, whether or not **HM** = **SHHS**. On the other hand, it is also natural to suggest that an analogue of the classic theorem of Martin holds, also asked by Kummer in [11]. In this paper we will answer these questions by demonstrating that the classes **HHHS** and **SM** are incompatible.

Theorem 3.1 *There is a c.e. set A which is semi-maximal, and for all c.e. $B \equiv_T A$, B is not hemi-hyperhypersimple.*

Theorem 4.1 *There is a c.e. set A which is hemi-hyperhypersimple, and for all c.e. $B \equiv_T A$, B is not semi-maximal.*

As a corollary we have that none of the implications in the previous diagram can be reversed and that the analogues of Martin's theorem fail in both cases (semi- and hemi-). Despite the fact that all the results known so far about the Turing degrees of these classes suggest that Martin's results can be generalized to these classes, we are able to demonstrate otherwise.

Corollary 1.4 *While the degrees of maximal and hyperhypersimple sets are the same, when we prefix hemi- or semi-, they are not the same.*

In Section 3 we will construct a c.e. degree in **SM** but not in **HHHS**. The idea of streaming is introduced and explained in Section 3 and forms the central ingredient in the proof of the main result in Section 4.

2 Preliminaries

Our notation is standard and follows Soare [15]. The reader is assumed to have familiarity with standard tree arguments. In this paper our construction trees grow downward. In Section 3 the construction is a fairly standard tree argument, with modules replacing the subrequirements. The construction in Section 4, however, is a little different in the following sense. In usual tree arguments, once a follower is appointed and later abandoned, it is never reused. In this case we will need to reuse certain followers which have been discarded. This will be coordinated by the top nodes in the spirit of $\varnothing'''$-arguments. How and when this is carried out is explained in further detail in Section 4.

We will drop the stage number from the notations if the context is clear. Within a stage s of the construction, several actions may take place and will change the value of an expression P when evaluated at different points within stage s. We will use $P[s]$ and also sometimes P_s to denote the value evaluated at the instance within stage s when it is mentioned. We adopt the convention of using uppercase Greek letters to denote functionals and lowercase Greek letters for their use. That is, $\gamma_e(x)$ refers to the use of $\Gamma_e^{U_e}(x)$, whereas $\delta_e(x)$ refers to the use of $\Delta_e^A(x)$. The use refers to the largest bit of the oracle accessed during the computation. Since we are only concerned with reductions which are total, we may assume that the use functions are nondecreasing. That is, $\gamma_e(x) \leq \gamma_e(x+1)$ for every x. Also the use of any convergent computation at s is less than s.

3 A Semi-maximal Set Whose Degree Does Not Contain a Hemi-hyperhypersimple Set

Theorem 3.1 *There is a c.e. set $A >_T \varnothing'$ which is semi-maximal, and for all c.e. $B \equiv_T A$, B is not hemi-hyperhypersimple.*

3.1 Requirements

We build a c.e set A satisfying the following requirements:

$$\mathcal{R}_e \quad : \quad \text{If } X_e \cap Y_e = \varnothing, \text{ then one of } X_e \cap \bar{A} \text{ or } Y_e \cap \bar{A} \text{ is c.e.}$$
$$\mathcal{Q}_e \quad : \quad \text{If } \Gamma_e^{U_e} = A \text{ and } \Delta_e^A = U_e, \text{ and } U_e \cap V_e = \varnothing, \text{ then } U_e \sqcup V_e \text{ is not hyperhypersimple.}$$

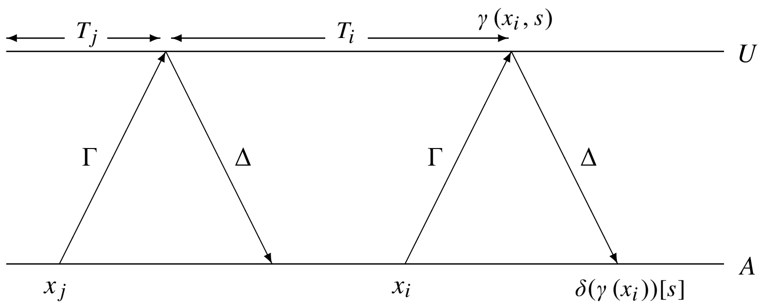
We let $\langle \Gamma_e, \Delta_e, X_e, Y_e, U_e, V_e \rangle_{e \in \mathbb{N}}$ be an effective list of all tuples $\langle \Gamma, \Delta, X, Y, U, V \rangle$ such that Γ, Δ are Turing functionals, and X, Y, U, V are c.e. sets. It is clear that A built this way is of neither computable nor of high degree.

The strategy for requirement $\mathcal{Q}_e$ builds the disjoint weak array $\{T_i\}_{i \in \mathbb{N}}$ witnessing the fact that $U_e \sqcup V_e$ is not hyperhypersimple. The *ith module of* $\mathcal{Q}_e$ ensures that $T_i \cap \overline{U_e \sqcup V_e} \neq \varnothing$, and we will try to ensure that all of the $\mathcal{Q}_e$-modules are successful.

3.2 Description of strategy

In this section, we discuss the strategy for each requirement in isolation. Hence, we will drop the subscript and write $\mathcal{R}$ or $\mathcal{Q}$ when discussing the respective strategies.

The basic strategy used to build a set A which is not of hemi-hyperhypersimple degree can be found in [5; 7]; we describe it here briefly for the benefit of the reader. The strategy for requirement $\mathcal{Q}$ builds $\{T_i\}_{i \in \mathbb{N}}$ and wants to ensure that $T_i \cap \overline{U \sqcup V} \neq \varnothing$ through the ith module, which we will call M_i. The action for each M_i is the following: it starts by picking a follower x_i targeted at A and waits for $l_\Gamma(s) > x_i$ and $l_\Delta(s) > \gamma(x_i, s)$, where l denote the respective lengths of agreement. Once that happens, we would enumerate into T_i all the numbers $y \leq \gamma(x_i, s)$ such that $y \notin U_s \sqcup V_s$, and y is not already in some other T_j. We would now freeze $A \restriction_{\delta(\gamma(x_i))[s]}$ to preserve the computations in both directions. The picture below summarizes the situation.



Note that M_i would be satisfied temporarily, for it would have made $T_i[s] \cap \overline{U_s \sqcup V_s} \neq \varnothing$. M_i would not need to do anything else until all the numbers we put in T_i had also entered V (they could not have entered U due to the A-restraint we imposed). When that happens, we would then put x_i into A and freeze $A \restriction_{x_i}$. Now some time in the future, U would have to respond with a change below $\gamma(x_i, s)$. Since U and V are disjoint, this cannot be a number we had placed in T_i at s. Hence, it has to be a number $< \gamma(x_j, s)$ which would be impossible since we always hold $A \restriction_{x_i}$. To summarize, the action of M_i is to

1. pick a follower x_i large enough; wait for l_Γ and $l_\Delta(s)$ to grow;
2. satisfy $T_i[s] \cap \overline{U_s \sqcup V_s} \neq \varnothing$ temporarily and impose A-restraint;
3. if ever $T_i[t] \cap \overline{U_t \sqcup V_t} = \varnothing$, enumerate x_i into A and impose A-restraint; if we ever enter this state, we would have a global win on the requirement $\mathcal{Q}$.

Since each module only imposes finite A-restraint and enumerates at most once, it is easy to see that all modules of the requirements $\mathcal{Q}_0, \mathcal{Q}_1, \ldots$ can be arranged in the style of a finite injury method if we only wanted to build such a set A with no additional property. However, the presence of the $\mathcal{R}$-type requirements forces us to have to be careful about the choice of followers x_i for a $\mathcal{Q}$-module, as described below.

We will now discuss the plan to satisfy a single requirement $\mathcal{R}$. Suppose we were trying to construct a maximal set: one maintains a set of markers $a_0[s], a_1[s], \ldots$ which are all pointing at elements in $\bar{A}_s$. We try to maximize the state of each marker a_i by letting $a_i[s+1]$ occupy the location of some $a_j[s]$ for some $j > i$ if such an action increases the state of the marker a_i.

Things would be very bad for the $\mathcal{Q}$-requirements, if we had to do the above, for each movement of a marker a_i is accompanied by the enumeration of all the values $a_i[s], a_{i+1}[s], \ldots, a_{j-1}[s]$ into A. Fortunately for us, we are allowed to have infinitely many elements in $\bar{A}$ whose states are never maximized. All we need is to ensure that either $X \cap \bar{A}$ or $Y \cap \bar{A}$ is c.e. Note that the set $X \cap \bar{A}$ is 2-c.e. since a number might first enter X, then later enter A. What we want to do is to prevent the latter from happening infinitely often without our permission on one of the two sides $X \cap \bar{A}$ or $Y \cap \bar{A}$.

More precisely, it is perfectly all right for us to have an alternating sequence of numbers $n_0 < n_1 < n_2 < \cdots$ such that $n_{2k} \in Y \cap \bar{A}$ and $n_{2k+1} \in X \cap \bar{A}$. What we do is that each time a new n_{2k+2} shows up (i.e., enters Y), we will freeze all numbers strictly between n_{2k} and n_{2k+2} (i.e., keep them out of A). We claim this does the job for $\mathcal{R}$: if we only freeze finitely many intervals, then $Y \cap \bar{A}$ is finite. If we freeze infinitely many intervals, then $X \cap \bar{A}$ is c.e., because if some $(X \cap \bar{A})(n)$ flips from 0 to 1 after n is frozen, then it can never flip back to 0.

As a side note, we remark that the strategy used to build a hemi-maximal (hemi-hyperhypersimple) set is very similar to the strategy used to build a semi-maximal (semi-hyperhypersimple) set except that we are not allowed any injuries to each separate requirement. For example, in the above, we are allowed to make A-enumerations within a frozen interval, and provided this happens finitely often our semi-maximal requirement is still satisfied. If we were instead trying to make A hemi-maximal by building the other half C of the splitting of a maximal set $A \sqcup C$, then once we freeze an A-interval (x_1, x_2) (which means we dump the entire interval (x_1, x_2) into C to maximize some states), then we cannot allow A to change within the frozen interval (x_1, x_2) anymore.

3.3 Interaction between strategies and the streaming procedure The construction is to take place on a subtree of $2^{<\omega}$, and we think of the construction tree as growing downward. To implement the above strategy at an $\mathcal{R}$-node α, we use a process called *streaming*. This term will be used throughout this proof and the next and is a crucial ingredient in helping $\mathcal{Q}$-requirements in their selection of followers. We maintain a list S^α of numbers which are *streamed by* α where α is an $\mathcal{R}$-node. The way to think about streaming is the following. One pictures $\mathbb{N}$ as a collection of infinite points on a line, and at the beginning $S^\alpha = \varnothing$. We think of S^α as acting as a sort of a gate or barrier situated at the node α. As more numbers get streamed (i.e., enter S^α), these numbers on the line fall through the gate and drop down to the $\mathcal{Q}$-nodes extending $\alpha^\frown\infty$ (which stands for infinitely many α-expansionary stages), who can then pick an appropriate follower from this list. Here $\alpha^\frown\infty$ and $\alpha^\frown f$ are the two immediate successors of α on the tree, and an α-expansionary stage s is a stage where more numbers are streamed by α. At each α-expansionary stage we ensure that there are new, fresh numbers waiting for the gate to open. If there are infinitely many α-expansionary stages, then the gate allows infinitely many numbers through. If a $\mathcal{Q}$-node extends $\alpha_0^\frown\infty$ and $\alpha_1^\frown\infty$, then it only appoints a follower x if x falls through both gates S^{α_0} and S^{α_1}. More specifically, the rule for streaming is the following. Each time a new large number enters Y and is not yet in A, we put it in S^α. Furthermore, all $\mathcal{Q}$-modules of a lower priority believing that α streams infinitely many numbers (i.e., α-outcome ∞) will pick their x-followers from S^α, while $\mathcal{Q}$-modules believing that α streams finitely many numbers (i.e., α-outcome f) will pick followers larger than $\max S^\alpha[s]$ and be initialized each time S^α grows. This restriction ensures that numbers which are missed out during α-streaming are never later enumerated into A. This corresponds to "freezing the interval" between two streamed numbers, as mentioned previously.

And if β is an $\mathcal{R}_k$-node below $\alpha^\frown\infty$, then β will only stream numbers which have already been streamed by α. This ensures that $\mathcal{Q}$-nodes below $\beta^\frown\infty$ get a continuous stream of numbers which they may use as followers. It follows similarly (now taking into account the growth of S^α) that

$$|S^\beta| < \infty \quad \Rightarrow \quad Y_k \cap \bar{A} \text{ is c.e.}$$
$$|S^\beta| = \infty \quad \Rightarrow \quad X_k \cap \bar{A} \text{ is c.e.}$$

The most direct way of arranging the requirements on the construction tree is as follows. There will be levels on the construction tree devoted to the $\mathcal{R}$-requirements which will have two outcomes ∞, f. We also need to put a top node τ for each $\mathcal{Q}$-requirement. Each τ has infinitely many τ-modules, which we denote by $M_0^\tau, \ldots$. Each module is treated as a subrequirement of τ and is assigned to the nodes on a level below $|\tau|$. Note that this layout is slightly different from the actual layout in the formal construction.

We see that having two different modules M_i^τ and M_j^τ of the same $\mathcal{Q}$-node τ act on different $\mathcal{R}$-guesses immediately produces a problem. Take, for instance, the module M_0^τ assigned to some node σ below $\alpha^\frown\infty$, where α is an $\mathcal{R}$-node. Now at non-α-expansionary stages where S^α does not grow, we might have some τ-daughter node σ' (assigned module M_j^τ) below $\alpha^\frown f$ running its basic strategy. Namely, it will pick follower $x > \max S^\alpha[s]$ and put all the numbers $z \le \gamma(x)$ into T_j. The danger is that at the next α-expansionary stage, we might have to stream x into S^α. If this happens at each α-expansionary stage for $j > 0$, then σ would not be able to pick any number in S^α as a follower since the array $T_0, T_1, \ldots$ has to be disjoint.

This obstacle is by no means an impossible one, because σ never needs to enumerate anything into A unless the enumeration also produces a global win for τ. Even though σ is of lower local priority (since σ's position on the tree is lower than the position of α), its global priority is higher than that of α (because σ is assigned a module working for τ, which is above α). Hence it is all right for σ to pick a follower not in S^α and basically ignores the streaming strategy of α. The requirement α will only sustain a finite amount of injury coming in this fashion (at most finitely often for each master $\mathcal{Q}$-node above α).

Finally, we will explain exactly how we intend to arrange the construction. The construction takes place on a subtree of $2^{<\omega}$. Nodes of even length $|\alpha| = 2e$ are assigned the requirement $\mathcal{R}_e$ with two outcomes $\infty <_{\text{left}} f$. Nodes of odd length $|\alpha| = 2e + 1$ are assigned the requirement $\mathcal{Q}_e$ with only a single outcome 0.

In the light of the above discussion, we observe that each $\mathcal{Q}$-node τ only need to enumerate at most once if it is not injured anymore. Furthermore, all τ-modules ignore streaming nodes lying between τ and itself, so it is not necessary for us to spread the τ-modules out on the nodes below τ. Rather, we will run all the τ-modules at the node τ itself. Since each module only requires a finite amount of processing time, we will finish off with one module before moving on to the next module. If at any point in time τ makes an A-enumeration (via one of its modules), it will impose a final A-restraint and be done. We will, however, still need to determine local priority among the modules of two different $\mathcal{Q}$-nodes $\tau_1 \subset \tau_2$. This is elaborated in Section 3.5 and will be used to determine which τ_2-modules are allowed to injure which τ_1-modules.

Notice that the $\mathcal{Q}$-strategies above cannot be modified to construct a semi-maximal set not of semi-hyperhypersimple degree. This is because we will need to modify the $\mathcal{Q}$-strategies to diagonalize all c.e. sets, and the $\mathcal{Q}$-subnodes will now make separate enumerations into A without a global $\mathcal{Q}$-win for us. This is similar to the situation we will face in Theorem 4.1. We also cannot modify the $\mathcal{R}$-strategies to construct a hemi-maximal set not of hemi-hyperhypersimple degree, because the $\mathcal{Q}$-strategies now cannot enumerate inside a frozen zone (any interval frozen by $\mathcal{R}$ during streaming is permanently frozen since we have to make A half of a maximal set).

We make a note here about the theorem. The $\mathcal{Q}$-requirements actually show something slightly stronger; it shows that $U \sqcup V$ is not even finitely strongly hypersimple. This is because almost every number is enumerated into one of the T_i (unless already in $U \sqcup V$), and each T_i is finite. A characteristic index for T_i can be easily computed.

3.4 Notations Let $\alpha <_{\text{left}} \beta$ denote that α is strictly to the left of β under lexicographic ordering. We write $\alpha \subset \beta$ to mean that α is a strict initial segment of β, and $\alpha \subseteq \beta$ to mean $\alpha \subset \beta$ or $\alpha = \beta$. We say that α is an *$\mathcal{S}$-node* if α is assigned the requirement $\mathcal{S}$.

For each $\mathcal{Q}_e$-node α, we build a weak array $\{T_i^\alpha\}_{i\in\mathbb{N}}$ and write T^α for $\sqcup_{i\in\mathbb{N}} T_i^\alpha$. The ith module of α, called the *(α, i)-module*, is responsible for making sure that $T_i^\alpha \cap \overline{U_e \sqcup V_e} \neq \varnothing$ provided the premises in $\mathcal{Q}_e$ hold. We also denote the (α, i)-module by M_i^α. For each $i \in \mathbb{N}$, we let x_i^α denote the follower targeted at A that M_i^α has picked. We also let F_i^α denote the state of M_i^α. This may either be 0 (meaning that the module is pending action), or it can be 1 (meaning that the module has already ensured that $T_i^\alpha \cap \overline{U_e \sqcup V_e} \neq \varnothing$ is at least temporarily satisfied). We introduce a global parameter called $\text{SAT}(e)$, for each $e \in \mathbb{N}$. This starts off initially as $\text{SAT}(e) = 0$, and when some $\mathcal{Q}_e$-node makes an enumeration into A, we will declare $\text{SAT}(e) = 1$ to record the fact that $\mathcal{Q}_e$ has been satisfied (via the falsification of the premise). We will subsequently prevent all other $\mathcal{Q}_e$-nodes at the same level from acting, since these nodes no longer need to act, provided that the appropriate restraint is held.

If α is an $\mathcal{R}_e$-node, we let S^α denote the set of numbers streamed by α for use by the $\mathcal{Q}$-nodes extending $\alpha^\frown\infty$. A stage s is *α-expansionary* if either $s = 0$ or else α is visited by the construction at stage s, and there are at least $t + 1$ distinct numbers $y_0, \dots, y_t$ such that t is the previous α-expansionary stage, and for all i,

1. $y_i > \max S^\alpha[s]$,
2. $y_i \in Y_{e,s} \cap \bar{A}_s$,
3. for all $\mathcal{R}$-nodes β such that $\beta^\frown\infty \subset \alpha$, we have $y_i \in S^\beta[s]$.

At each such α-expansionary stage $s > 0$, we will stream the numbers $y_0, \dots, y_t$ into the set S^α. Each time we stream numbers into S^α, we want to put more numbers into S^α than the previous time. Condition 1 states that we only stream numbers in increasing order. Condition 3 ensures that S^α is a refinement of the set S^β.

When we *initialize an (α, i)-module* for some $\mathcal{Q}$-node α, we mean that we set $x_i^\alpha \uparrow$ and $F_i^\alpha = 0$. To *initialize the $\mathcal{Q}$-node* α, we initialize all α modules and reset the definition of the weak array $\{T_i^\alpha\}_{i\in\mathbb{N}}$ by setting $T_i^\alpha = \varnothing$ for all i. To *initialize the $\mathcal{R}$-node* α, we set $S^\alpha = \varnothing$.

3.5 The local priority ordering Suppose α is a $\mathcal{Q}_e$-node. The basic strategy of each α-module has both a positive component (in the sense that it might change A) and a negative component (it wants to prevent changes to A). If any α-module changes A, then *every* module of α will not need to do any more work. Therefore, conceptually it makes more sense to think of the α-modules as having only strictly negative action. If any α-module sees a chance to change A, we will let α take over, change A, and freeze every α-module. Hence any positive activity is an action of the node α, having priority α (even though the module which was lucky enough to discover this fact had very low local priority). In more complicated $\varnothing'''$-arguments, this would correspond to a subnode σ of some top node τ witnessing the chance of a global win for α.

If α is a $\mathcal{Q}_e$-node, then we define the set of modules with a *lower local α-priority* to be all the M_i^β for some $\mathcal{Q}$-node $\beta \subset \alpha$ and $i > e$. That is, these modules might have higher "global priority," but we want to arrange for a secondary ordering in which we place almost all of β's modules below α. All the modules with higher α-priority have negative restraint which will not be injured by an A-enumeration made by α. During the construction if α makes an enumeration into A, it will initialize all nodes $\eta \supset \alpha$ since they have lower "global priority," as well as initialize all modules of lower local priority. On the other hand, for each M_i^β, there are only finitely many levels in the construction tree which have higher local priority than it (namely, the levels up until $\mathcal{Q}_{i-1}$), so the number of injuries it sustains is finite.

3.6 The construction During the construction, when we say that $\gamma_e(n)[s] \downarrow$ or $\delta_e(n)[s] \downarrow$, we not only mean that the respective computations have converged, but also that $\Gamma_e^{U_e}(m)[s] = A_s(m)$ and, respectively, $\Delta_e^A(m)[s] = U_{e,s}(m)$ holds for all $m \leq n$. That is, the respective lengths of agreements are sufficiently long, and if a computation converges without agreement, we treat it as being divergent.

At stage $s = 0$, we initialize all nodes, set $\mathrm{SAT}(e) = 0$ for all e, and do nothing else. Let $s > 0$. We define the stage s approximation to the true path, TP_s of length s inductively. We say that a node τ is *visited* at stage s if $TP_s \supset \tau$. Suppose that $\alpha = TP_s\restriction_d$ is defined. There are two cases.

Case 1 α is an $\mathcal{R}_e$-node: if s is not α-expansionary, let $TP_s(d) = f$, and do nothing. Otherwise, let $TP_s(d) = \infty$ and enumerate into S^α all the y_is satisfying conditions 1–3 for an α-expansionary stage above.

Case 2 α is a $\mathcal{Q}_e$-node: let $TP_s(d) = 0$. If $\mathrm{SAT}(e) = 1$, do nothing. Otherwise, if there is some i such that $F_i^\alpha = 1$, $\delta_e(\gamma_e(x_i^\alpha))[s] \downarrow$, and $T_i^\alpha[s] \cap \overline{U_{e,s} \sqcup V_{e,s}} = \varnothing$, we do the following:

- x_i^α into A,
- set $\mathrm{SAT}(e') = 0$ for all $e' > e$,
- initialize *all nodes* β such that $|\beta| > |\alpha|$,
- initialize M_i^β for each $\mathcal{Q}$-node $\beta \subset \alpha$ and $i > e$ (i.e., initialize all modules of a lower local priority),
- set $\mathrm{SAT}(e) = 1$, indicating that $\mathcal{Q}_e$ is satisfied.

Finally, if neither of the above applies, we look for the smallest i such that $F_i^\alpha = 0$, and we take actions for M_i^α. There are two possibilities.

(a) $x_i^\alpha \uparrow$ is currently undefined: check if there is some $x \notin A_s$ satisfying
   (i) $x > \delta_e(\gamma_e(x^\star))[s]$ (which we wait for convergence), where $x^\star =$ largest follower picked by any α-module so far,
   (ii) $x >$ any number used or mentioned prior to the end of stage s^-, where $s^- \leq s$ is the stage where α was last initialized,
   (iii) $x \in S^\beta[s]$ for all $\mathcal{R}$-nodes β such that $\beta^\frown\infty \subseteq \alpha$, and
   (iv) $x > \max S^\beta[s]$ for all $\mathcal{R}$-nodes β such that $\beta^\frown f \subseteq \alpha$ or $\beta <_{\mathrm{left}} \alpha$.

If there is such x, we set $x_i^\alpha \downarrow= x$ for the least such x and initialize all $\mathcal{Q}_{e'}$-nodes $\supset \alpha$, for $e' \geq i$ (i.e., initialize all nodes $\supset \alpha$ which do not have higher local priority. This is important because we do not want the future actions of these nodes to injure M_i^α).

(b) $x_i^\alpha \downarrow$ is currently defined: check if $\delta_e(\gamma_e(x_i^\alpha))[s] \downarrow$. If so, we enumerate all y satisfying $y \leq \gamma_e(x_i^\alpha)[s]$, $y \notin U_{e,s} \sqcup V_{e,s}$, and $y \notin T^\alpha[s]$ into T_i^α. Initialize all $\mathcal{Q}_{e'}$-nodes $\supset \alpha$, for $e' \geq i$. Set $F_i^\alpha = 1$ to indicate that we have temporarily satisfied M_i^α.

This concludes the inductive definition of TP_s. Finally, initialize all nodes $\beta >_{\text{left}} TP_s$ and go to the next stage.

3.7 Verification The true path of the construction is defined as usual to be the leftmost path visited infinitely often during the construction.

Lemma 3.2 *Each α on the true path is initialized only finitely often. If α is a $\mathcal{Q}$-node, then each of its modules is also initialized finitely often.*

Proof We restrict our attention to $\mathcal{Q}$-nodes on the true path. Let α_e denote the $\mathcal{Q}_e$-node on the true path. It will be sufficient to prove the following sentences:

φ_e : $\alpha_0, \dots, \alpha_e$ are initialized only finitely often;

θ_e : only finitely many enumerations can be made into A by a node β of length $|\beta| \leq |\alpha_e|$;

ψ_e : $M_j^{\alpha_i}$ is initialized only finitely often, for $i \leq e$ and $j \leq e+1$, by induction on e; this follows from the fact that φ_0 is clearly true and that $\varphi_e \Rightarrow \theta_e \Rightarrow \psi_e \Rightarrow \varphi_{e+1}$. □

Lemma 3.3 *All $\mathcal{R}$-nodes on the true path are satisfied.*

Proof Let α be an $\mathcal{R}_e$-node on the true path such that $X_e \cap Y_e = \varnothing$. Let o be the true outcome of α and s_0 be a stage after which $\alpha^\frown o$ is never initialized. Suppose $o = f$ is the true outcome of α; then we claim that $Y_e \cap \bar{A}$ is c.e. by specifying a c.e. set $R =^* Y_e \cap \bar{A}$.

Given $x > x_0 := \max S^\alpha[s_0]$, we enumerate x into R if there is a stage $t > s_0$ such that α is visited and $x \in Y_{e,t} \cap \bar{A}_t$. Furthermore, we also require that there is some $\mathcal{R}$-node α^- of maximal length such that $\alpha^{-\frown}\infty \subset \alpha$; for this node α^-, we require that $\max S^{\alpha^-}[t] > x$, and $x \notin S^{\alpha^-}[t]$.

We claim that this enumeration describes $Y_e \cap \bar{A}$. First, take $x \in Y_e \cap \bar{A}$ and $x > x_0$. Note that α^- must exist, otherwise $Y_e \cap \bar{A}$ will be finite. So, the only reason why x is never put in R after it shows up in Y_e must be because $x \in S^{\alpha^-}$. There can only be at most s_0 many such x, since the last α-expansionary stage is before s_0, and consequently $Y_e \cap \bar{A} \subseteq^* R$. Next we consider an $x \in R$. Such an x is put in R at stage $t > s_0$, and we want to show that x does not enter A after stage t. Since x never enters S^{α^-}, it clearly cannot be enumerated by a $\mathcal{Q}$-node $\beta \supseteq \alpha^{-\frown}\infty$. Neither can x be enumerated by $\beta >_{\text{left}} \alpha^{-\frown}\infty$ since such an enumeration has to take place after stage t, but at stage t we would have initialized β (since at t we visited α). Thus x cannot enter A after stage t, so that $R \subseteq Y_e \cap \bar{A}$.

Now suppose $o = \infty$ is the true outcome of α. We will build a c.e. set $\tilde{R} = X_e \cap \bar{A}$: we enumerate x into $\tilde{R}$ if there is some stage $t > s_0$ such that $\alpha^\frown\infty$ is visited, $x \in X_{e,t} \cap \bar{A}_t$, and $\max S^\alpha[t] > x$. We claim that $\tilde{R} = X_e \cap \bar{A}$. The direction $\supseteq$ is obvious, and if some x is enumerated in $\tilde{R}$ at stage t, then necessarily x never enters S^α (since $X_e \cap Y_e = \varnothing$). So for reasons similar to the ones above, x cannot enter A after stage t. □

Lemma 3.4 *Let M_i^α be a module of a $\mathcal{Q}_e$-node α which is not necessarily on the true path. After F_i^α is set to 1, no enumeration can be made into A below $\delta_e(\gamma_e(x_i^\alpha))$ unless either α enumerates or else M_i^α is initialized in the same stage.*

Proof Suppose on the contrary that β is a $\mathcal{Q}$-node which does the enumeration. The only possibility is that either $\beta = \alpha$ or we have $\beta \supset \alpha$. In the latter case, if β is not of higher local priority, it would be initialized at the same time when F_i^α is set to 1. On the other hand, if β is of higher local priority, it would initialize M_i^α whenever it makes an enumeration. □

Lemma 3.5 *All $\mathcal{Q}$-nodes on the true path are satisfied.*

Proof Let α be a $\mathcal{Q}_e$-node on the true path, where the premise is true via the sets U_e and V_e. Let s_0 be the first stage after which α is never initialized. $\{T_i^\alpha\}$ is a weak array since it is never reset after s_0, and it is clearly disjoint by step 2b of the construction.

We first claim that SAT(e) never equals 1 after stage s_0. Suppose the contrary. Thus at some stage t_0 we have a module M_i^β for some $\mathcal{Q}_e$-node β making an A-enumeration. We may assume that β is never initialized after t_0. Also we have at some largest stage $t_1 < t_0$, β is visited and flips F_i^β to 1. By Lemma 3.4, no enumeration in A below $\delta_e(\gamma_e(x_i^\beta))[t_1]$ can be made until β enumerates x_i^β at stage t_0. After β's action at t_0, no number below x_i^β can ever enter A.

Because of the enumeration made by β, we now have the disagreement $A(x_i^\beta) = 1 \neq A_{t_1}(x_i^\beta) = \Gamma_e^{U_e}(x_i^\beta)[t_1]$. Since we know that $\Gamma_e^{U_e} = A$, there must be a change, say $U_e(p)$, in U_e below $\gamma_e(x_i^\beta)[t_1]$ *after β's action at* t_0. But β's action at stage t_0 was due to the fact that it saw $T_i^\beta[t_0] \cap \overline{U_{e,t_0} \sqcup V_{e,t_0}} = \varnothing$, which means that $p \notin T_i^\beta[t_0]$. Clearly, $p \notin U_{e,t_1} \sqcup V_{e,t_1}$, which means that the only reason why it was not put into T_i^β at stage t_1 must be because we already have $p \in T^\beta[t_1]$. Hence $x_i^\beta > \delta_e(p)[t_1]$ by condition 2(a)(i) of the construction so that $p \notin U_e$, giving a contradiction.

Since SAT(e) is never 1 after s_0, α is not blocked from action each time it is visited after s_0. Fix an i, and we need to show that M_i^α eventually succeeds in making $T_i^\alpha \cap \overline{U_e \sqcup V_e} \neq \varnothing$. It will be sufficient to argue that each module state F_i^α eventually settles down to 1. This is because we never put anything into T_i^α unless $F_i^\alpha = 0$, so not every number in $\lim T_i^\alpha$ can enter $U_e \sqcup V_e$ lest SAT(e) is set to 1.

Now we argue inductively that $\lim F_i^\alpha = 1$; assume all $M_{i'}^\alpha$ are eventually in state 1 for $i' < i$. If $F_i^\alpha = 0$ at a sufficiently large stage then M_i^α will receive attention at each visit to α; in fact, all we do at each subsequent visit to α is to give it attention until F_i^α becomes 1. Now a follower will eventually be picked for x_i^α, because conditions 2(a)(i), (ii), and (iv) specify lower bounds for x_i^α which do not increase until x_i^α is picked. Condition (iii) will be satisfied eventually since S^β increases at each visit to α, and since each β only streams numbers into S^β, which are not yet in A. Once x_i^α receives a definition, it will never be canceled. F_i^α will be set to 1 when $\delta_e(\gamma_e(x_i^\alpha)) \downarrow$ (and subsequently never goes back to 0). □

4 A Hemi-hyperhypersimple Set Whose Degree Does Not Contain a Semi-maximal Set

Theorem 4.1 *There is a c.e. set A which is hemi-hyperhypersimple, and for all c.e. $B \equiv_T A$, B is not semi-maximal.*

4.1 Requirements We build disjoint c.e. sets A and C satisfying the following requirements:

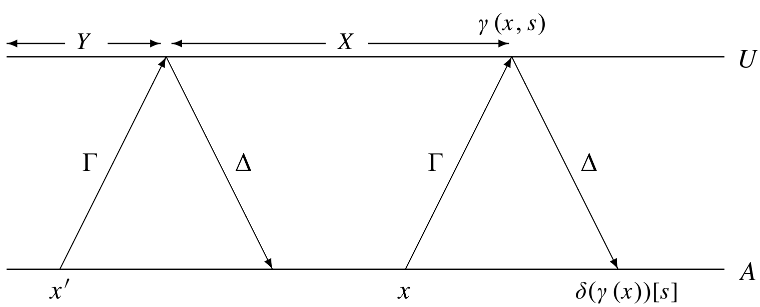
$$\begin{aligned}
\mathcal{S}_e &\quad : \quad \{V_x^e\}_{x\in\mathbb{N}} \text{ is disjoint} \Rightarrow \exists x \text{ such that } V_x^e - (A \sqcup C) \text{ is finite;}\\
\mathcal{Q}_e &\quad : \quad \text{if } \Gamma_e^{U_e} = A \text{ and } \Delta_e^A = U_e, \text{ build disjoint c.e. sets}\\
&\qquad\quad X, Y \text{ such that both } X - U_e \text{ and } Y - U_e \text{ are not c.e.}
\end{aligned}$$

We satisfy requirement $\mathcal{Q}_e$ via the following subrequirements:

$$\begin{aligned}
\mathcal{Q}_{e,2k} &\quad : \quad \text{if the } \mathcal{Q}_e\text{-premises hold, ensure that } X - U_e \neq W_k;\\
\mathcal{Q}_{e,2k+1} &\quad : \quad \text{if the } \mathcal{Q}_e\text{-premises hold, ensure that } Y - U_e \neq W_k.
\end{aligned}$$

Here we let $\langle \Gamma_e, \Delta_e, U_e\rangle_{e\in\mathbb{N}}$ be an effective list of all tuples $\langle \Gamma, \Delta, U\rangle$ such that Γ, Δ are Turing functionals and U is a c.e. set. Also $\{V_x^e\}_{x\in\mathbb{N}}$ stands for the eth uniformly c.e. sequence. We assume a listing where $\{V_x^e\}_{x\in\mathbb{N}}$ is a disjoint sequence for every e. As usual, we use uppercase Greek letters to denote functionals and lowercase Greek letters for their corresponding use. The $\mathcal{Q}$-requirements ensure that A is noncomputable. Also, C is automatically noncomputable; otherwise A has high degree and thus there will be a maximal set $U \equiv_T A$.

4.2 Description of an isolated strategy We first describe a single strategy used to meet $\mathcal{Q}_e$ and make A not of semi-maximal degree. In this section, we may occasionally drop the subscripts since we are describing strategies in isolation. The first try would be to proceed in roughly the same fashion as the $\mathcal{Q}$-strategies in Theorem 3.1. We now build disjoint c.e. sets X, Y and monitor the lengths of agreement at $\mathcal{Q}_e$. The subrequirement $\mathcal{Q}_{e,2k}$ (similarly $\mathcal{Q}_{e,2k+1}$) will ensure that for some p, we have $(X - U_e)(p) \neq W_k(p)$. To do this, we appoint a follower x for the $\mathcal{Q}_{e,2k}$-strategy and wait for the nested length of agreement to exceed x. We would then enumerate all $p \leq \gamma(x, s)$ into X, provided $p \notin Y$, and freeze $A\restriction_{\delta(\gamma(x))}$ to preserve the computations both ways. We will have a similar diagram as before.



Again, note that $\mathcal{Q}_{e,2k}$ would be satisfied temporarily until the opponent makes $(X - U_e)\restriction_{\gamma(x,s)} = W_k\restriction_{\gamma(x,s)}$ true. That is, every number we had put in X and not in U_e would also have entered W_k. When that happens, we put x into A and freeze $A\restriction_x$. The resulting U-change will cause a disagreement with W_k that lasts forever.

If we followed exactly this, then the atomic strategy of each $\mathcal{Q}_{e,2k}$ is the same as before; however, the reader will observe a significantly different effect on the rest of the construction: a single $\mathcal{Q}$-module in Theorem 3.1 will make an A-enumeration for a $\mathcal{Q}$-win. In this case, however, the $\mathcal{Q}$-subrequirements may each enumerate a finite number of times without getting a global $\mathcal{Q}$-win. Hence some modification to the atomic strategy will have to be made.

As the reader might recall, the streaming strategies in Theorem 3.1 work only because each streaming node ignores the A-enumerations made by $\mathcal{Q}$-modules which are of a higher global priority than it. This is obviously a problem here, since now there can be infinitely many such enumerations of higher global priority. Fortunately, this time we do not need to make A semi-maximal, but hemi-hyperhypersimple. Informally this helps because instead of being allowed only two states (in or out of some c.e. set), we are now allowed three or more different states (being in $V_0, V_1, V_2, \ldots$).

4.3 Interaction between two conflicting strategies The construction will take place on a finite branching tree. For the rest of this section and the next, we consider a $\mathcal{Q}$-node τ of a higher priority than an $\mathcal{S}$-node σ. σ has a rightmost finitary outcome (call it f for the time being, although this has a different label in the formal construction) and two infinitary outcomes to the left (it will be clear later why we have two). The subrequirements of a $\mathcal{Q}$-node τ will be assigned to nodes extending τ. These subrequirement nodes are called τ-daughter nodes. The main difficulty here analogous to the one we outlined in Theorem 3.1 was the following: it might be that while σ is waiting for numbers to show up in the array $\{V_x\}$, some τ-daughter node in the region $\supset \sigma^\frown f$ picks a follower x and enumerates all $p \leq \gamma(x)$ into X. We say that the number x is X*-used.* This can happen for various different x; that is, a number of different x might become X-used or Y-used while σ is waiting. Suppose next, some number shows up in $\{V_x\}$ causing σ to wake up and apply some streaming strategy which takes us to one of the left outcomes, call it i. We cannot control which numbers enter $V_0, V_1, \ldots$, and it may be the case that after streaming, the only surviving numbers left are all X-used (or all Y-used). Then the τ-daughter nodes $\supset \sigma^\frown i$ will only have numbers which have already been X-used to choose from, and so the τ-daughter nodes working for, say $\mathcal{Q}_{e,2k+1}$, will be unable to appoint a suitable follower. In the light of this discussion, it is clear now what we need to incorporate into the construction:

(F1) First, we need to "recycle followers" from right to left as described in the following scenario: a number x had been appointed a follower by some $\alpha \supset \sigma^\frown f$. When x undergoes σ-streaming, and assuming it survives the streaming, it will be available for nodes $\supset \sigma^\frown i$ to choose from. In this case, α relinquishes control of x (assuming α hasn't enumerated x in yet), and when α is next visited it will appoint another follower larger than x. x will now be available to nodes $\supset \sigma^\frown i$ (of the correct type, of course) for the rest of the construction. x might be recycled again a second time if there is another $\mathcal{S}$-node which does the above, but in any case x always migrates from the right to the left.

(F2) Each time σ applies a streaming strategy, it needs to make sure that there are at least two numbers $z_1 \neq z_2$ such that z_1 is not yet X-used and z_2 is not yet Y-used, and both z_1, z_2 survive the streaming. This ensures that both types of τ-daughter nodes $\supset \sigma^\frown i$ are able to appoint followers when they are visited.

How should we carry out streaming at σ then in order to have (F2)? We will have a single finitary outcome and two infinite outcomes corresponding to two different streaming strategies (B_0 and B_1). We will see that three outcomes is enough. The outer streaming strategy B_0 will look for two numbers $z_1 \neq z_2$ such that z_1 is not yet X-used and z_2 is not yet Y-used, and both z_1, z_2 are in $V_1 \sqcup V_2$. It will then kill all other numbers $\neq z_1, z_2$ by dumping them in C. The inner streaming strategy B_1 will be active (and carries out its own actions) while B_0 is waiting for new numbers to show up in $V_1 \sqcup V_2$, and B_1 is reset each time B_0 finds new numbers for streaming. B_1's actions are the following: it looks for two numbers $z_1 \neq z_2$ such that z_1 is not yet X-used and z_2 is not yet Y-used, and *either* z_1 *or* z_2 is in V_2. It then dumps all other numbers which have not yet been B_0-streamed. Since B_1 is active only when B_0 fails to find the required numbers, it follows that each time B_1 acts with z_1, z_2, and if $z_1 \in V_2$, then either z_2 hasn't appeared in $V_0 \cup V_1 \cup V_2$ or else already we have $z_2 \in V_0$.

It is clear that if B_0 finds infinitely many numbers, then $V_0 - (A \sqcup C)$ is finite, since B_0 kills every number which does not survive streaming. If B_0 gets stuck at some stage but B_1 manages to find infinitely many numbers, then $V_1 - (A \sqcup C)$ is finite because every number which survives B_1-streaming is either already in V_2 or it never enters V_1 (else together with its companion will be streamed by B_0). Last, if both B_0 and B_1 get stuck, then $V_2 - (A \sqcup C)$ is finite.

4.4 Technical considerations It should be clear that the above works for σ, and also that the τ-daughter nodes below σ on the true path will always have followers of the correct type to choose from. This ensures (F2). Also, it shows us that we can actually prove something slightly stronger. We do not actually need to consider full infinite arrays in the $\mathcal{S}$-requirements but only that the $\mathcal{S}$-requirements consider triples $\langle V_0, V_1, V_2\rangle$ of disjoint c.e. sets. This is the best we can do, since if we only consider pairs $\langle V_0, V_1\rangle$ of disjoint c.e. sets in the $\mathcal{S}$-requirements, then the set $A \sqcup C$ produced would be both r-maximal and hyperhypersimple, and hence maximal. The problem with only having a pair of disjoint c.e. sets is that we have too few states to play with.

There are various technical difficulties in arranging for (F1). For instance, the number x in the diagram in Section 4.2 is X-used but not Y-used. However, if some number $\leq x'$ enters A in future, then x cannot be used by any τ-daughter node anymore, because upon recovery of the computations, we may now have $\delta(p) > x$ for some $p \in Y$. To make things less messy and improve readability in the formal construction, we propose to organize the construction in the following manner.

Followers appointed by a node assigned the even requirements $\mathcal{Q}_{e,2k}$ will have to be even numbers, while followers appointed by the odd nodes assigned $\mathcal{Q}_{e,2k+1}$ are odd numbers. This eliminates the need to flag a number as being X-used or Y-used. To keep track of whether or not a number x is suitable for use by τ-children, we will flag x as being *τ-confirmed* when the nested length of agreement is $> x$, and x is currently appointed a follower by some τ-daughter node α. Instead of enumerating numbers into X (or Y if x is odd) only when α is next visited, we will instead enumerate the appropriate numbers into X immediately when x receives τ-confirmation (when τ is visited). This τ-confirmation will tell an $\mathcal{S}$-node $\sigma \supset \tau$

which numbers to consider for streaming; if a number can no longer be used for streaming then the τ-confirmation on it must be removed to avoid confusing the streaming node σ. To illustrate the interaction between confirmation and streaming, we present the following example.

Example 4.2 Suppose $\tau \subset \sigma$ where τ is a $\mathcal{Q}$-node, and σ is an $\mathcal{S}$-node and suppose α is a τ-daughter node such that $\alpha \supset \sigma^\frown f$. α starts by appointing a follower x (of the correct parity). At the next τ-expansionary stage s when τ is visited, we will have $\delta(\gamma(x))[s] \downarrow$, and by convention is less than s. We will declare x as τ-confirmed, and perform the following two actions:

1. enumerate all corresponding $p \leq \gamma(x)$ into X (or Y depending on the parity of x),
2. cancel all current followers y where $x < y < s$.

This confirmation tells the rest of the nodes below τ that x can be used as a follower by *any τ-daughter node of the correct parity*. This is true as long as $A\lceil_{\delta(\gamma(x))}$ does not change, and even when σ plays an infinite outcome to the left of f, as long as x survives the streaming, x will be released by α and can continue to be used by other τ-daughter nodes. On the other hand, when $A\lceil_{\delta(\gamma(x))}$ changes, then it has to be due to an $A\lceil_x$-change (because of (2) above). So confirmation on x can be removed, and x will no longer be considered.

The reader may also recognize this strategy as being similar to the cancelation and confirmation strategy used in the construction of a contiguous c.e. degree as presented in [1] and [4]. Note that the requirements actually imply that there cannot be three disjoint elements in $\mathcal{L}^*(A \sqcup C)$ so that the lattice of supersets only consists of four elements. Hence $A \sqcup C$ is quasi-maximal. Define a set to be $k+1$-maximal if it is the intersection of two k-maximal sets, where 1-maximal sets are the maximal sets. Then by nesting more streaming strategies in the $\mathcal{S}$-requirements, one could modify the proof below to show that for any $k \geq 1$, there is a hemi-$k+1$-maximal set whose degree does not contain a semi-k-maximal set. There are several possible classes of sets which one may investigate, defined by various finite restrictions on the weak arrays.

4.5 Construction tree layout The construction is organized on a finite branching tree, which grows downward. For each requirement $\mathcal{R}$, we say that α is an *$\mathcal{R}$-node* if α is assigned the requirement $\mathcal{R}$. The assignment of nodes is as follows.

Nodes of length $|\alpha| = 2\langle e, 0\rangle$ are assigned the requirement $\mathcal{Q}_e$ with two outcomes $\infty <_{\text{left}} f$. The left outcome ∞ stands for infinitely many α-expansionary stages in which the lengths of agreement increase, while outcome f stands for the guess that there are only finitely many α-expansionary stages. In the region below $\alpha^\frown f$, there will be no need for any action to be taken for the subrequirements of $\mathcal{Q}_e$. Nodes of length $|\alpha| = 2\langle e, k+1\rangle$ will be assigned the requirement $\mathcal{Q}_{e,k}$ with a single outcome 2, since the action of each $\mathcal{Q}_{e,k}$ is finitary. The reason why we choose the number 2 is to keep it consistent with the "finitary" outcome of the $\mathcal{S}_e$-nodes; see below.

We say that τ is a *top node* if τ is a $\mathcal{Q}_e$-node for some e. If $\alpha \supset \tau$ where α is a $\mathcal{Q}_{e,k}$-node and τ is a $\mathcal{Q}_e$-node, we say that α is a *daughter node of τ*. In this case, we also refer to τ as the top node of α, denoted by $\tau = \tau(\alpha)$. Furthermore, if k is even we call α a (τ, X)-daughter node; otherwise, we say it is a (τ, Y)-daughter node. α and β are called *sibling nodes* if they have the same top. Note that we also label τ-daughter nodes $\alpha \supset \tau^\frown f$ even though α never needs to act; this is to keep the construction tree layout and labeling of nodes clear and less confusing.

Nodes of odd length $|\alpha| = 2e+1$ are assigned the requirement $\mathcal{S}_e$. The node α has 3 outcomes, labeled $0 <_{\text{left}} 1 <_{\text{left}} 2$. Outcome n stands for '$V_n^e - (A \sqcup C)$ is finite'. Let $\alpha <_{\text{left}} \beta$ denote that α is strictly to the left of β under the usual lexicographic ordering. We write $\alpha \subseteq \beta$ to mean $\alpha \subset \beta$ or $\alpha = \beta$. As mentioned above, $\mathcal{Q}_e$-nodes are referred to as top nodes. A $\mathcal{Q}_{e,i}$-node will be referred to as a $\mathcal{Q}$-node, while an $\mathcal{S}_e$-node is known as an $\mathcal{S}$-node (when we do not want to be specific about the index e).

4.6 Notations At the $\mathcal{Q}_e$-node τ, we build the disjoint c.e. sets X_τ and Y_τ. We will only concern ourselves with the τ-computations which converge correctly both ways, so we monitor the *nested length of agreement* via $l_\tau[s]$, defined as the largest $x < s$ such that

1. for all $y < x$, we have $\Gamma_e^{U_e}(y)[s] \downarrow = A_s(y)$, and
2. for all $z \leq \gamma_e(x-1, s)$, we have $\Delta_e^A(z)[s] \downarrow = U_{e,s}(z)$.

In particular, if $x < l_\tau[s]$ at a stage s, then $\delta_e(\gamma_e(x))[s] \downarrow$, and by restraining $A \upharpoonright_{\delta_e(\gamma_e(x))[s]}$, we will be able to preserve the computations below $\Delta_e^A(\gamma_e(x))[s]$. Hence, U_e cannot change below $\gamma_e(x)[s]$ before we remove the restraint on A; otherwise we would get a τ-win by continuing to hold the same restraint on A. We sometimes write $\gamma_\tau, \delta_\tau, \Gamma_\tau, \Delta_\tau$ in place of $\gamma_e, \delta_e, \Gamma_e, \Delta_e$ to avoid cumbersome notations.

If α is a τ-daughter node, we use $x_\alpha[s]$ to denote the stage s follower that α has appointed, which might be put into A some time in the future to force changes in U_e. If α is a (τ, X)-daughter node, then it appoints even followers (i.e., even values for x_α), while if α is a (τ, Y)-daughter node then it appoints odd followers. This restriction is to help in streaming at $\mathcal{S}$-nodes—an $\mathcal{S}$-node σ will make sure that there is a continuous stream of numbers both even and odd for use by nodes below.

A stage s is *τ-expansionary* if either $s = 0$ or else τ is visited by the construction at stage s, and

1. $l_\tau[s] > l_\tau[s^-]$ where s^- is the previous τ-expansionary stage, and
2. $l_\tau[s] > x_\alpha[s]$ for every $\alpha \supseteq \tau^\frown\infty$.

Note that in (2) above, we wait for l_τ to exceed x_α for every $\alpha \supseteq \tau^\frown\infty$ including those α which are not daughters of τ. Undefined values count as 0. At τ-expansionary stages, we will take the least $x = x_\alpha$ in (2), enumerate all numbers y such that $x < y < s$ into C, and declare x as *τ-confirmed*. The purpose of τ-confirming a number x is to ensure that x can be used as a follower by any (τ, X)-daughter node (supposing x is even, similarly for x odd) so long as it is not yet killed. This feature is necessary so that $\mathcal{S}$-nodes below τ can run their streaming strategies compatible with τ.

Suppose that σ is an $\mathcal{S}_e$-node. There are two parameters, B_0^σ and B_1^σ, corresponding to the outcomes 0 and 1, respectively. These contain numbers which have been successfully streamed by σ. By convention, we fix $B_2^\sigma = \mathbb{N}$. If α is any node, then we let $\text{Avail}_\alpha[s] := \cap\{B_i^\sigma[s] \mid \sigma \text{ is an } \mathcal{S}\text{-node such that } \sigma^\frown i \subseteq \alpha\}$. This represents all the numbers which are currently available to α. We say that two numbers are of *different parity* if one of them is even and the other is odd. At any time there is a basic restraint that applies to σ. Denote this by $r_\sigma[s] :=$, the least number larger than

1. $\max\{x_\beta[t] \mid (\beta \subset \sigma \ \vee\ \beta <_{\text{left}} \sigma) \text{ and } t \leq s\}$ (all current and past followers),
2. s^- where s^- is the previous stage such that $TP_{s^-} <_{\text{left}} \sigma$.

That is, any number handled by σ has to be at least larger than r_σ. We say that a number z is *σ-good* at stage s if the following hold:

1. $z \in \overline{A_s \sqcup C_s} \cap \text{Avail}_\sigma$,
2. z is τ-confirmed for every top node τ such that $\tau^\frown\infty \subseteq \sigma$,
3. $z > r_\sigma[s]$.

The stage s approximation to the true path is denoted by TP_s and will be defined during the construction. The idea is that σ will only consider numbers which are σ-good for the purpose of streaming. If a number is larger than r_σ but is not σ-good, then it is useless for streaming, and σ will get rid of these numbers.

We state now what it means to *initialize* a node α at stage s. If α is a top node, then we set $X_\alpha = Y_\alpha = \varnothing$ and remove all α-confirmations. If α is a daughter node, then we set $x_\alpha =\uparrow$. If α is an $\mathcal{S}$-node, then we set $B_0^\alpha = B_1^\alpha = \varnothing$. To *remove all confirmations* on a number z means to remove any τ-confirmation currently on z for every top node τ. To *dump* a number n at a stage s means to enumerate n into C unless $n \in A_s \sqcup C_s$.

4.7 The construction At stage $s = 0$, we initialize all nodes and do nothing else. Let $s > 0$. We define the stage s approximation to the true path, TP_s of length $< s$ inductively. During any time (in a stage s), if the construction encounters HALT, we will immediately cease all further action, end the current stage s, and go to stage $s+1$. As usual, we say that a node α is *visited* at stage s if $TP_s \supset \alpha$. Suppose that $\alpha = TP_s\restriction_d$ is defined. We want to state the action for α and specify the outcome taken by α. There are three cases.

Case 1 *α is a $\mathcal{Q}_e$-node*: If s is not α-expansionary, let $TP_s(d) = f$ and do nothing. Otherwise, let $TP_s(d) = \infty$ and do the following in order.

1. For every $\beta >_{\text{left}} \alpha^\frown\infty$, we do the following: if β is a $\mathcal{Q}$-node, we remove all confirmations on x_β if it is defined. If β is an $\mathcal{S}$-node, then we remove all confirmations on z for every $z \in B_0^\beta \sqcup B_1^\beta$.
2. Initialize every $\beta >_{\text{left}} \alpha^\frown\infty$.
3. Pick the smallest number z such that $z = x_\beta$ for some $\beta \supseteq \alpha^\frown\infty$, and z is not α-confirmed. If z exists and is even, we enumerate all y satisfying $y \leq \gamma_e(z)[s]$ and $y \notin X_{\alpha,s} \sqcup Y_{\alpha,s}$ into X_α. If z exists and is odd we enumerate all such y into Y_α instead. Finally, declare z as α-confirmed.
4. If $\delta_e(\max X_\alpha \sqcup Y_\alpha)[s] \geq s^-$, then HALT where s^- is the previous α-expansionary stage.

Case 2 *α is a $\mathcal{Q}_{e,2k}$-node*: Let $TP_s(d) = 2$. If α is a $\mathcal{Q}_{e,2k+1}$-node, then exactly the same steps described below are to be taken, except that we replace X by Y and even with odd. If $\alpha \supset \tau(\alpha)^\frown f$, do nothing for α. Otherwise there are three subcases; pick the one that applies and take the actions described.

1. *x_α is currently undefined*: we need to pick a new follower for x_α by the following. Check if there is some even number $x \notin A_s \sqcup C_s$ satisfying
   (a) for every top node τ such that $\tau^\frown\infty \subseteq \alpha$, either x is currently τ-confirmed or else $x > \delta_\tau(\max X_\tau \sqcup Y_\tau)[s]$,
   (b) $x > \max\{x_\beta[t] \mid \beta \subset \alpha \ \vee\ \beta <_{\text{left}} \alpha \text{ and } t \leq s\}$,

(c) $x > s^-$, where s^- is the previous stage such that $TP_{s^-} <_{\text{left}} \alpha$.
(d) $x \in \text{Avail}_\alpha$.

If there is such x, we set $x_\alpha \downarrow = x$ for the least such x. In any case, initialize all nodes $\beta \supset \alpha$ and HALT.

2. *x_α is currently defined, but $x_\alpha \notin A_s \sqcup C_s$*: as we will see later in Lemma 4.3(8), x_α must be $\tau(\alpha)$-confirmed. This is the waiting phase. See if $(X_{\tau(\alpha)} - U_e)\lceil_{\gamma_e(x_\alpha)} = W_k\lceil_{\gamma_e(x_\alpha)}$. If they are not equal, do nothing. Otherwise, we do the following.
   (a) Enumerate x_α into A.
   (b) For every $z > x_\alpha$, remove all confirmations on z.
   (c) Initialize all nodes $\beta \supset \alpha$ and HALT.
3. *x_α is currently defined, and $x_\alpha \in A_s \sqcup C_s$*: do nothing, since α has been successful.

Case 3 *α is an $\mathcal{S}_e$-node*: First, we get rid of all the numbers which are not α-good. More specifically, for every $z_1 < z' < z_2$ such that z_1 and z_2 are α-good, z_1 is larger than max $B_0^\alpha \sqcup B_1^\alpha$, and z' is not α-good, dump z'. Next there are three subcases; pick the first in the list that applies. Let $s^\star = \max\{s, 1+$ the largest number mentioned so far$\}$.

1. *there are α-good numbers z_1, z_2 of different parity such that $z_i \in (V_1^e \sqcup V_2^e)$ and* max B_0^α *$< z_i < s$ for both i*: for every number $z \neq z_1, z_2$ (Any choice for z_1 and z_2 is fine.) such that $\max\{B_0^\alpha, r_\alpha\} < z < s^\star$, we dump z. Add z_1, z_2 to B_0^α, and let $TP_s(d) = 0$. Initialize all nodes $\beta >_{\text{left}} \alpha^\frown 0$ and set $B_1^\alpha = \varnothing$.
2. *there are α-good numbers z_1, z_2 of different parity such that* max $B_0^\alpha \sqcup B_1^\alpha$ *$< z_i < s$ for both i, and $z_i \in V_2^e$ for some i*: for every number $z \neq z_1, z_2$ such that $\max\{B_0^\alpha \sqcup B_1^\alpha, r_\alpha\} < z < s^\star$, we dump z. Add z_1, z_2 to B_1^α. Next, we review the quality of numbers in B_1^α; namely, for every $z \in B_1^\alpha$ such that z is no longer α-good and $z > r_\alpha$, we will dump z. Let $TP_s(d) = 1$. Initialize all nodes $\beta >_{\text{left}} \alpha^\frown 1$.
3. *otherwise*: do nothing, and let $TP_s(d) = 2$.

This concludes the inductive definition of TP_s. If the construction encounters HALT during stage s, then take TP_s to be whatever that was defined so far. Go to the next stage. Note that in step 3(a) we require *both of* z_1, z_2 to be in $V_1^e \sqcup V_2^e$ while in step 3(b) we only require *one of* z_1, z_2 to be in V_2^e.

4.8 Verification The true path TP of the construction is defined as usual to be the leftmost path visited infinitely often during the construction. That is, for every n, $TP\lceil_n$ is visited infinitely often and $TP_s <_{\text{left}} TP\lceil_n$ only finitely often. We say that a number z is *currently in use by α* at some stage s if $z = x_\alpha[s]$ (if α is a $\mathcal{Q}$-node) or $z \in B_0^\alpha \sqcup B_1^\alpha[s]$ (if α is an $\mathcal{S}$-node). During the verification process we will occasionally refer to a certain step in the construction; to reduce confusion we will prepend "S" to the step number. For instance, we write S2(a)(ii) to refer to step 2(a)(ii) of the construction.

The following lemma lists a few facts about the construction. (4) says that any number which is confirmed must be currently in use. (5) tells us that if x_α is confirmed, then that confirmation cannot be removed until α is initialized. (8) says that if some $\mathcal{Q}$-node picks a follower at a τ-expansionary stage, then at the next τ-expansionary stage this follower will receive τ-confirmation (i.e., there is no delay). (11) describes a key fact about confirmation—once a number x is τ-confirmed, then $A\lceil_{\delta_\tau(\gamma_\tau(x))}$ will be held until the confirmation is removed.

Lemma 4.3 *In the following, τ is a top node, and σ is an $\mathcal{S}$-node.*

1. *Enumerations into A are made only by $\mathcal{Q}$-nodes; enumerations into C are made only by $\mathcal{S}$-nodes.*
2. *$A \cap C = \varnothing$.*
3. *If $\alpha <_{\text{left}} \beta$ or $\alpha^\frown 2 \subseteq \beta$, and z_1, z_2 are in use by α, β, respectively, then $z_1 < z_2$.*
4. *If z is currently τ-confirmed and $z \notin A_s \sqcup C_s$, then z must currently be in use by some $\alpha \supseteq \tau^\frown\infty$.*
5. *If $x_\alpha \downarrow$ and is currently τ-confirmed, then $\tau^\frown\infty \subseteq \alpha$. Furthermore, this τ-confirmation on x_α cannot be removed unless α is initialized (either by the same action or before).*
6. *If a $z \in B_i^\sigma$ for some z and $i < 2$, and z is currently τ-confirmed, then either $\tau^\frown\infty \subseteq \sigma$ or $\tau \supseteq \sigma^\frown i$.*
7. *If τ is a top node, then at any time there can be at most one $\beta \supseteq \tau^\frown\infty$ with a follower x_β that is not τ-confirmed.*
8. *Suppose $\tau^\frown\infty \subseteq \alpha$ and α appoints a follower which is currently not τ-confirmed. Then the follower will be τ-confirmed at the next τ-expansionary stage, provided that α is not initialized in the meantime.*
9. *The true path of the construction exists.*
10. *Each node on the true path is initialized finitely often.*
11. *Once a number z is τ-confirmed at s, then $A\lceil_{\delta_\tau(\gamma_\tau(z))[s]}$ does not change unless either z is enumerated or the τ-confirmation on z is removed (either by the same action or before).*

Proof **(1)+(2)+(3)** The first two are trivial. For (3), observe that z_2 will be considered by β only after α has started using z_1. If α is a $\mathcal{Q}$-node, then it is clear. If α is an $\mathcal{S}$-node, use the fact that α only enumerates a new z in $B_0^\alpha \sqcup B_1^\alpha$ at stage s, if $z < s$.

(4) We argue by induction on the stage number. More specifically, we break down each stage s into separate actions by the different nodes on TP_s (this is assumed to be the case in the rest of the verification).

When z receives its τ-confirmation, it must clearly be in use by some $\mathcal{Q}$-node $\supseteq \tau^\frown\infty$. Assume that z is currently in use by some $\alpha \supseteq \tau^\frown\infty$ and now it is β which gets to act at stage s. If $\beta \supset \alpha$ or $\beta >_{\text{left}} \alpha$, then β's action will have no effect on α and its parameters. If $\beta <_{\text{left}} \alpha$ then α would have been initialized earlier in the stage and z cannot be in use by α. Hence we may assume $\beta \subseteq \alpha$ in the following three cases.

1. β is a top node: to have any effect on α we must have $\beta^\frown f \subseteq \alpha$, and all confirmation on z is removed.
2. β is a $\mathcal{Q}$-node: if $\beta = \alpha$ then z remains in use by α. So, $\beta \subset \alpha$ and therefore when β acts, we must have $x_\beta \downarrow$ (use the fact that we always halt in S2(a)). By (3) we have $x_\beta < z$, and so if β is to initialize α at s, it has to also remove all confirmation on z.

3. β is an $\mathcal{S}$-node: if $\beta = \alpha$ then the only consideration is when $z \in B_1^\alpha$ and β plays outcome 0. We must have $\max\{B_0^\alpha, r_\alpha\} < z < s$ holds at s, and hence we will either dump z, or we add z to B_0^α (and so z continues to be in use by α). If $\beta \subset \alpha$ then α has to be initialized when β acts at s (else β's action has no effect on the induction), and so $\tau^\frown\infty \subseteq \beta$ (otherwise τ will get initialized and all τ-confirmations are removed). Furthermore, we have $i = \alpha(|\beta|) > 0$, and β plays outcome $< i$ after acting at s. We first claim that $\max\{B_{i-1}^\beta, r_\beta\} < z < s^\star$ holds at s. Note that z is in use by α, so it must be that at the point when α appoints z as a follower (if α is a $\mathcal{Q}$-node) or when α streams z (if α is an $\mathcal{S}$-node), z has to be larger than the stage number of the previous visit to the left of α. A second fact to pay attention to is that if β puts z' into B_j^β at stage t, then $z' < t$. These two facts give $z > \max B_{i-1}^\beta[s]$. The fact that $z > r_\beta[s]$ follows by chasing the definition. Finally since the bounds on z are as such, we must have z is either dumped or continues to be in use by β after β acts at s.

(5) Note that in order for a number z to become τ-confirmed, z has to be first appointed a follower by some $\mathcal{Q}$-node $\beta \supseteq \tau^\frown\infty$. Thus we cannot have $\tau <_{\text{left}} \alpha$ nor can we have $\tau^\frown f \subseteq \alpha$. On the other hand, it is clear that we cannot have $\tau \supset \alpha$ nor $\tau >_{\text{left}} \alpha$. To see this, assume $\tau \supset \alpha$ or $\tau >_{\text{left}} \alpha$ and consider the following. If α appoints this number z as a follower after z receives τ-confirmation, then α will initialize τ when appointing z and remove the confirmation on z. Else α has to appoint z before z receives τ-confirmation. However, when z later receives τ-confirmation we must also have $z = x_\beta$ for some $\beta \supseteq \tau^\frown\infty$, which is not possible by (3). This shows the first part.

We can only remove this τ-confirmation on $z = x_\alpha$ if τ itself is initialized (in which case α is initialized as well) or directly through the actions of some node η later on. We show that when η acts and removes τ-confirmation on z, the same action also initializes α. Suppose η is a top node. Now η will remove all confirmations on z under S1(a). If $z = x_\beta$ then it must be that $\beta = \alpha$ by (3), and so α will be initialized immediately in S1(b). If $z \in B_0^\beta \sqcup B_1^\beta$ then we must have $\beta \subset \alpha$ once again by (3) and considering the stage where α appointed z relative to when β streams z. Hence α will also be initialized immediately in S1(b). If η is a $\mathcal{Q}$-node, then it is easy. η being an $\mathcal{S}$-node is impossible.

(6) We cannot have $\tau <_{\text{left}} \sigma^\frown i$ nor can we have $\tau^\frown f \subseteq \sigma$ due to similar reasons as in (5). Also we cannot have $\tau >_{\text{left}} \sigma^\frown i$ because only numbers $< s$ are placed in B_i^σ at stage s.

(7) Suppose the contrary, and fix a stage s, τ, and $\beta_1, \beta_2 \supseteq \tau^\frown\infty$ such that both x_{β_1} and x_{β_2} are not τ-confirmed at s. Assume that s is the least stage where this holds for τ. Since we always halt when a $\mathcal{Q}$-node appoints a follower, it follows that this current incarnation of x_{β_1} and x_{β_2} are appointed at different τ-expansionary stages, say $s_1 < s_2 \leq s$, respectively. By (5), it follows that x_{β_1} is not τ-confirmed when τ acts at s_2. Furthermore, x_{β_1} cannot be given τ-confirmation at s_2, which means that some $z < x_{\beta_1}$ has to be given τ-confirmation instead. This contradicts the minimality of s.

(8) Suppose that x_α is appointed at stage s, and s^+ is the next τ-expansionary stage. At s^+ we would τ-confirm x_α unless some $z < x_\alpha$ is given τ-confirmation instead, contradicting (7).

(9)+(10) We argue simultaneously by induction. Suppose $\alpha \in TP$ exists and is initialized finitely often. If α is an $\mathcal{S}$-node then it is clear that one of its outcomes is visited infinitely often. We consider α to be a top node. It is a problem only if there are infinitely many α-expansionary stages. We need to see that there are infinitely many α-expansionary stages where we do not encounter HALT at α. Suppose s is large enough such that the construction halts at α (if s does not exist then we are done).

Let $s^+ > s$ be the next α-expansionary stage. At stage s^+ we claim that there cannot be a number z receiving τ-confirmation under S1(c): if there was, then we must have $z = x_\eta[s^+]$ for some $\eta \supseteq \alpha^\frown\infty$. Since we encountered a HALT at stage s, it follows that η can only have appointed z as its follower before stage s, which means that by (5) and (8), z would be already τ-confirmed by the stage s^+, a contradiction. Hence z doesn't exist, and one can conclude that the value $\max X_\alpha \sqcup Y_\alpha$ is unchanged between s and s^+. We have $\delta_e(\max X_\alpha \sqcup Y_\alpha)[s] < s$, so we will be done if we can show that $A_s\upharpoonright_s = A_{s^+}\upharpoonright_s$, because then the construction does not halt at S1(d) at s^+. Between s and s^+, which $\mathcal{Q}$-node β can possibly enumerate below s? Since α is not initialized (we assume s is large enough), it follows that $\beta >_{\text{left}} \alpha^\frown\infty$. This is not possible too, since β is initialized at s, and consequently picks its follower larger than s.

Suppose now α is a $\mathcal{Q}$-node. Since α is initialized finitely often, it follows that eventually if $x_\alpha \downarrow$, then it will be final. Hence we would be done if we can show that α does not halt in S2(a) infinitely often. It will also follow that α initializes $\alpha^\frown 2$ finitely often. To complete the induction, we will need to show that if α is visited at a sufficiently large stage s in which $x_\alpha \uparrow$, it will be able to find a suitable follower for appointment.

Let σ be the maximal $\mathcal{S}$-node such that $\sigma^\frown i \subseteq \alpha$ for some $i < 2$. If σ does not exist, then it is clear that α has no problems appointing a follower at s because $\text{Avail}_\alpha = \mathbb{N}$. The main trouble that α faces in choosing followers comes from the restriction in Avail_α, because it has to conform to streaming strategies from above. Since σ played outcome $i < 2$ when it was visited at s, it follows that there are some σ-good numbers z_1, z_2 which are newly added to B_i^σ. Our task therefore is to show that both z_1 and z_2 satisfy the conditions in S2(a) to be appointed a follower of α—in that case α could then appoint one of z_1, z_2 of the correct parity.

Certainly $z_1, z_2 \notin C_s$ because there are no $\mathcal{S}$-nodes between σ and α. Also $z_1, z_2 \notin A_s$ since α is not initialized. Clearly, we have z_1, z_2 satisfies S2(a)(iv). As for S2(a)(ii) and S2(a)(iii), observe that the bounds reach a limit since α is on the true path, so if s is large enough then z_1, z_2 will be larger than the required lower bounds in S2(a)(ii) and S2(a)(iii). Finally, we show that z_1, z_2 satisfy S2(a)(i) for τ. If $\tau \subset \sigma$ then z_1, z_2 are τ-confirmed since they are σ-good, and this confirmation cannot be removed as we travel from σ down to α (because z_1, z_2 are newly streamed and thus cannot be in use by anyone yet). So we may assume that $\sigma^\frown i \subset \tau^\frown\infty \subseteq \alpha$. The fact that we did not halt at τ as we travel from σ down to α means that we have $\delta_\tau(\max X_\tau \sqcup Y_\tau) < s^-$, where s^- is the previous stage where $\sigma^\frown i$ is visited. We have $z_1, z_2 > \max B_i^\sigma[s^-]$ which implies that $z_1, z_2 \geq s^- > \delta_\tau(\max X_\tau \sqcup Y_\tau)$ because we would have dumped all the useless numbers $< s^-$ when σ acted at s^-. Hence, both z_1 and z_2 are available for α to choose from at stage s.

(11) Any A-change has to be effected by some $\mathcal{Q}$-node β which enumerates $x_\beta < \delta_\tau(\gamma_\tau(z))[s]$. What are the possible positions of β relative to τ? It is not hard to see that $\beta >_{\text{left}} \tau^\frown\infty$ is impossible. If $\beta \subset \tau$ or $\beta <_{\text{left}} \tau$, then β's action (when it enumerates x_β) also initializes τ. Therefore, we must have $\beta \supseteq \tau^\frown\infty$. By (5) and (8), it follows that, at the instance when β enumerates x_β, it must be that $y = x_\beta$ is already τ-confirmed. So we have that both y and z are τ-confirmed. If $y = z$ then it is trivial so we assume $y \neq z$. If y receives τ-confirmation after z does, and since τ is not initialized between the confirmations of z and y, it follows that $y > \delta_\tau(\gamma_\tau(z))[s]$ because $y = x_\beta$ has to satisfy S2(a)(i). Hence we must have y receives τ-confirmation before z. But then $y < z$ (for similar reasons), and we are done because β will remove all confirmation on z at the same time it enumerates y. $\square$

The $\mathcal{Q}$-strategies succeed Fix a $\mathcal{Q}_{e,2k}$-node α on the true path, and let $\tau = \tau(\alpha)$. Also assume that $\Gamma_e^{U_e} = A$ and $\Delta_e^A = U_e$. Hence there are obviously infinitely many τ-expansionary stages, and so $\alpha \supseteq \tau^\frown\infty$. Clearly $X_\tau \cap Y_\tau = \varnothing$. A similar argument as the one below will follow for (τ, Y)-daughter nodes. By Lemma 4.3(10) it follows that x_α will receive a final definition $x_\alpha = x$, at stage s_0, where x is even.

Lemma 4.4 *Suppose $\alpha, \tau, e, k,$ and x are as above, and suppose further that $x \in A \sqcup C$. Then there is some number p such that $p \in U_e \cap X_\tau \cap W_k$.*

Proof It is not hard to see that after α appoints x as its follower, x cannot be dumped by any $\mathcal{S}$-node nor can it be enumerated into A by any $\mathcal{Q}$-node (other than α itself): the only nontrivial case to consider is when we have some $\mathcal{S}$-node σ such that $\sigma^\frown i \subseteq \alpha$ which dumps x (for $i = 1$ or 2). In this case, by Lemma 4.3(5) and (8), it follows that when σ is next visited after s_0, we must have that x is σ-good. Furthermore, x stays σ-good forever and so σ cannot possibly dump x after s_0.

Since $x \in A \sqcup C$, α will eventually enumerate x at some stage $s_1 > s_0$. Let $t < s_1$ be the stage when x receives τ-confirmation. Before the next τ-expansionary stage $s_2 > s_1$, some number $p < \gamma_e(x)[s_1]$ must enter U_e. By Lemma 4.3(11) and the fact that both t and s_1 are τ-expansionary stages, we have $\gamma_e(x)[t] = \gamma_e(x)[s_1]$. Suppose for a contradiction that at stage t when x receives τ-confirmation, we already have $p \in X_\tau \sqcup Y_\tau$. It is easy to see that at stage t, we must have $\delta_e(p)[t] < x$ (by considering when p could have been put in $X_\tau \sqcup Y_\tau$). Since x must remain τ-confirmed until stage s_1, it follows that there is no change in $A\restriction_x$ between t and s_1 and also between s_1 and s_2. Thus $\delta_e(p)[s_2] = \delta_e(p)[t] < x$, and since p has entered U_e by s_2, it follows that $U_e(p) \neq \Delta_e(p)[s_2]$, a contradiction to the fact that s_2 is τ-expansionary. Hence at stage t we do not already have $p \in X_\tau \sqcup Y_\tau$, and consequently when x is given τ-confirmation at t, we will place p in X_τ, since x is even. Finally, since α takes S2(b) at stage s_1, it follows that $W_k(p) = (X_\tau - U_e)(p) = 1$ holds at s_1. $\square$

It is clear that if $x \notin A \sqcup C$, then $X_\tau - U_e \neq W_k$ since $\gamma_e(x)$ settles. On the other hand, if $x \in A \sqcup C$, then $X_\tau - U_e \neq W_k$ by Lemma 4.4.

The $\mathcal{S}$-strategies succeed Fix an $\mathcal{S}_e$-node σ on the true path, with true outcome i, and assume that V_o^e, V_1^e, V_2^e are pairwise disjoint. Let s_0 be large enough so that $\sigma^\frown i$

is never initialized. We say a number y is *σ-excellent* if there is a stage $t > s_0$ such that y is σ-good at every visit to $\sigma^\frown i$ after t.

Lemma 4.5 *There are infinitely many even numbers in B_i^σ, and infinitely many odd numbers in B_i^σ, which are σ-excellent.*

Proof We consider the even case; a similar argument follows for the odd case. Fix an arbitrary number $M > \lim r_\sigma$. We can then consider indices e and k large enough such that $W_k = \varnothing$, $U_e = A$, and Γ_e, Δ_e are constant on $\mathcal{P}(\mathbb{N})$ and such that the $\mathcal{Q}_{e,2k}$-node α on the true path appoints a final follower $x > M$ for some even x, and $\alpha \supset \sigma^\frown i$. We show that x is the required even number. By Lemma 4.4 it follows that $x \notin A \sqcup C$. It is also obvious that $x \in \mathrm{Avail}_\sigma$ forever. Also by Lemma 4.3(5) and (8) it follows that x will be τ-confirmed for any $\tau^\frown\infty \subseteq \sigma$ and stays τ-confirmed forever. □

There are three cases; we first consider the case when $i = 2$. We claim that $V_2^e - (A \sqcup C)$ is finite. Suppose for a contradiction that there is some $p > \max\{B_0^\sigma \sqcup B_1^\sigma, \lim r_\sigma\}$ and $p \in V_2^e - (A \sqcup C)$. We may assume that $\max B_0^\sigma \sqcup B_1^\sigma < z_1 < p < z_2$ for two σ-excellent numbers z_1, z_2 of different parity from Lemma 4.5. Hence p must also be σ-excellent; otherwise p will be dumped. In that case, when the conditions become right, S3(a) or S3(b) will apply to bring us to the left of the true outcome, a contradiction.

Now suppose that $i = 1$. We claim that $V_1^e - (A \sqcup C)$ is finite. Note that $|B_1^\sigma| = \infty$. Suppose once again for a contradiction that there are $p_2 > p_1 > \max\{B_0^\sigma, \lim r_\sigma\}$ and $p_1, p_2 \in V_1^e - (A \sqcup C)$ exists. Now p_1 and p_2 must both be put in B_1^σ lest they be dumped. Furthermore, p_1 and p_2 have to be σ-excellent; otherwise they will also be dumped after entering B_1^σ. We may assume that p_2 is put in B_1^σ after p_1 shows up in V_1^e. Hence, when p_2 was placed in B_1^σ at stage t, there must be a companion number q of opposite parity placed in B_1^σ together with p_2 such that $q \in V_2^e[t]$. If p_1 and p_2 are of the same parity, then p_1 and q are of different parity and S3(a) would apply instead of S3(b) at stage t. So it must be that p_1 and p_2 are of different parity. In that case, when p_2 eventually shows up in V_1^e, then S3(a) would apply to give another contradiction.

Finally, consider the case $i = 0$. We show that $V_0^e - (A \sqcup C)$ is finite. Again note that $|B_0^\sigma| = \infty$. If $p > \lim r_\sigma$ and $p \in V_0^e$, then p cannot be put in B_0^σ, and hence would be dumped when a large enough number goes into B_0^σ.

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