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Finitary Set Theory

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Abstract I argue for the use of the adjunction operator (adding a single new element to an existing set) as a basis for building a finitary set theory. It allows a simplified axiomatization for the first-order theory of hereditarily finite sets based on an induction schema and a rigorous characterization of the primitive recursive set functions. The latter leads to a primitive recursive presentation of arithmetical operations on finite sets.

1 Introduction

What does set theory tell us about the finite sets? This may seem an odd question, because the explication of the infinite is the raison d'être of set theory. That's how it originated in Cantor's work. The universalist, or reductionist, claim of set theory—its claim to provide a foundation for all of mathematics—came later. Nevertheless, I propose to take seriously the picture that set theory provides of the (or a) universe and apply it to the finite sets. In any case, set theory reaches the infinite by building it upon the finite.

The set-theoretic view of the universe of sets has different aspects that apply to the hereditarily finite sets:

1. the universalist claim, inasmuch as it presumably says that the hereditarily finite sets provide sufficient means to express all of finitary mathematics;
2. in particular, the subsuming of arithmetic within finite set theory by the identification of the natural numbers with the finite (von Neumann) ordinals: although this is, in principle, not the only representation one could choose, it is hegemonic because it is the most practical and graceful;
3. the cumulative hierarchy which starts with the empty set and generates all sets by iterating the power set operator.

I shall take for granted the first two items, as well as the general idea of generating all sets from the empty set but propose a different generating principle: the binary

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operation of *adjunction* (or *adduction*[1])

$$\langle x, y\rangle \mapsto x \cup \{y\}$$

which adds a single new element to an already existing set.

It is well known that the hereditarily finite sets can be characterized as those sets which are members of every class X such that the empty set is in X and X is closed under adjunction; see Section 2 for some historical remarks.

In [10], I investigated a hierarchy on the hereditarily finite sets based on the adjunction operator. In this paper I show how the notion of adjunction is a natural starting point for expressing set-theoretic forms of first-order mathematical induction and for defining the primitive recursive set functions. Adjunction, as a generator, will be seen to be intrinsically finitist in character.

An interesting contrast between arithmetic and set theory is that the former is based upon functions (successor, addition, multiplication) whereas the latter, in its usual formalization, is based upon a relation (the membership relation).[2]

By making the notion of adjunction fundamental, we can base set theory upon a function which is a generalization of the successor function of arithmetic. This brings out in a more direct way the parallels between finite set theory and arithmetic. In fact, the fundamental generator for arithmetic, the successor function, will be seen as the *diagonalization* of the adjunction operator.

In Section 2, building on work of Previale, I define a first-order version of an induction principle based on adjunction which is due to Tarski and Givant. This is used to present a simplified version of Previale's Peano set theory PS and some subtheories of PS.

In Section 3, I survey the primitive recursive set functions and use adjunction to simplify and clarify some previous expositions thereof. In particular, I introduce a criterion called respectfulness which ensures that a primitive recursive definition is consistent. I discuss primitive recursive definitions of basic set-theoretic operators.

This allows me, in the brief Section 4, to state suitable versions of different kinds of set-theoretic induction that also hold in subtheories of PS.

In another paper [9], I defined generalizations to all sets of ordinal addition and multiplication. In Section 5, I sketch the development of this arithmetic for the hereditarily finite sets. I show how, in the finite case, addition and multiplication of sets have natural primitive recursive definitions in terms of adjunction, and I sketch how their arithmetic is developed within subsystems of PS.

2 Peano Set Theory

We work in a language $\mathcal{L}(0\,;)$ for set theory which differs from the usual language $\mathcal{L}(0 \in)$. $\mathcal{L}(0\,;)$ has a constant symbol 0, which will be used for the empty set and a binary function symbol written $x; y$ or $[x; y]$.[3] The intended interpretation of $a; p$ is $a \cup \{p\}$. Informally, I shall write $a; p, q$ for $[a; p]; q$.

$p \in a$ is defined to mean $a; p = a$. So, in fact, $\mathcal{L}(0 \in)$ and $\mathcal{L}(0\,;)$ have the same expressive power: each of the symbols is definable in terms of the other in a common extension $\mathcal{L}(0 \in ;)$.

Following the usual development of set theory, the finite (von Neumann) ordinals are defined by iterating the *successor* operator $x \mapsto [x; x]$. V_ω denotes the ωth level of the usual cumulative hierarchy, which is obtained by iterating the power set operator. V_ω is equal to the set of hereditarily finite sets. Following a standard abuse of notation, I shall also denote by V_ω the structure for $\mathcal{L}(0\,;)$ whose domain is V_ω with the restriction of the adjunction operator to V_ω.

For closed terms s, t of $\mathcal{L}(0\,;)$, define $s \equiv t \Leftrightarrow s^{V_\omega} = t^{V_\omega}$, where t^{V_ω} is the interpretation of the closed term t in V_ω. Equivalently, $s \equiv t \Leftrightarrow V_\omega \models s = t$. A syntactical equivalent for this concept is given in [10], along with more details of these preliminaries.

Let PS_0 be the theory consisting of the universal closures of the following axioms:

$$0;\, x \neq 0. \tag{1}$$

$$x;\, y, y = x;\, y. \tag{2}$$

$$x;\, y, z = x;\, z, y. \tag{3}$$

$$x;\, y, z = x;\, y \leftrightarrow x;\, z = x \;\vee\; z = y. \tag{4}$$

PS_0 is true in V_ω. Montagna and Mancini [13] have shown how to interpret Robinson's Q in a system consisting of the axioms (1) and (4).

In 1909, Zermelo [27] proposed a form of induction on finite sets. In *Principia Mathematica*, Whitehead and Russell proved that the sets obtainable from the empty set by repeated adjunctions are precisely the finite sets.[4] In 1924 Tarski [21] included this idea among several notions of finiteness which he proved equivalent. He used a formulation in terms of an induction that was closer in spirit to Peano's induction axiom. Recast as a first-order schema, and in the notation of the present paper, Tarski's 1924 induction is this:

$$\textit{Weak induction} \quad \varphi(0) \wedge \forall xy(\varphi(x) \rightarrow \varphi([x;\, y])) \rightarrow \forall x \varphi(x), \tag{5}$$

where φ may also contain parameters. But what turns out to be a more fruitful form of induction was proposed half a century later by Givant and Tarski [6]:

$$\textit{Induction} \quad \varphi(0) \wedge \forall xy(\varphi(x) \wedge \varphi(y) \rightarrow \varphi([x;\, y])) \rightarrow \forall x \varphi(x). \tag{6}$$

As Givant and Tarski pointed out, this version of induction incorporates foundation (see Proposition 2.6 below). And it corresponds to generating the hereditarily finite sets rather than the finite sets.

Following, but simplifying, Previale [17], I denote by PS (for Peano set theory) the axioms (1)–(4) together with the induction schema (6) for first-order formulas, with parameters. After Lemma 2.5, I shall discuss the equivalence of my version of PS with Previale's.

An equivalent axiom system was given by Tarski and Givant in [23, p. 223], and a modification was developed by Świerczkowski [19] to provide a framework for the Incompleteness Theorems in which sequences do not need to be Gödel coded.

Now I review the Levy hierarchy of formulas of set theory: $\Delta_0 = \Sigma_0 = \Pi_0$ is the class of formulas all of whose quantifiers are bounded, that is, of form $\forall y \in x$ or $\exists y \in x$. A Σ_{n+1} formula consists of a block of unbounded existential quantifiers followed by a Π_n formula, and a Π_{n+1} formula consists of a block of unbounded universal quantifiers followed by a Σ_n formula. This hierarchy of formulas can be used to define a hierarchy of subsystems of PS, analogous to the subsystems $I\Sigma_n$ and $I\Pi_n$ of first-order Peano arithmetic PA which were first studied by Parsons [15], [16]. $I\Sigma_n S$ will denote PS with the induction schema (6) restricted to Σ_n formulas, and similarly for $I\Pi_n S$ and $I\Delta_0 S$. I shall denote by *Weak* $I\Sigma_n S$ and *Weak* $I\Pi_n S$ the corresponding theories using weak induction (5). Basic predicate logic shows that $I\Sigma_n S \vdash \textit{Weak}\, I\Sigma_n S$.

Analogously to the case of arithmetic, we can define a theory $B\Sigma_n S$ which consists of *Weak* $I\Delta_0 S$ together with the Σ_n-*collection* schema:

$$\forall x \in y\, \exists z\, \varphi \rightarrow \exists u\, \forall x \in y\, \exists z \in u\, \varphi,$$

for φ in Σ_n. Then we follow what was done for arithmetic in Parsons [15] and Paris-Kirby [14].

Lemma 2.1 *Any formula of form* $\forall x \in y\ \theta$ *with* θ *in* Σ_n *is, provably in* $B\Sigma_n S$, *equivalent to a* Σ_n *formula.*

A block of like unbounded quantifiers can be replaced by a single quantifier, either by developing a pairing function, or, as the referee has pointed out, by observing that it is provable in PS_0 that

$$\exists x_1 \ldots \exists x_n\, \varphi \leftrightarrow \exists x\, \exists x_1 \in x \ldots \exists x_n \in x\, \varphi,$$

so if $\varphi \in \Pi_n$, then by Lemma 2.1 there is a Π_n formula ψ such that

$$B\Sigma_n S \vdash \exists x_1 \ldots \exists x_n\, \varphi \leftrightarrow \exists x\, \psi.$$

Lemma 2.2 *Weak* $I\Sigma_n S \vdash B\Sigma_n S$, *if* $n > 0$.

Proof By induction on n. The idea is much like the corresponding proof for arithmetic, but I give some details to indicate how weak induction is enough. Suppose that either $n = 1$ or $n > 1$ with the result proven for $n-1$. Suppose $\forall x \in a \exists z \varphi$ where φ is a Σ_n formula, say, φ is $\exists y \theta(x, y, z)$ with θ in Π_{n-1}. It will be enough to prove $\forall u\, \psi(u)$, where $\psi(u)$ is

$$(u \subseteq a) \rightarrow (\exists t\, \forall x \in u\, \exists z \in t\, \exists y \in t\, \theta(x, y, z)).$$

If $n = 1$, then ψ is Σ_1. If $n > 1$, then by the inductive hypothesis we have $B\Sigma_{n-1}S$, so by Lemma 2.1 we may assume that ψ is Σ_n. Now we apply weak induction to ψ. Suppose $\psi(u)$ and $u \subseteq a$, so that for some b, $\forall x \in u\, \exists z \in b\, \exists y \in b\, \theta$. We need to show that $\psi(u; v)$ for any v. But we may assume that $v \in a$ so that for some c, d, $\theta(v, c, d)$. Then $\forall x \in [u; v]\, \exists z \in [b; c, d]\, \exists y \in [b; c, d]\, \theta(x, y, z)$. □

Weak $I\Sigma_1 S$ proves the existence for any a, p of $a \setminus \{p\}$, where $\setminus$ denotes the set-theoretic difference.

Lemma 2.3 *Weak* $I\Sigma_1 S \vdash \forall xy\, \exists z\, ((y \notin x \wedge z = x) \vee (y \in x \wedge z; y = x \wedge y \notin z))$.

Proof For given p, show by weak induction on a that $\exists z((p \notin a \wedge z = a) \vee (p \in a \wedge z; p = a \wedge p \notin z))$, or as we may informally write, $z = a \setminus \{p\}$ exists. If $a = 0$, then $z = 0$. If $z = a \setminus \{p\}$ is given, then we can define $z' = [a; q] \setminus \{p\}$: if $p = q$, then $z' = z$, and otherwise $z' = z; q$. □

Next I note that the ordinals of a model of $I\Sigma_n S$ armed with the successor function form a model of the arithmetical theory $I\Sigma_n$: let $\mathrm{Ord}(x)$ be a Δ_0 formula denoting 'x is transitive as are all elements of x', so that this predicate represents the ordinals of a model of PS_0. Then in $I\Sigma_n S$ one can prove that for a Σ_n formula φ,

$$\varphi(0) \wedge \forall x (\mathrm{Ord}(x) \rightarrow (\varphi(x) \rightarrow \varphi(x; x)) \rightarrow \forall x (\mathrm{Ord}(x) \rightarrow \varphi(x)).$$

Lemma 2.4 *Weak* $I\Sigma_n S \vdash$ *Weak* $I\Pi_n S$.

Proof Suppose that φ is a counterexample to *Weak* $I\Pi_n S$: $\varphi(0) \wedge \forall xy(\varphi(x) \rightarrow \varphi([x;y]))$ but $\neg\varphi(a)$, with $\varphi \in \Pi_n$. Let $\psi(x)$ be a Σ_n formula representing $\neg\varphi(a \setminus x)$: formally, use Lemmas 2.1 and 2.2 to obtain $\psi(x)$ equivalent to

$$\exists y(\forall z \in y\,(z \in a) \wedge \forall z \in a(z \in y \leftrightarrow z \notin x) \wedge \neg\varphi(y)).$$

Then $\psi(0)$ and $\neg\psi(a)$, and suppose $\psi(x)$. Then for any y, $\psi(x;y)$: for if $y \in x$, then this is trivial, and if $y \notin x$, then $a \setminus x = [(a \setminus [x;y]);y]$ and so the assumption of weak induction for φ gives $\neg\varphi(a \setminus x) \rightarrow \neg\varphi(a \setminus [x;y])$. So ψ is a counterexample to *Weak* $I\Sigma_n S$. $\square$

Extensionality follows from the particular instances of it in axioms (1) and (4).

Lemma 2.5

(i) $\mathrm{PS}_0 \vdash \forall xy\,[x;y] \neq 0$.
(ii) *Weak* $I\Delta_0 S \vdash \exists y(y \in a) \leftrightarrow a \neq 0$.
(iii) *Weak* $I\Sigma_1 S \vdash \forall y(y \in a \leftrightarrow y \in b) \rightarrow a = b$.

Proof (i) If $a;p = 0$, then using axiom (2), $0;p = a;p$, $p = a;p = 0$, contradicting axiom (1).

(ii) If $p \in a$, then $a = a;p$ so by (i) $a \neq 0$. A weak induction on a shows that $a \neq 0 \rightarrow \exists y \in a\,(y = y)$.

(iii) A proof that extensionality follows from (ii) and the above axioms for PS is essentially contained in Previale [17, Section 5]. We prove by weak induction on a that $\forall z(\forall y(y \in a \leftrightarrow y \in z) \rightarrow a = z)$. (This can be written as a Π_1 statement, so we are appealing to Lemma 2.4.) The case $a = 0$ follows from (ii). Suppose the result is proven for a and show it for $a;p$. If $p \in a$, then $a;p = a$, so there is nothing to prove, so assume $p \notin a$. Assume for arbitrary z: $\forall y(y \in a;p \leftrightarrow y \in z)$. Take $z_0 = z \setminus \{p\}$ given by Lemma 2.3. So $p \notin z_0$ and $z_0;p = z$. Thus for any y,

$$z;y = z \leftrightarrow z_0;p,y = z_0;p \leftrightarrow z_0;y = z_0 \vee y = p \quad \text{(axiom (4))};$$

that is, $y \in z \leftrightarrow y \in z_0 \vee y = p$. Since $y \in a;p \leftrightarrow y \in a \vee y = p$, and recalling that $p \notin a$ and $p \notin z_0$, it follows that $y \in z_0 \leftrightarrow y \in a$. By inductive hypothesis, $z_0 = a$ and hence $z = z_0;p = a;p$ using (2). $\square$

Alternatively, as the referee has suggested, one can prove extensionality by developing the concept of rank in $I\Sigma_1 S$ and using induction on rank (which I shall mention later in Section 4).

Now I discuss the relationship of the present system to that of Previale [17]. Since extensionality guarantees the uniqueness of $x \setminus \{y\}$, we may choose, as Previale does, to include in our language a binary function symbol for $x \setminus \{y\}$, with as its definition,

$$x \setminus \{y\} = z \leftrightarrow (y \notin x \wedge z = x) \vee (y \in x \wedge z;y = x \wedge y \notin z)$$

(cf. Lemma 2.3).

Previale has another symbol in his language: $x < y$, corresponding to 'x is an element of the transitive closure of y'. But this symbol can be eliminated by definition in $I\Sigma_1 S$ and stronger theories, because $I\Sigma_1 S \vdash \mathrm{TC}$, where TC is the *axiom of transitive containment* which states that every set is a subset of a transitive set, or equivalently that every set x has a transitive closure $\mathrm{TC}(x)$. (The proof of this from $I\Sigma_1 S$ is straightforward, and it may be instructive to observe where induction, rather than weak induction, is used; cf. equation (11) below.) Thus the theory PS in Previale's extended language is a conservative extension of our PS.

Proposition 2.6 (Previale)

(i) *($\in$-induction)* $\mathrm{PS} \vdash \forall x((\forall y \in x\, \varphi(y) \to \varphi(x)) \to \forall x \varphi(x)$.

(ii) *(Foundation schema)* $\mathrm{PS} \vdash \exists x \varphi(x) \to \exists x(\varphi(x) \wedge \forall y \in x\, \neg\varphi(y))$.

(iii) $\mathrm{PS} \vdash \mathrm{ZF} + \neg\infty$*; that is, Zermelo-Fränkel set theory with the axiom of infinity replaced by its negation.*

Here $\mathrm{ZF} + \neg\infty$ may just as well be considered as formulated in $\mathcal{L}(0\,;)$ rather than in the usual language of set theory. For the proof of this proposition the reader is referred to [17]. Note that (ii) is just the contrapositive of (i) with φ replaced by $\neg\varphi$. (i) and (ii) will be refined to subsystems of PS in Section 4 below.

The following is an extension of (iii).

Proposition 2.7 PS *is logically equivalent to* $\mathrm{ZF} + \neg\infty + \mathrm{TC}$.

Sketch of a proof It was mentioned above that $\mathrm{PS} \vdash \mathrm{TC}$. For the converse, which is a variation of the well-known (or at least known) fact that TC is needed to prove $\in$-induction in the absence of infinity, notice that if we have a counterexample to induction in the shape $\varphi(0)$, $\forall xy(\varphi(x) \wedge \varphi(y) \to \varphi([x; y]))$, but $\neg\varphi(a)$; then the set $\{x \in \mathrm{TC}(\{a\}) \mid \varphi(x)\}$ can be used to obtain the axiom of infinity.[5] □

Closely related to all this is the "folklore" result that PA and $\mathrm{ZF} + \neg\infty$ are mutually faithfully interpretable. The argument can be summarized as follows, in two converse directions.

On the one hand, a coding first introduced by Ackermann [1] gives an interpretation of finite set theory in number theory, which is exposed in detail, for example, by Wang [25] and Kaye and Wong [8]. Essentially, if we already have natural numbers $n_1, \ldots, n_k$ coding the finite sets $a_1, \ldots, a_k$, then the set $\{a_1, \ldots, a_k\}$ is coded by $2^{n_1} + \cdots + 2^{n_k}$. Ackermann's coding will be used in Section 3 below. And on the other hand, as in the discussion preceding Lemma 2.4, the ordinals of a model of $\mathrm{ZF} + \neg\infty$ form a model of PA. For detailed discussion of this, including the fact that TC needs to be added to $\mathrm{ZF} + \neg\infty$ to get interpretations which are inverse to each other, see Kaye and Wong [8].

3 Primitive Recursive Set Functions

Tait [20] expounds the thesis that "finitist reasoning is essentially primitive recursive reasoning in the sense of Skolem":

> We discern finite sequences in our experience—sequences of words on a page, of peals of a bell, of people in a room ordered by age or size or simply by counting. ([20], p. 529)

Tait calls the "form" of finite sequences "Number," and continues:

> . . . we are taking Number in its ordinal sense. If we wished to take it in the sense of cardinals we would only need to replace 'finite sequence' in the discussion with 'finite set' without any essential difference.

So, while it is upon the basis of Number that Tait defends his thesis, it applies equally well to finitist reasoning in set theory. The primitive recursive set functions are the finitistic set operations. In this section I shall show how the language of adjunction allows a simple definition and development of the primitive recursive set functions on V_ω, using recursive definitions of form

$$f(0, \vec{z}) = g(\vec{z})\,, \quad f([a; p], \vec{z}) = h(a, p, f(a, \vec{z}), f(p, \vec{z}), \vec{z}). \tag{7}$$

Before laying this out in detail, I survey some previous expositions of the primitive recursive set functions.

Rödding [18] defines the primitive recursive set functions as the smallest class of functions from V_ω^k to V_ω containing as initial functions the constant function $\tilde{0}(\vec{x}) = 0$, the projections $P_{n,i}(x_1, \ldots, x_n) = x_i$, singleton operator $x \mapsto \{x\}$, union $x \cup y$, and intersection $x \cap y$, and closed under *substitutions*

$$f(\vec{x}) = g(h_1(\vec{x}), \cdots, h_k(\vec{x}))$$

and *recursive definitions* of form

$$\begin{aligned} f(0, \vec{z}) &= g(\vec{z}), \\ f(\{a\}, \vec{z}) &= h_1(a, f(a, \vec{z})), \vec{z}), \\ f(a \cup b, \vec{z}) &= h_2(a, b, f(a, \vec{z}), f(b, \vec{z}), \vec{z}). \end{aligned}$$

In my exposition below, two of Rödding's generating operators, union and singleton, will be replaced by the single adjunction operator $[x; y]$. Indeed, it is easy to define adjunction in terms of union and singleton, and singleton in terms of adjunction, and I shall give a primitive recursive definition of union in terms of adjunction.

In addition, Mahn [12] shows that the intersection operator may be omitted from the initial functions in Rödding's system, and below I shall adapt his proof to obtain a primitive recursive definition of intersection.

Jensen and Karp [7] define a more general class of (transfinite) primitive recursive set functions on a broad class of transitive classes (so that V_ω is a special case). Following, they say, Gandy, they use adjunction among their initial functions in place of singleton and union, and instead of intersection they use the cases function:

$$C(x, y, u, v) = \begin{cases} x & \text{if } u \in v, \\ y & \text{otherwise.} \end{cases} \tag{8}$$

And Jensen and Karp have yet another recursion schema:

$$f(x, \vec{z}) = g(\bigcup \{f(u, \vec{z}) \,|\, u \in x\}, x, \vec{z}).$$

I omit the technical proof that this schema is equivalent in the case of finitary set theory to Rödding's and to (7). Below I shall indicate how the cases function can be obtained from the other initial functions.

Informally, since Ackermann's coding surely fits our intuition of a primitive recursive translation procedure between sets and numbers, primitive recursive set theory is essentially the same as primitive recursive arithmetic. This is formally proved by Rödding [18] who shows that the primitive recursive set functions are identical, modulo Ackermann's coding, with the number-theoretic primitive recursive functions. (See also [7, Section 3].)

In the above recursion schemas, both (7) and the Rödding version, I glossed over a problem with primitive recursions on sets. The problem is that there are many ways to build up a hereditarily finite set using adjunction (or using singleton and union); that is, for nonempty $a \in V_\omega$ there are many closed terms t of $\mathcal{L}(0\,;)$ such that $t^{V_\omega} = a$, and we need to make sure that the result of the recursive procedure does not depend on the order in which the set is built up.[6] Rödding deals with this problem by stipulating that the three clauses of his recursive definition are not mutually contradictory (*nicht untereinander widerspruchsvoll*) and leaving it at that. But this is not clear. I now provide a precise and sufficient criterion for such consistency.

Definition 3.1 Let $h(x, y, u, v, \vec{z})$ be a function from V_ω^{k+4} to V_ω, where k is the arity of $\vec{z}$. h is *respectful* if the universal closures of the following statements are true:

$$h([x; y], y, h(x, y, u, v, \vec{z}), v, \vec{z}) = h(x, y, u, v, \vec{z}) \tag{9}$$

and

$$h([x; y], y', h(x, y, u, v, \vec{z}), v', \vec{z}) = h([x; y'], y, h(x, y', u, v', \vec{z}), v, \vec{z}). \tag{10}$$

Definition 3.2 The set $\mathrm{PR}(V_\omega)$ of *primitive recursive set functions* on V_ω is the smallest set of functions from V_ω^k to V_ω containing as initial functions the constant function $\tilde{0}(\vec{x}) = 0$, the projections $P_{n,i}(x_1, \ldots, x_n) = x_i$, and adjunction $[x; y]$, and closed under *substitutions*

$$f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_k(\vec{x}))$$

where g and h_i are primitive recursive, and *recursion* of form (7) where g and h are primitive recursive and h is respectful.

Later it will be convenient to distinguish concisely the primitive recursive set functions in $\mathrm{PR}(V_\omega)$ from the classical (number-theoretic) primitive recursive functions on ω, so I shall denote the latter by $\mathrm{PR}(\omega)$.

We need to justify Definition 3.2 by showing that such an f is well defined. The following informal argument will show that respectfulness of h suffices for building up f without ambiguity.

Suppose f is defined recursively from $f(0)$ and $f([a; p]) = h(a, p, f(a), f(p))$ with h respectful. (Here and later I omit parameters.) The value of $f(\{a_1, \ldots, a_n\})$ is obtained by iterations of the recursion scheme that yield a sequence consisting of the values of $f(0)$, $f(\{a_1\})$, $f(\{a_1, a_2\}), \ldots, f(\{a_1, \ldots, a_n\})$. The value of $f(\{a_1, \ldots, a_k\})$ depends, via the function h, on the value of $f(\{a_1, \ldots, a_{k-1}\})$ and the value of $f(a_k)$, and the latter is likewise obtained by considering a list of the members of a_k. So to show that the value assigned to $f(\{a_1, \ldots, a_n\})$ is unique it is enough to show that whenever $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_m\}$, the two sequences of values end at the same value.

The second clause (10) of respectfulness implies that we can always transpose two adjacent members of a sequence; that is, iterations over the sequences

$$\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\} \text{ and } \{a_1, \ldots, a_{i+1}, a_i, \ldots, a_n\}$$

give the same final value. Since any permutation of $\{a_1, \ldots, a_n\}$ is obtainable as a product of such transpositions, any permutation of the original sequence will give the same final value. The first clause (9) implies that repeated elements can be omitted: $\{a_1, \ldots, a_i, a_i, a_{i+2}, \ldots, a_n\}$ and $\{a_1, \ldots, a_i, a_{i+2}, \ldots, a_n\}$ give the same final value. So any two listings of a set yield the same final value.

The above argument can be formalized syntactically by working with functions on terms of $\mathcal{L}(0\,;)$ rather than on sets and showing that if $s \equiv t$ then a respectfully defined function f has $f(s) \equiv f(t)$.

3.1 Primitive recursive set functions and $I\Sigma_1 S$ Just as the number-theoretic primitive recursive functions are those which are provably total in $I\Sigma_1$ (a result due to Parsons [15]), analogously the primitive recursive set functions are those provably total in $I\Sigma_1 S$. Before getting to the nuts and bolts of constructing some primitive recursive set functions, it will be useful to sketch one half of this analogue of Parsons' theorem for set theory, namely, that the class of definable functions of $I\Sigma_1 S$ is closed under primitive recursion, so that any primitive recursive set function is definable in $I\Sigma_1 S$ by a Σ_1 formula. This will justify the use in $I\Sigma_1 S$ of recursion over already-defined primitive recursive functions to define new ones.

Suppose f is defined from g and h by the primitive recursion scheme (7), with h respectful, and that we already have Σ_1 formulas φ and ψ defining the functions g and h: $I\Sigma_1 S \vdash \forall\vec{z}\exists! y\varphi(y,\vec{z})$ and $I\Sigma_1 S \vdash \forall\vec{z}xyuv\exists! w\psi(x,y,u,v,w,\vec{z})$. Informally, we write $g(\vec{z}) = w$ for $\varphi(w,\vec{z})$ and $h(x,y,u,v,\vec{z}) = w$ for $\psi(x,y,u,v,w,\vec{z})$. We need to show that there is a Σ_1 formula θ such that $I\Sigma_1 S \vdash \forall\vec{z}\forall x\exists! u\theta(x,u,\vec{z})$ and $I\Sigma_1 S$ also proves $\forall\vec{z}\theta(0, g(\vec{z}),\vec{z})$ and $\forall\vec{z}\forall xy\theta([x;y], h(x,y,f(x,\vec{z}), f(y,\vec{z}),\vec{z})$. This θ defines f.

In $I\Sigma_1 S$, sequences can be coded in the usual way, so that a set s represents a sequence $\langle s_0,\dots,s_{l(s)}\rangle$, where $l(s)$ is one less than the length of s.

Let 's builds x' be the formula

$$s_0 = 0 \ \wedge\ s_{l(s)} = x \ \wedge\ \forall i \le l(s)\ (i = 0 \ \vee\ \exists jk < i\ \ s_i = s_j; s_k).$$

Then it is straightforward to show that

$$I\Sigma_1 S \vdash \forall x\ \exists s\ (s \text{ builds } x).$$

Now let $\theta(x,u,\vec{z})$ be the formula

$$\begin{aligned}&\exists st\ \big(l(s) = l(t) \wedge\ s \text{ builds } x\ \wedge\ \varphi(t_0,\vec{z})\\ &\qquad \wedge\ \forall i \le l(t)\,\forall jk < i\ (s_i = [s_j; s_k] \to \psi(s_j, s_k, t_j, t_k, t_i, \vec{z}))\ \wedge\ t_{l(t)} = u\big).\end{aligned}$$

This θ has the required properties. And, as above, the respectfulness of h guarantees the uniqueness of $u = f(x,\vec{z})$, regardless of which sequence s building x is used.

The other half of the analogue of Parsons' theorem—that the provably total functions of $I\Sigma_1 S$ are primitive recursive—can be verified by examining proofs of Parsons' theorem which use general consequences of Herbrand's theorem that apply also to our set-theoretic setting, for example, those by Avigad [2] and Ferreira [3]. In particular, constructions of primitive recursive set functions such as will be given below can be construed as existence proofs in $I\Sigma_1 S$.

3.2 Some examples of primitive recursive set functions In the remainder of this section, I shall run through the primitive recursive definitions of some familiar basic set operations.

If t is a closed term of $\mathcal{L}(0\,;)$, then the constant function to t^{V_ω} is in PR. Any open term $t(\vec{x})$ of $\mathcal{L}(0;\,)$ gives rise to a function t^{V_ω} in PR, and we simply write t for this function.

The binary union operator $f(x,y) = x \cup y$ is defined by

$$a \cup 0 = a,\ \ a \cup [b;p] = [(a \cup b);p].$$

Here the recursion step of (7) is carried out using the function $h(x, y, u, v) = u; y$ which is respectful because, working in PS_0,

$$h([x; y], y, h(x, y, u, v), v) = h([x; y], y, [u; y], v) \\ = [u; y]; y = u; y = h(x, y, u, v),$$

and similarly

$$h([x; y], y', h(x, y, u, v,), v') = u; y, y' = u; y', y \\ = h([x; y'], y, h(x, y', u, v'), v).$$

When using a primitive recursive definition of a standard operation in PS, we need to justify the name by verifying the properties of the function in PS. Thus for the union we note the following.

Lemma 3.3 *The universal closures of the following statements are provable in* $I\Sigma_1 S$*:*

(i) $0 \cup a = a$*;*
(ii) $x \in a \cup b \leftrightarrow x \in a \vee x \in b$*.*

Proof (i) By induction on a.

$$\begin{aligned} 0 \cup [a; p] &= (0 \cup a); p && \text{by the definition of } \cup \\ &= a; p && \text{by inductive hypothesis.} \end{aligned}$$

(ii) By induction on b for fixed a:

$$\begin{aligned} x \in a \cup [b; p] = (a \cup b); p &\leftrightarrow x \in a \cup b \ \vee\ x = p && \text{by Axiom (4)} \\ &\leftrightarrow x \in a \ \vee\ x \in b \ \vee\ x = p && \text{by inductive hypothesis} \\ &\leftrightarrow x \in a \ \vee\ x \in [b; p] && \text{by Axiom (4) again.} \end{aligned}$$

□

(The attentive reader may notice that both the inductions in this proof were weak inductions. But as in the discussion above, $I\Sigma_1 S$ rather than *Weak* $I\Sigma_1 S$ is needed to establish good behavior of recursive definitions; other properties of standard set operators do require $I\Sigma_1 S$, as does the proof of Lemma 3.4 below.)

Standard properties of the binary union operator, such as commutativity and associativity, follow.

The unary union operator $\bigcup x$ is defined by

$$\bigcup 0 = 0, \ \bigcup [a; p] = \bigcup a \cup p.$$

Here we need to check that the defining function $h(x, y, u, v) = u \cup y$ is respectful. This is because $(a \cup p) \cup p = a \cup p$ and $(a \cup p) \cup q = (a \cup q) \cup p$. In the following examples of standard operators, verification of respect, and of the standard properties, is left to the reader where it is straightforward.

The transitive closure is defined by

$$\mathrm{TC}(0) = 0, \ \mathrm{TC}(a; p) = [(\mathrm{TC}(a) \cup \mathrm{TC}(p)); p]. \tag{11}$$

I omit proofs that $a \subseteq \mathrm{TC}(a)$, $y \in x \in \mathrm{TC}(a) \to y \in \mathrm{TC}(a)$, and $\mathrm{TC}(a) = \bigcup\{\mathrm{TC}(x) \mid x \in a\} \cup a$. The following strengthening of the axiom of foundation is proved, for PS, by Previale [17, Section 3, Proposition 8].

Lemma 3.4 $I\Sigma_1 S \vdash \forall x(x \notin \mathrm{TC}(x))$.

The power set operator Px is defined in two stages: first define an operation $a \triangleright b = \{[x; b] \mid x \in a\}$ by

$$0 \triangleright b = 0,\ [a; p] \triangleright b = (a \triangleright b); [p; b].$$

Now $P0 = 1,\ P[a; p] = Pa \cup (Pa \triangleright p)$.

The set ${}^y x$ of functions from y to x: first define $a \triangleleft_p b = \{[a; \langle p, y\rangle] \mid y \in b\}$ by

$$a \triangleleft_p 0 = 0,\ a \triangleleft_p [b; q] = (a \triangleleft_p b); [a; \langle p, q\rangle].$$

(The ordered pair $\langle p, q\rangle$ is defined as usual: $\langle p, q\rangle = 0; [0; p], [0; p, q]$). Then

$${}^0 x = 1,\ {}^{[a;p]}x = \bigcup\{z \triangleleft_p x \mid z \in {}^a x\}.$$

The Cartesian product $a \times b$: first define $a \times \{b\}$ by

$$0 \times \{b\} = 0,\ [a; p] \times \{b\} = (a \times \{b\}); \langle p, b\rangle.$$

Now $a \times 0 = 0,\ a \times [b; p] = (a \times b) \cup (a \times \{p\})$.

The operation $x \Gamma y = x \cup y \cup \{ [u; v] \mid u \in x \wedge v \in y\}$ is in $\mathrm{PR}(V_\omega)$: first define $a \star q$ by

$$0 \star q = 0,\ [a; p] \star q = (a \star q); [p; q].$$

Then $a\Gamma 0 = 0,\ a\Gamma[b; p] = (a\Gamma b \cup a \star p); p$.

Now the adjunctive hierarchy introduced in [10] can be defined by

$$A_0 = \{0\},\ A_{n+1} = A_n \Gamma A_n,$$

and the results concerning the A_n in [10] can be formalized as proofs in PS.

If $\varphi(\vec{x})$ is a formula, we write $\chi_\varphi(\vec{x})$ for the characteristic function of φ: $\chi_\varphi(\vec{x}) = 1$ if $V_\omega \models \varphi(\vec{x})$, $= 0$ if not. A relation is said to be primitive recursive if and only if its characteristic function is. The function $\chi_{x\neq 0}(x)$ can be primitively recursively defined by

$$\chi_{x\neq 0}(0) = 0,\ \chi_{x\neq 0}([a; p]) = 1$$

and is in $\mathrm{PR}(V_\omega)$ since the constant function $h(x, y, u, v) = 1$ is easily seen to be respectful. Likewise for $\chi_{x=0}(x)$.

It is almost as easy to write the recursion scheme for a simple cases function, though we don't yet have the full Jensen-Karp cases function (8).

Lemma 3.5 *Suppose*

$$f(\vec{x}) = \begin{cases} g(\vec{x}) & \text{if } k(\vec{x}) = 0, \\ h(\vec{x}) & \text{otherwise} \end{cases}$$

where g, h, and k are primitive recursive. Then f is primitive recursive.

Proof Let $\eta(x, 0) = 0,\quad \eta(x, [y; z]) = x$. Then

$$f(\vec{x}) = \eta(h(\vec{x}), k(\vec{x})) \cup \eta(g(\vec{x}), \chi_{x=0}(k(\vec{x}))).$$

I shall next indicate how to obtain primitive recursive definitions of such functions as the Jensen-Karp cases function, intersection, and $x \setminus \{y\}$. These operators do not allow of straightforward primitive recursive definitions (as the union operator did, for example), and especially for the cases function which is nonmonotonic, it is not immediately obvious how to proceed. We shall draw on some of the machinery of the classical theory of the primitive recursive functions on the natural numbers, $\mathrm{PR}(\omega)$.

The natural numbers are firmly identified here, in our set-theoretic framework, with the finite von Neumann ordinals (elements of ω).

But the crucial trick, due to Mahn [12], is to use also a different representation of the natural numbers within V_ω. Define $\underline{0} = 0$ and $\underline{n+1} = \{\underline{n}\}$, and $\underline{\omega} = \{\underline{n} \mid n \in \omega\} \subseteq V_\omega$. Then for any function $f : \omega^k \to \omega$, define $\underline{f} : \underline{\omega}^k \to \underline{\omega}$ by $\underline{f}(\vec{\underline{n}}) = \underline{m}$ if and only if $f(\vec{n}) = m$. □

Lemma 3.6 *Let $f : \omega^k \to \omega$ be in* $\mathrm{PR}(\omega)$*. Then there is a function $\underline{\underline{f}}$ in* $\mathrm{PR}(V_\omega)$*, that is, a primitive recursive set function $\underline{\underline{f}} : V_\omega^k \to V_\omega$, such that $\underline{\underline{f}} \supset \underline{f}$.*

Proof By induction on the way $\mathrm{PR}(\omega)$ functions are generated. The initial functions (zero, successor, and projections) and composition are easily dealt with; for example; if S is the successor function then $\underline{\underline{S}}(x) = \{x\}$. Suppose f is generated from g and h by (suppressing parameters) $f(0) = g$, $f(n+1) = h(n, f(n))$, with $\underline{\underline{g}}$ and $\underline{\underline{h}}$ already obtained. Then define

$$\underline{\underline{f}}(0) = \underline{\underline{g}}, \quad \underline{\underline{f}}([x; y]) = \begin{cases} \underline{\underline{h}}(y, \underline{\underline{f}}(y)) & \text{if } x = 0, \\ \underline{\underline{f}}(x) \cup \underline{\underline{h}}(y, \underline{\underline{f}}(y)) & \text{otherwise.} \end{cases}$$

This is a respectful primitive recursion, with the aid of Lemma 3.5. □

Next, let $\gamma : V_\omega \to \omega$ be Ackermann's bijection coding hereditarily finite sets by natural numbers,[7] so that $x \in y$ if and only if $2^{\gamma(x)}$ occurs in the binary expansion of $\gamma(y)$. By standard techniques in $\mathrm{PR}(\omega)$ functions, the function

$$e(m, n) = \begin{cases} 0 & \text{if } \gamma^{-1}(m) \in \gamma^{-1}(n), \\ 1 & \text{otherwise,} \end{cases}$$

is in $\mathrm{PR}(\omega)$: Rödding ([18], p. 18) gives details.

Similarly, the binary operation α which simulates adjunction on Ackermann codes

$$\alpha(\gamma(x), \gamma(y)) = \gamma([x; y])$$

is in $\mathrm{PR}(\omega)$:

$$\alpha(m, n) = \begin{cases} \alpha(m) & \text{if } e(m, n) = 0 \\ \alpha(m) + 2^n & \text{otherwise.} \end{cases}$$

We define

$$G(0) = 0, \quad G([x; y]) = \underline{\underline{\alpha}}(G(x), G(y)).$$

It can be verified that this is a respectful definition so that G is in $\mathrm{PR}(V_\omega)$, and (by induction on x) that $G(x) = \underline{\gamma(x)}$.

Now if C is the cases function (8), then

$$C(x, y, u, v) = \begin{cases} x & \text{if } \underline{\underline{e}}(G(u), G(v)) = 0, \\ y & \text{otherwise,} \end{cases}$$

and Lemma 3.5 tells us that C is primitive recursive.

The natural primitive recursion step in defining intersection

$$x \cap [y; z] = \begin{cases} (x \cap y); z & \text{if } z \in x, \\ x \cap y & \text{if } z \notin x, \end{cases}$$

can similarly be made to conform to Lemma 3.5. Similarly for $x \setminus \{y\}$, cf. the proof of Lemma 2.3. The function assigning rank in the cumulative hierarchy is also now available:

$$\varrho([x; y]) = C(\varrho(x), [\varrho(y); \varrho(y)], \varrho(y), \varrho(x)). \tag{12}$$

And the cardinality function $|x|$ is just as easy.

To see that $\chi_{x=y}$ is primitive recursive, use the PR(ω) function $d(m, n) = (m \dot{-} n) + (n \dot{-} m)$. So $\chi_{x=y}(a, b) = \chi_{x=0}(\underline{\underline{d}}(G(a), G(b)))$.

For $\chi_{x\in y}$, observe that

$$\chi_{x\in y}(a, 0) = 0\ , \ \ \chi_{x\in y}(a, [b; p]) = \chi_{x\in y}(a, b) \cup \chi_{x=y}(b, p).$$

4 Other Forms of Induction

From Section 3, I draw, in particular, the facts that the transitive closure and rank functions are definable and provably total in $I\Sigma_1 S$: see (11) and (12). It is straightforward to prove their standard properties in $I\Sigma_1 S$. In particular, the sets V_u of the cumulative hierarchy can be defined for ordinals u and shown to have the property

$$x \in V_u \leftrightarrow \varrho(x) \in u.$$

This enables us to formulate and prove in subsystems of PS suitable versions of $\in$-induction (refining Proposition 2.6), and of induction on rank.

Proposition 4.1 ($n > 0$)

(i) *Let $\varphi(x)$ be a Σ_n formula. Then $I\Sigma_n S \vdash \forall x((\forall y \in x\, \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$.*

(ii) *Let $\varphi(x)$ be a Π_n formula. Then $I\Sigma_n S \vdash \exists x \varphi(x) \rightarrow \exists x(\varphi(x) \wedge \forall y \in x\, \neg\varphi(y))$.*

(iii) *Let $\varphi(x)$ be a Σ_n formula. Then the following is provable in $I\Sigma_n S$: if $\varphi(0)$ and for all ordinals u, $\forall x(\varrho(x) \in u \rightarrow \varphi(x)) \rightarrow \forall x(\varrho(x) = u \rightarrow \varphi(x))$, then $\forall x \varphi(x)$.*

The proof of (i) proceeds by proving the apparently stronger statement

$$\forall x(\forall y \in \mathrm{TC}(x)\, \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x).$$

To do this, assume $\forall x(\forall y \in \mathrm{TC}(x)\, \varphi(y) \rightarrow \varphi(x))$ and prove by induction on x the formula $\forall y \in \mathrm{TC}(x)\, \varphi(y)$, which can be considered to be Σ_n because of Lemmas 2.1 and 2.2.

5 Generalizing Arithmetical Operations

The techniques of Section 3 could be used, in particular, to obtain extensions of the basic ordinal operations of arithmetic (addition and multiplication) to primitive recursive functions on all hereditarily finite sets. But there are natural PR(V_ω) extensions of these basic functions of arithmetic, more transparently related to the originals than what the general techniques of Section 3 would give.

The fact that (infinitary) ordinal addition has a natural generalization to all sets was observed by Tarski [22], and Scott (unpublished) did the same for multiplication and exponentiation. Garcia [5] independently and later rediscovered these generalizations, as did I later still. In [9] I explored the properties of ordinal addition and multiplication of sets, in a framework which includes both finitary (PS) and infinitary (ZF) set theory.

My purpose here is to sketch the development of addition and multiplication of sets in the finite case, giving only enough detail to indicate how they have natural primitive recursive definitions whose basic properties are provable in subtheories of PS. In [9] definitions and proofs were by $\in$-induction. Here, induction (6) is used. This section may be read independently of [9], although the reader is referred there for more details and further development.

Addition of sets is defined primitive recursively as follows:

$$a + 0 = a, \quad a + [b;\, p] = [(a + b);\, (a + p)]. \tag{13}$$

The notation is suitable because when $a, b \in \omega$, $a + b$ under this definition agrees with the usual addition of finite ordinals, so that it generalizes arithmetical addition. For example, $2 + 1 = 2 + [0;\, 0] = (2 + 0);\, (2 + 0) = 2;\, 2 = 3$. For any a, note that $a + 1 = [a;\, a]$.

We need to check that the defining function for the recursion step in the definition of addition, namely, $h(x, y, u, v) = [u;\, v]$, is respectful: for (9),

$$h([x;\, y], y, h(x, y, u, v), v) = [h(x, y, u, v);\, v] = [u;\, v];\, v = h(x, y, u, v)$$

and similarly (10) can be verified, using axiom (3).

Notice that addition of sets is not commutative even in our finite case: for example, $1 + \{1\} = \{0, 2\}$ whereas $\{1\} + 1 = \{1, \{1\}\}$.

Tarski's original definition of addition of sets, which was used in [9], is different, being by recursion on the membership relation: $a + b = a \cup \{a + x \mid x \in b\}$. In the finite case, the two definitions are easily seen to be equivalent: this is in (iii) and (iv) of Proposition 5.1.

A closely related function is the *lift* function $\lambda_a(b)$ defined by

$$\lambda_a(0) = 0, \;\; \lambda_a([b;\, p]) = [\lambda_a(b);\, (a + p)].$$

Thus $\lambda_a(1) = [0;\, a]$, and it is straightforward to prove by induction that $\lambda_0(a) = a$.

I now review some basic properties of addition of sets, giving only some sample proofs.

Proposition 5.1 *The universal closures of the following are provable in $I\Sigma_1 S$:*

(i) $0 + a = a$.

(ii) *Addition is associative.*

(iii) $a + b = a \cup \lambda_a(b)$.

(iv) $\lambda_a(b) = \{a + x \mid x \in b\}$.

(v) $a + b \notin \mathrm{TC}(a)$.

(vi) $\mathrm{TC}(a) \cap \lambda_a(b) = 0$. *(Hence $a \cap \lambda_a(b) = 0$.)*

(vii) $\lambda_a(\lambda_b(c)) = \lambda_{a+b}(c)$.

In particular, $a + b$ is the disjoint union of a and $\lambda_a(b)$. For example,

$$5 = 3 + 2 = 3 \cup \lambda_3(2) = \{0, 1, 2\} \cup \{3, 4\}.$$

Proof of (ii) By induction on c, for fixed a and b, that $a + (b + c) = (a + b) + c$. The case $c = 0$ is easy. Suppose the desired conclusion holds for both c and p. Then

$$\begin{aligned}(a + b) + [c;\, p] &= ((a + b) + c);\, ((a + b) + p) = (a + (b + c));\, (a + (b + p)) \\ &= a + [(b + c);\, (b + p)] = a + (b + [c;\, p]).\end{aligned}$$

Proof of (iii) By weak induction on b: the induction step is

$$a + [b; p] = (a + b); (a + p) = (a \cup \lambda_a(b)); (a + p)$$
$$= a \cup [\lambda_a(b); (a + p)] = a \cup \lambda_a([b; p]).$$

The penultimate equality uses the definition of $\cup$. $\square$

Two further brief remarks about 5.1: (iv) is used as the $\in$-inductive definition of the lift function in [9], and the case $b = 0$ of (v) is Lemma 3.4.

Left cancellation of addition needs a little care.

Proposition 5.2 *The following are provable in $I\Sigma_1 S$:*

(i) $\forall xyz(\lambda_x(y) = \lambda_x(z) \to y = z)$;
(ii) $\forall xyz(x + y = x + z \to y = z)$.

Sketch of a proof It suffices to prove (i), since (ii) then follows using Proposition 5.1(iii) and (vi). For fixed a, prove by induction on rank u (see Section 4) that $\forall yz \in V_u\, (\lambda_a(y) = \lambda_a(z) \to y = z)$. So suppose $y, z \in V_u$ and $\lambda_a(y) = \lambda_a(z)$. For any $v \in y$, $a + v \in \lambda_a(y)$ and so $a + v = a + w$ for some $w \in z$. The inductive hypothesis gives $v = w$. Hence $y \subseteq z$, and similarly $z \subseteq y$. $\square$

The following is similar.

Proposition 5.3 *The following are provable in $I\Sigma_1 S$:*

(i) $|\lambda_a(b)| = |b|$;
(ii) $|a + b| = |a| + |b|$.

Multiplication Define[8]

$$a \cdot 0 = 0, \quad a \cdot [b; p] = (a \cdot b) \cup \lambda_{a \cdot p}(a).$$

The defining function $h(x, y, u, v, a) = u \cup \lambda_v(a)$ is respectful. To see that this agrees with the usual multiplication on finite ordinals, observe that for any a and b,

$$a \cdot (b + 1) = a \cdot [b; b] = (a \cdot b) \cup \lambda_{a \cdot b}(a) = a \cdot b + a,$$

using Proposition 5.1(iii). It is immediate that $a \cdot 1 = a$ and $a \cdot 2 = a + a$. Multiplication is not commutative: for example, $\{1\} \cdot 2 = \{1, \{1, \{1\}\}\}$, whereas $2 \cdot \{1\} = \{2, 3\}$.

Before proving properties of multiplication, note this simple consequence of Proposition 5.1(iv).

Lemma 5.4 $I\Sigma_1 S \vdash \lambda_x(y \cup z) = \lambda_x(y) \cup \lambda_x(z)$.

In [9], multiplication of sets is defined by $a \cdot b = \{a \cdot q + r \mid q \in b \wedge r \in a\}$, which was Scott's original definition. Equivalently, $a \cdot b = \bigcup\{\lambda_{a \cdot x}(a) \mid x \in b\}$. Part (iii) of the next Proposition says that this definition agrees, for finite sets, with the one above.

Proposition 5.5 *The universal closures of the following are provable in $I\Sigma_1 S$:*

(i) $0 \cdot a = 0$.
(ii) $1 \cdot a = a$.
(iii) $a \cdot b = \bigcup\{\lambda_{a \cdot x}(a) \mid x \in b\}$.
(iv) $a \cdot (b \cup c) = a \cdot b \cup a \cdot c$.

(v) $a \cdot \lambda_b(c) = \lambda_{a \cdot b}(a \cdot c)$.

(vi) *Left distributivity:* $a \cdot (b + c) = a \cdot b + a \cdot c$.

(vii) *Multiplication is associative.*

As a sample proof here is the inductive step for (vii):

$$\begin{aligned}(a \cdot b) \cdot [c; p] &= (a \cdot b) \cdot c \cup \lambda_{(a \cdot b) \cdot p}(a \cdot b) & \\ &= a \cdot (b \cdot c) \cup \lambda_{a \cdot (b \cdot p)}(a \cdot b) & \text{by inductive hypothesis} \\ &= a \cdot (b \cdot c) \cup a \cdot \lambda_{b \cdot p}(b) & \text{by (v)} \\ &= a \cdot (b \cdot c \cup \lambda_{b \cdot p}(b)) & \text{by (iv)} \\ &= a \cdot (b \cdot [c; p]). &\end{aligned}$$

There is more about multiplication in [9].

This arithmetic of sets allows a straightforward generalization of a standard result about number-theoretic primitive recursive functions:

Proposition 5.6 *$\chi_{\varphi(\vec{x})}$ is primitive recursive for every Δ_0 formula $\varphi(\vec{x})$.*

The case where φ is atomic was discussed at the end of Section 3. The inductive proof on the structure of φ follows the number-theoretic case. Thus, for example, given a function $f(x, \vec{y})$ we can use multiplication of sets and primitive recursion to define a product function $\prod_{i \in x} f(i, \vec{y})$, and this is used to obtain $\chi_{\forall z \in x\, \varphi(z, \vec{y})}$ from $\chi_{\varphi(x, \vec{y})}$.

Notes

1. I used this latter term in [9]. Świerczkowski [19] suggests "x eats y."
2. The referee has pointed out that the first to propose an axiomatization of set theory based on function symbols rather than the membership relation was von Neumann [24], although his system is quite different from that of the present paper.
3. This notation is adapted from Lavine [11, p. 401].
4. In their sense of the word finite, that is, sets of finite cardinality: [26, Vol. II, *120.24].
5. See [4, pp. 176ff.] for a general discussion of the role of the axiom TC.
6. The referee has pointed out that the nonuniqueness problem already occurs in the classical recursive functions and has been dealt with there by stipulating that the function be single-valued or by using the smallest terminating computation.
7. Mentioned at the end of Section 2.
8. By what criteria do I claim that this definition, and the definition of addition, are "natural" generalizations of the operations on ordinals? Because they are inductive definitions using simple formulas, which transparently diagonalize when $b = p$ to the operations on ordinals, just as adjunction diagonalizes to the successor operator, and the elementary algebraic properties are proved using straightforward inductions. And because they are, as far as I know, the only generalizations with these desirable features. Exponentiation can also be generalized similarly, but there is more than one way to do it and either the definitions and proofs get messier or some of the elementary algebraic properties fail, so I regard it as less natural.

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