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Structural Completeness in Fuzzy Logics

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Abstract Structural completeness properties are investigated for a range of popular t-norm based fuzzy logics—including Łukasiewicz Logic, Gödel Logic, Product Logic, and Hájek's Basic Logic—and their fragments. General methods are defined and used to establish these properties or exhibit their failure, solving a number of open problems.

1 Introduction

A logic—viewed as a finitary consequence relation—is *structurally complete* ($\mathcal{SC}$ for short) if each of its proper extensions admits new theorems. Or put another way, every rule that is admissible (preserves the set of theorems) should also be derivable (already belong to the consequence relation). A logic is *hereditarily structurally complete* ($\mathcal{HSC}$ for short) if all of its extensions (including the logic itself) are $\mathcal{SC}$. From an algebraic perspective, a variety is structurally complete if each of its subquasivarieties generates a proper subvariety, and it is primitive if each of its subquasivarieties is a variety. Classical Logic and its most common fragments are structurally complete. However, the property fails for Intuitionistic Logic and many other nonclassical logics. Moreover, even when it does hold for a logic, structural completeness can be very sensitive to changes in the language.

The notion of structural completeness is due to Pogorzelski [23] and has been investigated by many authors including Prucnal [24], Makinson [19], and (from an algebraic perspective) Prucnal and Wronski [25] and Bergman [4]. Admissible rules have been studied intensively for intermediate and modal logics by Rybakov [26]. For the purposes of this paper, two strands of research on these topics are particularly relevant. First, there have been several papers on structural completeness in many-valued logics, including Łukasiewicz logics, studied by Wojtylak ([29], [30], [31]) and Tokarz [27], and Gödel logics (also known as Gödel-Dummett logics) by Dzik and Wronski [13]. Secondly, a recent paper of Olson et al. [22] provides a wealth

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of new results on structural completeness for substructural logics and their algebraic counterparts, residuated lattices. Other related work includes a paper of Dzik [12] on unification in BL-algebras (the algebras of Hájek's Basic Logic [18]) and hoops.

In this paper we investigate structural completeness for *fuzzy logics*. Such logics, developed in particular by Hájek in [18], are typically (but not always) defined on the real unit interval [0, 1] with conjunction and implication connectives interpreted by left-continuous t-norms and their residua. Among logics based on continuous t-norms, Łukasiewicz Logic Ł, Gödel Logic G, and Product Logic Π are treated as fundamental since any continuous t-norm is an ordinal sum of the corresponding three t-norms (see, e.g., [18]). Also important in this context are Hájek's Basic Logic BL [18] and Esteva and Godo's Monoidal t-Norm Logic MTL [14], characterized by validity in all logics based on continuous and left-continuous t-norms, respectively. Other fuzzy logics featured prominently in the literature include extensions of MTL and BL characterized by properties such as cancellation, n-contraction, and having a strict or involutive negation (see, e.g., [14] and the survey [10]). All such "t-norm based" logics admit the weakening theorem $A \to (B \to A)$ and correspond to classes of residuated lattices satisfying integrality. Interesting "uninorm based" fuzzy logics without weakening have been studied in [20]; however, we will leave structural completeness for such logics as a topic for future research.

Positive and negative (hereditary) $\mathcal{SC}$ results are established here for fuzzy logics and their fragments using three methods adapted from the literature, taking advantage of algebraization to consider equivalent quasivarieties as required. First, we show that $\mathcal{SC}$ can be established for a quasivariety $\mathcal{Q}$ by defining (partial) embeddings from generating algebras for $\mathcal{Q}$ into the free algebra for $\mathcal{Q}$ with countably many generators. This approach has been used (implicitly) to show structural completeness for Łukasiewicz logics [31] and Gödel logics [13] and is extended here to Product Logic Π, the related Cancellative Hoop Logic CHL, and Basic Logic BL. Secondly, we extend "Prucnal's trick," used to establish $\mathcal{SC}$ for a range of implicational logics in [24], to establish $\mathcal{HSC}$ for (fragments of) logics obeying an n-contraction condition. The key tool is a correspondence between $\mathcal{HSC}$ and a "hereditary version" of the deduction theorem. Finally, we investigate the weaker notion of *passive structural completeness* ($\mathcal{PSC}$ for short), studied by Rybakov in [26] and called nonoverflow completeness (in an algebraic context) by Wronski (in, e.g., his presentation "Overflow rules and a weakening of structural completeness" at the *51st Conference on the History of Logic*, Krakow, 2005). $\mathcal{PSC}$ is satisfied for a logic if all rules with nonunifiable premises are derivable. We use the nonderivability of such rules both to establish a range of negative results and also to obtain positive $\mathcal{PSC}$ results for a particular class of logics.

Aside from introducing useful general methods, the paper contains the following new results:

1. the $\{\to, \cdot\}$ fragment of Ł (the logic of Wajsberg hoops) is not $\mathcal{SC}$ (Theorem 3.15);
2. all fragments of CHL and Π are $\mathcal{SC}$ (Theorems 3.17 and 3.21) and the full logics are $\mathcal{HSC}$ (Corollaries 3.18 and 3.22);
3. the $\{\to, \cdot\}$ fragment of BL (the logic of basic hoops) is not $\mathcal{SC}$ (Theorem 3.27), but the implicational fragment is $\mathcal{SC}$ (Theorem 3.26);
4. all extensions of Strict Monoidal t-Norm Logic SMTL are $\mathcal{PSC}$ (Theorem 5.11).

2 Basic Notions

2.1 Logics and consequence relations We begin by recalling some general definitions concerning the theory of logical calculi (for more details see, e.g., [28]). The notions of a *propositional language* $\mathcal{L}$ (a set of *connectives* with specified finite arities) and set of $\mathcal{L}$*-formulas* $\mathbf{Fm}_{\mathcal{L}}$ over a fixed countably infinite set of propositional variables are defined in the usual manner. An $\mathcal{L}$*-theory* T is just a set of $\mathcal{L}$-formulas and an $\mathcal{L}$*-substitution* σ is an endomorphism on $\mathbf{Fm}_{\mathcal{L}}$ (understood here as the formula (or term) algebra), writing $\sigma(T)$ for $\{\sigma\varphi \mid \varphi \in T\}$. A *logic* L in the language $\mathcal{L}$ is a *finitary* structural consequence relation (in the sense of Tarski) on $\mathbf{Fm}_{\mathcal{L}}$, omitting $\mathcal{L}$ if the language is clear from the context. That is, L is a relation between theories and formulas (writing $T \vdash_{\mathrm{L}} \varphi$ instead of $(T, \varphi) \in \mathrm{L}$ and $T \vdash_{\mathrm{L}} T'$ instead of $T \vdash_{\mathrm{L}} \varphi$ for each $\varphi \in T'$) satisfying the following conditions:

1. if $\varphi \in T$, then $T \vdash_{\mathrm{L}} \varphi$;
2. if $T' \vdash_{\mathrm{L}} T$ and $T \vdash_{\mathrm{L}} \psi$, then $T' \vdash_{\mathrm{L}} \psi$;
3. if $T \vdash_{\mathrm{L}} \varphi$, then there is a finite set $T' \subseteq T$ such that $T' \vdash_{\mathrm{L}} \varphi$;
4. if $T \vdash_{\mathrm{L}} \varphi$, then $\sigma(T) \vdash_{\mathrm{L}} \sigma(\varphi)$ for each $\mathcal{L}$-substitution σ.

An L*-theorem* is a formula φ such that $\varnothing \vdash_{\mathrm{L}} \varphi$ (abbreviated as $\vdash_{\mathrm{L}} \varphi$).

A logic L_2 in a language $\mathcal{L}_2 \supseteq \mathcal{L}_1$ is said to be an *expansion* of L_1 in $\mathcal{L}_1$ if for each $\mathcal{L}_1$-theory T and $\mathcal{L}_1$-formula φ, $T \vdash_{\mathrm{L}_1} \varphi$ implies $T \vdash_{\mathrm{L}_2} \varphi$. The expansion is *conservative* if also $T \vdash_{\mathrm{L}_2} \varphi$ implies $T \vdash_{\mathrm{L}_1} \varphi$ for each $\mathcal{L}_1$-theory T and $\mathcal{L}_1$-formula φ. In this case we say that L_1 is the $\mathcal{L}_1$-fragment of L_2 and denote L_1 by $\mathrm{L}_2{\restriction}\mathcal{L}_1$, observing that the $\mathcal{L}_1$-fragment of L_2 is uniquely determined. If the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ coincide, then we speak of an *extension* rather than an expansion.

We can now make the notions of structural completeness and hereditary structural completeness for a logic precise.

Definition 2.1 A logic L is *structurally complete* ($\mathcal{SC}$ for short) if all of its extensions have new theorems and *hereditarily structurally complete* ($\mathcal{HSC}$ for short) if all of its extensions are $\mathcal{SC}$.

We also recall an alternative well-known characterization of structural completeness (see, e.g., [22] and [26]) that makes use of the notion of a rule.

Definition 2.2 A *rule* for a language $\mathcal{L}$ is an ordered pair, written $T \rhd \varphi$, where T is a *finite* $\mathcal{L}$-theory and φ is an $\mathcal{L}$-formula, called an *axiom* if $T = \varnothing$. A rule $T \rhd \varphi$ is *derivable* in the logic L if $T \vdash_{\mathrm{L}} \varphi$ and *admissible* in L if for each substitution σ, $\vdash_{\mathrm{L}} \sigma(T)$ implies $\vdash_{\mathrm{L}} \sigma\varphi$.

Theorem 2.3 *A logic* L *is* $\mathcal{SC}$ *if and only if all of its admissible rules are derivable.*

Example 2.4 Intuitionistic Logic is famously not $\mathcal{SC}$, having admissible but non-derivable rules such as

$$\neg p \to (q \vee r) \rhd (\neg p \to q) \vee (\neg p \to r).$$

However, for certain restrictions of the language (e.g., to implicational formulas) all the admissible rules are derivable, and so these fragments of Intuitionistic Logic are $\mathcal{SC}$.

We will also make use of a characterization of hereditary structural completeness proved in [22]. Let L be a logic and $\mathcal{R}$ a set of rules. By $\mathrm{L} + \mathcal{R}$ we denote the weakest extension of L where all rules from $\mathcal{R}$ are derivable (i.e., the intersection of

all such extensions). We say that L' is an *axiomatic* extension of L if there is a set of axioms $\mathcal{A}$ such that $\mathrm{L}' = \mathrm{L} + \mathcal{A}$.

Theorem 2.5 ([22]) *The following are equivalent for any logic* L*:*

(1) L *is* $\mathcal{HSC}$*;*
(2) *every axiomatic extension of* L *is* $\mathcal{SC}$*;*
(3) *every extension of* L *is an axiomatic extension of* L*.*

2.2 Algebraic semantics It will be helpful to also have an algebraic formulation of the above structural completeness properties. We assume familiarity with some standard notions from Universal Algebra as may be found in a good reference book such as [9]. For convenience, we will use the same terminology for algebras as for logics, speaking of a language rather than a type (or signature), formulas rather than terms, and the formula (rather than term) algebra $\mathbf{Fm}_{\mathcal{L}}$.

Recall that a quasi-identity has the form $\varphi_1 \approx \psi_1 \wedge \cdots \wedge \varphi_m \approx \psi_m \Rightarrow \varphi \approx \psi$. A *(quasi)variety* is a class of algebras of the same language axiomatizable by a set of (quasi-)identities. For a class $\mathcal{K}$ of algebras of the same language, the *variety* $\mathbb{V}(\mathcal{K})$ *and quasivariety* $\mathbb{Q}(\mathcal{K})$ *generated by* $\mathcal{K}$ are the smallest variety and quasivariety containing $\mathcal{K}$, respectively.

Definition 2.6 A quasivariety $\mathcal{Q}$ is *structurally complete* ($\mathcal{SC}$ for short) if each of its proper subquasivarieties generates a proper subvariety of $\mathbb{V}(\mathcal{Q})$ and is *primitive* (deductive) if each of its subquasivarieties is $\mathcal{SC}$.

Notice that a variety $\mathcal{V}$ is $\mathcal{SC}$ if and only if each proper subquasivariety of $\mathcal{V}$ generates a proper subvariety of $\mathcal{V}$ and primitive (the algebraic version of $\mathcal{HSC}$, attributed to Pigozzi in [4]) if and only if each of its subquasivarieties is a variety.

The logical and algebraic definitions are connected using the comprehensive theory of algebraizable logics of [6]. Let us call a homomorphism from $\mathbf{Fm}_{\mathcal{L}}$ into an algebra $\mathbf{A}$ for $\mathcal{L}$, an $\mathbf{A}$-*evaluation*. For a class of algebras $\mathcal{K}$ of the same language $\mathcal{L}$ and set of $\mathbf{Fm}_{\mathcal{L}}$-identities $\Sigma \cup \{\varphi \approx \psi\}$, we write $\Sigma \models_{\mathcal{K}} \varphi \approx \psi$ if for any $\mathbf{A} \in \mathcal{K}$ and $\mathbf{A}$-evaluation e, $e(\varphi') = e(\psi')$ for all $(\varphi' \approx \psi') \in \Sigma$ implies $e(\varphi) = e(\psi)$.

A logic L is *elementarily algebraizable* (from now on, omitting the adjective "elementarily") if there is a quasivariety $\mathcal{K}$, a set of identities in one variable $E(p)$, and a set of formulas in two variables $\Delta(p, q)$ such that

(i) $T \vdash_{\mathrm{L}} \varphi$ iff $\bigcup\{E(\psi) \mid \psi \in T\} \models_{\mathcal{K}} E(\varphi)$;
(ii) $\Sigma \models_{\mathcal{K}} \varphi \approx \psi$ iff $\bigcup\{\Delta(\varphi', \psi') \mid \varphi' \approx \psi' \in \Sigma\} \models_{\mathrm{L}} \Delta(\varphi, \psi)$;
(iii) $\varphi \dashv\vdash_{\mathrm{L}} \bigcup\{\Delta(\varphi, \psi) \mid \varphi \approx \psi \in E(\varphi)\}$;
(iv) $\varphi \approx \psi \dashv\models_{\mathcal{K}} \bigcup\{E(\psi) \mid \psi \in \Delta(\varphi, \psi)\}$.

The class $\mathcal{K}$ is then called the *equivalent quasivariety* of L. In our setting (finitary logics and elementary algebraizability) we can assume that both sets $E(p)$ and $\Delta(p, q)$ are finite.

The following theorem is easily established, using the above conditions to translate between the logical and algebraic definitions of structural completeness.

Theorem 2.7 ([22]) *For any algebraizable logic* L *with equivalent quasivariety* $\mathcal{Q}$*,*

(a) L *is* $\mathcal{SC}$ *iff* $\mathcal{Q}$ *is* $\mathcal{SC}$*;*
(b) L *is* $\mathcal{HSC}$ *iff* $\mathcal{Q}$ *is primitive.*

We will also make quite heavy use of the following theorem to deal with fragments of logics. Recall that $\mathcal{L}$-subreducts of an algebra are just subalgebras of the $\mathcal{L}$-reduct of that algebra.

Theorem 2.8 ([6]) *Let* L *be an algebraizable logic for a language* $\mathcal{L}'$ *with equivalent quasivariety* $\mathcal{Q}$ *and translations* E *and* Δ *in a sublanguage* $\mathcal{L} \subseteq \mathcal{L}'$. *Then* $\mathrm{L}\restriction\mathcal{L}$ *is algebraizable with the same translations and an equivalent quasivariety* $\mathcal{Q}\restriction\mathcal{L}$ *consisting of all* $\mathcal{L}$*-subreducts of algebras from* $\mathcal{Q}$.

2.3 Residuated lattices and fuzzy logics We move on now from general definitions to our particular realm of interest: the class of residuated lattices. These structures, investigated in detail in, for example, [17] and [8], provide a suitable algebraic framework for a wide range of substructural logics, including a broad family of fuzzy logics. Since algebras for the most popular of these fuzzy logics are both integral and commutative (the logics themselves are said to admit *weakening* and *exchange* rules), we restrict our attention in this paper to the following definitions.

An *integral commutative residuated lattice* (ICRL for short) is an algebra

$$\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, \top \rangle$$

with binary operations $\wedge, \vee, \cdot, \to$, and a constant $\top$ such that

1. $\langle A, \wedge, \vee \rangle$ is a lattice with top element $\top$,
2. $\langle A, \cdot, \top \rangle$ is a commutative monoid,
3. $x \cdot y \leq z$ iff $x \leq y \to z$ for all $x, y, z \in A$.

A *bounded integral commutative residuated lattice* (BICRL for short) is an algebra $\langle A, \wedge, \vee, \cdot, \to, \bot, \top \rangle$ such that $\langle A, \wedge, \vee, \cdot, \to, \top \rangle$ is an ICRL with bottom element $\bot$. The classes of ICRLs and BICRLs both form varieties (see, e.g., [8] for proofs).

Observe that $\top = x \to x$ for all x in any (B)ICRL. We also define the following useful abbreviations both for formulas and elements of a (B)ICRL:

$$\begin{array}{rcll} \neg x & =_{\mathrm{def}} & x \to \bot & \\ x \leftrightarrow y & =_{\mathrm{def}} & (x \to y) \wedge (y \to x) & \\ x^0 & =_{\mathrm{def}} & \top & \\ x \to^0 y & =_{\mathrm{def}} & y & \\ x^{n+1} & =_{\mathrm{def}} & x \cdot x^n & (n \in \mathbb{N}) \\ x \to^{n+1} y & =_{\mathrm{def}} & x \to (x \to^n y) & (n \in \mathbb{N}). \end{array}$$

Given a sublanguage $\mathcal{L}$ of the language of BICRLs containing $\to$ and a quasivariety $\mathcal{Q}$ of $\mathcal{L}$-subreducts of (B)ICRLs, we define a logic $\mathrm{L}^{\mathcal{Q}}$ as

$$T \vdash_{\mathrm{L}^{\mathcal{Q}}} \varphi \qquad \text{iff} \qquad \{\psi \approx \top \mid \psi \in T\} \models_{\mathcal{Q}} \varphi \approx \top.$$

Clearly, $\mathcal{Q}$ is the equivalent quasivariety of $\mathrm{L}^{\mathcal{Q}}$ with translations $E = \{p \approx \top\}$ and $\Delta = p \leftrightarrows q =_{\mathrm{def}} \{p \to q, q \to p\}$.

The implicational subreducts of ICRLs are called BCK-algebras, and their logic is called BCK. All logics $\mathrm{L}^{\mathcal{Q}}$ defined as above are expansions of BCK. Conversely, any expansion L of BCK in a sublanguage of the language of BICRLs is algebraizable with the given translations, and its equivalent quasivariety, denoted by $\mathcal{Q}^{\mathrm{L}}$, consists of algebras $\mathbf{A}$ satisfying

$$\{\psi \approx \top \mid \psi \in T\} \models_{\mathbf{A}} \varphi \approx \top \qquad \text{whenever} \qquad T \vdash_{\mathrm{L}} \varphi.$$

Table 1 Properties of Integral Commutative Residuated Lattices

Label	Name	Condition
(prl)	prelinearity	$\top \approx (x \to y) \vee (y \to x)$
(div)	divisibility	$x \cdot (x \to y) \approx y \cdot (y \to x)$
(can)	cancellation	$x \to (x \cdot y) \approx y$
(rcan)	restricted cancellation	$\top \approx \neg x \vee ((x \to (x \cdot y)) \to y)$
(inv)	involution	$\neg\neg x \approx x$
(pc)	pseudo-complementation	$\bot \approx x \wedge \neg x$
(c_n)	n-contraction	$x^n \approx x^{n-1}$

Clearly, for any quasivariety $\mathcal{Q}$ of $\mathcal{L}$-subreducts of BICRLs, $\mathcal{Q}^{\mathrm{L}^{\mathcal{Q}}} = \mathcal{Q}$, and for any expansion L of BCK in a sublanguage of the language of BICRLs, $\mathrm{L}^{\mathcal{Q}^{\mathrm{L}}} = \mathrm{L}$. Hence to recap from the previous section: a logic L is structurally complete if and only if the quasivariety $\mathcal{Q}^{\mathrm{L}}$ is structurally complete and the quasivariety $\mathcal{Q}$ is structurally complete if and only if the logic $\mathrm{L}^{\mathcal{Q}}$ is structurally complete.

ICRLs provide algebraic semantics for the logic FL^{+}_{ew}, where adding $\bot$ gives the logic FL_{ew} (also known as Monoidal Logic ML). We obtain algebras for other substructural logics, including the most popular fuzzy logics, by adding further conditions such as those listed in Table 1.

The fuzzy logics investigated in this paper are displayed in Table 2. We also consider *Cancellative Hoop Logic* CHL, whose equivalent variety is the class of ICRLs satisfying (prl), (div), and (can). Our selection includes the fundamental fuzzy logics Ł, G, and Π, and the core logics MTL and BL of left-continuous and continuous t-norms, respectively. The choice of the remaining logics reflects interesting differences in structural completeness properties established by our methods. Results for other logics widely studied in the literature such as IMTL and ΠMTL (characterized by BICRLs satisfying (prl) and (inv), and BICRLs satisfying (prl) and (rcan), respectively) follow from our theorems, but are not sufficiently different to merit special attention.

Table 2 Logics and Their Equivalent Varieties

Label	Logic	Class of BICRLs satisfying
MTL	Monoidal t-Norm Logic	(prl)
SMTL	Strict MTL	(prl), (pc)
$\mathrm{C_n}$MTL	n-Contractive MTL ($n \geq 2$)	(prl), (c_n)
BL	Basic Logic	(prl), (div)
$\mathrm{C_n}$BL	n-Contractive BL ($n \geq 2$)	(prl), (div), (c_n)
G	Gödel Logic	(prl), (c_2)
Ł	Łukasiewicz Logic	(prl), (div), (int)
Π	Product Logic	(prl), (div), (rcan)

For convenience,

1. we denote the class $\mathcal{Q}^{\mathrm{L}}$ (where no confusion can occur) simply by L;
2. we call the algebras from $\mathcal{Q}^{\mathrm{L}}$, L-*algebras*;
3. we call linearly ordered L-algebras, L-*chains*;
4. we denote the $\bot$-*free fragment* of L, obtained by removing $\bot$ from the language, by L^{+}.

We remark that there exist different names for many of these classes of algebras. For example, Ł-algebras are term-equivalent to MV-algebras, while G-algebras are term-equivalent to Heyting algebras satisfying (prl). Also, these quasivarieties are typically generated by certain distinguished subclasses (see, e.g., the survey paper on completeness in fuzzy logics [10]):

1. each class of L-algebras in Table 2 is generated both by the class of L-chains and the class of L-algebras with lattice reduct $\langle [0, 1], \min, \max\rangle$ (also, the class of CHL-algebras is generated by CHL-chains and CHL-algebras with lattice reduct $\langle (0, 1], \min, \max\rangle$);
2. in particular, MTL-algebras and BL-algebras are generated by all such algebras where $\cdot$ is a *left-continuous t-norm* and *continuous t-norm*, respectively;
3. the classes of algebras for G, Ł, Π, CHL, and BL are generated by one particular algebra (see Section 3 below for details).

In this paper, we will study the properties of (hereditary) structural completeness not only for the logics themselves but also for their various *fragments* (containing at least implication). However, many of these fragments coincide. To avoid repetition, let us mention some general patterns. Consider a sublanguage $\mathcal{L}$ of the language $\{\to, \cdot, \wedge, \vee, \bot\}$ of BICRLs containing implication:

1. the $\mathcal{L} \setminus \{\bot\}$ fragments coincide for MTL and SMTL;
2. the $\mathcal{L} \cup \{\wedge\}$ and $\mathcal{L} \cup \{\wedge, \vee\}$ fragments coincide for all logics extending MTL;
3. the $\mathcal{L} \cup \{\cdot\}$ and $\mathcal{L} \cup \{\cdot, \wedge, \vee\}$ fragments coincide for all logics extending BL.

Of course, in particular logics, there may be a further collapse of fragments; we will remark on this in the relevant sections below.

3 Structural Completeness via Embeddings into Free Algebras

Our aim in this section will be to formalize a condition for structural completeness already employed implicitly in the literature and use it to establish new results for (fragments of) fuzzy logics. Let us denote the free algebra with countably many generators for a quasivariety $\mathcal{Q}$ by $\mathbf{F}_{\mathcal{Q}}$. Our approach will be based upon the following key fact.

Theorem 3.1 ([4]) *A quasivariety $\mathcal{Q}$ is $\mathcal{SC}$ if and only if $\mathcal{Q} = \mathbb{Q}(\mathbf{F}_{\mathcal{Q}})$.*

Intuitively, the idea will be to show that a quasivariety $\mathcal{Q}$ is $\mathcal{SC}$ by proving that a set of algebras generating $\mathcal{Q}$ as a quasivariety can all be embedded into $\mathbf{F}_{\mathcal{Q}}$. It then follows that any quasi-identity failing in one of the generating algebras must fail in $\mathbf{F}_{\mathcal{Q}}$. To get us started, we will consider a typical example. We will show that the variety CHL of CHL-algebras (term-equivalent to cancellative hoops), and hence also the corresponding logic, is $\mathcal{SC}$. First, we identify a useful generating algebra (noting that $\mathbb{Z}^{-} = \{0, -1, -2, \ldots\}$).

Theorem 3.2 ([15]) $\mathrm{CHL} = \mathbb{Q}(\mathbf{Z}^{-})$ *for* $\mathbf{Z}^{-} = \langle \mathbb{Z}^{-}, \min, \max, +, \to, 0\rangle$ *and* $x \to y = \min(0, y - x)$.

To establish structural completeness, we need to show that if a quasi-identity fails in $\mathbf{Z}^-$, then it fails in $\mathbf{F}_{\mathrm{CHL}}$. But this will be the case if we can find a function that maps every element of $\mathbb{Z}^-$ to an element of $\mathbf{F}_{\mathrm{CHL}}$ (i.e., an equivalence class $[\varphi]_{\mathrm{CHL}}$ of formulas) and respects the operations of $\mathbf{Z}^-$. That is, we seek an *embedding* of $\mathbf{Z}^-$ into $\mathbf{F}_{\mathrm{CHL}}$.

Recall that $q^0 = \top$ and $q^{n+1} = q \cdot q^n$ for $n \in \mathbb{N}$, and consider the mapping $\alpha^{\mathrm{CHL}} : \mathbb{Z}^- \to \mathbf{F}_{\mathrm{CHL}}$ defined by

$$\alpha^{\mathrm{CHL}}(i) = [q^{-i}]_{\mathrm{CHL}}.$$

To show that α^{CHL} is an embedding, we first need $\models_{\mathrm{CHL}} q^0 \approx \top$ which is true by definition, and the following for all $i, j \in \mathbb{Z}^-$:

(i) $\models_{\mathrm{CHL}} q^{-(i+j)} \approx q^{-i} \cdot q^{-j}$;
(ii) $\models_{\mathrm{CHL}} q^{-\min(0,j-i)} \approx q^{-i} \to q^{-j}$.

(i) is almost immediate. For (ii), notice that if $i \leq j$, then $q^{-\min(0,j-i)} = q^0 = \top$ and $\models_{\mathrm{CHL}} q^{-i} \to q^{-j} \approx \top$. If $i > j$, then $q^{-\min(0,j-i)} = q^{i-j}$ and $q^{-i} \to q^{-j} = q^{-i} \to (q^{-i} \cdot q^{i-j})$. So by cancellativity, $\models_{\mathrm{CHL}} q^{-i} \to q^{-j} \approx q^{i-j}$ as required. Finally, $[q^{-i}] = [q^{-j}]$ if and only if $i = j$ (also by cancellativity), so α^{CHL} is one-to-one and hence an embedding.

In what follows, we generalize this reasoning in several directions. We treat quasivarieties generated by classes of algebras (rather than just a single algebra), we allow partial embeddings (rather than embeddings), and we deal uniformly with several fragments of a logic simultaneously.

3.1 General conditions For two algebras $\mathbf{A}$ and $\mathbf{B}$ of the same language $\mathcal{L}$, $\mathbf{A}$ is *partially embeddable* into $\mathbf{B}$ when each finite subset F of A can be partially embedded into $\mathbf{B}$. That is, there is a mapping $f : F \to \mathbf{B}$ such that for each $c \in \mathcal{L}$ and elements $\overrightarrow{a} \in F$ satisfying $c^{\mathbf{A}}(\overrightarrow{a}) \in F$, $f(c^{\mathbf{A}}(\overrightarrow{a})) = c^{\mathbf{B}}(\overrightarrow{f(a)})$. A class $\mathcal{K}$ of algebras is *(partially) embeddable into* $\mathbf{B}$ if every member of $\mathcal{K}$ is (partially) embeddable into $\mathbf{B}$. Obviously, embeddability implies partial embeddability.

We show that to prove the structural completeness of a quasivariety $\mathcal{Q}$, it is sufficient to show that for some set of algebras $\mathcal{K}$ that generate this quasivariety, each member of $\mathcal{K}$ can be (partially) embedded into $\mathbf{F}_{\mathcal{Q}}$.

Theorem 3.3 *Suppose that $\mathcal{Q} = \mathbb{Q}(\mathcal{K})$. If $\mathcal{K}$ is partially embeddable into $\mathbf{F}_{\mathcal{Q}}$, then $\mathcal{Q}$ is $\mathcal{SC}$.*

Proof We make use of Theorem 3.1. Suppose that a quasi-identity γ fails in $\mathcal{Q}$. Then it fails in some $\mathbf{A} \in \mathcal{K}$. That is, there is an $\mathbf{A}$-evaluation e that witnesses this failure. Consider the set $F \subseteq A$ of values assigned by e to the subformulas occurring in γ. Let f be the partial embedding of F into $\mathbf{F}_{\mathcal{Q}}$. Then the evaluation $f \circ e$ witnesses the failure of γ in $\mathbf{F}_{\mathcal{Q}}$. □

We now extend this result to sublanguages. Let $\mathcal{Q}$ be a quasivariety based on a language $\mathcal{L}$. Recall that the free algebra $\mathbf{F}_{\mathcal{Q}}$ consists of equivalence classes of formulas, denoted $[\varphi]_{\mathcal{Q}}$ for a formula φ. We will make use of the *canonical morphism* $h_{\mathcal{Q}} : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{F}_{\mathcal{Q}}$ defined by $h_{\mathcal{Q}}(\varphi) = [\varphi]_{\mathcal{Q}}$. Recall that by $\mathcal{Q}{\restriction}\mathcal{L}$ we denote the class of all $\mathcal{L}$-subreducts of algebras from $\mathcal{Q}$, and if $\mathcal{Q}$ is a quasivariety, then so is $\mathcal{Q}{\restriction}\mathcal{L}$.

Lemma 3.4 *Let $\mathcal{Q}$ be a quasivariety for the language $\mathcal{L}'$ and let $\mathcal{L} \subseteq \mathcal{L}'$. Then $h_{\mathcal{Q}}(\varphi) = h_{\mathcal{Q}}(\psi)$ if and only if $h_{\mathcal{Q}\restriction\mathcal{L}}(\varphi) = h_{\mathcal{Q}\restriction\mathcal{L}}(\psi)$ for all $\varphi, \psi \in \mathbf{Fm}_{\mathcal{L}}$.*

Proof For $\varphi, \psi \in \mathbf{Fm}_{\mathcal{L}}$, $h_{\mathcal{Q}}(\varphi) = h_{\mathcal{Q}}(\psi)$ if and only if $\models_{\mathcal{Q}} \varphi \approx \psi$ if and only if $\models_{\mathcal{Q}\restriction\mathcal{L}} \varphi \approx \psi$ if and only if $h_{\mathcal{Q}\restriction\mathcal{L}}(\varphi) = h_{\mathcal{Q}\restriction\mathcal{L}}(\psi)$. □

We now formulate our key theorem in such a way that $\mathcal{SC}$ can be established uniformly for several sublanguages of a given language at the same time. The basic idea is that our mappings from the generating algebras should be to formulas of a minimal language $\mathcal{L}_m$, but the canonical morphism should then take these formulas to equivalence classes in a maximal language $\mathcal{L}'$. $\mathcal{SC}$ will then hold for any language $\mathcal{L}$ such that $\mathcal{L}_m \subseteq \mathcal{L} \subseteq \mathcal{L}'$.

Theorem 3.5 *Let $\mathcal{Q} = \mathbb{Q}(\mathcal{K})$ be a quasivariety for the language $\mathcal{L}'$ and let $\mathcal{L}_m \subseteq \mathcal{L}'$. Suppose that for each finite subset $F \subseteq A$ for $\mathbf{A} \in \mathcal{K}$, there is a mapping $m_F^{\mathbf{A}} : F \to \mathbf{Fm}_{\mathcal{L}_m}$ such that $h_{\mathcal{Q}} \circ m_F^{\mathbf{A}}$ is a partial embedding of F into $\mathbf{F}_{\mathcal{Q}}$. Then $\mathcal{Q}\restriction\mathcal{L}$ is $\mathcal{SC}$ whenever $\mathcal{L}_m \subseteq \mathcal{L} \subseteq \mathcal{L}'$.*

Proof We make use of Theorem 3.3. Since $\mathcal{Q} = \mathbb{Q}(\mathcal{K})$, the quasivariety $\mathcal{Q}\restriction\mathcal{L}$ is generated as a quasivariety by the class $\mathcal{K}_{\mathcal{L}} = \{\mathbf{A}\restriction\mathcal{L} \mid \mathbf{A} \in \mathcal{K}\}$. Let us consider an algebra $\mathbf{A}\restriction\mathcal{L} \in \mathcal{K}_{\mathcal{L}}$ and a finite subset $F \subseteq A$. We know that there is a mapping $m_F^{\mathbf{A}}$ whose range is a subset of $\mathbf{Fm}_{\mathcal{L}}$. Thus the mapping $f = h_{\mathcal{Q}\restriction\mathcal{L}} \circ m_F^{\mathbf{A}}$ is a mapping of F into $\mathbf{F}_{Q\restriction\mathcal{L}}$.

We show that f is a partial morphism. Consider a (without loss of generality) binary connective $c \in \mathcal{L}$ and $a, b \in F$ such that $c^{\mathbf{A}\restriction\mathcal{L}}(a, b) \in F$. Thus also $c^{\mathbf{A}}(a, b) \in F$ and we can use the fact that $h_{\mathcal{Q}} \circ m_F^{\mathbf{A}}$ and $h_{\mathcal{Q}}$ are (partial) morphisms to obtain

$$h_{\mathcal{Q}} \circ m_F^{\mathbf{A}}(c^{\mathbf{A}}(a, b)) = c^{\mathbf{F}_{\mathcal{Q}}}(h_{\mathcal{Q}} \circ m_F^{\mathbf{A}}(a), h_{\mathcal{Q}} \circ m_F^{\mathbf{A}}(b)) = h_{\mathcal{Q}}(c(m_F^{\mathbf{A}}(a), m_F^{\mathbf{A}}(b))).$$

Hence, by Lemma 3.4, we obtain

$$h_{\mathcal{Q}\restriction\mathcal{L}} \circ m_F^{\mathbf{A}}(c^{\mathbf{A}}(a, b)) = h_{\mathcal{Q}\restriction\mathcal{L}}(c(m_F^{\mathbf{A}}(a), m_F^{\mathbf{A}}(b))) = c^{\mathbf{F}_{\mathcal{Q}\restriction\mathcal{L}}}(h_{\mathcal{Q}\restriction\mathcal{L}} \circ m_F^{\mathbf{A}}(a), h_{\mathcal{Q}\restriction\mathcal{L}} \circ m_F^{\mathbf{A}}(b)).$$

Finally, we show that f is one-to-one. Assume that $f(a) = f(b)$. Then $h_{\mathcal{Q}} \circ m_F^{\mathbf{A}}(a) = h_{\mathcal{Q}} \circ m_F^{\mathbf{A}}(b)$ (since $m_F^{\mathbf{A}}(a), m_F^{\mathbf{A}}(b) \in \mathbf{Fm}_{\mathcal{L}}$ we can use Lemma 3.4). Since $h_{\mathcal{Q}} \circ m_F^{\mathbf{A}}$ is a one-to-one mapping, $a = b$ as required. □

Finally, we make an interesting observation. For quasivarieties generated by residuated chains (as is the case for our fuzzy logics), with some restrictions on the language, we can obtain a converse to Theorem 3.3.

Theorem 3.6 *Let $\mathcal{L}$ be a sublanguage of* BICRL*s containing $\to$ and $\vee$, and let $\mathcal{Q}$ be a quasivariety of $\mathcal{L}$-subreducts of* BICRL*s generated by a class of linearly ordered algebras $\mathcal{K}$. Then $\mathcal{K}$ is partially embeddable into $\mathbf{F}_{\mathcal{Q}\restriction\mathcal{L}}$ if and only if $\mathcal{Q}\restriction\mathcal{L}$ is $\mathcal{SC}$.*

Proof One direction is Theorem 3.3. For the opposite direction, let us take $\mathbf{A} \in \mathcal{K}$ and a finite set $F \subseteq A$. Consider a set of pairwise distinct variables $\{p_a \mid a \in F\}$

and a quasi-identity γ:

$$\bigwedge_{c\in\mathcal{L};a_1,\ldots,a_n,c^{\mathbf{A}}(a_1,\ldots,a_n)\in F} c(p_{a_1},\ldots,p_{a_n}) \approx p_{c^{\mathbf{A}}(a_1,\ldots,a_n)} \Rightarrow \top \approx \bigvee_{a,b\in F;a\not\leq b} (p_a \to p_b).$$

Clearly, γ is not valid in $\mathcal{Q}$: just consider the algebra $\mathbf{A}$ and $\mathbf{A}$-evaluation $v(p_a) = a$ (notice that $v(p_a \to p_b) \neq \top^{\mathbf{A}}$ for $a \not\leq b$ and since $\mathbf{A}$ is linearly ordered, $\top^{\mathbf{A}}$ is finitely join irreducible). But $\mathcal{Q}$ is generated by $\mathbf{F}_{\mathcal{Q}}$ (by Theorem 3.1), so γ is not valid in $\mathbf{F}_{\mathcal{Q}}$.

Let e be an $\mathbf{F}_{\mathcal{Q}}$-evaluation satisfying the premises of γ but not its conclusion. Define a mapping $f : F \to \mathbf{F}_{\mathcal{Q}}$ by $f(a) = e(p_a)$. Obviously, f is a partial homomorphism (as it validates the premises of γ). We show that f is one-to-one. If $a, b \in F$ and $a \neq b$, then $a \not\leq b$ or $b \not\leq a$. Assuming the former case, without loss of generality,

$$f(a) \to^{\mathbf{F}_{\mathcal{Q}}} f(b) = e(p_a) \to^{\mathbf{F}_{\mathcal{Q}}} e(p_b) = e(p_a \to p_b) < \top^{\mathbf{F}_{\mathcal{Q}}}.$$

The first equality is by definition, the second is trivial, and the third follows from the fact that $e(p_a \to p_b) \leq e(\bigvee_{a,b\in F;a\not\leq b}(p_a \to p_b)) < \top^{\mathbf{F}_{\mathcal{Q}}}$. Hence, $f(a) \neq f(b)$ as required. □

This theorem may also be generalized beyond the scope of this paper. We do not need to assume commutativity (any of the two division operations of a residuated lattice would work) or integrality. We would just need an extra assumption that L is algebraizable with a translation $E(p)$ containing just one identity (all known algebraizable finitary logics have this property) and to change the quasi-identity γ to

$$\bigwedge_{c\in\mathcal{L};a_1,\ldots,a_n,c^{\mathbf{A}}(a_1,\ldots,a_n)\in F} c(p_{a_1},\ldots,p_{a_n}) \approx p_{c^{\mathbf{A}}(a_1,\ldots,a_n)} \Rightarrow E(\bigvee_{a,b\in F;a\not\leq b} (p_a \to p_b)).$$

A further generalization would be to replace linearly ordered algebras with finitely subdirectly irreducible algebras in the formulation of the theorem. The only place that linearity is used in the above proof is to get that $\top$ is finitely join irreducible, but this can also be shown true for finitely subdirectly irreducible algebras (essentially by following Proposition 1.3.4 of [11]).

3.2 Gödel Logic As preparation for more complicated cases, let us begin by revisiting (and extending a little) the structural completeness proofs of Dzik and Wroński for Gödel logic G in [13]. First, we recall the following well-known result for G, noting that for convenience we reduce the language of BICRLs by removing $\cdot$ (since this coincides here with $\wedge$).

Theorem 3.7 $\mathrm{G} = \mathbb{Q}(\{\mathbf{G_n} \mid 2 \leq n \in \mathbb{N}\}) = \mathbb{Q}(\mathbf{G}_\infty)$ *where*

$\mathbf{G_n} = \langle\{1,\ldots,n\}, \min, \max, \to, 1, n\rangle$ *with* $x \to y = y$ *if* $x > y$, *and* n *otherwise.*

$\mathbf{G}_\infty = \langle[0,1], \min, \max, \to, 0, 1\rangle$ *with* $x \to y = y$ *if* $x > y$, *and* 1 *otherwise.*

Let us now consider the embedding defined in [13] for Gödel Logic with the full language.

Proposition 3.8 ([13]) *For $2 \leq n \in \mathbb{N}$ and distinct variables $p_1, \ldots, p_n$, let*

$$\begin{array}{lcl} \varphi_1 & = & \bot, \\ \varphi_2 & = & p_2 \vee (p_2 \to p_1), \\ \varphi_i & = & p_i \vee (p_i \to \varphi_{i-1}) \quad \text{for } i = 3 \ldots n-1, \\ \varphi_n & = & p_1 \to p_1. \end{array}$$

Then $\alpha_n^{\mathrm{G}} : \mathbf{G_n} \to \mathbf{F}_{\mathrm{G}}$ defined by $\alpha_n^{\mathrm{G}}(i) = [\varphi_i]_{\mathrm{G}}$ is an embedding.

Structural completeness for Gödel Logic follows immediately from Theorem 3.3.

Theorem 3.9 ([13]) *G is $\mathcal{SC}$.*

Notice, however, that the embedding defined in Proposition 3.8 makes essential use of $\to$, $\vee$, and $\bot$. So we cannot yet conclude that all fragments of Gödel Logic are $\mathcal{SC}$. We require embeddings that make no mention of the unwanted connectives. For positive fragments, this can be achieved by adapting the previous embedding as follows.

Proposition 3.10 *For $2 \leq n \in \mathbb{N}$ and distinct variables $p_1, \ldots, p_n$, let*

$$\begin{array}{lcl} \varphi_1 & = & p_1, \\ \varphi_i & = & ((p_i \to \varphi_{i-1}) \to p_i) \to p_i \quad \text{for } i = 2 \ldots n-1, \\ \varphi_n & = & p_1 \to p_1. \end{array}$$

Then $\alpha_n^{\mathrm{G}^+} : \mathbf{G}_\mathbf{n}^+ \to \mathbf{F}_{\mathrm{G}^+}$ defined by $\alpha_n^{\mathrm{G}^+}(i) = [\varphi_i]_{\mathrm{G}^+}$ is an embedding.

Proof First we establish the following:

(i) $\models_{\mathrm{G}} \varphi_i \wedge \varphi_j \approx \varphi_i$ for $1 \leq i \leq j \leq n$;
(ii) $\models_{\mathrm{G}} \varphi_i \to \varphi_j \approx \varphi_j$ for $1 \leq j < i \leq n$;
(iii) $\models_{\mathrm{G}} \varphi_i \to \varphi_j \approx \top$ for $1 \leq i \leq j \leq n$.

For (i), it is enough to check the case where $j = i + 1$. For $i = n - 1$, the claim is immediate, so suppose that $1 \leq i < n - 1$. Then for any $\mathbf{G}_\infty$-evaluation e, as required,

$$e(\varphi_i) \leq ((e(p_{i+1}) \to e(\varphi_i)) \to e(p_{i+1})) \to e(p_{i+1}) = e(\varphi_{i+1}).$$

For (ii), it is enough to check the case where $i = j + 1$. If $i = n$, the claim is immediate, so suppose that $1 \leq i \leq n - 1$. We want to show

$$\models_{\mathrm{G}} (((p_i \to \varphi_{i-1}) \to p_i) \to p_i) \to \varphi_{i-1} \approx \varphi_{i-1}.$$

For any $\mathbf{G}_\infty$-evaluation e, if $e(p_i) \leq e(\varphi_{i-1})$, then we are done. If $e(p_i) > e(\varphi_{i-1})$, then

$$\begin{aligned} &e((((p_i \to \varphi_{i-1}) \to p_i) \to p_i) \to \varphi_{i-1}) \\ &\qquad = e(((\varphi_{i-1} \to p_i) \to p_i) \to \varphi_{i-1}) = e(p_i \to \varphi_{i-1}) = e(\varphi_{i-1}). \end{aligned}$$

For (iii), we use (i) to get that $\models_{\mathrm{G}} \varphi_i \to \varphi_j \approx (\varphi_i \wedge \varphi_j) \to \varphi_j \approx \top$, as required.

Now we can show that $\alpha_n^{\mathrm{G}^+}$ is an embedding. (i)–(iii) show that the operations $\wedge$ and $\to$ are preserved, and the same holds for $\vee$ since it is definable using $\wedge$ and $\to$. Finally, $\alpha_n^{\mathrm{G}^+}$ is one-to-one. If $[\varphi_i] = [\varphi_j]$, then $i = j$, since for any $\mathbf{G}_\infty$-evaluation e where $e(p_i) < e(p_{i+1})$ for $i = 1 \ldots n - 1$, we get $e(\varphi_i) = e(p_i) < e(p_{i+1}) = e(\varphi_{i+1})$ for $i = 1 \ldots n - 1$. □

Now we can make use of Theorem 3.5. Our quasivariety $\mathcal{Q}$ is G^+, generated as a quasivariety by $\{\mathbf{G}_\mathbf{n}^+ \mid 2 \leq n \in \mathbb{N}\}$. Our minimal sublanguage is $\mathcal{L}_m = \{\to\}$ and we want to show that $\mathrm{G}{\restriction}\mathcal{L}$ is $\mathcal{SC}$ for any $\mathcal{L}$ such that $\mathcal{L}_m \subseteq \mathcal{L} \subseteq \mathcal{L}'$. But now by the previous proposition, for each $n = 2, 3, \ldots$, we have that $\alpha_n^{\mathrm{G}^+}$ is an embedding (no need for partial embeddings here) of $\mathbf{G}_\mathbf{n}^+$ into $\mathbf{F}_{\mathrm{G}^+}$. Hence $\mathrm{G}{\restriction}\mathcal{L}$ is $\mathcal{SC}$.

Notice that this reasoning cannot be extended to cover Theorem 3.9 since the embedding of Proposition 3.10 fails when we have $\bot$ in the language. In this case we are unable to prove $[p_1]_\mathrm{G} = [\bot]_\mathrm{G}$, so we no longer have an embedding into the appropriate free algebra. However, a small change—replacing p_1 with $\bot$ in the definition of φ_1—does the trick. Hence we get the following theorem.

Theorem 3.11 *$\mathrm{G}{\restriction}\mathcal{L}$ is $\mathcal{SC}$ for $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \vee, \bot\}$.*

Note that we will see an alternative way to prove structural completeness (in fact, hereditary structural completeness) for all fragments of G in Section 4.

3.3 Łukasiewicz Logic It will be easy to show (see Section 5) that Łukasiewicz Logic Ł is not structurally complete. However, it has been proved by Wojtylak that the property does hold for certain positive fragments [31] (proofs for this logic concentrating on different notions of structural completeness may be found in [27], [29], and [30]). We will revisit (and simplify) this proof in our framework. First, recall the following well-known result, noting that for the finite-valued Łukasiewicz logics $\mathrm{Ł}_{n+1}$ ($n = 1, 2, \ldots$) we transfer the usual semantics on the set $\{0, 1/n, \ldots, (n-1)/n, 1\}$ to the (for this paper) more convenient set $\{-n, \ldots, 0\}$.

Lemma 3.12 *$\mathrm{Ł} = \mathbb{Q}(\{\mathbf{Ł_{n+1}} \mid 1 \leq n \in \mathbb{N}\}) = \mathbb{Q}(\mathbf{Ł}_\infty)$ where*

$\mathbf{Ł_{n+1}} = \langle \{-n, \ldots, 0\}, \min, \max, \cdot, \to, -n, 0 \rangle$

$$\text{with} \qquad \begin{aligned} x \cdot y &= \max(-n, x + y) \\ x \to y &= \min(0, y - x); \end{aligned}$$

$\mathbf{Ł}_\infty = \langle [0, 1], \min, \max, \cdot, \to, 0, 1 \rangle$

$$\text{with} \qquad \begin{aligned} x \cdot y &= \max(0, x + y - 1) \\ x \to y &= \min(1, 1 - x + y). \end{aligned}$$

Note also that each individual algebra $\mathbf{Ł_{n+1}}$ for $1 \leq n \in \mathbb{N}$ generates a variety of algebras for the $n+1$-valued Łukasiewicz logic $\mathrm{Ł}_{n+1}$.

Since in Ł disjunction $\varphi \vee \psi$ is definable using implication as $(\varphi \to \psi) \to \psi$ and strong conjunction $\varphi \cdot \psi$ is definable using implication and negation as $\neg(\varphi \to \neg\psi)$, many of our usual fragments coincide. Hence we can restrict our attention to the sublanguages $\{\to\}$, $\{\to, \wedge\}$, $\{\to, \cdot\}$, and $\{\to, \bot\}$, leaving the latter (the full language) to Section 5. We begin, following the spirit but not the details of the proof in [31], with the first two cases.

Proposition 3.13 *Let $1 \leq n \in \mathbb{N}$ and $m_n^\mathbf{Ł} : \mathbf{Ł_{n+1}}{\restriction}\{\to, \wedge\} \to \mathbf{Fm}_{\{\to\}}$ be defined by $m_n^\mathbf{Ł}(i) = \varphi_i$ where*

$$\begin{aligned} \varphi &= ((p \to^{n-1} q) \to p) \to p \\ \varphi_i &= (\varphi \to^{-i} q) \to q \qquad\qquad \text{for } i = 0, -1, \ldots, -n. \end{aligned}$$

Then $h_{\mathrm{Ł}{\restriction}\{\to,\wedge\}} \circ m_n^\mathbf{Ł} : \mathbf{Ł_{n+1}}{\restriction}\{\to, \wedge\} \to \mathbf{F}_{\mathrm{Ł}{\restriction}\{\to,\wedge\}}$ is an embedding.

Proof To show that $h_{\mathbf{Ł}\restriction\{\to,\wedge\}} \circ m_n^{\mathbf{Ł}}$ is a morphism it is enough to show the following:

(i) $\models_{\mathbf{Ł}} \varphi_i \to \varphi_j \approx \top$ for $-n \leq i \leq j \leq 0$;
(ii) $\models_{\mathbf{Ł}} \varphi_i \to \varphi_j \approx \varphi_{j-i}$ for $-n \leq j < i \leq 0$;
(iii) $\models_{\mathbf{Ł}} \varphi_i \wedge \varphi_j \approx \varphi_{\min(i,j)}$ for all $-n \leq i, j \leq 0$.

First, however, let us establish a couple of useful properties.

Claim 1 $\models_{\mathbf{Ł}} q \vee \varphi^n \approx \varphi^n$ and hence $\models_{\mathbf{Ł}} \varphi_i \approx \varphi^{-i}$ for $i = 0, -1, \ldots, -n$.

Proof of claim Let e be an $\mathbf{Ł}_\infty$-evaluation. Then

$$e(\varphi^n) = e((((p^{n-1} \to q) \to p) \to p)^n) = \max(e((p^{n-1} \to q)^n), e(p^n)).$$

If $e(p^{n-1}) \leq e(q)$, then $e(\varphi^n) = 1$ and we are done. If $e(q) \leq e(p^n)$, then since $e(p^n) \leq e(\varphi^n)$ we are also done. So suppose that $e(p^{n-1}) > e(q) > e(p^n)$. It follows that $e(p) \leq e(p^{n-1} \to q)$. But now

$$\begin{aligned} e(q) &= e(p^{n-1} \wedge q) \\ &= e(p^{n-1} \cdot (p^{n-1} \to q)) \\ &\leq e((p^{n-1} \to q)^{n-1} \cdot (p^{n-1} \to q)) \\ &= e((p^{n-1} \to q)^n) \\ &= e(\varphi^n). \end{aligned}$$

Finally, using the above, for any $\mathbf{Ł}_\infty$-evaluation e and $0 \leq i \leq n$, $e(\varphi^i) \geq e(\varphi^n) \geq e(q)$ and so $e(\varphi_i) = \max(e(\varphi^i), e(q)) = e(\varphi^i)$, as required. □

Claim 2 For any $\mathbf{Ł}_\infty$-evaluation e, any formula ψ, and any $k > 0$,

$$e(\psi^k) = \begin{cases} ke(\psi) - k + 1 > 0 & \text{if } e(\psi) > (k-1)/k \\ e(\psi^k) = 0 & \text{otherwise.} \end{cases}$$

Proof of claim A simple computation in $\mathbf{Ł}_\infty$. □

Claim 3 For any $\mathbf{Ł}_\infty$-evaluation e, $e(\varphi^n) = 0$ if and only if ($e(q) = 0$ and $e(p) = (n-1)/n$).

Proof of claim For the right-to-left direction, if $e(q) = 0$ and $e(p) = (n-1)/n$, then $e(\varphi) = \max(e(p), 1 - e(p^{n-1})) = \max(e(p), 1 - 1/n) = (n-1)/n$. So $e(\varphi^n) = 0$. For the left-to-right direction, suppose that $e(\varphi^n) = 0$. Then $e(q) = 0$ by Claim 1. Also $e(p^n) = 0$ and $e((p^{n-1} \to q)^n) = 0$. So $e(p) \leq (n-1)/n$ and $1 - e(p^{n-1}) \leq (n-1)/n$ (by Claim 2). From the latter inequality we get $e(p^{n-1}) \geq 1/n$ and so $e(p^{n-1}) = (n-1)e(p) - (n-1) + 1 \geq 1/n$ (by Claim 2 again). Rearranging, $e(p) \geq (n-1)/n$ and thus together $e(p) = (n-1)/n$. □

Parts (i) and (iii) follow immediately from Claim 1. For part (ii), we show that for any $\mathbf{Ł}_\infty$-evaluation e,

$$e(\varphi^{-i} \to \varphi^{-j}) = 1 - e(\varphi^{-i}) + e(\varphi^{-j}) = e(\varphi^{i-j}).$$

First, assume that $e(\varphi^{-j}) \neq 0$. Then using the fact that $x \to (x \cdot y) = y$ in $\mathbf{Ł}_\infty$ for $x \cdot y > 0$, we get $e(\varphi^{-i} \to \varphi^{-j}) = e(\varphi^{-i} \to \varphi^{-i+i-j}) = e(\varphi^{i-j})$. If $e(\varphi^{-j}) = 0$, then $e(\varphi^n) = 0$ and so by Claim 3, $e(q) = 0$ and $e(p) = (n-1)/n$. But (looking at the first part of the proof of Claim 3) in this case $e(\varphi) = (n-1)/n$ and so $j = -n$ (otherwise $e(\varphi^{-j}) \neq 0$). As the case of $i = 0$ is simple, we can assume

that $e(\varphi^{-i}) > 0$ and $e(\varphi^{i+n}) > 0$. Hence the proof is completed using Claim 2 and observing that

$$1 - (-i\frac{n-1}{n} - (-i) + 1) = (n+i)\frac{n-1}{n} - (n+i) + 1.$$

Finally, from (ii) and the fact that $\not\models_{\text{Ł}} \varphi_i \approx \top$ for $0 < i \leq n$ we obtain that $h_{\text{Ł}\restriction\{\to,\wedge\}} \circ m_n^{\text{Ł}}$ is one-to-one. □

Hence, by Theorem 3.5, we arrive at the following result.

Theorem 3.14 ([31]) *Ł$\restriction\mathcal{L}$ is $\mathcal{SC}$ for $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \vee\}$.*

In Proposition 3.13, we are not able to prove that the embedding preserves strong conjunction. This leaves one open case: the $\{\to, \cdot\}$-fragment of Ł, or algebraically, the variety of Wajsberg hoops. This has perhaps been overlooked due to the fact that Wojtylak refers in [31] to the $\{\to, \wedge, \vee\}$-fragment as the positive fragment. Here we answer the question negatively.

Theorem 3.15 *Ł$\restriction\{\to, \cdot\}$ is not $\mathcal{SC}$.*

Proof We give a direct proof. It is easily seen that the following rule is not derivable in Ł (just consider, for example, the $\mathbf{Ł}_\infty$-evaluation that takes p to 0):

$$p \to (p \cdot p) \rhd p.$$

However, suppose that $\vdash_{\text{Ł}\restriction\{\to,\cdot\}} \sigma p \to (\sigma p \cdot \sigma p)$ for some $\{\to, \cdot\}$-substitution σ. This means that the $\{\to, \cdot\}$-formula σp is idempotent and hence for every $\mathbf{Ł}_\infty$-evaluation e, always $e(\sigma p) = 0$ or $e(\sigma p) = 1$. But the functions corresponding to the formulas of Ł are continuous in $\mathbf{Ł}_\infty$. Also $\bot$ cannot be defined in the language $\{\to, \cdot\}$ (consider, e.g., the $\mathbf{Ł}_\infty$-evaluation that takes every variable to 1). So σp must be a theorem of Ł$\restriction\{\to, \cdot\}$. That is, the rule is admissible in Ł$\restriction\{\to, \cdot\}$, and Ł$\restriction\{\to, \cdot\}$ is not $\mathcal{SC}$. □

3.4 Cancellative Hoop Logic Structural completeness for CHL was already established as an example at the beginning of this section. However, we can now utilize the embeddings for Łukasiewicz Logic as *partial* embeddings to give a uniform proof that all fragments of this logic are $\mathcal{SC}$.

Proposition 3.16 *For a finite subset F of $\mathbb{Z}^-$ where $n = \min F$, we define $m_F^{\mathbb{Z}^-} : F \to \mathbf{Fm}_{\{\to\}}$ by $m_F^{\mathbb{Z}^-}(i) = \varphi_i$ where*

$$\begin{array}{rcll} \varphi & = & ((p \to^{n-1} q) \to p) \to p & \\ \varphi_i & = & (\varphi \to^{-i} q) \to q & \text{for } i = 0, -1, \ldots, -n. \end{array}$$

Then $h_{\text{CHL}} \circ m_F^{\mathbb{Z}^-}$ is a partial embedding of F into $\mathbf{F}_{\text{CHL}}$.

Proof Since the $\{\to, \wedge\}$-fragment of Ł is a sublogic of CHL, we obtain the following from the proof of Theorem 3.13:

(i) $\models_{\text{CHL}} \varphi_i \to \varphi_j \approx \top$ for $-n \leq i \leq j \leq 0$;
(ii) $\models_{\text{CHL}} \varphi_i \to \varphi_j \approx \varphi_{j-i}$ for $-n \leq j < i \leq 0$;
(iii) $\models_{\text{CHL}} \varphi_i \wedge \varphi_j \approx \varphi_{\min(i,j)}$ for all $-n \leq i, j \leq 0$.

Furthermore, by Claim 1 of the proof of Proposition 3.13 and cancellation, easily

(iv) $\models_{\text{CHL}} \varphi_i \cdot \varphi_j \approx \varphi_{i+j}$ for all $-n \leq i + j \leq 0$.

Hence $h_{\mathrm{CHL}} \circ m_F^{\mathbf{Z}^-}$ is a partial mapping of F into $\mathbf{F}_{\mathrm{CHL}}$. Moreover, notice that $\not\models_{\mathrm{CHL}} \varphi_i \approx \top$ for $0 < i \leq n$ and hence using (ii), $h_{\mathrm{CHL}} \circ m_F^{\mathbf{Z}^-}$ must also be one-to-one, that is, a partial embedding. □

So, by Theorem 3.5, we obtain immediately the following.

Theorem 3.17 CHL$\restriction\mathcal{L}$ *is* $\mathcal{SC}$ *for* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \vee, \cdot\}$.

Moreover, from [1], Theorem 6.3, we know that CHL has only trivial subvarieties. That is, its axiomatic extensions are CHL and the inconsistent logic. Since both are $\mathcal{SC}$, by Theorem 2.5 we obtain the following corollary.

Corollary 3.18 CHL *is* $\mathcal{HSC}$.

Note that this latter result follows also directly from Theorem 2.7(b) using the fact established in [16], Chapter 3, Proposition 2.17, that the variety of CHL-algebras has only trivial subquasivarieties. On the other hand, we can see this corollary and Theorem 2.7(b) as an alternative proof of [16], Chapter 3, Proposition 2.17.

3.5 Product Logic We turn our attention now to the third fundamental fuzzy logic, Product Logic Π. As shown in [18], the variety of Π-algebras is generated as a quasivariety by the algebra

$$\langle [0,1], \min, \max, *, \to, 0, 1\rangle \quad \text{with } x * y = x \cdot y$$
$$\text{and } x \to y = y/x \text{ if } x > y, 1 \text{ otherwise.}$$

However, for our purposes it will be more useful to consider an alternative generator, closely related to the generating algebras investigated above for Ł and CHL.

Lemma 3.19 ([18]) $\Pi = \mathbb{Q}(\mathbf{Z}_\perp^-)$ *where*

$$\mathbf{Z}_\perp^- = \langle \mathbb{Z}^- \cup \{-\infty\}, \min, \max, +, \to, 0, -\infty\rangle \text{ with } x \to y = \min(0, y - x).$$

We show structural completeness for fragments of Π following the same basic idea as for CHL. However, here we have to be careful about the bottom element: in the above algebra $-\infty$. Our strategy will be to deal with it in three different ways depending on the language at hand. First, we can map the bottom element to $\perp$ if this constant is in the language. If not, but we have $\cdot$ in the language, then we make use of a formula $(f \to (f \cdot f)) \to f$ that acts as a "$\perp$-surrogate" taking the value $-\infty$ if f is $-\infty$, and 0 otherwise. Finally, in the implicational fragment, our "$\perp$-surrogate" is just a variable f, taking care that all our other elements are mapped to formulas above f in the formula algebra.

Since the core of the proof remains the same in all three cases, we combine these methods to make a more complicated but uniform formulation.

Proposition 3.20 *For a finite subset* F *of* $\mathbb{Z}^-$ *where* $n = \min F$, *let*

$$\begin{aligned} \varphi &= ((p \to^{n-1} q) \to p) \to p \\ \varphi_i &= (\varphi \to^{-i} q) \to q \quad \text{for } i = 0, -1, \ldots, -n. \end{aligned}$$

Take a variable f *different from* p *and* q *and define the substitution* σ *by*

$$\sigma r = ((r \to f) \to f) \to ((f \to r) \to r).$$

For $c \in \{\to, \cdot, \bot\}$, define $m^c_F : F \cup \{-\infty\} \to \mathbf{Fm}_{\{\to,c\}}$ by

$$m^c_F(i) = \sigma\varphi_i \text{ for } i > -\infty$$

$$m^c_F(-\infty) = \begin{cases} f & \text{if } c = \to \\ (f \to (f \cdot f)) \to f & \text{if } c = \cdot \\ \bot & \text{if } c = \bot. \end{cases}$$

Then

(a) $h_{\Pi\restriction\{\to,\wedge,\vee\}} \circ m^{\overrightarrow{}}_F$ *is a partial embedding of* $F \cup \{-\infty\}$ *into* $\mathbf{F}_{\Pi\restriction\{\to,\wedge,\vee\}}$;

(b) $h_{\Pi\restriction\{\to,\cdot\}} \circ m^{\cdot}_F$ *is a partial embedding of* $F \cup \{-\infty\}$ *into* $\mathbf{F}_{\Pi\restriction\{\to,\cdot\}}$;

(c) $h_\Pi \circ m^{\bot}_F$ *is a partial embedding of* $F \cup \{-\infty\}$ *into* $\mathbf{F}_\Pi$.

Proof We start by noting the following (easy to prove) properties for each $\mathbf{Z}^-_\bot$-evaluation e:

(1) $e(\sigma r) = \begin{cases} e(r) & \text{if } e(f) = -\infty \text{ and } e(r) > -\infty \\ 0 & \text{otherwise;} \end{cases}$

(2) $e(m^{\cdot}_F(-\infty)) = \begin{cases} -\infty & \text{if } e(f) = -\infty \\ 0 & \text{otherwise;} \end{cases}$

(3) if $e(f) > -\infty$, then $e(\sigma\varphi) = 0$ and thus also $e(\sigma\varphi_i) = 0$;

(4) if $e(f) = -\infty$, then $e(\sigma\varphi) > -\infty$ and thus also $e(\sigma\varphi_i) > -\infty$.

It then follows easily from (1) (reasoning in the algebras $\mathbf{Z}^-$ and $\mathbf{Z}^-_\bot$) that for any formulas ψ_1 and ψ_2 not containing f or $\bot$,

$$\models_{\mathrm{CHL}} \psi_1 \approx \psi_2 \quad \text{iff} \quad \models_\Pi \sigma\psi_1 \approx \sigma\psi_2.$$

As the formulas φ and φ_i are exactly those from the proof of Proposition 3.16 we obtain

(i) $\models_\Pi \sigma\varphi_i \to \sigma\varphi_j \approx \top$ for $-n \le i \le j \le 0$;

(ii) $\models_\Pi \sigma\varphi_i \to \sigma\varphi_j \approx \sigma\varphi_{j-i}$ for $-n \le j < i \le 0$;

(iii) $\models_\Pi \sigma\varphi_i \cdot \sigma\varphi_j \approx \sigma\varphi_{i+j}$ for $-n \le i + j \le 0$.

For $c \in \{\to, \cdot, \bot\}$ we use (2)–(4) to obtain

(i′) $\models_\Pi m^c_F(-\infty) \to \sigma\varphi_j \approx \top$ for $-n \le j \le 0$;

(ii′) $\models_\Pi \sigma\varphi_i \to m^c_F(-\infty) \approx m^c_F(-\infty)$ for $-n \le i \le 0$.

As the cases for $\vee$ and $\wedge$ are corollaries of (i) and (i′) we have that $h_{\Pi\restriction\{\to,\wedge,\vee\}} \circ m^{\overrightarrow{}}_F$ is a partial morphism. To show that the other two mappings are partial morphisms we use (2)–(4) to show that, for $c \in \{\cdot, \bot\}$,

(iii′) $\models_\Pi \sigma\varphi_i \cdot m^c_F(-\infty) \approx m^c_F(-\infty)$ for $-n \le i \le 0$;

(iii″) $\models_\Pi m^c_F(-\infty) \cdot m^c_F(-\infty) \approx m^c_F(-\infty)$ for $-n \le i \le 0$.

Finally, to complete the proof we need to show that all three morphisms are one-to-one. But this follows from (ii) and (ii′) and the simple fact that $\not\models_\Pi m^c_F(-\infty) \approx \top$ for any $c \in \{\to, \cdot, \bot\}$ and $\not\models_\Pi \varphi_i \approx \top$ for any $-n \le i \le 0$. □

Hence combining this proposition with Theorem 3.5, we obtain the following.

Theorem 3.21 $\Pi\restriction\mathcal{L}$ *is* $\mathcal{SC}$ *for* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \vee, \cdot, \bot\}$.

Moreover, since the only extensions of Product Logic are Classical Logic or the inconsistent logic, we have the following.

Corollary 3.22 Π *is* $\mathcal{HSC}$.

3.6 Basic Logic Our final and most complicated case is Hájek's Basic Logic BL, the logic of continuous t-norms. As we will see in Section 5, BL itself is not structurally complete. However, we will show here that certain fragments do have this property. First, let us consider an appropriate generating set of algebras for the equivalent quasivariety BL based on a particular ordinal sum construction (see [2]).

Definition 3.23 Let $(\mathcal{I}, \leq)$ be a linearly ordered set with bottom element 1 and let $\mathbf{B}$ be a nontrivial BL-chain. The ordinal sum $\bigoplus_{i\in\mathcal{I}} \mathbf{B}$ is defined as

$$\bigoplus_{i\in\mathcal{I}} \mathbf{B} = \langle (\mathcal{I} \times (\mathbf{B} - \{\top^{\mathbf{B}}\})) \cup \{\top\}, \min, \max, \cdot, \to, (1, \bot^{\mathbf{B}}), \top\rangle$$

with top element $\top$, min and max defined lexicographically, and

$$(i,x)\cdot(j,y) = \begin{cases} (i, x \cdot^{\mathbf{B}} y) & \text{if } i = j \\ \min((i,x),(j,y)) & \text{otherwise} \end{cases} \qquad \begin{aligned} (i,x)\cdot\top &= (i,x) \\ \top\cdot(j,y) &= (j,y) \\ \top\cdot\top &= \top \end{aligned}$$

$$(i,x)\to(j,y) = \begin{cases} \top & \text{if } (i,x)\leq(j,y) \\ (i, x \to^{\mathbf{B}} y) & \text{if } i = j \text{ and } x > y \\ (j,y) & \text{otherwise} \end{cases} \qquad \begin{aligned} (i,x)\to\top &= \top \\ \top\to(j,y) &= (j,y) \\ \top\to\top &= \top. \end{aligned}$$

Lemma 3.24 ([21]) $\mathrm{BL} = \mathbb{Q}(\{\mathbf{BL_n} \mid 2 \leq n \in \mathbb{N}\}) = \mathbb{Q}(\mathbf{BL}_\infty)$ *where*

$$\mathbf{BL_n} = \bigoplus_{i\in\{1,\ldots,n\}} \mathbf{Ł_n} \quad \textit{and} \quad \mathbf{BL}_\infty = \bigoplus_{i\in\{1,2,\ldots\}} \mathbf{Ł}_\infty.$$

We will speak of the copies of $\mathbf{Ł_n}$ and $\mathbf{Ł}_\infty$ as *components* of the algebras $\mathbf{BL_n}$ and $\mathbf{BL}_\infty$, respectively, with the understanding that $\top$ is a common element of all components. We will refer to elements being in higher, lower, or the same component with respect to the index set $\{1,\ldots,n\}$ or $\{1,2,\ldots\}$.

For convenience, we introduce an auxiliary connective $\varphi \prec \psi =_{\mathrm{def}} (\varphi \to \psi) \to \psi$ and observe that, for each $\mathbf{BL}_\infty$-evaluation e,

$$e(\varphi\prec\psi) = \begin{cases} \top & \text{if } e(\varphi) \text{ is in a higher component than } e(\psi) \\ \max(e(\varphi), e(\psi)) & \text{otherwise.} \end{cases}$$

Proposition 3.25 *For distinct variables* $p_1, \ldots, p_n$ *let*

$$\begin{aligned} \varphi_1 &= p_1, \\ \varphi_i &= (p_i \prec \varphi_{i-1}) \to (\varphi_{i-1} \prec p_i) \quad \textit{for } 2 \leq i \leq n, \end{aligned}$$

and for distinct variables $q_1, \ldots, q_n$ *let*

$$\begin{aligned} \psi_1 &= (\varphi_1 \to^{n-1} q_1) \prec (\varphi_1 \prec q_1) \\ \psi_i &= \psi_{i-1} \prec ((\varphi_i \to^{n-1} q_i) \prec (\varphi_i \prec q_i)) && \textit{for } 2 \leq i \leq n \\ \psi_{(i,x)} &= (\psi_i \to^{-x} q_i) \to q_i && \textit{for } 1 \leq i \leq n \textit{ and } -n \leq x \leq -1. \end{aligned}$$

Define $m^n_{\mathrm{BL}} : \mathbf{BL_n}\restriction\{\to\} \to \mathbf{Fm}_{\{\to\}}$ *by* $m^n_{\mathrm{BL}}(\top) = \top$ *and* $m^n_{\mathrm{BL}}((i,x)) = \psi_{(i,x)}$. *Then* $h_{\mathrm{BL}\restriction\{\to,\wedge\}} \circ m^n_{\mathrm{BL}} : \mathbf{BL_n}\restriction\{\to,\wedge\} \to \mathbf{F}_{\mathrm{BL}\restriction\{\to,\wedge\}}$ *is an embedding.*

Proof We begin with a series of claims analyzing the formulas defined above.

Claim 1 For $2 \leq i \leq n$ and an arbitrary $\mathbf{BL}_\infty$-evaluation e,

$$e(\varphi_i) = \begin{cases} e(p_i) & \text{if } e(p_i) \text{ is in a higher component than } e(\varphi_{i-1}) \\ \top & \text{otherwise.} \end{cases}$$

Hence, for each $j > i$, either $e(\varphi_j) = \top$ or $e(\varphi_j)$ is in a higher component than $e(\varphi_i)$.

Proof of claim If $e(p_i)$ is in a higher component than $e(\varphi_{i-1})$, then

$$e(\varphi_i) = e((p_i \prec \varphi_{i-1}) \to (\varphi_{i-1} \prec p_i)) = e(\varphi_{i-1} \prec p_i) = e(p_i).$$

If $e(p_i)$ is in a lower component than $e(\varphi_{i-1})$, then $e(\varphi_{i-1} \prec p_i) = \top$, so $e(\varphi_i) = \top$. Finally, if they are in the same component, then $e(p_i \prec \varphi_{i-1}) = e(\varphi_{i-1} \prec p_i)$, so $e(\varphi_i) = \top$. The second part of the claim is obvious. □

Hence if k is minimal such that $e(\varphi_k) = \top$ (set $k = n+1$ if there is no such number), then $e(\varphi_j)$ is in a higher component than $e(\varphi_i)$ for any $k > j > i$ and $e(\varphi_j) = \top$ for any $j \geq k$.

Claim 2 For $1 \leq i \leq n$, let us say that a $\mathbf{BL}_\infty$-evaluation e is i-good if, for each $j \leq i$,

1. $e(\varphi_j) \neq \top$;
2. $e(\varphi_j)$ and $e(q_j)$ are in the same component;
3. $e(q_j) < e(\varphi_j)^{n-1}$.

Then

$$e(\psi_i) = \begin{cases} e((\varphi_i \to^{n-1} q_i) \prec \varphi_i) \neq \top & \text{if } e \text{ is } i\text{-good} \\ \top & \text{otherwise.} \end{cases}$$

Hence, $e(\psi_i)$ and $e(\psi_{(i,x)})$ (for each $-n \leq x \leq -1$) are in the same component as $e(\varphi_i)$.

Proof of claim We proceed by induction on i. For the base case, if $e(\varphi_1)^{n-1} \leq e(q_1)$, then $e(\varphi_1 \to^{n-1} q_1) = \top$ and $e(\psi_1) = \top$ as required. If $e(\varphi_1) = \top$ or $e(\varphi_1)$ is in a higher component than $e(q_1)$, then $e(\varphi_1 \prec q_1) = \top$ and $e(\psi_1) = \top$ as required. Suppose now that e is 1-good. Then $e(\varphi_1 \prec q_1) = e(\varphi_1)$ and we are done. The proof of the second part of the claim is trivial.

For the induction step, suppose that $i > 1$. If e is not $(i-1)$-good, then $e(\psi_{i-1}) = \top$ and so $e(\psi_i) = \top$ as required. If e is $(i-1)$-good but not i-good, then we proceed as in the case $i = 1$. Hence assume that e is i-good. Observe that $e((\varphi_i \to^{n-1} q_i) \prec (\varphi_i \prec q_i)) = e((\varphi_i \to^{n-1} q_i) \prec \varphi_i) \neq \top$ is in the same component as $e(\varphi_i)$ and $e(\psi_{i-1}) \neq \top$ is in the same component as $e(\varphi_{i-1})$ (using the induction hypothesis). Since $e(\varphi_i) \neq \top$, we can use Claim 1 to see that $e(\psi_{i-1})$ is in a lower component than $e((\varphi_i \to^{n-1} q_i) \prec \varphi_i)$ and so the proof is done. (Notice that we crucially use the fact here that $e(\psi_{i-1}) \neq \top$.) Again, the proof of the second part of the claim is trivial. □

Claim 3 For $1 \leq i \leq n$ and $-n \leq x \leq -1$,

$$\models_{\mathrm{BL}} q_i \vee \psi_i^n \approx \psi_i^n \qquad \text{and} \qquad \models_{\mathrm{BL}} \psi_{(i,x)} \approx \psi_i^{-x}.$$

Proof of claim We establish the claim for an arbitrary $\mathbf{BL}_\infty$-evaluation e. We use Claim 2 to notice that the only nontrivial case is when e is i-good. However, here we can proceed as in the proof of Claim 1 occurring in the proof of Proposition 3.13 for Ł. □

Now to show that our mapping is a morphism it is enough (the case of $\wedge$ is then an easy corollary) to show

(i) $\models_{\mathrm{BL}} \psi_{(i,x)} \to \psi_{(j,y)} \approx \top$ for $(i, x) \leq (j, y)$;

(ii) $\models_{\mathrm{BL}} \psi_{(i,x)} \to \psi_{(j,y)} \approx \psi_{(j,y)}$ for $i > j$;

(iii) $\models_{\mathrm{BL}} \psi_{(i,x)} \to \psi_{(i,y)} \approx \psi_{(i,y-x)}$ for $x > y$.

Again, we establish the claims for an arbitrary $\mathbf{BL}_\infty$-evaluation e. First assume that e is neither i-good nor j-good. Then parts (i)–(iii) immediately follow from Claims 2 and 3.

Now assume that e is both i-good and j-good. Part (iii) is then a corollary of Claims 2 and 3 and the corresponding part of the proof of Proposition 3.13 for Ł. To prove part (i), first notice that the case of $i = j$ is a simple corollary of Claim 2. For $i < j$ we use Claim 1 to get that $e(\varphi_i)$ is in a lower component than $e(\varphi_j)$ (since $e(\varphi_j) \neq \top$). As by Claim 2, $e(\psi_{(i,x)})$ is in the same component as $e(\varphi_i)$ and $e(\psi_{(j,x)})$ is in the same component as $e(\varphi_j)$, the proof is done. The proof of part (ii) is analogous.

Finally to show injectivity, observe that $\not\models_{\mathrm{BL}} \psi_{(i,x)} \approx \top$ (any i-good evaluation would provide a counterexample and such an evaluation clearly exists) and using parts (ii) and (iii), we have that $\not\models_{\mathrm{BL}} \psi_{(i,x)} \approx \psi_{(j,y)}$ for all $(i, x) \neq (j, y)$. □

Hence by Theorem 3.5, we arrive at the following result.

Theorem 3.26 $\mathrm{BL}\restriction\mathcal{L}$ *is* $\mathcal{SC}$ *for* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \vee\}$.

Finally, as might be expected from the failure of structural completeness for the $\{\to, \cdot\}$-fragment of Ł, the same result also holds for this fragment of BL.

Theorem 3.27 $\mathrm{BL}\restriction\{\to, \cdot\}$ *is not* $\mathcal{SC}$.

Proof We use the same underivable rule as in the corresponding case for Ł, namely, $p \to (p \cdot p) \rhd p$. Suppose that $\vdash_{\mathrm{BL}\restriction\{\to,\cdot\}} \sigma p \to (\sigma p \cdot \sigma p)$ for some $\{\to, \cdot\}$-substitution σ. Then the $\{\to, \cdot\}$-formula σp is idempotent. But now using the characterization of the free n-generated BL-algebra given in Theorem 6.1 of [3] we know that any such idempotent formula must be equivalent to $\top$. So $\vdash_{\mathrm{BL}} \sigma p$ and the rule is admissible. □

4 $\mathcal{HSC}$ and the Hereditary Deduction Theorem

We now turn our attention to a different method of establishing (hereditary) structural completeness for logics, an extension of the well-known "Prucnal's trick" used for several implicational propositional calculi in [24]. To illustrate the trick, consider any extension L of the implication-conjunction fragment $\mathrm{G}\restriction\{\wedge, \to\}$ of Gödel Logic. Suppose that the rule $\varphi_1, \ldots, \varphi_n \rhd \psi$ is admissible in L. Consider a substitution $\sigma p = (\varphi_1 \wedge \cdots \wedge \varphi_n) \to p$ and prove (using some key theorems of $\mathrm{G}\restriction\{\wedge, \to\}$)

that $\vdash_{\mathrm{L}} \sigma\varphi \leftrightarrow ((\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \varphi)$ for any implication-conjunction formula φ. Since clearly $\vdash_{\mathrm{L}} \sigma\varphi_i$ for $i = 1, \ldots, n$, the admissibility of the rule gives $\vdash_{\mathrm{L}} \sigma\psi$. So $\vdash_{\mathrm{L}} (\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \psi$ and by properties of $\wedge$ and $\rightarrow$, we obtain $\varphi_1, \ldots, \varphi_n \vdash_{\mathrm{L}} \psi$. Hence L is $\mathcal{SC}$ and $\mathrm{G}\restriction\{\wedge, \rightarrow\}$ is $\mathcal{HSC}$.

We use this trick here not to establish $\mathcal{HSC}$ directly, but rather to show that the logic in question and all its extensions share a "hereditary version" of the local (global) deduction theorem that is (under certain conditions) equivalent to $\mathcal{HSC}$. Besides establishing a further property of the logic, this method also avoids the need for $\wedge$ in the language and works in fragments where the trick cannot be applied (see Theorem 4.8).

4.1 General conditions First, we recall the notion of a local (or global) deduction theorem for a logic.

Definition 4.1 A logic L has the *local deduction theorem* $\mathcal{LDT}$ with respect to a set $\mathcal{E}$ of sets of formulas in two variables if, for each theory T and formulas φ, ψ,

$$T, \varphi \vdash_{\mathrm{L}} \psi \text{ iff } (\exists \Delta \in \mathcal{E})(T \vdash_{\mathrm{L}} \Delta(\varphi, \psi)).$$

L has the *global deduction theorem* $\mathcal{GDT}$ with respect to $\mathcal{E}$ if $\mathcal{E}$ contains just one set of formulas.

Example 4.2 All the fuzzy logics considered in this paper have the $\mathcal{LDT}$ with respect to the set

$$\mathcal{E} = \{\{p \rightarrow^n q\} \mid n \in \mathbb{N}\}.$$

Notice that a logic can have the $\mathcal{LDT}$ with respect to different sets. For example, in addition to the $\mathcal{LDT}$ mentioned in the previous example, Gödel Logic has the $\mathcal{GDT}$ with respect to $\mathcal{E} = \{\{p \rightarrow q\}\}$.

We now introduce the new notion of a "hereditary" deduction theorem, essentially saying that the appropriate deduction theorem is preserved (with the same set) for all extensions of the logic.

Definition 4.3 A logic L has the *hereditary local deduction theorem* $\mathcal{HLDT}$ (*hereditary global deduction theorem* $\mathcal{HGDT}$) with respect to a set $\mathcal{E}$ of sets of formulas in two variables if each of its extensions has the $\mathcal{LDT}$ ($\mathcal{GDT}$) with respect to $\mathcal{E}$.

We can now formulate the crucial theorem of this section.

Theorem 4.4 *Let* L *be a logic with the* $\mathcal{LDT}$ *with respect to* $\mathcal{E}$. *Then* L *has the* $\mathcal{HLDT}$ *with respect to* $\mathcal{E}$ *if and only if* L *is* $\mathcal{HSC}$.

Proof Right-to-left direction. Let L′ be any extension of L. Since L is $\mathcal{HSC}$, by Theorem 2.5, L′ must be an *axiomatic* extension of L. Let us denote the set of additional axioms (closed under substitution) by $\mathcal{A}$. Then $T \vdash_{\mathrm{L}'} \varphi$ if and only if $T, \mathcal{A} \vdash_{\mathrm{L}} \varphi$. In particular, to see that L′ has the $\mathcal{LDT}$ with respect to $\mathcal{E}$, note that

$$\begin{aligned} T, \varphi \vdash_{\mathrm{L}'} \psi \quad &\text{iff} \quad T, \mathcal{A}, \varphi \vdash_{\mathrm{L}} \psi \\ &\text{iff} \quad T, \mathcal{A} \vdash_{\mathrm{L}} \Delta(\varphi, \psi) \quad \text{for some } \Delta \in \mathcal{E} \\ &\text{iff} \quad T \vdash_{\mathrm{L}'} \Delta(\varphi, \psi) \quad \text{for some } \Delta \in \mathcal{E}. \end{aligned}$$

Left-to-right direction. Using Theorem 2.5 it is enough to show that any extension of L is an *axiomatic* extension of L. Let L′ be an extension of L with a set of rules S.

Since L has the $\mathcal{HLDT}$ with respect to $\mathcal{E}$, L′ has the $\mathcal{LDT}$ with respect to $\mathcal{E}$. Hence, let us denote by $\Delta^{\psi}_{T,\varphi}$ some (arbitrary but fixed) set in $\mathcal{E}$ existing when $T, \varphi \vdash_{\mathrm{L}'} \psi$ such that $T \vdash_{\mathrm{L}'} \Delta^{\psi}_{T,\varphi}(\varphi, \psi)$.

For a rule $R = \varphi_1, \ldots \varphi_n \rhd \psi$, we define sets $A^R_{n+1}, \ldots A^R_1$ of formulas by induction:

1. $A^R_{n+1} = \{\psi\}$;
2. $A^R_i = \{\chi(\varphi_i, \delta) \mid \chi \in \Delta^{\delta}_{\{\varphi_1,\ldots\varphi_{i-1}\},\varphi_i}$ and $\delta \in A^R_{i+1}\} = \bigcup_{\delta \in A^R_{i+1}} \Delta^{\delta}_{\{\varphi_1,\ldots\varphi_{i-1}\},\varphi_i}(\varphi_i, \delta)$ for $i = n \ldots 1$.

Roughly speaking, A^R_1 is the set of formulas resulting from applying the local deduction theorem of L′ exhaustively to R. If L′ has the global deduction theorem and $\mathcal{E} = \{\{\chi\}\}$, then $A^R_1 = \{\chi(\varphi_1, \chi(\ldots\chi(\varphi_{n-1}, \chi(\varphi_n, \psi))\ldots)\}$.

Let $\bar{\mathrm{L}}$ be the extension of L with the set of axioms $\mathcal{A} = \bigcup_{R\in S} A^R_1$. We show that $\bar{\mathrm{L}}$ is L′. It is easy to see that L′ proves all the formulas from $\mathcal{A}$, so $\bar{\mathrm{L}} \subseteq \mathrm{L}'$. To prove the converse, it is sufficient to show that $\varphi_1 \ldots \varphi_n \vdash_{\bar{\mathrm{L}}} \psi$ for each rule $R = \varphi_1 \ldots \varphi_n \rhd \psi$ in S. We prove $\varphi_1, \ldots, \varphi_{i-1} \vdash_{\bar{\mathrm{L}}} A^R_i$ by induction on i. The base case $i = 1$ is simple. Observe that $\Delta(\varphi, \psi), \varphi \vdash_{\mathrm{L}} \psi$ since $\Delta(\varphi, \psi) \vdash_{\mathrm{L}} \Delta(\varphi, \psi)$ for any $\Delta \in \mathcal{E}$. Thus $\Delta^{\delta}_{\{\varphi_1,\ldots\varphi_{i-1}\},\varphi_i}(\varphi_i, \delta), \varphi_i \vdash_{\mathrm{L}} \delta$ for each $\delta \in A^R_{i+1}$. So $A^R_i, \varphi_i \vdash_{\mathrm{L}} A^R_{i+1}$, and by the induction hypothesis, we get $\varphi_1, \ldots, \varphi_{i-1}, \varphi_i \vdash_{\bar{\mathrm{L}}} A^R_{i+1}$ as required. □

Notice in fact that in the proof of the left-to-right direction, a (seemingly) weaker property than $\mathcal{HLDT}$ suffices; namely, that any logic L′ extending L has $\mathcal{LDT}$ with respect to *some* set of sets of formulas $\mathcal{E}^{\mathrm{L}'}$ such that $\Delta(\varphi, \psi), \varphi \vdash_{\mathrm{L}} \psi$ for each $\Delta \in \mathcal{E}^{\mathrm{L}'}$.

Corollary 4.5 *Let* L *be a logic with the* $\mathcal{GDT}$ *with respect to* $\mathcal{E}$. *Then* L *has the* $\mathcal{HGDT}$ *with respect to* $\mathcal{E}$ *if and only if* L *is* $\mathcal{HSC}$.

Note that a similar connection between $\mathcal{HSC}$ and deduction theorems in the special case of (fragments of) intermediate logics was announced in [32].

4.2 Applications We now present some applications of Theorem 4.4 for "n-contractive" logics such as $\mathrm{C_n}$MTL and $\mathrm{C_n}$BL ($n \geq 3$). Many of the results obtained are already known in the literature. However, as will be explained below, the uniformity of our approach offers certain (small) advantages over previous work in the area.

Recall that FL^{+}_{ew} is the logic of all ICRLs. We also define *Hoop Logic* HL (since HL-algebras are termwise equivalent to hoops [5]) as the logic of ICRLs satisfying (div) (see Table 1). Moreover, we denote the extensions of these (and other) logics such that their algebras additionally satisfy (c_n) for $2 \leq n \in \mathbb{N}$ by adding the prefix C_n.

We note the following useful identities.

Lemma 4.6 *Let* $1 \leq n \in \mathbb{N}$. *Then*

(i) $\models_{\mathrm{C}_{n+1}\mathrm{FL}_{ew}} \varphi \to^n (\psi \to \chi) \approx (\varphi \to^n \psi) \to (\varphi \to^n \chi)$;
(ii) $\models_{\mathrm{FL}_{ew}} \varphi \to (\psi \wedge \chi) \approx (\varphi \to \psi) \wedge (\varphi \to \chi)$;
(iii) $\models_{\mathrm{MTL}} \varphi \to (\psi \vee \chi) \approx (\varphi \to \psi) \vee (\varphi \to \chi)$;
(iv) $\models_{\mathrm{C}_{n+1}\mathrm{HL}} \varphi \to^n (\psi \cdot \chi) \approx (\varphi \to^n \psi) \cdot (\varphi \to^n \chi)$.

The cases of (i)–(iii) are well known (see, for example, [17]) and (iv) is established in [16], Lemma 1.10.

Theorem 4.7 *The following logics are $\mathcal{HSC}$ for $2 \leq n \in \mathbb{N}$:*

(a) $\mathrm{C_nFL}_{ew}{\restriction}\mathcal{L}$ *for* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge\}$;

(b) $\mathrm{C_nMTL}{\restriction}\mathcal{L}$ *for* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \vee\}$;

(c) $\mathrm{C_nHL}{\restriction}\mathcal{L}$ *for* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \cdot\}$;

(d) $\mathrm{C_nBL}{\restriction}\mathcal{L}$ *for* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \vee, \cdot\}$.

Proof We just prove this theorem for $\mathrm{C_nMTL}{\restriction}\{\to, \wedge\}$ ($2 \leq n \in \mathbb{N}$) since the other cases follow a very similar pattern. We will show that this logic has the hereditary global deduction theorem $\mathcal{HGDT}$ and hence, by Theorem 4.4, is $\mathcal{HSC}$. Let L be any logic extending $\mathrm{C_nMTL}{\restriction}\{\to, \wedge\}$. We show that L has the global deduction theorem with respect to $\mathcal{E} = \{\{p \to^n q\}\}$. First observe that (easily) $T \vdash_{\mathrm{L}} \varphi \to^n \psi$ implies $T, \varphi \vdash_{\mathrm{L}} \psi$.

Now suppose that $T, \varphi \vdash_{\mathrm{L}} \psi$. We define the substitution $\sigma p = \varphi \to^n p$. By a simple induction on formula complexity, we can use the identities of Lemma 4.6 to obtain $\vdash_{\mathrm{L}} \sigma\psi \leftrightarrows (\varphi \to^n \psi)$ (since $\sigma\psi \approx \varphi \to^n \psi$ holds in any L-algebra). Using the substitution invariance of L we obtain $\sigma(T), \sigma\varphi \vdash_{\mathrm{L}} \sigma\psi$. Since (using weakening) $T \vdash_{\mathrm{L}} \sigma(T)$ and $\vdash_{\mathrm{L}} \sigma\varphi$, we obtain $T \vdash_{\mathrm{L}} \varphi \to^n \psi$. □

In particular, Intuitionistic Logic is $\mathrm{C_2FL}_{ew}$ so part (a) above is just the well-known result (see, e.g., [24]) that the implication (and conjunction) fragment of Intuitionistic Logic is $\mathcal{HSC}$. Note in fact that our results for $\mathcal{L} = \{\to, \wedge\}$ are corollaries of [22], Corollary 6.8, and parts (b) and (d) for $\mathcal{L} = \{\to, \wedge, \vee\}$ are corollaries of [22], Corollary 8.4 (or of the fact that in these logics $\vee$ is definable in terms of $\to$ and $\wedge$). Parts (c) and (d) for $\mathcal{L} = \{\to, \wedge, \cdot\}$ are corollaries of results by Ferreirim [16] that the variety of n-potent hoops is primitive. The results for $\mathcal{L} = \{\to\}$ are corollaries of [7], Theorem 4.4, where the variety of n-contractive BCK-algebras is shown to be primitive. Finally, the results for BL were proved independently of these other sources in [12].

Nevertheless, our method used to obtain all these results has the virtue of uniformity, gives the $\mathcal{HGDT}$ for the logics, and can be seen as an alternative proof of the primitivity of their equivalent quasivarieties. Moreover, it can be used to deal with tricky cases. As an example, let us prove that all (not just the $\bot$-free) fragments of G are $\mathcal{HSC}$. An alternative proof, given for the full language in [13], consists of showing that each extension of the fragment—that is, a fragment of an n-valued Gödel logic—is $\mathcal{SC}$.

Theorem 4.8 $\mathrm{G}{\restriction}\mathcal{L}$ *is* $\mathcal{HSC}$ *for* $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \vee, \bot\}$.

Proof The only remaining cases to be shown are those involving $\bot$. Let us denote by φ_f the formula resulting from φ by replacing all occurrences of $\bot$ by a new fixed variable f, and let $T_f = \{\varphi_f \mid \varphi \in T\}$. Define a substitution $\sigma(q) = \bot$ for $q = f$ and q otherwise. Then clearly $\sigma(\varphi_f) = \varphi$. Now consider a logic L extending G (the proofs for fragments are analogous). We will show that L has the $\mathcal{GDT}$ with respect to $\{\{p \to q\}\}$. It is sufficient to prove the following.

Claim If $T \vdash_{\mathrm{L}} \psi$, then $\{f \to q \mid q \text{ occurring in } T, \psi\} \cup T_f \vdash_{\mathrm{L}} \psi_f$.

We then just apply the claim to $T, \varphi \vdash_{\mathrm{L}} \psi$. Since $\bot$ does not occur in the second expression we can use the $\mathcal{HGDT}$ of the $\bot$-free fragment of L established in the previous theorem to obtain

$$\{f \to q \mid q \text{ occurring in } T, \varphi, \psi\} \cup T_f \vdash_{\mathrm{L}\restriction\{\to,\vee,\wedge\}} \varphi_f \to \psi_f.$$

But then also $\sigma(\{f \to q \mid q \text{ occurring in } T, \varphi, \psi\} \cup T_f) \vdash_{\mathrm{L}} \sigma(\varphi_f \to \psi_f)$. That is, $T \vdash_{\mathrm{L}} \varphi \to \psi$.

Proof of claim Let S denote the set of all variables occurring in T and ψ. Proceeding contrapositively, consider an L-algebra $\mathbf{A} = (A, \to, \wedge, \vee, \bot, \top)$ and $\mathbf{A}$-evaluation e such that $e(f) \leq e(q)$ for all $q \in S$, $e(T_f) \subseteq \{\top\}$, and $e(\psi_f) \neq \top$. We show that in this case $T \nvdash_{\mathrm{L}} \psi$.

We construct an algebra $\mathbf{A}' = (\{x \in A \mid e(f) \leq x\}, \to_f, \wedge_f, \vee_f, e(f), \top)$ by restricting the operations of $\mathbf{A}$. Clearly $\mathbf{A}'$ is a subalgebra of $\mathbf{A}$ and so is an L-algebra. Define an $\mathbf{A}'$-evaluation e' such that $e'(q) = e(q)$ for any $q \in S \cup \{f\}$ and notice that $e'(\chi) = e(\chi_f)$ for any $\chi \in T \cup \{\psi\}$. Thus $e'(T) \subseteq \{\top\}$ and $e'(\psi) \neq \top$; that is, $T \nvdash_{\mathrm{L}} \psi$. □

5 Passive Structural Completeness

In this section we investigate a weaker notion of structural completeness, obtaining new methods for proving or disproving that a logic is $\mathcal{SC}$. We then make use of some simple observations regarding derivable and nonderivable rules to fill in some of the remaining gaps for fuzzy logics.

As a motivating example, consider the three-valued Łukasiewicz logic Ł$_3$ (see Section 3). There is no substitution σ such that the formula $\sigma(p \leftrightarrow \neg p)$ is derivable in Ł$_3$ (or even classically derivable). So the rule $p \leftrightarrow \neg p \rhd q$ is Ł$_3$-admissible. On the other hand, the rule is not Ł$_3$-derivable, since (recalling the definitions of Section 3.3) when $e(p) = e(q) = -1$ for an **Ł$_3$**-evaluation e, we have $e(p \leftrightarrow \neg p) = 0$. Hence (see, e.g., [31]) Ł$_3$ is not $\mathcal{SC}$. However, this same proof also works to show that Łukasiewicz Logic is not $\mathcal{SC}$. Any theorem of Ł is a theorem of Classical Logic, and the three-valued algebra **Ł$_3$** is an Ł-algebra. So again the rule is Ł-admissible but not Ł-derivable. Indeed, this proof will work for *any* logic satisfying these conditions. Let us see now how these ideas can be generalized.

5.1 General conditions

We introduce the following key notions.

Definition 5.1 A rule $T \rhd \varphi$ is *passive* for a logic L if there is no substitution σ such that $\vdash_{\mathrm{L}} \sigma(T)$. We say that L is *passively structurally complete* ($\mathcal{PSC}$) if each passive rule of L is L-derivable.

Notice that a rule for a logic L is passive exactly when the premises of the rule fail to be L-*unifiable*. Passive rules and L-unifiability are considered extensively by Rybakov in [26], while the notion of passive structural completeness has been investigated from an algebraic perspective by Wronski and called nonoverflow completeness.

Example 5.2 Some (fragments of) logics are trivially $\mathcal{PSC}$. Clearly, each $\mathcal{SC}$ logic is $\mathcal{PSC}$ (as each passive rule is admissible). Moreover, for any extension L of FL$_{ew}$ and $\{\to\} \subseteq \mathcal{L} \subseteq \{\to, \wedge, \vee, \cdot\}$, the substitution $\sigma(p) = \top$ (defined, recall as $q \to q$)

makes any formula of $\mathrm{L}\restriction\mathcal{L}$ a theorem. So there are no passive rules, and the logic is $\mathcal{PSC}$.

Example 5.3 A more interesting example is Intuitionistic Logic IL which is $\mathcal{PSC}$ but not $\mathcal{SC}$. The failure of structural completeness is well known (see, e.g., [26]). For the weaker notion, suppose that $T \rhd \varphi$ is a nonderivable rule of IL. Then clearly $T \nvdash_{\mathrm{IL}} \bot$, and so also by the Deduction and Glivenko theorems, $T \nvdash_{\mathrm{CL}} \bot$. Now let e be a classical evaluation satisfying T. Define a substitution by $\sigma(p) = \bot$ if $e(p) = \bot$ and $\top$ otherwise. Clearly, $\vdash_{\mathrm{CL}} \sigma(\psi) \leftrightarrow \top$ for each $\psi \in T$ and since IL proves the same variable-free formulas as CL, $\vdash_{\mathrm{IL}} \sigma(\psi) \leftrightarrow \top$ for all $\psi \in T$. So $T \rhd \varphi$ is not passive in IL. Note that this result was reported by Wronski at the 51st Conference on the History of Logic, Krakow, 2005.

We now show that the notion of $\mathcal{PSC}$ has many nice properties not shared by the usual notion of structural completeness. In particular, it is preserved *upward*, that is, by going to stronger logics.

Theorem 5.4 ($\mathcal{PSC}$ upward) *Any extension of a $\mathcal{PSC}$ logic is $\mathcal{PSC}$.*

Proof Let L' be an extension of a $\mathcal{PSC}$ logic L. If $T \rhd \varphi$ is nonderivable in L', then it is also nonderivable in L. Since L is $\mathcal{PSC}$, the rule is not passive in L. That is, there is a substitution σ such that $\vdash_{\mathrm{L}} \sigma(T)$. But then also $\vdash_{\mathrm{L}'} \sigma(T)$, so the rule is not passive in L'. □

Next we show that $\mathcal{PSC}$ is preserved also *downward*, that is, by restricting to certain fragments. Since the situation is not as simple as in the upward case, we begin by introducing an auxiliary notion.

Definition 5.5 $\mathrm{L}\restriction\mathcal{L}$ is a *passive fragment* of L if each passive rule in $\mathrm{L}\restriction\mathcal{L}$ is passive in L.

The notion of a passive fragment is tailored to obtain the theorem below. Not all fragments of logics are passive; for example, $\mathrm{Ł}\restriction\{\wedge, \vee, \bot\}$ is not a passive fragment of Ł (just consider the rule $q \rhd \bot$). Later, however, we will give a sufficient condition useful for recognizing passive fragments.

Theorem 5.6 ($\mathcal{PSC}$ downward) *Any passive fragment of a $\mathcal{PSC}$ logic is $\mathcal{PSC}$.*

Combining $\mathcal{PSC}$ upward and downward, we have that any extension of a passive fragment of a $\mathcal{PSC}$ logic is $\mathcal{PSC}$. Moreover, formulated negatively, we obtain a useful tool for disproving $\mathcal{PSC}$ (and hence $\mathcal{SC}$).

Corollary 5.7 *If some logic extending a passive fragment of* L *is not $\mathcal{PSC}$, then* L *is not $\mathcal{PSC}$.*

Alternatively, if $T \nvdash_{\mathrm{L}} \varphi$ and $T \rhd \varphi$ is passive in an extension of some passive fragment of L, then L is not $\mathcal{PSC}$. We now give three conditions for identifying passive fragments.

Lemma 5.8 *Let* L *be an algebraizable logic in the language $\mathcal{L}$ and $\{\rightarrow\} \subseteq \mathcal{L}' \subseteq \mathcal{L}$. Then $\mathrm{L}\restriction\mathcal{L}'$ is a passive fragment of* L *if one of the following cases holds:*

(a) *for each finite set of $\mathcal{L}$-formulas $\varphi_1, \ldots, \varphi_n$, there is an $\mathcal{L}$-substitution σ and $\mathcal{L}'$-formulas $\psi_1, \ldots, \psi_n$ such that $\models_{\mathrm{L}} \sigma(\varphi_i) \approx \psi_i$ for $i = 1 \ldots n$;*

(b) *there exists an $\mathcal{L}$-substitution σ such that for each $\mathcal{L}$-formula φ there is an $\mathcal{L}'$-formula ψ satisfying $\models_{\mathrm{L}} \sigma(\varphi) \approx \psi$;*
(c) *there is a set of $\mathcal{L}'$-formulas Ψ such that for each n-ary connective $c \in \mathcal{L}$ and formulas $\psi_1, \ldots, \psi_n \in \Psi$ there is $\psi \in \Psi$ satisfying $\models_{\mathrm{L}} c(\psi_1, \ldots, \psi_n) \approx \psi$.*

Proof It is enough to show that (a) is a sufficient condition (since (a) follows immediately from (b) and (b) can be obtained from (c) by setting $\sigma p = \psi$ for any p and any $\psi \in \Psi$).

Let $T \rhd \varphi$ be a passive rule in $\mathrm{L}\restriction\mathcal{L}'$. We will show that it is passive in L. Proceeding contrapositively, assume that there is an $\mathcal{L}$-substitution σ such that $\vdash_{\mathrm{L}} \sigma(T)$. Suppose that $q_1, \ldots, q_n$ are the variables of T, written $T(q_1, \ldots, q_n)$. For a set of variables $p_1, \ldots, p_n$ disjoint from $q_1, \ldots, q_n$, define $\sigma_1 q_i = p_i$ for $i = 1 \ldots n$, leaving other variables as they are.

By assumption, for $\sigma q_1, \ldots, \sigma q_n$ there is an $\mathcal{L}$-substitution σ_2 and $\mathcal{L}'$-formulas $\psi_1, \ldots, \psi_n$ such that $\models_{\mathrm{L}} \sigma_2 \sigma q_i \approx \psi_i$. We define our $\mathcal{L}'$-substitution as $\sigma' q_i = \sigma_1 \psi_i$ (this is a sound definition since each formula $\sigma_1 \psi_i$ for $i = 1 \ldots n$ does not contain any of the $q_1, \ldots, q_n$). So using our assumptions (and also the structurality and algebraizability of L),

$$\models_{\mathrm{L}} \{E(\chi) \mid \chi \in T(\sigma_1\sigma_2\sigma q_1, \ldots, \sigma_1\sigma_2\sigma q_n)\}$$
$$\text{and} \qquad \models_{\mathrm{L}} \sigma_1\sigma_2\sigma q_i \approx \sigma_1 \psi_i \quad \text{for } i = 1 \ldots n.$$

Hence also by replacement and algebraizability again, $\vdash_{\mathrm{L}} T(\sigma_1\psi_1, \ldots, \sigma_1\psi_n)$. That is, $\vdash_{\mathrm{L}} \sigma'(T)$. Since $\sigma'(T)$ consists of $\mathcal{L}'$-formulas, we obtain $\vdash_{\mathrm{L}\restriction\mathcal{L}'} \sigma'(T)$. So $T \rhd \varphi$ is not passive in $\mathrm{L}\restriction\mathcal{L}'$. □

Example 5.9 As an example of case (c) above, consider an extension L of $\mathrm{FL}_{ew}\restriction\mathcal{L}$ where $\mathcal{L} \supseteq \{\to, \bot\}$. Let us take $\Psi = \{\top, \bot\}$. Then for each $\psi_1, \psi_2 \in \{\bot, \top\}$ and $\# \in \mathcal{L}$, easily $\models_{\mathrm{L}} \psi_1 \# \psi_2 \approx \varphi$ for some $\varphi \in \{\top, \bot\}$. So $\mathrm{L}\restriction\mathcal{L}'$ is a passive fragment of L for any $\{\to, \bot\} \subseteq \mathcal{L}' \subseteq \mathcal{L}$.

5.2 Applications We now use these general methods to settle some cases for our particular class of logics. First, we tackle extensions of the logic SMTL. The crucial aspect here is the fact that the negation $\neg x =_{\mathrm{def}} x \to \bot$ in an SMTL-chain is *strict*. That is, $x \to \bot$ is $\top$ if $x = \bot$ and $\bot$ otherwise. Observe then that $x \to y > \bot$ implies $x = \bot$ or $y > \bot$, and $x \cdot y = \bot$ implies $x = \bot$ or $y = \bot$.

Lemma 5.10 *The logic* SMTL *is $\mathcal{PSC}$.*

Proof Suppose that a rule $T \rhd \varphi$ is not derivable in SMTL. Then there is an SMTL-chain $\mathbf{A}$ and $\mathbf{A}$-evaluation e such that $e(\psi) = \top^{\mathbf{A}}$ for all $\psi \in T$. We define a substitution σ by $\sigma p = \top$ if $e(p) > \bot^{\mathbf{A}}$ and $\sigma p = \bot$ otherwise. To establish the lemma, it is sufficient to prove the following.

Claim $\vdash_{\mathrm{SMTL}} \sigma\psi \leftrightarrow \top$ whenever $e(\psi) > \bot^{\mathbf{A}}$ and $\vdash_{\mathrm{SMTL}} \sigma\psi \leftrightarrow \bot$ whenever $e(\psi) = \bot^{\mathbf{A}}$.

Just recall that $e(\psi) = \top^{\mathbf{A}}$ and so $\vdash_{\mathrm{SMTL}} \sigma\psi$ for all $\psi \in T$. Hence the rule $T \rhd \varphi$ is not passive.

Proof of claim We proceed by induction on the complexity of ψ, recalling that SMTL proves the same variable-free formulas as Classical Logic. The base cases

are immediate. For the induction step, $\wedge$ and $\vee$ are easy, and the other cases are handled as follows.

1. $\psi = \varphi_1 \cdot \varphi_2$. Suppose that $e(\varphi_1 \cdot \varphi_2) = \bot^{\mathbf{A}}$. Then $e(\varphi_i) = \bot^{\mathbf{A}}$ for some $i \in \{1, 2\}$ ($\mathbf{A}$ is an SMTL-chain). Without loss of generality, let $i = 1$. So by the induction hypothesis, $\vdash_{\mathrm{SMTL}} \sigma\varphi_1 \leftrightarrow \bot$. Hence also $\vdash_{\mathrm{SMTL}} \sigma(\varphi_1) \cdot \sigma(\varphi_2) \leftrightarrow \bot \cdot \sigma(\varphi_2)$ and $\vdash_{\mathrm{SMTL}} \sigma(\varphi_1 \cdot \varphi_2) \leftrightarrow \bot$. Now suppose that $e(\varphi_1 \cdot \varphi_2) > \bot^{\mathbf{A}}$. So easily, $e(\varphi_i) > \bot^{\mathbf{A}}$ and by the induction hypothesis, $\vdash_{\mathrm{SMTL}} \sigma\varphi_i \leftrightarrow \top$ for $i \in \{1, 2\}$. Hence also $\vdash_{\mathrm{SMTL}} \sigma(\varphi_1) \cdot \sigma(\varphi_2) \leftrightarrow \top \cdot \top$ and $\vdash_{\mathrm{SMTL}} \sigma(\varphi_1 \cdot \varphi_2) \leftrightarrow \top$.
2. $\psi = \varphi_1 \to \varphi_2$. Suppose that $e(\varphi_1 \to \varphi_2) = \bot^{\mathbf{A}}$. So easily, $e(\varphi_1) > \bot^{\mathbf{A}}$ and $e(\varphi_2) = \bot^{\mathbf{A}}$, and by the induction hypothesis, $\vdash_{\mathrm{SMTL}} \sigma\varphi_1 \leftrightarrow \top$ and $\vdash_{\mathrm{SMTL}} \sigma\varphi_2 \leftrightarrow \bot$. Hence also $\vdash_{\mathrm{SMTL}} (\sigma(\varphi_1) \to \sigma(\varphi_2)) \leftrightarrow (\top \to \bot)$ and $\vdash_{\mathrm{SMTL}} \sigma(\varphi_1 \to \varphi_2) \leftrightarrow \bot$. Now suppose that $e(\varphi_1 \to \varphi_2) > \bot^{\mathbf{A}}$. So $e(\varphi_1) = \bot^{\mathbf{A}}$ or $e(\varphi_2) > \bot^{\mathbf{A}}$ ($\mathbf{A}$ is an SMTL-chain) and by the induction hypothesis, $\vdash_{\mathrm{SMTL}} \sigma\varphi_1 \leftrightarrow \bot$ or $\vdash_{\mathrm{SMTL}} \sigma\varphi_2 \leftrightarrow \top$. Hence also $\vdash_{\mathrm{SMTL}} (\sigma(\varphi_1) \to \sigma(\varphi_2)) \leftrightarrow (\bot \to \sigma(\varphi_2))$ or $\vdash_{\mathrm{SMTL}} (\sigma(\varphi_1) \to \sigma(\varphi_2)) \leftrightarrow (\sigma(\varphi_1) \to \top)$. In either case, $\vdash_{\mathrm{SMTL}} \sigma(\varphi_1 \to \varphi_2) \leftrightarrow \top$ as required. □

As a corollary of this lemma, the $\mathcal{PSC}$ upward and downward, and Example 5.9 we obtain (for fragments without $\bot$ the claim is trivial).

Theorem 5.11 *Let* L *be an extension of* SMTL *and* $\mathcal{L} \supseteq \{\to\}$. *Then* $\mathrm{L}{\restriction}\mathcal{L}$ *is* $\mathcal{PSC}$.

Example 5.12 The logics SMTL, SBL, ΠMTL, and Π (which we already know to be $\mathcal{SC}$) all extend SMTL and are hence $\mathcal{PSC}$.

Finally, we can use our methods to generalize the example of the three-valued Łukasiewicz logic presented at the beginning of this section.

Lemma 5.13 *The n-valued Łukasiwicz logic* $\mathrm{Ł_n}$ *is not* $\mathcal{PSC}$ *for* $3 \leq n \in \mathbb{N}$.

Proof We fix $3 \leq n \in \mathbb{N}$ and consider the rule $p \to^n \bot, \neg p \to p \rhd \bot$. The formula $p \to^n \bot$ is classically equivalent to $p \to \neg p$. So clearly there is no substitution that makes the premises of our rule classical theorems or, indeed, theorems of $\mathrm{Ł_n}$. That is, the rule is passive for $\mathrm{Ł_n}$. But in the n-valued algebra $\mathbf{Ł_n}$, we can evaluate p as -1, and an easy computation shows that the rule is nonderivable in $\mathrm{Ł_n}$. □

Hence by Proposition 5.7 (recall that $\mathrm{Ł_n}$ is term equivalent to $\mathrm{Ł_n}{\restriction}\{\to, \bot\}$) and Example 5.9, we obtain a general result.

Theorem 5.14 *Let* L *be an extension of* FL_{ew} *such that* $\mathrm{Ł_n}$ *is an extension of* L *for some* $3 \leq n \in \mathbb{N}$. *Then* $\mathrm{L}{\restriction}\mathcal{L}$ *is not* $\mathcal{PSC}$ *for any* $\mathcal{L} \supseteq \{\to, \bot\}$.

Example 5.15 In particular, the $\mathcal{L}$-fragments of the following logics for $\mathcal{L} \supseteq \{\to, \bot\}$ and $n \geq 3$ are not $\mathcal{PSC}$ (and hence not $\mathcal{SC}$): FL_{ew}, $\mathrm{C}_n\mathrm{FL}_{ew}$, MTL, C_nMTL, IMTL, BL, C_nBL, Ł.

Theorem 5.14 is formulated to deal specifically with a range of fuzzy logics. To end this section, let us take a slight detour to show how the theorem can be strengthened to obtain an "alternative" proof of [22], Proposition 10.5. In fact, we prove slightly more by showing that the logics are not even $\mathcal{PSC}$ but the core of our proof is the same as in the original. However, to derive the conclusion, we can use our $\mathcal{PSC}$

method rather than appeal to Bergman's theorem (see [4]) relating the $\mathcal{SC}$ of a variety and validity of positive existential sentences in that variety.

Theorem 5.16 *Let* L *be an extension of* FL_{ew} *such that there is a simple* L*-algebra satisfying n-contraction for some* $n \geq 3$ *different from the two-valued Boolean algebra* **2**. *Then* $\mathrm{L}\restriction\mathcal{L}$ *is not* $\mathcal{PSC}$ *for any* $\mathcal{L} \supseteq \{\rightarrow, \bot\}$.

Proof Let **A** be a simple L-algebra satisfying n-contraction different from **2** and let L′ be the logic defined as $\mathrm{L}^{\mathbb{Q}(\{\mathbf{A}\})}$. Then clearly L′ extends L and it is sufficient to show that $\mathrm{L}'\restriction\{\rightarrow, \bot\}$ is not $\mathcal{PSC}$. Inspecting the proof of Lemma 5.13, observe that $p \rightarrow^n \bot, \neg p \rightarrow p \rhd \bot$ is passive in $\mathrm{L}'\restriction\{\rightarrow, \bot\}$ by the same argument. If we show that there is an $x \in A$ such that $x^n = 0$ and $\neg x \leq x$, then $p \rightarrow^n \bot, \neg p \rightarrow p \rhd \bot$ is not derivable in L′ and the proof is done. For this, we refer to the proof of [22], Proposition 10.5. □

6 Summary for Fuzzy Logics

The most interesting cases of our results for fuzzy logics are summarized in Tables 3 and 4, making use of the following key:

$\mathcal{HSC}$	The corresponding fragment is $\mathcal{HSC}$.
$\mathcal{SC}$	The corresponding fragment is $\mathcal{SC}$; whether it is $\mathcal{HSC}$ is unknown.
$\mathcal{PSC}$	The corresponding fragment is $\mathcal{PSC}$; whether it is $\mathcal{SC}$ is unknown.
$\neg\mathcal{PSC}$	The corresponding fragment is not $\mathcal{PSC}$ (hence also not $\mathcal{SC}$).
$\neg\mathcal{SC}$	The corresponding fragment is not $\mathcal{SC}$; whether it is $\mathcal{PSC}$ is unknown.
$\neg\mathcal{HSC}$	The corresponding fragment is not $\mathcal{HSC}$; whether it is $\mathcal{SC}$ is unknown.

Clearly, some of these results are more informative than others. Indeed, since all $\bot$-free fragments of our logics are trivially $\mathcal{PSC}$ (there are no passive rules), we do not mention this fact in Table 3.

Let us identify our sources. The various positive and negative $\mathcal{SC}$ and $\mathcal{HSC}$ results for Ł, CHL, Π, and BL are established in the relevant subsections of Section 3. $\mathcal{HSC}$ for the $\bot$-free fragments of $\mathrm{C_n}$MTL and $\mathrm{C_n}$BL (including $\mathrm{G} = \mathrm{C_2MTL} = \mathrm{C_2BL}$) follows from Theorem 4.7. The negative $\mathcal{HSC}$ results for MTL and $\mathrm{C_n}$MTL in Table 3 follow, respectively, from the fact that the corresponding fragments of Ł are not $\mathcal{SC}$, and Theorem 10.6 of [22]. The negative $\mathcal{PSC}$ results for logics with $\bot$ were established in Example 5.15, while $\mathcal{PSC}$ for extensions of SMTL follows from Theorem 5.11. Finally, $\mathcal{HSC}$ for fragments of G with $\bot$ was obtained in Theorem 4.8.

Of the problems left open, perhaps the most interesting is the case of structural completeness for the positive fragments of MTL. Unlike stronger logics such as BL, there is no (known) easily manipulated set of generating algebras for the corresponding variety. Moreover, although just one example of an admissible but nonderivable rule is needed to disprove structural completeness, finding such examples can also require a deep understanding of the logic in question (as, e.g., for the $\{\cdot, \rightarrow\}$ fragment of BL above). Regarding fragments of logics such as Ł, P, and CHL that are known to be $\mathcal{SC}$ where $\mathcal{HSC}$ is unknown, a better understanding of the relevant subvarieties is required: if we can show that all of these are $\mathcal{SC}$, then the logic is $\mathcal{HSC}$.

Table 3 Fragments with $\rightarrow$ and without $\bot$

Logics	$\rightarrow$	$\rightarrow, \wedge, \vee$	$\rightarrow, \vee$	$\rightarrow, \cdot$	$\rightarrow, \cdot, \wedge, \vee$
MTL = SMTL	?	?	?	$\neg\mathcal{HSC}$	$\neg\mathcal{HSC}$
C_nMTL ($n \geq 3$)	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\neg\mathcal{HSC}$	$\neg\mathcal{HSC}$
BL	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\neg\mathcal{SC}$	$\neg\mathcal{SC}$
C_nBL ($n \geq 3$)	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$
G	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$
Ł	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\neg\mathcal{SC}$	$\neg\mathcal{SC}$
Π	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$
CHL	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{HSC}$	$\mathcal{HSC}$

Table 4 Fragments with $\rightarrow$ and $\bot$

Logics	$\rightarrow, \bot$	$\rightarrow, \wedge, \vee, \bot$	$\rightarrow, \vee, \bot$	$\rightarrow, \cdot, \bot$	$\rightarrow, \cdot, \bot, \wedge, \vee$
MTL, BL, C_nMTL, C_nBL ($n \geq 3$)	$\neg\mathcal{PSC}$	$\neg\mathcal{PSC}$	$\neg\mathcal{PSC}$	$\neg\mathcal{PSC}$	$\neg\mathcal{PSC}$
SMTL	$\mathcal{PSC}$	$\mathcal{PSC}$	$\mathcal{PSC}$	$\mathcal{PSC}$	$\mathcal{PSC}$
G	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$
Ł	$\neg\mathcal{PSC}$	$\neg\mathcal{PSC}$	$\neg\mathcal{PSC}$	$\neg\mathcal{PSC}$	$\neg\mathcal{PSC}$
Π	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{HSC}$	$\mathcal{HSC}$

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