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Growing Commas. A Study of Sequentiality and Concatenation

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Abstract In his paper "Undecidability without arithmetization," Andrzej Grzegorczyk introduces a theory of concatenation TC. We show that pairing is not definable in TC. We determine a reasonable extension of TC that is sequential, that is, has a good sequence coding.

1 Introduction

The supervenience of structured objects on strings of symbols is one of the central facts of human life. It underlies writing and speech. The possibility of this supervenience is based on mathematical facts.

One way to study these mathematical facts is to focus on the true theory of the free monoid in two or more generators. However, such a study does not give us much insight into what principles are precisely essential in deriving certain facts. To do a more refined study, we consider weak systems for strings and concatenation. This allows us to study the fine-structure of reasoning about this supervenience. Weak systems have more models, so we obtain greater generality. In fact, the results of this paper are also valid for certain systems of *decorated linear order types*. The most salient of these is the system of *all* decorated linear order types with at least two elements in the class of decorations.

The focus of this paper is the question *How do finite sets emerge out of strings?*

1.1 The editor axiom and its consequences The central principle of string reasoning we will study is the editor axiom, a.k.a. Tarski's law. This axiom tells us that if a given string s is partitioned into (x, y), that is, $x * y = s$, and into (u, v), that is, $u * v = s$, then there is a common refinement partition (t, w, z), that is, either $t = u$, $t * w = x$, $w * z = v$, $z = y$, or $t = x$, $t * w = u$, $w * z = y$, $z = v$.

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The editor axiom was introduced by Tarski in 1935. The basic theory of the editor axiom considered here is TC, a theory introduced by Grzegorczyk.[1] We can think of this theory as having two classes of "standard models," to wit: the finite strings of at least two letters and the decorated linear order types for classes of letters or colors of at least two elements.[2]

Remark 1.1 In TC we do not have a good notion of *occurrence*. We would like to define an occurrence of v in w as a pair (u, v), where $w = u * v * z$, for some z. However, since TC does not exclude that u is a proper initial segment of itself, we cannot pin down a uniquely determined place in w in this way. A secondary target of this paper is to understand how to reason in the absence of a good notion of occurrence. We will see that it is possible to simulate some of the usual reasoning involving occurrences.

We will show that TC is sufficient for verifying, for an appropriate coding of finite sets, that new elements can be added in a nonambiguous way (adjunction), that is, in such a way that if we form a set $x \cup \{y\}$ no extra unwanted elements slip in. This is done in Section 4. However, these principles are not enough to guarantee the totality of adjunction. We will give a model theoretic argument to show that we cannot even produce a total pairing operation from these principles. See Section 5. We show that addition of a suitable collection principle is sufficient to give us the desired existence axiom.

Let us call a string of as *a tally number*. The collection principle, needed to guarantee that the operation of adjunction is total, says that, for any string x, there is a tally number y such that for any tally number z that is a substring of x, z is strictly contained in y; that is, z is a substring of y, but not vice versa. This collection principle is valid in our "standard models." We show that TC plus the collection principle is interpretable in TC. As a corollary we obtain yet another proof that Robinson's Arithmetic Q is interpretable in TC.

1.2 Sequentiality As outlined in Subsection 1.1, we will show that we can code finite sets in such a way that TC plus a collection principle is able to verify the basic properties of adjunction or "adding an element." One can rephrase this property by saying that there is a *direct interpretation* of Adjunctive Set Theory in TC plus collection. A direct interpretation is an interpretation without domain relativization, where identity is translated as identity. Adjunctive Set Theory is the theory stating the properties of adjunction.

Theories directly interpreting Adjunctive Set Theory are *sequential theories*. Such theories satisfy many important metamathematical properties such as interpreting all their own restricted consistency statements. We refer the reader to the detailed discussion in Subsection 4.1.

1.3 Sets from strings We briefly consider the problem to define arbitrary finite sets of strings from strings. How can we do it? Consider the set $\{w_0, \ldots, w_{n-1}\}$, where the w_i are arbitrary strings. A first attempt would be to consider the string $w_0 \ldots w_{n-1}$, but clearly we cannot retrieve the w_i unambiguously from this string. A second attempt is to choose a separator string or "comma" w^* and represent our set as $w_0 w^* w_1 \ldots w_{n-2} w^* w_{n-1}$. However, this won't work either, since the w_i are arbitrary and nothing prevents w^* from occurring as a substring of some of the w_i. Fortunately, we have one more degree of freedom: we may choose our comma *after* the w_i are given. This is what Quine does in his classical paper [23].[3] Quine's commas are of the form $\mathtt{b}w^*\mathtt{b}$, where w^* is a string of $\mathtt{a}$s longer than any string of $\mathtt{a}$s occurring in the w_i.

The context of Quine's study, however, is the true theory of (at least) binary strings. From the computational point of view, adjunction is somewhat awkward for Quine's coding. Suppose s codes a set and we want to adjoin w. Suppose w contains a string of $\mathtt{a}$s longer than the comma of s. In that case we have to update all the commas in s before we can adjoin w in the obvious way. In the context of weak theories this is a complex operation. For this reason we will employ a variant of Quine's basic idea. We will allow the commas to grow during the construction of the set in such a way that we need not update s in adjoining w. We discuss the ins and outs of the construction in more detail in Subsection 4.2.

We show that TC itself is not sequential: it does not even have a pairing function.

1.4 A brief history In this subsection, we give an admittedly incomplete sketch of the history of research on string theory or theory of concatenation. The story starts around 1935 with Tarski's famous paper [28]. Here he gives a second-order axiomatization of the theory of syntax based on concatenation. See pages 173 and 174 of [30]. Tarski's axiomatization is in a style appropriate for the theory of the free monoid. The editor axiom is the left-to-right direction of Tarski's Axiom 4. (The right-to-left direction is equivalent to associativity of concatenation.) Independently of Tarski, a second-order axiomatization of string theory was given by Hermes in [16]. The axiomatization by Hermes is in the style appropriate for the axiomatization of the free algebra of a number of unary operations (multiple successor arithmetic), over which concatenation is defined by recursion. Corcoran, Frank, and Maloney in their paper [4] show that these two systems are definitionally equivalent (or synonymous).[4] They also show that, for any finite number of generators, these systems are definitionally equivalent with second-order arithmetic.

In 1946, Quine studies, in [23], the true first-order theory of concatenation for a finite alphabet with at least two letters. In this context we do not have the convenient automatic coding of sequences using pregiven sets. Quine shows that true first-order arithmetic can be interpreted in the true first-order theory of concatenation for a finite alphabet with at least two letters in two ways. The first way is to represent the numbers as tally numbers, that is, to represent them as strings of $\mathtt{a}$s for a chosen generator $\mathtt{a}$. This interpretation does not use the whole domain, and thus, is not direct. Quine calls it *unilateral*. Quine also provides a direct interpretation. He speaks of a *bilateral* interpretation.[5] Here the strings are given an ω-ordering using the familiar length-first ordering.[6] Via this ordering we identify strings with numbers.

To define multiplication on tally numbers one only needs to code sequences of tally numbers. This is comparatively easy. However, to define addition and multiplication on length-first numbers, you need sequences of arbitrary strings.[7]

Quine specifies an inverse for his direct interpretation of arithmetic in the length-first numbers. Thus, he shows that the true first-order theory of concatenation (for a fixed number of generators larger or equal than two) is synonymous or definitionally equivalent to true first-order arithmetic.[8]

Quine's basic vocabulary consisted of atoms and concatenation. Variations of this basic repertoire are studied in [31].

Our story goes on in 1953. The focus of interest shifts to the study of weak undecidable theories. According to [29], pp. 86 and 87, Szmielew and Tarski proved that Robinson's Arithmetic Q is interpretable in a weak theory of concatenation F. One consequence of this is the essential undecidability of F. The theory F consists roughly of the semigroup axioms plus axioms telling us that there are two atoms plus the cancellation laws. See Appendix D for the precise axioms. Regrettably no proof was published of the result and it is a bit of a mystery how they did it. The problem is not that we cannot find a proof of the result: in fact, stronger results are known now. The problem is that the known proofs use a method invented by Solovay in 1976. See [25]. This method is so rich in consequences that it is hard to see why Szmielew and Tarski would not have proved much more if they indeed had found this method.

Not really part of our story, but worth mentioning, is the work of Büchi and Senger [1] on existentially definable relations in the true theory of concatenation.

Tarski's program of studying weak theories of concatenation was taken up again by Grzegorczyk around 2005. In his paper [13], he shows that the weak theory of concatenation TC is undecidable. The axioms of TC are given in Section 2. This result is improved in the paper [14] by Grzegorczyk and Zdanowski.

Grzegorczyk's work inspired further research into the interpretability of Q in weak theories of concatenation. Sterken proves in her master's thesis that a weak theory $\mathsf{Q}^{\mathsf{bin}}$ is mutually interpretable with Q.[9] For a definition of $\mathsf{Q}^{\mathsf{bin}}$ see Section 2. Both Ganea (in [12]) and Švejdar (in [26]) prove independently the interpretability of Q in $\mathsf{TC}^{\varepsilon}$, a variant of TC.

Finally, I want to mention a related line of research. In 1986, Buss published [2]. In this book, among other things, a weak theory of arithmetic S^1_2 is developed that is designed for representing the p-time definable functions. In many respects S^1_2 is the natural theory to develop the arithmetization needed for the Second Incompleteness Theorem.[10] Later, two beautiful alternative versions of S^1_2 were given. One is a theory of sets and numbers provided by Zambella. See [40]. The other is a theory of strings developed by Ferreira. See [8] and [9]. We know that Buss's, Zambella's, and Ferreira's theories are mutually interpretable. I am convinced that these three theories are definitionally equivalent. However, this never has been verified in detail. For a careful description of an interpretation of S^1_2 in Ferreira's theory, see [10].

1.5 On pitfalls Errors are everywhere dense in this business. It is very easy to make mistakes in reasoning in the very weak theories. A good example is the following. In the Szmielew & Tarski system F consider the tally numbers. These are the strings of $\mathtt{a}$s.[11] Surely the tally numbers are closed under concatenation?[12] One can show that this need not be so by producing a model where you can concatenate a certain string β_0 of $\mathtt{a}$s with itself, creating ex nihilo a rogue $\mathtt{b}$ in the resulting string. The example is described in Appendix D.

2 Theories of Concatenation

In his paper [13], Grzegorczyk introduces a theory of concatenation TC. Grzegorczyk's theory is in essence an earlier theory due to Tarski plus axioms guaranteeing the existence of at least two letters or atoms. We will call Tarski's theory TC_0.

The theory TC has a binary function symbol $*$ for concatenation and two constants $\mathtt{a}$ and $\mathtt{b}$. The theory is axiomatized as follows.

TC1 $\vdash (x * y) * z = x * (y * z)$

TC2 $\vdash x * y = u * v \to ((x = u \wedge y = v) \vee$
$\exists w\ ((x * w = u \wedge y = w * v) \vee (x = u * w \wedge y * w = v)))$

TC3 $\vdash x * y \neq \mathsf{a}$

TC4 $\vdash x * y \neq \mathsf{b}$

TC5 $\vdash \mathsf{a} \neq \mathsf{b}$

Grzegorczyk calls axiom TC2 "the editor axiom." Tarski's theory TC_0 has only concatenation in its signature and is axiomatized by TC1 and TC2.

Grzegorczyk and Zdanowski have shown that TC is essentially undecidable. See [14]. This result can be strengthened by showing that Robinson's Arithmetic Q is mutually interpretable with TC. See below. Note that TC_0 is undecidable—since it has an extension that parametrically interprets TC—but that TC_0 is not essentially undecidable: it is satisfied by a one-point model. It also has an extension that is a definitional extension of the theory of pure identity.

The theories TC and TC_0 are theories for concatenation without the empty string, in other words, without the unit element ε. We find it more convenient to work in a theory with unit. Our variant TC^ε of TC with empty string added looks as follows.

$\mathsf{TC}^\varepsilon 1$ $\vdash \varepsilon * x = x \wedge x * \varepsilon = x$

$\mathsf{TC}^\varepsilon 2$ $\vdash (x * y) * z = x * (y * z)$

$\mathsf{TC}^\varepsilon 3$ $\vdash x * y = u * v \to \exists w\ ((x * w = u \wedge y = w * v) \vee (x = u * w \wedge y * w = v))$

$\mathsf{TC}^\varepsilon 4$ $\vdash \mathsf{a} \neq \varepsilon$

$\mathsf{TC}^\varepsilon 5$ $\vdash x * y = \mathsf{a} \to (x = \varepsilon \vee y = \varepsilon)$

$\mathsf{TC}^\varepsilon 6$ $\vdash \mathsf{b} \neq \varepsilon$

$\mathsf{TC}^\varepsilon 7$ $\vdash x * y = \mathsf{b} \to (x = \varepsilon \vee y = \varepsilon)$

$\mathsf{TC}^\varepsilon 8$ $\vdash \mathsf{a} \neq \mathsf{b}$

The theories TC and TC^ε are bi-interpretable. See Appendix A.[13] We will also consider the theory $\mathsf{TC}_0^\varepsilon$, axiomatized by $\mathsf{TC}^\varepsilon 1, 2, 3$.

There is a somewhat different salient theory of concatenation which is in many respects a direct analogue of Robinson's Arithmetic Q. We call this theory $\mathsf{Q}^{\mathsf{bin}}$. The axioms of this theory are as follows.

$\mathsf{Q}^{\mathsf{bin}}1$ $\vdash \mathsf{S}_{\mathsf{a}} x \neq \varepsilon,$

$\mathsf{Q}^{\mathsf{bin}}2$ $\vdash \mathsf{S}_{\mathsf{b}} x \neq \varepsilon,$

$\mathsf{Q}^{\mathsf{bin}}3$ $\vdash \mathsf{S}_{\mathsf{a}} x \neq \mathsf{S}_{\mathsf{b}} y,$

$\mathsf{Q}^{\mathsf{bin}}4$ $\vdash \mathsf{S}_{\mathsf{a}} x = \mathsf{S}_{\mathsf{a}} y \to x = y,$

$\mathsf{Q}^{\mathsf{bin}}5$ $\vdash \mathsf{S}_{\mathsf{b}} x = \mathsf{S}_{\mathsf{b}} y \to x = y,$

$\mathsf{Q}^{\mathsf{bin}}6$ $\vdash x * \varepsilon = x,$

$\mathsf{Q}^{\mathsf{bin}}7$ $\vdash x * \mathsf{S}_{\mathsf{a}} y = \mathsf{S}_{\mathsf{a}}(x * y),$

$\mathsf{Q}^{\mathsf{bin}}8$ $\vdash x * \mathsf{S}_{\mathsf{b}} y = \mathsf{S}_{\mathsf{b}}(x * y),$

$\mathsf{Q}^{\mathsf{bin}}9$ $\vdash x = \varepsilon \vee \exists y\ (x = \mathsf{S}_{\mathsf{a}} y \vee x = \mathsf{S}_{\mathsf{b}} y).$

In Appendix B, we show that $\mathsf{Q}^{\mathsf{bin}}$ and TC^ε are mutually interpretable.

3 Basics of $\mathsf{TC}^{\varepsilon}$

In this section we provide some basic facts concerning $\mathsf{TC}^{\varepsilon}$. We define

$$\begin{aligned}
\mathsf{atom}(x) &:\leftrightarrow x \neq \varepsilon \wedge \forall y, z\ (y * z = x \to (x = \varepsilon \vee y = \varepsilon)),\\
x \subseteq y &:\leftrightarrow \exists u, v\ y = u * x * v,\\
x \subset y &:\leftrightarrow \exists u, v\ (y = u * x * v \wedge (u \neq \varepsilon \vee v \neq \varepsilon)),\\
x \subset^{+} y &:\leftrightarrow x \subseteq y \wedge \neg\, y \subseteq x,\\
x \subseteq_{\mathsf{ini}} y &:\leftrightarrow \exists v\ x * v = y,\\
x \subseteq_{\mathsf{end}} y &:\leftrightarrow \exists u\ u * x = y,\\
y : \mathsf{N}_x &:\leftrightarrow \forall z {\subseteq} y\ (x \subseteq z \vee z = \varepsilon).
\end{aligned}$$

We will call a y in N_x an x-string. If $\prec$ is one of our preorderings, then we define

$$y \prec_x z :\leftrightarrow \forall u : \mathsf{N}_x\ (u \subseteq y \to \exists v {\subseteq} z\ u \prec v).$$

Fact 3.1 The theory $\mathsf{TC}^{\varepsilon}$ proves the following facts.

1. (a) $(\mathsf{atom}(x) \wedge \mathsf{atom}(y) \wedge u * x = v * y) \to (u = v \wedge x = y)$.
   (b) $(\mathsf{atom}(x) \wedge \mathsf{atom}(y) \wedge x * u = y * v) \to (u = v \wedge x = y)$.
2. Suppose $\mathsf{atom}(x)$ and $u * v = x * w$. Then, either $u = \varepsilon$ or there is a u_0 such that $u = x * u_0$ and $u_0 * v = w$. Similarly for $u * v = w * x$.

Proof We reason in $\mathsf{TC}^{\varepsilon}$. Suppose x is an atom.

Ad 1: We treat (a). Suppose $\mathsf{atom}(y)$ and $u * x = v * y$. Then, by the editor axiom, we have a w such that (i) $u * w = v$ and $x = w * y$ or (ii) $u = v * w$ and $w * x = y$. In case (i), we have $w = \varepsilon$ and hence $u = v$ and $x = y$. Case (ii) is similar. Item (b) is similar.

Ad 2: Suppose $u * v = x * w$. By the editor axiom, there is a z such that (i) $(u * z = x$ and $v = z * w)$ or (ii) $(u = x * z$ and $z * v = w)$.

In case (i), either (i1) $u = \varepsilon$—and we are done—or (i2) $z = \varepsilon$ and $u = x$. In case (i2), we have $u = x$ and $v = w$. So we can take $u_0 := \varepsilon$. In case (ii), we can take $u_0 := z$. □

Fact 3.2 We have in $\mathsf{TC}^{\varepsilon}$ the following facts.

1. The relation $\subseteq_{\mathsf{ini}}$ is a weak partial preordering with minimal element ε. The atoms in our sense are also atoms of this preordering.[14] Our preordering is linear when restricted to the initial substrings of an element x.
2. The relation $\subseteq_{\mathsf{end}}$ is a weak partial preordering with minimal element ε. The atoms in our sense are also atoms of this weak preordering. Our preordering is linear when restricted to the final substrings of x.
3. The relation $\subseteq$ is a partial preordering on the substrings of x with minimal element ε. The atoms in our sense are precisely the atoms of the preordering.
4. $x \subseteq y * z \to (x \subseteq y \vee x \subseteq z \vee \exists x_0, x_1\ (x = x_0 * x_1 \wedge x_0 \subseteq_{\mathsf{end}} y \wedge x_1 \subseteq_{\mathsf{ini}} z))$.
5. The relation $\subset$ is a partial preordering. The relation $\subset^{+}$ is a strong ordering. We have $x \subset^{+} y \to x \subset y$.[15]

Proof We reason in $\mathsf{TC}^{\varepsilon}$. We only treat (4). Suppose $x \subseteq y * z$. So, for some u, v, we have $u * x * v = y * z$. By the editor axiom, there is a w such that (a) $u * w = y$ and $x * v = w * z$ or (b) $u = y * w$ and $w * x * v = z$. In case (b), we have $x \subseteq z$

and we are done. We treat case (a). By the editor axiom, we have an r such that (a1) $x * r = w$ and $v = r * z$ or (a2) $x = w * r$ and $r * v = z$. In case (a1) we have $x \subseteq w \subseteq y$, so $x \subseteq y$, and we are done. In case (a2), we take $x_0 := w$ and $x_1 := r$. □

Definition Let $I(x)$ be a formula. We treat $\{x \mid I(x)\}$ as a virtual class. Par abus de langage, we write I for $\{x \mid I(x)\}$. We take $\mathsf{DC}(I)(x) :\leftrightarrow \forall y{\subseteq}x\ I(y)$.

Fact 3.3 In $\mathsf{TC}^{\varepsilon}$, we have the following. Suppose (the virtual class) I is closed under concatenation. Let $J := \mathsf{DC}(I)$. Then, J is closed under concatenation and downward closed under $\subseteq$.

Proof Reason in $\mathsf{TC}^{\varepsilon}$. Suppose I is closed under concatenation. Let $J := \mathsf{DC}(I)$. Clearly, J is downward closed under $\subseteq$. Suppose x_0 and x_1 are in J. To show $x_0 * x_1$ is in J, suppose $y \subseteq x_0 * x_1$. By Fact 3.2(4), we have that either (a) $y \subseteq x_0$ or (b) $y \subseteq x_1$ or (c) for some y_0, y_1, $y = y_0 * y_1$ and $y_0 \subseteq_{\mathsf{end}} x_0$ and $y_1 \subseteq_{\mathsf{ini}} x_1$. In cases (a) and (b), we immediately have that y is in I. In case (c), we find that y_0 is in I and y_1 is in I. Hence, by the closure of I under concatenation, y is in I. □

Fact 3.4 We have in $\mathsf{TC}^{\varepsilon}$ the following facts.

1. N_x is closed under ε and concatenation. Moreover, it is downward closed under taking substrings.
2. N_x is nontrivial, that is, not equal to $\{\varepsilon\}$, if and only if it contains x. If x is an atom, then N_x is nontrivial. (In case $x = \varepsilon$, then N_x consists of all strings, and is, ipso facto, nontrivial.)

Proof We only treat the case that N_x is closed under concatenation. Let

$$I_x(y) :\leftrightarrow y = \varepsilon \vee x \subseteq y.$$

Clearly, I_x is closed under concatenation and $\mathsf{N}_x = \mathsf{DC}(I_x)$. The desired result now follows from Fact 3.3. □

Our final fact follows an idea of Pudlák. Consider any model of $\mathsf{TC}^{\varepsilon}_0$. Fix an element w. We call a sequence $(w_0, \ldots, w_k)$ a *partition* of w if we have that $w_0 * \cdots * w_k = w$. The partitions of w form a category with the following morphisms. $f : (u_0, \ldots, u_n) \to (w_0, \ldots, w_k)$ if and only if f is a surjective and weakly monotonic function from $n + 1$ to $k + 1$ such that, for any $i \leq k$, $w_i = u_s * \cdots * u_\ell$, where $\{j \mid f(j) = i\} = \{j \mid s \leq j \leq \ell\}$. We write $(u_0, \ldots, u_n) \leq (w_0, \ldots, w_k)$ for $\exists f\ f : (u_0, \ldots, u_n) \to (w_0, \ldots, w_k)$. In this case, we say that $(u_0, \ldots, u_n)$ is *a refinement* of $(w_0, \ldots, w_k)$.

Fact 3.5 We work in any model of $\mathsf{TC}^{\varepsilon}_0$. Consider a w in the model. Then any two partitions of w have a common refinement.

Proof Fix any model of $\mathsf{TC}^{\varepsilon}_0$. We first prove that, for all w, all pairs of partitions $(u_0, \ldots, u_n)$ and $(w_0, \ldots, w_k)$ of w have a common refinement, by induction of $n + k$.

If either n or k is zero, this is trivial. Suppose that $(u_0, \ldots, u_{n+1})$ and $(w_0, \ldots, w_{k+1})$ are partitions of w. We have, by the editor axiom, that there is a v such that (a) $u_0 * \cdots * u_n * v = w_0 * \cdots * w_k$ and $u_{n+1} = v * w_{k+1}$ or (b) $u_0 * \cdots * u_n = w_0 * \cdots * w_k * v$ and $v * u_{n+1} = w_{k+1}$. By symmetry, we only need to treat case (a). By the induction hypothesis, there is a common refinement $(x_0, \ldots x_m)$

of $(u_0, \ldots, u_n, v)$ and $(w_0, \ldots, w_k)$. Let this be witnessed by f, respectively, g. It is easily seen that $(x_0, \ldots x_m, w_{k+1})$ is the desired refinement with witnessing functions f' and g', where $f' := f[m+1 : n+1]$, $g' := g[m+1 : k+1]$. □

Note that the length of the common refinement produced by our proof is the sum of the lengths of our original partitions minus one.

We will use refinements to simulate the presence of occurrences. Instead of working with occurrences in an absolute sense, we will treat them as places in a sufficiently fine refinement.

4 TC^ε and Sequentiality

In this section, we introduce the notion of sequentiality and give an extension of TC^ε that is extensional.

4.1 What is sequentiality? Adjunctive Set Theory AS is the theory in the language with $\in$ and $=$, which is axiomatized as follows.

AS1 $\vdash \exists x\ \forall y\ y \notin x$ (empty set axiom)

AS2 $\vdash \forall u, v\ \exists x\ \forall y\ (y \in x \leftrightarrow (y \in u \vee y = v))$ (adjunction axiom)

A theory is *sequential* if and only if it directly interprets adjunctive set theory AS. Direct interpretability means interpretability without relativization of quantifiers, that sends identity to identity. Said differently, a theory is sequential if we can define a predicate $\in$ provably satisfying the axioms of AS.

Remark 4.1 The notion of sequential theory was introduced by Pudlák in his paper [21]. Pudlák uses his notion for the study of the degrees of local multi-dimensional parametric interpretability. He proves that sequential theories are prime in this degree structure. In [22], sequential theories provide the right level of generality for theorems about consistency statements.

The notion of sequential theory was independently invented by Friedman who called it *adequate theory*. See Smoryński's survey [24].[16] Friedman uses the notion to provide the Friedman characterization of interpretability among finitely axiomatized sequential theories. (See also [33] and [34].) Moreover, he shows that ordinary interpretability and faithful interpretability among finitely axiomatized sequential theories coincide. (See also [35] and [37].)

Adjunctive Set Theory is mutually interpretable with Q. For the interpretability of AS in Q, see, for example, [20] or [15]. Here is the story of the interpretability of Q in AS in a nutshell.

1. In [27], Szmielew and Tarski announce the interpretability of Q in AS plus extensionality. See also [29], p. 34.
2. A new proof of the Szmielew-Tarski result is given by Collins and Halpern in [3].
3. Montagna and Mancini, in [18], give an improvement of the Szmielew-Tarski result. They prove that Q can be interpreted in an extension of AS in which we stipulate the functionality of empty set and adjunction of singletons.
4. In Appendix III of [19], Mycielski, Pudlák, and Stern provide the ingredients of the interpretation of Q in AS.

In a forthcoming paper we will provide another proof of the interpretability of Q in AS. For further work concerning sequential theories, see, for example, [22], [24], [19], [15], [35], [36], [17], [37], [39].

4.2 Sequentiality from concatenation We pick up the discussion of Subsection 1.3 and, in part, repeat it in more detail. Our problem is to define appropriate commas to code finite sets. If we had a fresh letter, we would be done. However, our rules dictate that we must be able to make sets of all possible strings—so we have no extra letter available. One idea to create commas would be to employ a tally length function.

Remark 4.2 Consider the following list of properties: $\Lambda_{\mathsf{a}}\varepsilon := \varepsilon$, $\Lambda_{\mathsf{a}}x := \mathsf{a}$, if x is an atom, $\Lambda_{\mathsf{a}}(x * y) := \Lambda_{\mathsf{a}}x * \Lambda_{\mathsf{a}}y$. A function that satisfies these properties is called a *tally length function*.[17]

In the model of finite strings, the tally length function is uniquely determined and has a very efficient computation on the two tape Turing machine. In the model of decorated linear order types, we also have a tally length function. (I don't know whether it is necessarily unique.) Thus, it seems to me a very reasonable function to add as a primitive.[18]

If we extend our language with a tally length function, there are several possibilities to define sets. For example, we could create room for a comma by replacing the w_i by the result of doubling each atom in w_i. Define

(i) $u \equiv_{\mathsf{a}} v :\leftrightarrow \Lambda_{\mathsf{a}}u = \Lambda_{\mathsf{a}}v$,

(ii) $\mathsf{dubb}(w, \tilde{w}) :\leftrightarrow \forall u, x, v\ ((w = u * x * v \wedge \mathsf{atom}(x)) \rightarrow$
$\exists \tilde{u}, \tilde{v}\ (\tilde{u} \equiv_{\mathsf{a}} u*u \wedge \tilde{v} \equiv_{\mathsf{a}} v*v \wedge \tilde{w} = \tilde{u}*x*x*\tilde{v}))$.

Now we can represent $\{w_0, \ldots, w_{n-1}\}$ by

$$\tilde{w}_0 * \mathsf{a} * \mathsf{b} * \tilde{w}_1 * \cdots * \mathsf{a} * \mathsf{b} * \tilde{w}_{n-1},$$

where $\mathsf{dubb}(w_i, \tilde{w}_i)$, for each i. A second idea is to represent the set $\{w_0, \ldots, w_{n-1}\}$ by

$$w_0 * \mathsf{a} * w_1 * \cdots \mathsf{a} * w_{n-1} * w_0 * \mathsf{b} * w_1 * \cdots * \mathsf{b} * w_{n-1}.$$

To retrieve the w_i, we clearly need a relation like $\equiv_{\mathsf{a}}$. A third idea is to represent $\{w_0, \ldots, w_{n-1}\}$ by

$$\Lambda_{\mathsf{a}}w_0 * \mathsf{a} * \mathsf{b} * \Lambda_{\mathsf{a}}w_1 * \cdots * \mathsf{a} * \mathsf{b} * \Lambda_{\mathsf{a}}w_{n-1} * \mathsf{a} * \mathsf{b} * \mathsf{b} *$$
$$w_0 * \mathsf{a} * \mathsf{b} * w_1 * \cdots * \mathsf{a} * \mathsf{b} * w_{n-1}.$$

To make any of these ideas work we will need additional axioms over $\mathsf{TC}^{\varepsilon}$ plus the tally axioms.

In our treatment we will not use a tally length function. For one thing, it is nicer, of course, to avoid expanding the signature. More seriously, it seems to me that each of the ideas involving the tally length function involve the notion of occurrence of a substring, which we do not have in $\mathsf{TC}^{\varepsilon}$. Can we avoid this presupposition? We remind the reader of Quine's way of representing sets. See [23]. He represents $\{w_0, \ldots, w_{n-1}\}$ by

$$w_0 * \mathsf{b} * u * \mathsf{b} * w_1 * \cdots * \mathsf{b} * u * \mathsf{b} * w_{n-1}.$$

Here u is an a-string strictly longer than all a-strings that are substrings of the w_i. This idea works perfectly in the context of a sufficiently strong theory, but it has the following disadvantage. If we want to adjoin an element to a set, we may have to update all commas in the given representation. This is a complex operation. It is also a costly operation in terms of growth. Suppose we have only $\mathtt{a}$ and $\mathtt{b}$ in our alphabet. Consider the Quine representation of the set of all strings of length n. This will have as length approximately 2^n. Now we adjoin a string of $\mathtt{a}$s of length 2^n. The result of adjunction will have length approximately 2^{2n}. So adjunction will give rise to multiplication of lengths and thus grow faster than concatenation. We employ a variant of Quine's idea. We represent $\{w_0, \ldots, w_{n-1}\}$ by

$$\mathtt{b} * u_0 * \mathtt{b} * w_0 * \mathtt{b} * u_1 * \mathtt{b} * w_1 * \cdots \mathtt{b} \ * u_{n-1} * \mathtt{b} * w_{n-1},$$

where the u_i are $\mathtt{a}$-strings and $u_i \subseteq u_j$, if $i \leq j$. We demand that $w_i \subset^+_{\mathtt{a}} u_i$. This idea is derived from some lecture notes by Visser, de Moor, and Walsteijn of 1986, to wit [32]. Here is the formal realization. We define the following.

1. (u', u) is a *comma*, if u is an $\mathtt{a}$-string and $u' \subseteq_{\mathtt{a}} u$.
2. $x \in y$ if (i) there are commas (u', u) and (v', v) such that $(u', \mathtt{b}, u, \mathtt{b}, x)$ is a partition of v' and $(v', \mathtt{b}, v, \mathtt{b}, w)$ is a partition of y, for some w, and $x \subset^+_{\mathtt{a}} u$ or (ii) there is a comma (u', u) such that $(u', \mathtt{b}, u, \mathtt{b}, x)$ is a partition of y and $x \subset^+_{\mathtt{a}} u$.
3. $\varnothing := \varepsilon$.
4. $\mathsf{adj}(x, y, z)$ if and only if, for some c, (x, c) is a comma and $(x, \mathtt{b}, c, \mathtt{b}, y)$ is a partition of z, and $y \subset^+_{\mathtt{a}} c$.

   $\mathsf{adj}(x, y, z)$ stands for adjunction; that is, '$x \cup \{y\} = z$', without commitment to either the existence or the uniqueness of z.

4.3 Correctness of the definitions In this subsection, we show the correctness of the joint definitions of $\in$ and adj, in TC^ε.[19]

Theorem 4.3 *We have* $\mathsf{TC}^\varepsilon \vdash \mathsf{adj}(x, y, z) \to \forall w\ (w \in z \leftrightarrow (w \in x \vee w = y))$.

Proof We reason in any model of TC^ε. Suppose $\mathsf{adj}(x, y, z)$. This means that $\sigma := (x, \mathtt{b}, c, \mathtt{b}, y)$ is a partition of z, where (x, c) is a comma and $y \subset^+_{\mathtt{a}} c$.

Suppose $w \in x$. If w is in x by the first disjunct of the definition of $\in$, then, trivially, w is in z. Suppose w is in x by the second disjunct. So there is a comma (d', d) such that $(d', \mathtt{b}, d, \mathtt{b}, w)$ is a partition of x and $w \subset^+_{\mathtt{a}} d$. Since also (x, c) is a comma, we find that $w \in z$, by the first clause of the definition of $\in$.

Clearly, $y \in z$ by the second clause of the definition of $\in$.

Suppose that $w \in z$. First, we consider the case that this is true by the first clause of the definition of $\in$. So, we have commas (d', d), (e', e) such that $(d', \mathtt{b}, d, \mathtt{b}, w)$ is a partition of e' and $\tau := (d', \mathtt{b}, d, \mathtt{b}, w, \mathtt{b}, e, u)$ is a partition of z, for some u, and $w \subset^+_{\mathtt{a}} d$.

Let $\zeta := (z_0, \ldots, z_m)$ be a common partition of $\sigma = (x, \mathtt{b}, c, \mathtt{b}, y)$ and $\tau = (d', \mathtt{b}, d, \mathtt{b}, w, \mathtt{b}, e, u)$. Let f and g be the witnessing morphisms.

It will be pleasant to have a name, for example, for $\mathtt{b}$-as-occurring-at-place-3-in-σ. We will call this item $(\sigma, 3)$. Similarly, for other strings-as-occuring-at-places-in-a-partition.

It is easily seen that there is a unique i such that $z_i = \mathtt{b}$ and $f(i) = 3$. (There may be other i' with $f(i') = 3$, but, for such i', we must have $z_{i'} = \varepsilon$.) This i is the place of $(\sigma, 3)$ relative to the context (ζ, f, g). Note that the number i is just dependent on

ζ and f. However, par abus de langage, we will write it as 3_σ.[20] Similarly, we can define 5_τ as the unique j such that $g(j) = 5$ and $z_j = \mathtt{b}$. We distinguish a number of cases.

Case 1 Suppose $3_\sigma \leq 5_\tau$. It follows that $e \subseteq y$ and $c \subseteq e'$. Hence, $e \subset_{\mathtt{a}} y \subset^+_{\mathtt{a}} c \subseteq_{\mathtt{a}} e$. It follows that $e \subset^+ e$. Quod impossibile.

Case 2 Suppose that $1_\sigma < 5_\tau < 3_\sigma$. Since 2_σ is an occurrence of the $\mathtt{a}$-string c, we get a contradiction.

Case 3 Suppose $5_\tau = 1_\sigma$. In this case, we have $(d', \mathtt{b}, d, \mathtt{b}, w)$ is a partition of x, and (d', d) is a comma and $w \subset^+_{\mathtt{a}} d$. So $w \in x$, by the second clause of the definition of $\in$.

Case 4 Suppose $5_\tau < 1_\sigma$. We have $f(5_\tau) = 0$. Clearly, every j such that $g(j) = 5$ must be $< 1_\sigma$. But then the j such that $g(j) = 6$ must also be $< 0_\sigma$, since τ_6 is the $\mathtt{a}$-string e. It follows that, for some v, $(d', \mathtt{b}, d, \mathtt{b}, w, \mathtt{b}, e, v)$ is a partition of x. Moreover, (d', d) and (e', e) are commas and $w \subset^+_{\mathtt{a}} d$. So $w \in x$, by the first clause of the definition of $\in$.

Next we suppose that $x \in z$ by the second clause of the definition of $\in$. So, we have, for some comma (d', d), that $\nu := (d', \mathtt{b}, d, \mathtt{b}, w)$ is a partition of z and $w \subset_{\mathtt{a}} d$. Let $\zeta = (z_0, \ldots, z_m)$ be a common refinement of $\sigma = (x, \mathtt{b}, c, \mathtt{b}, y)$ and $\nu = (d', \mathtt{b}, d, \mathtt{b}, w)$. Let f and g be witnessing functions.

Suppose $3_\sigma < 3_\tau$. In this case we may show, reasoning as in Case 4 above, that $d \subseteq y$ and $c \subseteq d'$. So, $c \subseteq_{\mathtt{a}} d' \subseteq_{\mathtt{a}} d \subseteq_{\mathtt{a}} y \subset^+_{\mathtt{a}} c$. It follows that $c \subset^+ c$. A contradiction. Similarly, we may refute the supposition that $3_\sigma > 3_\tau$. We may conclude that $3_\sigma = 3_\tau$ and, thus, that $w = y$. □

4.4 Existence of adjuncts What we need to get existence is obviously something like a suitable collection principle. We define $\mathtt{a}$-collection as follows:

$\mathtt{a}$-**coll:** $\vdash \forall x\, \exists y : \mathsf{N}_{\mathtt{a}}\, x \subseteq_{\mathtt{a}} y$.

Note however that this not enough. We need our $\mathtt{a}$-string y strictly above the $\mathtt{a}$-strings of x. So we need strong $\mathtt{a}$-collection.

$\mathtt{a}$-**coll$^+$:** $\vdash \forall x\, \exists y : \mathsf{N}_{\mathtt{a}}\, x \subset^+_{\mathtt{a}} y$.

It is immediate that $\mathsf{TC}^\varepsilon + \mathtt{a}\text{-}\mathsf{coll}^+$ proves the existence clause of our adjunction axiom. Since the empty set axiom is trivial, we find $\mathsf{TC}^\varepsilon + \mathtt{a}\text{-}\mathsf{coll}^+$ is sequential.

We will now show that TC^ε interprets $\mathsf{TC}^\varepsilon + \mathtt{a}\text{-}\mathsf{coll}^+$. We will do this by first interpreting the strictness axiom (defined below) and then interpreting $\mathtt{a}$-coll. Here is the strictness axiom.

strict: $\vdash \forall u\; u \not\subset u$.

It is easily seen that strictness plus $\mathtt{a}$-coll implies $\mathtt{a}$-coll$^+$: if we have $x \subseteq_{\mathtt{a}} y$, then we have $x \subset^+_{\mathtt{a}} y * \mathtt{a}$. Note that we interpret more than necessary. In the decorated order types strictness fails, but we do have strong $\mathtt{a}$-collection.

Theorem 4.4 *The theory* TC^ε *interprets* $\mathsf{TC}^\varepsilon + \mathsf{strict}$ *on an initial segment.*

Proof Consider $I(u) :\leftrightarrow u \not\subset u$. We show that I is closed under concatenation. Suppose u_0 and u_1 are in I. Suppose, for some v_0, v_1, we have $v_0 * u_0 * u_1 * v_1 = u_0 * u_1$. By the editor axiom, there is a w such that (1) $v_0 * u_0 * w = u_0$ and $u_1 * v_1 = w * u_1$ or (2) $v_0 * u_0 = u_0 * w$ and $w * u_1 * v_1 = u_1$. Suppose we are in case (1). Since u_0 is in I, we find that $v_0 = w = \varepsilon$. It follows that $u_1 * v_1 = u_1$, and, hence, since u_1 is in I, that $v_1 = \varepsilon$.

By Fact 3.3, $J := \mathsf{DC}(I)$ is closed under concatenation and downward closed under substrings. Also clearly, J contains ε, $\mathtt{a}$, and $\mathtt{b}$. Noting that $\forall u\ u \not\subset u$ is universal, we find that relativization to J interprets $\mathsf{TC}^{\varepsilon} + \mathsf{strict}$. $\square$

Theorem 4.5 *We can interpret* $\mathsf{TC}^{\varepsilon} + \mathtt{a}\text{-}\mathsf{coll}$ *in* $\mathsf{TC}^{\varepsilon}$ *on an initial segment.*

Proof We work in $\mathsf{TC}^{\varepsilon}$. We first form a predicate $\mathsf{N}^{\star}_{\mathtt{a}}(x)$ such that (i) $\mathsf{N}^{\star}_{\mathtt{a}}$ is a subclass of $\mathsf{N}_{\mathtt{a}}$, (ii) $\mathsf{N}^{\star}_{\mathtt{a}}$ is closed under ε, $\mathtt{a}$, and concatenation, (iii) $\mathsf{N}^{\star}_{\mathtt{a}}$ is downward closed under $\subseteq$, and (iv) $\mathsf{N}^{\star}_{\mathtt{a}}$ satisfies 'for any x and y in $\mathsf{N}^{\star}_{\mathtt{a}}$, we have $x * y = y * x$'.

Let $I(x) :\leftrightarrow \mathtt{a} * x = x * \mathtt{a}$. Let $J(x) :\leftrightarrow \forall y{:}I\ x * y = y * x$. Since $\mathtt{a}$ is in I, we find that J is a subclass of I. It is easily seen that J is closed under concatenation. It follows that $K := \mathsf{DC}(J)$ is closed under concatenation and is downward closed under taking substrings. Clearly, K contains ε and $\mathtt{a}$. We take $\mathsf{N}^{\star}_{\mathtt{a}} := \mathsf{N}_{\mathtt{a}} \cap K$.

Suppose that, for $i = 0, 1$, we have $x_i \subseteq_{\mathtt{a}} y_i$ and $y_i : \mathsf{N}^{\star}_{\mathtt{a}}$. We show that $x_0 * x_1 \subseteq_{\mathtt{a}} y_0 * y_1$. Let $z : \mathsf{N}_{\mathtt{a}}$ be a substring of $x_0 * x_1$. We want to show that z is a substring of $y_0 * y_1$. We have either (1) $z \subseteq x_0$ or (2) $z \subseteq x_1$ or (3) for some z_0, z_1, $z = z_0 * z_1$, $z_0 \subseteq_{\mathsf{end}} x_0$, and $z_1 \subseteq_{\mathsf{ini}} x_1$. In cases (1) and (2), we are immediately done. We treat case (3). We find that $z_0 \subseteq y_0$ and $z_1 \subseteq y_1$. We have, for certain v_{ij}, $y_i = v_{i0} * z_i * v_{i1}$. Since $v_{ij} \subseteq y_i$ and $z_i \subseteq y_i$, we have that v_{ij} and z_i are in $\mathsf{N}^{\star}_{\mathtt{a}}$. Hence,

$$y_0 * y_1 = v_{00} * z_0 * v_{01} * v_{10} * z_1 * v_{11} = v_{00} * v_{01} * z_0 * z_1 * v_{10} * v_{11}.$$

So $z \subseteq y_0 * y_1$.

Let $L(x) :\leftrightarrow \exists y{:}\mathsf{N}^{\star}_{\mathtt{a}}\ x \subseteq_{\mathtt{a}} y$. Clearly, L is downward closed under substrings. By the above, L is closed under concatenation. It is easily seen that ε, $\mathtt{a}$, and $\mathtt{b}$ are in L. Finally, trivially, $\mathsf{N}^{\star}_{\mathtt{a}}$ is contained in L. Thus, restriction to L interprets $\mathsf{TC}^{\varepsilon} + \mathtt{a}\text{-}\mathsf{coll}$. $\square$

Theorem 4.6 $\mathsf{TC}^{\varepsilon}$ *interprets* $\mathsf{TC}^{\varepsilon} + \mathtt{a}\text{-}\mathsf{coll}^{+}$.

Proof First interpret $\mathsf{TC}^{\varepsilon} + \mathsf{strict}$ in $\mathsf{TC}^{\varepsilon}$. Then relativize to the class L of the previous theorem to obtain an interpretation of $\mathsf{TC}^{\varepsilon} + \mathtt{a}\text{-}\mathsf{coll}$. Since strict can be written in purely universal form as $\vdash \forall u, v, w\ (u = w * u * v \to (u = \varepsilon \wedge v = \varepsilon))$, we find that we will inherit strict in our interpretation. Finally, strict plus $\mathtt{a}\text{-}\mathsf{coll}$ implies $\mathtt{a}\text{-}\mathsf{coll}^{+}$. $\square$

We did not explore the tally length representations in the context of $\mathsf{TC}^{\varepsilon}$ and its extensions. It is very well possible that we can make a tally length representation work $\mathsf{TC}^{\varepsilon}$ plus axioms that are incomparable to strong $\mathtt{a}$-collection. If that is true, the tally representations and the growing commas representation would both have relative advantages and disadvantages.

5 $\mathsf{TC}^{\varepsilon}$ Does Not Have Pairing

We prove that $\mathsf{TC}^{\varepsilon}$ does not prove pairing by producing a model with "too many automorphisms." We first define what it is for a theory to have pairing. Let PAIR be the following theory.

PAIR 1 $\vdash (\mathsf{pair}(x, y, z) \wedge \mathsf{pair}(x', y', z)) \rightarrow (x = x' \wedge y = y')$

PAIR 2 $\vdash \forall x, y\, \exists z\ \mathsf{pair}(x, y, z)$

A theory has pairing if it directly interprets PAIR. In other words, if there we can define a predicate pair in the language of the theory that provably satisfies the axioms of PAIR. We first prove a lazy version of our result. Then we raise our standards and prove the strongest version I could think of.

5.1 A model construction A *concatenation structure* is a model of $\mathsf{TC}_0^\varepsilon$.[21] There are many concatenation structures. For example, all groups are concatenation structures. Also, if $(X, \leq)$ is a linear ordering with a minimal element, then (X, max) is a concatenation structure. The simplest example is any structure with $x * y = y$, if $y \neq \varepsilon$, and $x * y = x$, if $y = \varepsilon$.

We show that concatenation structures are closed under the operation $\circledast$ which is defined as follows. Consider concatenation structures $\mathcal{X}$ and $\mathcal{U}$. We will use $x, y, z, \ldots$ for the elements of $\mathcal{X}$ and $u, v, w, \ldots$ for the elements of $\mathcal{U}$. We employ p, q, r ambiguously for both. We use α, β, γ for the elements of $\mathcal{X} \circledast \mathcal{U}$.

The elements of $\mathcal{X} \circledast \mathcal{U}$ are the elements x of $\mathcal{X}$, and triples (y, u, z), where y and z are in $\mathcal{X}$ and u is in $\mathcal{U}$. We assume that the triples (y, u, z) are disjoint from the xs. We write $*$ for concatenation in $\mathcal{X}$ and $\mathcal{U}$ and $\star$ for the new concatenation. We define

$\star$	x'	(y', u', z')
x	$x * x'$	$(x * y', u', z')$
(y, u, z)	$(y, u, z * x')$	$(y, u * u', z')$

We prove that $\mathcal{X} \circledast \mathcal{U}$ is indeed a concatenation structure. Clearly, the unit of $\mathcal{X}$ functions as the new unit of $\mathcal{X} \circledast \mathcal{U}$. Here is the verification of associativity.

α	β	γ	$(\alpha \star \beta) \star \gamma$	$\alpha \star (\beta \star \gamma)$
x	x'	x''	$(x * x') * x''$	$x * (x' * x'')$
x	x'	(y'', u'', z'')	$(x * (x' * y''), u'', z'')$	$((x * x') * y'', u'', z'')$
x	(y', u', z')	x''	$(x * y', u', z' * x'')$	$(x * y', u', z' * x'')$
x	(y', u', z')	(y'', u'', z'')	$(x * y', u' * u'', z'')$	$(x * y', u' * u'', z'')$
(y, u, z)	x'	x''	$(y, u, (z * x') * x'')$	$(y, u, z * (x' * x''))$
(y, u, z)	x'	(y'', u'', z'')	$(y, u * u'', z'')$	$(y, u * u'', z'')$
(y, u, z)	(y', u', z')	x''	$(y, u * u', z' * x'')$	$(y, u * u', z' * x'')$
(y, u, z)	(y', u', z')	(y'', u'', z'')	$(y, (u * u') * u'', z'')$	$(y, u * (u' * u''), z'')$

Next we verify the editor axiom. Suppose $\alpha \star \beta = \gamma \star \delta$. We are looking for a witness of the editor axiom, say θ. We run though the possible cases.

Case 1 All four elements involved are in $\mathcal{X}$. By the editor axiom of $\mathcal{X}$, we are done.

Case 2 Three elements are in $\mathcal{X}$. This is impossible.

Case 3 Two elements are in $\mathcal{X}$ and they are both on the same side of the identity. This is impossible.

Case 4 Two elements are in $\mathcal{X}$ and they are either α and γ, or β and δ. We have, for example, $(y, u, z) \star x = (y', u', z') \star x'$. So $(y, u, z * x) = (y', u', z' * x')$. Let p be provided by the editor axiom for $\mathcal{X}$ such that, for example, $z * p = z'$ and $x = p * x'$. We take $\theta := p$. We have

$$(y, u, z) \star p = (y, u, z') = (y', u', z') \text{ and } x = p \star x'.$$

Case 5 Two elements are in $\mathcal{X}$ and they are either α and δ, or β and γ. We have, for example, $(y, u, z) \star x = x' \star (y', u', z')$. So $(y, u, z * x) = (x' * y', u', z')$. We take $\theta := (y', u, z)$. We have

$$(y, u, z) = x' \star (y', u, z) \text{ and } (y', u, z) \star x = (y', u', z').$$

Case 6 One element is in $\mathcal{X}$. We have, for example,

$$x \star (y, u, z) = (y', u', z') \star (y'', u'', z'');$$

that is, $(x * y, u, z) = (y', u' * u'', z'')$. We take $\eta := (y, u', z')$. We have

$$x \star (y, u', z') = (x * y, u', z') = (y', u', z'),$$

and

$$(y, u', z') \star (y'', u'', z'') = (y, u' * u'', z'') = (y, u, z).$$

Case 7 No elements are in $\mathcal{X}$. We have

$$(y, u, z) \star (y', u', z') = (y'', u'', z'') \star (y''', u''', z''').$$

So $(y, u * u', z') = (y'', u'' * u''', z''')$. Let p be provided by the editor axiom for $\mathcal{U}$ such that, for example, $u * p = u''$ and $u' = p * u'''$. Take $\theta := (y', p, z'')$. We have

$$(y, u, z) \star (y', p, z'') = (y, u * p, z'') = (y'', u'', z''),$$

and

$$(y', p, z) \star (y''', u''', z''') = (y', p * u''', z''') = (y', u', z').$$

An important property of the construction $\circledast$ is that automorphisms of $\mathcal{X}$ and $\mathcal{U}$ can be lifted in the obvious way to automorphisms of $\mathcal{X} \circledast \mathcal{U}$.

5.2 No pairing We show that PAIR is not directly interpretable in $\mathsf{TC}^{\varepsilon}$. We consider the case of one-dimensional interpretations with no parameters. In the next subsection, we will consider parameters, multidimensionality, and more.

Suppose there is a predicate pair in $\mathsf{TC}^{\varepsilon}$ satisfying the axioms of PAIR. Let $\mathcal{A}_2$ be the monoid on generators $\mathtt{a}$ and $\mathtt{b}$. Let D be a domain with at least four elements. We extend D to a concatenation structure $D^{\dagger}$ by stipulating that $d * e := e$ and by adding a unit to the structure. Note that any permutation of D is an automorphism of $D^{\dagger}$.

Consider $\mathcal{B} := \mathcal{A}_2 \circledast D^{\dagger}$. We identify the elements of $\mathcal{A}_2$ with their counterparts in the construction of $\mathcal{B}$ and the elements of $D^{\dagger}$ with the triples $(\varepsilon, d, \varepsilon)$. Note that ε of $\mathcal{A}$ maps to the unit of $\mathcal{B}$ and that $\mathtt{a}$ and $\mathtt{b}$ map to atoms of $\mathcal{B}$. So, $\mathcal{B}$ is a model of $\mathsf{TC}^{\varepsilon}$ (modulo expansion of the signature). (The unit of $D^{\dagger}$ does not map to the unit of $\mathcal{B}$.)

Let d and e be different elements of D. Suppose $\mathsf{pair}(d, e, \alpha)$. Suppose first that α is not a triple. In this case there is an automorphism of $\mathcal{B}$, mapping d to d', e to e, and α to α, where d' is in $D \setminus \{d, e\}$. We get $\mathsf{pair}(d', e, \alpha)$. A contradiction.

Suppose $\alpha = (x, e', y)$, where $e' \in D \cup \{\varepsilon\}$. Clearly, one of d, e is not identical to e'. Suppose it is, for example, d. Let $d' \in D \setminus \{d, e, e'\}$. There is an automorphism of $\mathcal{B}$, mapping d to d', e to e, and α to α. We get $\mathsf{pair}(d', e, \alpha)$. A contradiction.

We may conclude that $\mathsf{TC}^{\varepsilon}$ does not have pairing.

Remark 5.1 Let Λ^0_{a} be the usual tally length function on binary strings. Let φ be an automorphism of $D^{\dagger}$. Define $\Lambda^{\varphi}_{\mathsf{a}}\alpha := \Lambda^0_{\mathsf{a}}\alpha$, if α is in $\mathcal{A}_2$, and $\Lambda^{\varphi}_{\mathsf{a}}\alpha := (\Lambda^0_{\mathsf{a}}x, \varphi u, \Lambda^0_{\mathsf{a}}y)$, if α is (x, u, y), where x and y are in $\mathcal{A}_2$ and u is in $D^{\dagger}$.

It is easy to see that we have indeed defined a tally length function, since triples can never be atoms. It follows that even if we add a tally length function we still cannot define pairing. Note that no (x, u, y) is in N_{a}. So, the range of $\Lambda^{\varphi}_{\mathsf{a}}$ is not contained in N_{a}.

If D has at least two elements, our model allows more than one tally length function. We also see that tally length functions need not be idempotent.

5.3 The pro version of no pairing We adapt the proof of the previous subsection to prove a stronger result. We borrow some ideas from the proof of Lemma 6.5 of [11].

Let me explain what we are going to prove. A first step is to widen our concept of interpretation: we will consider multidimensional interpretations with parameters.[22]

A second step is that we widen the notion of direct interpretability. In the category of interpretations where we identify two interpretations whenever the interpreting theory proves that they are the same, direct interpretability can be defined as follows. Let ID be the pure theory of identity. Let $\imath_U : \mathsf{ID} \to U$ be the interpretation of ID in U, obtained by just reducing the signature. In other words, $\imath_U$ is the unique direct interpretation of ID in U. It is easy to see that $K : U \to V$ is direct if and only if $\imath_V = K \circ \imath_U$. We now take this characterization and reinterpret it in the category of interpretations where we count two interpretations as the same when the associated mappings of models are the same modulo isomorphism. We do not demand that these isomorphisms are definable in the interpreting theory. The notion that we obtain in this way is *cardinality preserving interpretation*. An interpretation $K : U \to V$ is *cardinality preserving* if, for any model $\mathcal{M}$ of V, the internal model $K(\mathcal{M})$ defined by K has the same cardinality as $\mathcal{M}$.

We will show that there is no cardinality preserving multidimensional interpretation with parameters of PAIR in $\mathsf{TC}^{\varepsilon}$. It follows that $\mathsf{TC}^{\varepsilon}$ is not weakly bi-interpretable with a theory that has pairing.

We again work in the the model $\mathcal{A}_2 \circledast D^{\dagger}$ of the previous subsection. However, now we demand that D is uncountable. We will call the unit of $D^{\dagger}$, $\varepsilon^{\dagger}$ to distinguish it from the unit ε of $\mathcal{A}_2$.

Suppose we have an n-dimensional interpretation K of PAIR in $\mathsf{TC}^{\varepsilon}$. Let P be a given finite set of parameters in the model. Let the domain of K be Δ (in the parameters). We may assume that P is a subset of (the embedded elements of) D plus $\varepsilon^{\dagger}$, since all other elements are definable from elements of D plus $\varepsilon^{\dagger}$.[23] Let E be the identity of K (given the parameters).

A *form* is an n-tuple $(t_0, \dots, t_{n-1})$, where each t_i is either (i) an a,b-string or (ii) a term of the form udu' where u and u' are a,b-strings and d is a parameter or (iii) a term of the form $u\varepsilon^{\dagger}u'$ where u and u' are a,b-strings or (iv) a term of the form uXu', where u and u' are a,b-strings and X is a variable. We identify forms modulo permutations of variables. An f-assignment σ is an injective mapping from variables to $D \setminus P$. We define σF, for F a form, in the obvious way. We define $\sigma[F]$ as the set of values of the variables of F.

Clearly, any n-tuple from the domain of $\mathcal{B}$ can be obtained as σF, for some σ and F. Conversely, every such element A uniquely determines σ and F such that $A = \sigma F$.

We call a permutation of D *permissible* if it leaves each parameter in place. Two n-tuples have the same form if and only if they are mapped to each other by an admissible permutation. It follows that either all elements of a given form are in Δ, or none is. We have the following simple lemma.

Lemma 5.2 *Suppose that σF is in Δ and $(\sigma F)E(\tau F)$, where $\sigma[F]$ and $\tau[F]$ are disjoint. Then there is only one instantiation of F, modulo E.*

Proof Suppose that σF is in Δ and $(\sigma F)E(\tau F)$, where $\sigma[F]$ and $\tau[F]$ are disjoint. Let σ' and τ' be any other pair with $\sigma'[F]$ and $\tau'[F]$ disjoint. We can find a permissible permutation φ such that $\varphi \circ \sigma = \sigma'$ and $\varphi \circ \tau = \tau'$. It follows that $(\sigma' F)E(\tau' F)$. Let ν and ρ be any f-assignments. Let θ be an f-assignment where $\theta[F]$ is disjoint from $\nu[F]$ and $\rho[F]$ on the variables of F. We find $(\nu F)E(\theta F)E(\rho F)$ and, hence, $(\nu F)E(\rho F)$. So F contains only one element modulo E. □

We show that any F has at most one instance in Δ modulo E. Suppose F has an instance in Δ. If F has no variables we are immediately done. So suppose F has at least one variable.

Let $A, B, \ldots$ range over Δ. We define a nonfunctional sequence coding as follows (for standard n):

1. $\mathsf{seq}_2(A_0, A_1, B) :\leftrightarrow \mathsf{pair}(A_0, A_1, B)$,
2. $\mathsf{seq}_{k+3}(A_0, \ldots, A_{k+2}, B) :\leftrightarrow \exists C$
$$(\mathsf{seq}_{k+2}(A_0, \ldots, A_{k+1}, C) \wedge \mathsf{pair}(C, A_{k+2}, B)).$$

Let $\sigma_0, \ldots, \sigma_n$ be a sequence of $n+1$ f-assignments such that the $\sigma_i[F]$ are pairwise disjoint. Suppose that $\mathsf{seq}_{n+1}(\sigma_0 F, \ldots, \sigma_n F, B)$, for some B. Say B is of form τG, for some form G. The number of variables of G is smaller or equal to n. So $\tau[G]$ is smaller or equal to n. It follows that $\tau[G]$ is disjoint from one of the $\sigma_i[F]$, say for i_0. Let φ be any admissible permutation that leaves the elements of $\tau[G]$ and of the $\sigma_i[F]$, for $i \neq i_0$, in place but moves the elements of $\sigma_{i_0}[F]$ to a set disjoint from the $\sigma_{i_0}[F]$. We find that $\mathsf{seq}_{n+1}(\varphi\sigma_0 F, \ldots, \varphi\sigma_n F, \varphi B)$. It follows that

$$\mathsf{seq}_{n+1}(\sigma_0 F, \ldots, \sigma_{i_0-1} F, \varphi\sigma_{i_0} F, \sigma_{i_0+1} F, \ldots, \sigma_n F, B).$$

So $(\sigma_{i_0} F)E(\varphi\sigma_{i_0} F)$. It follows, by the lemma, that F contains, modulo E, only one object in Δ.

Since there are only countable many forms, it follows that Δ is countable (modulo E). So K is *not* cardinality preserving. Note that our result still holds when we add a tally length function.

Appendix A Comparing TC and $\mathsf{TC}^{\varepsilon}$

We show that TC is *bi-interpretable* with a corresponding theory $\mathsf{TC}^{\varepsilon}$.[24] This means that there are interpretations $K : \mathsf{TC}^{\varepsilon} \to \mathsf{TC}$ and $M : \mathsf{TC} \to \mathsf{TC}^{\varepsilon}$ so that $K \circ M : \mathsf{TC} \to \mathsf{TC}$ is isomorphic to the interpretation $\mathsf{id}_{\mathsf{TC}}$ via a definable isomorphism F, and $M \circ K : \mathsf{TC}^{\varepsilon} \to \mathsf{TC}^{\varepsilon}$ is isomorphic to the interpretation $\mathsf{id}_{\mathsf{TC}^{\varepsilon}}$ via a definable isomorphism G.[25]

We can take K and M one-dimensional interpretations without parameters. We specify K, M, F, and G. We use C for the relational formulation of concatenation and E as an alternative way of writing identity.

1. $\delta_K(x) :\leftrightarrow x = \mathtt{a} \vee \exists x_0\, x = \mathtt{b} * x_0$,
2. $x \mathsf{E}_K y :\leftrightarrow x = y$,
3. $\mathsf{C}_K(x, y, z) :\leftrightarrow (x = \mathtt{a} \wedge y = z) \vee (y = \mathtt{a} \wedge x = z) \vee$
   $\exists x_0, y_0\, (x = \mathtt{b} * x_0 \wedge y = \mathtt{b} * y_0 \wedge z = \mathtt{b} * x_0 * y_0)$,
4. $\varepsilon_K := \mathtt{a}$,
5. $\mathtt{a}_K = \mathtt{b} * \mathtt{a}$,
6. $\mathtt{b}_K = \mathtt{b} * \mathtt{b}$,
7. $\delta_M(x) :\leftrightarrow x \neq \varepsilon$,
8. $x \mathsf{E}_M y :\leftrightarrow x = y$,
9. $x *_M y := x * y$,
10. $xFy :\leftrightarrow x = \mathtt{b} * y$,
11. $xGy :\leftrightarrow (x = \mathtt{a} \wedge y = \varepsilon) \vee x = \mathtt{b} * y$.

The verification that our definitions work is routine. We note the important fact that the presence of atoms in TC^ε implies that $x * y = \varepsilon \to (x = \varepsilon \vee y = \varepsilon)$.

The author thinks that he can prove the even stronger theorem, to wit that TC and TC^ε are definitionally equivalent, but the proof still has to be written up.

Appendix B Comparing TC^ε and $\mathsf{Q}^{\mathsf{bin}}$

We first show how to interpret $\mathsf{Q}^{\mathsf{bin}}$ in TC^ε. We work in TC^ε. Define

$$I(x) :\leftrightarrow \forall y \subseteq_{\mathsf{ini}} x\, (y = \varepsilon \vee \exists z\, (z = y * \mathtt{a} \vee y = z * \mathtt{b}).$$

It is easy to see that I is closed under ε, $\mathtt{a}$, and $\mathtt{b}$ and that it is downward closed under $\subseteq_{\mathsf{ini}}$. We show that it is closed under concatenation. Suppose that x_0 and x_1 are in I and that $y \subseteq_{\mathsf{ini}} x_0 * x_1$, say $y * w = x_0 * x_1$. We want to show $(\dagger_y)$ $y = \varepsilon$ or $\exists z\, (z = y * \mathtt{a} \vee y = z * \mathtt{b})$.

By the editor axiom, there is a u such that (1) $y * u = x_0$ and $w = u * x_1$ or (2) $y = x_0 * u$ and $u * w = x_1$. In the first case, $y \subseteq_{\mathsf{ini}} x_0$ and, hence, we have $\dagger_y$. In the second case, $u \subseteq_{\mathsf{ini}} x_1$. So we have $\dagger_u$. If $u = \varepsilon$, we find $y = x_0$. So $y \subseteq_{\mathsf{ini}} x_0$ and $\dagger_y$. Otherwise, for some z, (2.1) $u = z * \mathtt{a}$ or (2.2) $u = z * \mathtt{b}$. In case (1.1) it follows that $y = x_0 * u = x_0 * (z * \mathtt{a}) = (x_0 * z) * \mathtt{a}$. Case (1.2) is similar. We may conclude $\dagger_y$.

Our interpretation $K : \mathsf{Q}^{\mathsf{bin}} \to \mathsf{TC}$ is just relativization to I, where we set $\mathsf{S}_{\mathtt{a}} x := x * \mathtt{a}$ and $\mathsf{S}_{\mathtt{b}} x := x * \mathtt{b}$.

We provide the reverse interpretation $M : \mathsf{TC} \to \mathsf{Q}^{\mathsf{bin}}$. We work in $\mathsf{Q}^{\mathsf{bin}}$. Let $I(x) :\leftrightarrow \forall y, z\, (y * z) * x = y * (z * x)$. It is easy to see that I is closed under ε, $\mathsf{S}_{\mathtt{a}}$, $\mathsf{S}_{\mathtt{b}}$, concatenation and under the predecessor functions corresponding to $\mathsf{S}_{\mathtt{a}}$ and $\mathsf{S}_{\mathtt{b}}$.

So relativization to I will give us an interpretation of $\mathsf{Q}^{\mathsf{bin}}$ plus the associativity of concatenation. We proceed to work in this theory. Let

$$J(x) :\leftrightarrow \forall y, u, v\, (y{*}x = u{*}v \to \exists z\, ((y{*}z = u{\wedge}x = z{*}v){\vee}(y = u{*}z{\vee}z{*}x = v))).$$

We define $\mathtt{a} := \mathsf{S}_{\mathtt{a}}\varepsilon$ and $\mathtt{b} := \mathsf{S}_{\mathtt{b}}\varepsilon$. Clearly, J is closed under ε. Suppose $x = \mathtt{a}$, $y * \mathtt{a} = u * v$. If $v = \varepsilon$, we can take $z := \mathtt{a}$. We have $y * z = u$ and $x = z * v$. If $v = \mathsf{S}_{\mathtt{a}} v_0$, we can take $z := v_0$. We have

$$\mathsf{S}_{\mathtt{a}}(u * z) = u * v_0 * \mathtt{a} = u * v = y * \mathtt{a} = \mathsf{S}_{\mathtt{a}} y.$$

So $y = u * z$. Moreover, $z * x = v_0 * \mathtt{a} = v$. It is easily seen that the case that $v = \mathsf{S}_{\mathtt{b}} v_0$ leads to a contradiction. We may conclude that $\mathtt{a}$ is in J. By similar reasoning, we find that $\mathtt{b}$ is in J.

We now show that J is closed under concatenation. Suppose x_0 and x_1 are in J and $y * x_0 * x_1 = u * v$. For some z_0, we have (1) $y * x_0 * z_0 = u$ and $x_1 = z_0 * v$ or (2) $y * x_0 = u * z_0$ and $z_0 * x_1 = v$.

In case (1) we can take the desired $z := x_0 * z_0$. We have $y * z = y * x_0 * z_0 = u$ and $z * v = x_0 * z_0 * v = x_0 * x_1$.

In case (2), we have a z_1 such that (2.1) $y * z_1 = u$ and $x_0 = z_1 * z_0$ or (2.2) $y = u * z_1$ and $z_1 * x_0 = z_0$. In case (2.1), we can take $z := z_1$. We have $y * z = y * z_1 = u$ and $z * v = z_1 * v = z_1 * z_0 * x_1 = x_0 * x_1$. In case (2.2), we can take $z := z_1$. We have $u * z = u * z_1 = y$ and $z * x_0 * x_1 = z_1 * x_0 * x_1 = z_0 * x_1 = v$.

Finally, we define $J^\star(x) :\leftrightarrow \forall y \subseteq x\ J(y)$. It is easily seen that $J^\star$ is closed under ε, $\mathtt{a}$, $\mathtt{b}$, and downward closed under taking substrings. We show that $J^\star$ is closed under concatenation. Suppose x_0 and x_1 are in $J^\star$ and $y \subseteq x_0 * x_1$. We have, for some w_0, w_1, that $x_0 * x_1 = w_0 * y * w_1$. Since x_1 is in $J^\star$ and, a fortiori, in J, there is a z_0 such that (1) $x_0 * z_0 = w_0$ and $x_1 = z_0 * y * w_1$ or (2) $x_0 = w_0 * z_0$ and $z_0 * x_1 = y * w_1$. In case (1), we have that $y \subseteq x_1$; hence $y \in J$.

In case (2), we use again that x_1 is in J. We can find a z_1 such that (2.1) $z_0 * z_1 = y$ and $x_1 = z_1 * w_1$ or (2.2) $z_0 = y * z_1$ and $z_1 * x_1 = w_1$. In case (2.1), we note that $z_0 \subseteq x_0$ and $z_1 \subseteq x_1$. Hence, z_0 and z_1 are both in J. Thus $y = z_0 * z_1$ is also in J. In case (2.2), we find that $y \subseteq z_0 \subseteq x_0$, so $y \subseteq x_0$ and y is in J.

Since we can rewrite the editor axiom with substring-bounded quantifiers, it is easily seen that relativization to $J^\star$ interprets TC^ε in $\mathsf{Q}^{\mathsf{bin}}$ plus the associativity of concatenation.

Appendix C A Model without a Tally

Consider a model $\mathcal{M}$ of the true theory of concatenation for the alphabet $\{\mathtt{a}, \mathtt{b}\}$ that contains a nonstandard element. It follows that $\mathcal{M}$ contains a nonstandard string of $\mathtt{b}$s. We consider the class A of the strings of this model that contain only standardly many $\mathtt{a}$s. This class is closed under empty string, atoms, concatenation and is downward closed under substrings. Consider the submodel $\mathcal{A}$ of $\mathcal{M}$ determined by A. It is clear that $\mathcal{A}$ satisfies TC^ε and does not have a tally function.

We call a formula $\Delta_0^\subseteq$ if it only contains substring bounded quantifiers. A formula is $\Pi_1^\subseteq$ if it is given by a $\Delta_0^\subseteq$-formula preceded by universal quantifiers. We see that $\mathcal{A}$ satisfies all $\Pi_1^\subseteq$-sentences true in $\mathcal{M}$ and, hence, true in the standard model of binary strings. Thus, there is a model of TC^ε plus all $\Pi_1^\subseteq$-sentences which are true in the standard model of binary strings, in which there is no tally length function.

Appendix D How to Create a Letter ex Nihilo

The axioms of F are the following.

F1 $\vdash x * (y * z) = (x * y) * z$.

F2 $\vdash x * z = y * z \to x = y$.

F3 $\vdash z * x = z * y \to x = y$.

F4 $\vdash x * \mathtt{a} \neq y * \mathtt{b}$.

F5 $\vdash x = \mathtt{a} \vee x = \mathtt{b} \vee \exists y\ (x = y * \mathtt{a} \vee x = y * \mathtt{b})$.

Our axiomatization differs slightly from the one of Szmielew and Tarski. They do not have constants for $\mathtt{a}$ and $\mathtt{b}$ but have an axiom saying that two objects with the properties of $\mathtt{a}$ and $\mathtt{b}$ exist.

To construct a model, we define a rewrite system. Terms are strings of 'letters' (except the empty string) of the alphabet $\mathtt{a}$, $\mathtt{b}$, α_n, β_n. The intuition is that the α_n and the β_n stand for strings $\ldots \mathtt{aaa}$, but somehow for different ones. We have the following rewrite rules.

1. $\alpha_{n+1}\mathtt{a} \to \alpha_n$
2. $\beta_{n+1}\mathtt{a} \to \beta_n$
3. $\mathtt{b}\alpha_n \to \beta_0\beta_n$

We will assign to each of $\mathtt{a}$, α_n, β_n weight 1, and to $\mathtt{b}$ weight 2. The weight of a string is the sum of the weights of the occurrences in the string. Our system strongly terminates because in our reduction steps the weight is strictly decreasing. It is also Church-Rosser. The only critical pairs have the form $\mathtt{b}\alpha_{n+1}\mathtt{a}$ and we have $\mathtt{b}\alpha_{n+1}\mathtt{a} \to \beta_0\beta_{n+1}\mathtt{a} \to \beta_0\beta_n$ and $\mathtt{b}\alpha_{n+1}\mathtt{a} \to \mathtt{b}\alpha_n \to \beta_0\beta_n$. We may conclude that our system has unique normal forms.

We now define a model where the domain consists of the normal forms of our rewrite system and $x * y$ is the normal form of xy. *Notationally it is essential to distinguish carefully between xy which is the concatenation of x and y and $x * y$, which is the normal form of xy.* We verify the axioms.

Associativity F1 follows by the uniqueness of normal forms.

Axiom F4 tells us that $x * \mathtt{a} \neq y * \mathtt{b}$. It is easy to prove by induction on reduction sequences that any reduct of $x\mathtt{a}$ is of one of the forms $z\mathtt{a}$, $z\alpha_k$, or $z\beta_k$. Any reduct of $y\mathtt{b}$ is of the form $z\mathtt{b}$. So clearly the normal forms of $x\mathtt{a}$ and $y\mathtt{b}$ cannot be equal.

We verify axiom F5. In case the normal form x ends with $\mathtt{a}$, or with $\mathtt{b}$, we are immediately done. If $x = \alpha_k$ or $x = x'\alpha_k$, then $x = \alpha_{k+1} * \mathtt{a}$ or $x = x'\alpha_{k+1} * \mathtt{a}$. Similarly for the case that x ends with β_k.

We verify the right cancellation law F2; that is, for all $x, y, z, x * z = y * z \Rightarrow x = y$. We proceed by induction on the weight of $xzyz$. Suppose $x * z = y * z$.

Case 1 In case xz and yz are in normal form, we are done.

Case 2 Suppose one of xz or yz is not in normal form. By symmetry, we may assume that xz is not in normal form. Clearly, $z = \gamma$ or $z = z'\gamma$, where γ is a letter.

Case 2.1 Suppose $z = z'\gamma$. We have $(x * z') * \gamma = (y * z') * \gamma$.

Case 2.1.1 If one of xz', yz' is not in normal form, clearly, the weight of $(x * z')\gamma(y * z')\gamma$ is less than the weight of $xzyz$. So it follows that $x * z' = y * z'$. Again the weight of $xz'yz'$ is less than the weight of $xzyz$. Hence $x = y$.

Case 2.1.2 Now suppose that both xz' and yz' are in normal form. Note that both xz' and $z'\gamma$ are normal forms. It follows that $xz = xz'\gamma$ would be a normal form. This contradicts our assumption.

Case 2.2 We suppose that $z = \gamma$. So $x * \gamma = y * \gamma$.

Case 2.2.1 Suppose $\gamma = \mathtt{a}$. We have $x * \mathtt{a} = y * \mathtt{a}$. Since $x\mathtt{a}$ is not in normal form, x is either α_{n+1} or $x'\alpha_{n+1}$ or β_{n+1} or $x'\beta_{n+1}$. In the first case, $x * \mathtt{a}$ will be α_n. It follows that y must also be α_{n+1}. In the second case, $x * \mathtt{a}$ will be $x'\alpha_n$. (There can be no $\mathtt{b}$ at the end of x' since x is in normal form.) It follows that y must also be of the form $y'\alpha_{n+1}$. We find that $x'\alpha_n = y'\alpha_n$ and hence $x' = y'$ and thus $x = y$. (Note that this last step does not use the Induction Hypothesis: we are comparing normal forms.) The reasoning for the remaining cases is similar.

Case 2.2.2 Suppose $\gamma = \mathtt{b}$. Then $x\mathtt{b}$ would be a normal form. Quod non.

Case 2.2.3 Suppose $\gamma = \alpha_k$. We have $x * \alpha_k = y * \alpha_k$. Since $x\alpha_k$ is not a normal form, $x = \mathtt{b}$ or $x = x'\mathtt{b}$. In the second case, $x * \alpha_k = x'\mathtt{b} * \alpha_k = x'\beta_0\beta_k$. To make $x'\beta_0\beta_k = y * \alpha_k$ possible, we must have $y = y'\mathtt{b}$ and hence $y * \alpha_n = y'\beta_0\beta_n$. It follows that $x' = y'$ and thus $x = y$. The first case is similar.

Case 2.2.4 Suppose $\gamma = \beta_k$. Then $x\beta_k$ would be a normal form. Quod non.

We verify the left cancellation law $\mathsf{F3}$; that is, for all x, y, z, $z * x = z * y \Rightarrow x = y$. We proceed by induction on the weight of $zxzy$. Suppose $z * x = z * y$. We run through an argument that is the mirror image of the argument concerning right cancellation. The first point where a difference appears is 2.2.1.

Case 2.2.1 Suppose $\gamma = \mathtt{a}$. Clearly $\mathtt{a} * x$ would be in normal form. Quod non.

Case 2.2.2 Suppose $\gamma = \mathtt{b}$. Then x must be of the form α_k or $\alpha_k x'$. In the second case, $\mathtt{b}x = \beta_0\beta_k x'$. (In case $k > 0$, x' cannot begin with $\mathtt{a}$, since x is in normal form.) It follows that $y = \alpha_k y'$ and $\beta_0\beta_k x' = \beta_0\beta_k y'$, so $x' = y'$ and $x = y$. The first case is similar.

Case 2.2.3 Suppose $\gamma = \alpha_k$. We have $\alpha_k * x = \alpha_k * y$. We have either $x = \mathtt{a}^n$, where $n \neq 0$, or $x = \mathtt{a}^n x'$, where x' does not begin with $\mathtt{a}$ and n may be zero. Similarly, either $y = \mathtt{a}^m$, where $m \neq 0$, or $y = \mathtt{a}^m y'$, where y' does not begin with $\mathtt{a}$ and m may be zero. We will only treat the case that $x = \mathtt{a}^n x'$ and $y = \mathtt{a}^m y'$. The other cases are easier. We have $\alpha_k * x = \alpha_k * \mathtt{a}^n x' = \alpha_{k \dot{-} n}\mathtt{a}^{n \dot{-} k} x'$. Here $\dot{-}$ is cut-off subtraction. We find $\alpha_k * y = \alpha_k * \mathtt{a}^m y' = \alpha_{k \dot{-} m}\mathtt{a}^{m \dot{-} k} y'$. It follows that $k \dot{-} n = k \dot{-} m$ and $n \dot{-} k = m \dot{-} k$. So $n = m$. Hence $x' = y'$ and $x = y$.

Case 2.2.4 Suppose $\gamma = \beta_k$. This case is similar to the the previous case.

Consider β_0 in the model. It is easily seen that the substrings of β_0 are precisely β_0 and $\mathtt{a}^n$, for all $n > 0$. So β_0 is in $\mathsf{N}_{\mathtt{a}}$. On the other hand, $\mathtt{b}$ is a substring of $\beta_0 * \beta_0$. So $\mathtt{b}$ appears ex nihilo.

Notes

1. For more detailed historical remarks, see Subsection 1.4.

2. A decorated linear ordering is a linear ordering plus a function from the domain of the ordering to decorations (i.e., elements from a given class). An isomorphism between two decorated linear orderings (with decorations from a given class) is an isomorphism of linear orderings that preserves decorations. A decorated linear order type is the equivalence class modulo isomorphism of a decorated linear ordering. The definition of concatenation of decorated order types is as expected. For a study of decorated linear order types as a semantics for theories of concatenation, see [5].

3. In fact Quine defines relations, but the mechanism is the same.

4. The axiomatizations employed in [4] are not precisely the original ones.

5. The text admits the alternative reading that Quine intends a bilateral interpretation to be one half of a definitional equivalence. This reading fits better the symmetry suggested by *bilateral*.

6. Quine calls this a *lexicographic ordering*. As far as I can see the majority vote in the present day literature is to use *lexicographic ordering* for the ordering employed in ordinary dictionaries. In contrast, the length-first ordering is employed in dictionaries for crossword puzzles.

7. In fact, Quine defines multiplication of length-first numbers by defining the bijection between tally numbers and length-first numbers. He uses the bijection to induce the operations on the length-first numbers from those of the tally numbers.

8. The notion of logical synonymy or definitional equivalence was introduced by de Bouvère in 1965. See [6] and [7]. Clearly, Quine understood the notion in 1935 without having an explicit definition of it.

9. Since $\mathsf{Q}^{\mathsf{bin}}$ is, by Appendix B, interpretable in $\mathsf{TC}^{\varepsilon}$ (see Section 2), we find another proof of the interpretability of Q in $\mathsf{TC}^{\varepsilon}$ and hence in TC. It is easily seen that $\mathsf{Q}^{\mathsf{bin}}$ is interpretable in F, so this also gives a proof of the interpretability of Q in F.

10. The theory S^1_2 is mutually interpretable with Q. Thus, the salient property of representing the p-time definable functions is *not* preserved modulo mutual interpretability.

11. In F there are two equivalent definitions of a string of $\mathtt{a}$s. The first definition is "x is a string of $\mathtt{a}$s if and only if $\mathtt{b}$ is not a substring of x." The second definition is "x is string of $\mathtt{a}$s if and only if, for every substring y of x, $\mathtt{a}$ is a substring of y." (The system F is semigroup-style, without a unit.)

12. The example is real in the sense that I, for some time, naïvely assumed that the tally numbers of F were closed under concatenation.

13. Almost all desirable properties of theories are preserved modulo bi-interpretability—for example, finite axiomatizability, κ-categoricity, sequentiality. Moreover, bi-interpretability is a bisimulation with respect to theory extension: if U is bi-interpretable with V and $U \subseteq U'$, then there is a $V' \supseteq V$ which is bi-interpretable with V.

14. It is not difficult to produce a model to show that we cannot prove that the atoms in our sense are the only atoms of the preordering.

15. It is easy to produce a countermodel to show that the converse does not generally hold.

16. An important difference is that in the definition, as given by Smoryński, Elementary Arithmetic EA (a.k.a. $I\Delta_0 + \mathsf{EXP}$) is stipulated to be interpretable in adequate theories. This demand is evidently much too strong.

17. It is easy to produce a model of $\mathsf{TC}^{\varepsilon}$ that does not admit a tally length function. See Appendix C. One can also produce a model that admits different tally length functions and in which the range of these functions is *not* $\mathsf{N}_{\mathtt{a}}$. See Subsection 5.2.

18. Of course, we can define a tally length function in a sufficiently strong extension of $\mathsf{TC}^{\varepsilon}$. However, that is *after* we coded sequences. We are now precisely considering it as a tool to define sequences.

19. The fact that correctness can be verified in a weaker theory than the one we need for existence, by itself, does not give us much information. After all, we could define an alternative adjunction by
    - $\mathsf{adj}^{\star}(x, y, z) :\leftrightarrow \mathsf{adj}(x, y, z) \wedge \forall w\ (w \in z \leftrightarrow (w \in x \vee w = y))$.

    For *this* definition correctness would be trivial and the whole burden of verification would be shifted to the existence clause. Of course, such cleverness only shifts the work to other places.

20. So, note that it is possible that $3_{\sigma} \neq 3_{\tau}$, even if $\sigma = \tau$.

21. Warning: The presence of the unit is essential to make the construction work.

22. There is one further possible widening of our class of interpretations: we could consider piecewise interpretations, where the new domain is assembled out of possibly overlapping pieces. However, since $\mathsf{TC}^{\varepsilon}$ provides an infinity of closed terms that are pairwise provably different, one can show that piecewise interpretations, for the case at hand, can always be replaced by multidimensional ones.

23. In fact, the $\mathtt{a}$,$\mathtt{b}$-strings are definable in the model. It follows that $\varepsilon^{\dagger}$ is definable.

24. Gregorczyck and Zdanowski prove that TC interprets $\mathsf{TC}^{\varepsilon}$ in [14]. Our argument is a variation of their argument.

25. See [38] for detailed definitions.

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