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Generalized Halfspaces in the Mixed-Integer Realm

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Abstract In the ordered Abelian group of reals with the integers as a distinguished subgroup, the projection of a finite intersection of generalized halfspaces is a finite intersection of generalized halfspaces. The result is uniform in the integer coefficients and moduli of the initial generalized halfspaces.

1 Introduction

A halfspace in $\mathbb{R}^n$ is the solution set of a homogeneous weak linear inequality in n variables. Fundamental to the study of linear inequalities is the closure under projection of the class of finite intersections of halfspaces. This result fails when one turns from $\mathbb{R}$ to nondivisible ordered Abelian groups like $\mathbb{Z}$, where congruences modulo a positive integer become important. But one may recover over $\mathbb{Z}$ a version of this result on projections by exploiting a more liberal notion of generalized halfspace, which combines weak inequalities with congruences in a special way ([5], Section 4). Postponing to Section 2 the definition of generalized halfspace, one may give an example:

$$\{(x, y) \in \mathbb{Z}^2 : \exists u, v(x \leq u \leq v \leq y \wedge u \equiv_3 0 \wedge v \equiv_2 0)\}$$

is a generalized halfspace of $\mathbb{Z}^2$, where $u \equiv_3 0$ ($v \equiv_2 0$) means that u is divisible by 3 (v is divisible by 2) in $\mathbb{Z}$.

A different notion of congruence plays a role in Weispfenning's mixed real-integer quantifier elimination for a language $\mathcal{M}$ in which one may describe the reals as an ordered Abelian group $\mathcal{R}$ with the integers as a definable subgroup [8]. $\mathcal{M} = \{+, -, <, 0, 1, \lfloor\cdot\rfloor\} \cup \{\equiv_n : n \geq 2\}$[1] contains a language $\{+, -, <, 0\}$ for ordered Abelian groups, a constant symbol 1 for the number one, a symbol for the function obeying

$$\lfloor x \rfloor \in \mathbb{Z} \text{ and } 0 \leq x - \lfloor x \rfloor < 1,$$

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and binary relation symbols for extensions to the reals of the usual congruence relations on $\mathbb{Z}$: for integers $n \geq 2$ and reals x and y

$$x \equiv_n y \text{ iff } x = y + nz \text{ for some integer } z.$$

Thus a real x is an integer just in case $x = \lfloor x \rfloor$, that is, just in case $2x \equiv_2 0$.

This note will show that the projection theorem for $\mathbb{Z}$ of [5] continues to hold in the structure studied by Weispfenning if one substitutes his notion of congruence for the usual one. This claim does not follow immediately from his method of eliminating quantifiers because it may introduce disjunction (e.g., in case 2 of the proof of Theorem 3.1 in [8]). However, the effective proof in [6] of the projection result for $\mathbb{Z}$, with a slight modification at the start, continues to work in this new context, and as before the theorem is uniform in coefficients and moduli in a manner explained below.

Yet Section 3 points out a limitation to this uniformity result. It is expressed as a quantifier-elimination result in a two-sorted theory featuring a sort for scalars and a sort for elements of the ordered group: $\mathbb{Z}$, in [6], and $\mathbb{R}$ here. Once the group $\mathbb{Z}$ is replaced by $\mathbb{R}$, one might hope also to replace integer scalars by real scalars, together with a means of distinguishing real scalars from integer scalars, since only the latter seem relevant to the definition of congruences. But an example presented in Section 3 shows that a natural proposal for a structure of this kind cannot obey the obvious analogue of the uniform projection theorem established in Section 2.

The Conclusion, finally, suggests possible applications of the results of Section 2.

2 Projections

The formal counterpart of a generalized halfspace is a congruence inequality. If $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ are disjoint lists of distinct variables, $D(x)$ is a conjunction of weak inequalities $x_i \leq x_j$ for which the graph

$$\{(i, j) : x_i \leq x_j \text{ is a conjunct of } D\}$$

contains no cycles, and $H(x; y)$ is a conjunction of congruences between $\mathbb{Z}$-linear forms in the variables x, y, let $\leq^{D,H}$ be a new $(n+2)$-place relation symbol and $\mathcal{M}'$ be the expansion of $\mathcal{M}$ by all the $\leq^{D,H}$s. $\mathcal{R}'$ is the $\mathcal{M}'$-expansion of $\mathcal{R}$ obeying the axioms

$$\forall u, v, y(u \leq_y^{D,H} v \leftrightarrow \exists x(u \leq x \leq v \ \wedge \ D(x) \ \wedge \ H(x; y))).$$

If r, s, and $t_1, \ldots, t_n$ are $\mathbb{Z}$-linear forms in the variables $w = (w_1, \ldots, w_k)$, any formula

$$r \leq_{t_1,\ldots,t_n}^{D,H} s$$

is a congruence inequality in w. A generalized halfspace of $\mathbb{R}^k$ is the solution set in $\mathcal{R}'$ of a congruence inequality in k variables.

Note that while $x = \lfloor x \rfloor$ is in $\mathcal{R}'$ equivalent to $2x \equiv_2 0$, which amounts to a congruence inequality, $y = \lfloor x \rfloor$ is not equivalent in $\mathcal{R}'$ to any conjunction of congruence inequalities (even if parameters are allowed). For since weak inequalities between linear forms and congruences between linear forms define closed sets, and congruence inequalities result from systems of such weak inequalities and congruences by a form of bounded quantification, a generalized halfspace H in $\mathbb{R}^k$ is the projection of a closed set in $\mathbb{R}^{k+l}$ whose fibers over H are bounded, and so H is closed. Though the graph of $y = \lfloor x \rfloor$ may not be defined (with parameters) in $\mathcal{R}'$ by a conjunction of congruence inequalities, closely related sets are so definable. For example, if $\psi(x, y)$ is

$$\exists z(x \leq z \leq y \wedge 2z \equiv_2 0),$$

$\psi(x-1, y) \wedge \psi(y, x)$ defines the infinite staircase obtained from the graph of $y = \lfloor x \rfloor$ by filling in the missing vertical segments.

That the class of finite intersections of generalized halfspaces is closed under projection follows from Theorem 2.1.

Theorem 2.1 In $\mathcal{R}'$, the existential quantification of a conjunction of congruence inequalities is equivalent to a conjunction of congruence inequalities.

Proof Since existential quantifiers may be eliminated one by one, the following discussion will show that if $\varphi(w, v)$ is a conjunction of congruence inequalities in $(w_1, \ldots, w_k, v) = (w, v)$, then $\exists v\varphi$ is equivalent in $\mathcal{R}'$ to a conjunction of congruence inequalities in w.

Each conjunct of $\varphi(w, v)$ has the form

$$a + rv \leq^{D,H}_{c_1,\ldots,c_n} b + sv,$$

where a, b are $\mathbb{Z}$-linear forms in w, $c_1, \ldots, c_n$ are $\mathbb{Z}$-linear forms in (w, v), and r, s are integers. By repeatedly subtracting the same linear form from both sides and making corresponding changes to the system H of congruences, one may assume that $rs = 0$ and that if $r \neq 0$ ($s \neq 0$), it is positive and a (b) is identically zero. Because

$$x \equiv_n y \leftrightarrow kx \equiv_{nk} ky$$

when n and k are positive integers, one may multiply the arguments of each congruence inequality, and the moduli of the congruences in H, by suitable positive integers to ensure that for some integer $t > 0$, every $r \neq 0$ ($s \neq 0$) is t and v occurs in any H only as a multiple of tv. Let $\theta(w, v)$ result from $\varphi(w, v)$ through replacement of all occurrences of tv by v. Because the reals form a divisible group, $\exists v\varphi$ is equivalent in $\mathcal{R}'$ to $\exists v\theta$, and so one may assume that $t = 1$.

This last step—possible only in a divisible group—allows one to assume that $\varphi(w, v)$ is a conjunction $(\sharp) =$

$$\bigwedge_l 0 \leq^{D_l,H_l}_{w,v} b_l \wedge \bigwedge_m a_m \leq^{D_m,H_m}_{w,v} v \wedge \bigwedge_n v \leq^{D_n,H_n}_{w,v} e_n,$$

where the as, bs, and es are $\mathbb{Z}$-linear forms in w. Yet given such a φ, one may eliminate v from $\exists v\varphi$ exactly as in the Appendix to [6]; only the manner of reaching a formula $(\sharp)$ had to be different. $\square$

To express the uniformity, in coefficients of linear forms and moduli of congruences, of the passage from $\exists v\varphi(w, v)$ to a conjunction $\psi(w)$ of congruence inequalities, one may carry out the argument for Theorem 2.1 in a two-sorted theory featuring a new sort for scalars. $\mathcal{M}^{\mathrm{II}}$ is a two-sorted language with scalar variables $\rho, \sigma, \tau, \ldots$ and module variables $x, y, z, \ldots$. The module sort features a copy of $\{+, -, <, 0, 1, \lfloor\cdot\rfloor\}$, where the function symbols $+, -, \lfloor\cdot\rfloor$ take module terms as arguments and yield module terms and the relation symbol $<$ takes module terms as arguments. The scalar sort features a copy of $\{+, -, \cdot, <, 0, 1, g, \alpha, \beta, \gamma\}$, where the function symbols $+, -, \cdot, g, \alpha, \beta, \gamma$ take scalar terms as arguments and yield scalar terms and the relation symbol $<$ takes scalar terms as arguments. There is a binary function symbol $\cdot$ which takes a scalar term δ and a module term a to yield a module term $\delta \cdot a$. There is a ternary relation symbol $\equiv$ which takes two module terms a, b and a scalar term δ as arguments to yield an atomic formula $a \equiv_\delta b$ (called a congruence of modulus δ). And if $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ are disjoint lists of distinct module variables, $\sigma = (\sigma_1, \dots, \sigma_k)$ is a list of distinct scalar variables, $D(x)$ is a conjunction of weak inequalities $x_i \le x_j$ for which the graph

$$\{(i, j) : x_i \le x_j \text{ is a conjunct of } D\}$$

contains no cycles, and $H(x; y; \sigma)$ is a conjunction of congruences, between linear forms in x, y, with module variables from σ, then $\le^{D,H}$ is a relation symbol of $\mathcal{M}^{\mathrm{II}}$ with $n + 2$ module arguments and k scalar arguments. T^{II} is the $\mathcal{M}^{\mathrm{II}}$-theory having axioms with the following import (initial universal quantifiers may be dropped for legibility):

(1) axioms for ordered Abelian groups (in the module sort);
(2) axioms for ordered integral domains (in the scalar sort);
(3) $0 < 1$ (in the module sort);
(4) $0 < 1$ (in the scalar sort);
(5) $\rho\cdot(x+y) = (\rho\cdot x)+(\rho\cdot y) \wedge (\rho+\sigma)\cdot x = (\rho\cdot x)+(\sigma\cdot x) \wedge 1\cdot x = x \wedge (\rho\sigma)\cdot x = \rho\cdot(\sigma\cdot x) \wedge (\rho\cdot x = 0 \to \rho = 0 \vee x = 0) \wedge (\rho \le \sigma \wedge 0 \le x \to \rho\cdot x \le \sigma\cdot x) \wedge (0 \le \rho \wedge x \le y \to \rho\cdot x \le \rho\cdot y)$;
(6) $g(\rho, \sigma) = g(\sigma, \rho) \wedge \rho = \gamma(\rho, \sigma)g(\rho, \sigma) \wedge \sigma = \gamma(\sigma, \rho)g(\rho, \sigma) \wedge 1 = \alpha(\rho, \sigma)\gamma(\rho, \sigma) + \beta(\rho, \sigma)\gamma(\sigma, \rho)$;
(7) $\rho \ne 0 \to \exists y(x = \rho \cdot y)$;
(8) $0 \le x - \lfloor x \rfloor < 1$;
(9) $0 < \lfloor x \rfloor \to 1 \le \lfloor x \rfloor$;
(10) $0 = \lfloor 0 \rfloor \wedge 1 = \lfloor 1 \rfloor \wedge \lfloor \lfloor x \rfloor + \lfloor y \rfloor \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \wedge \lfloor -\lfloor x \rfloor \rfloor = -\lfloor x \rfloor \wedge \lfloor \rho \cdot \lfloor x \rfloor \rfloor = \rho \cdot \lfloor x \rfloor$;
(11) $0 < \rho \to \exists y(\lfloor x \rfloor \le \rho \cdot \lfloor y \rfloor < \lfloor x \rfloor + \rho \cdot 1)$;
(12) $x \equiv_\rho y \leftrightarrow \exists z(x = y + \rho \cdot \lfloor z \rfloor)$;
(13) $u \le^{D,H}_{y;\sigma} v \leftrightarrow \exists x(u \le x \le v \wedge D(x) \wedge H(x; y; \sigma))$ (for all pairs (D, H) as above).

A formula $a \le^{D,H}_{c;\delta} b$ in which a, b, the terms in $c = (c_1, \dots, c_n)$, and the terms in $\delta = (\delta_1, \dots, \delta_k)$ are $\mathcal{M}^{\mathrm{II}}$-terms not containing $\lfloor\cdot\rfloor$, is a congruence inequality whose moduli are the moduli of the congruences in $H(x; c; \delta)$.

The ordered $\mathbb{Z}$-module of reals yields a model of T^{II} in which the reals form the module domain and the integers form the scalar domain. Repeating the argument for Theorem 2.1 within T^{II}, one may prove Theorem 2.2.

Theorem 2.2 Let $\psi(v, w, \theta)$ be a conjunction of congruence inequalities with free variables among the module variables $v = (v_1, \dots, v_p)$ and $w = (w_1, \dots, w_q)$ and the scalar variables $\theta = (\theta_1, \dots, \theta_s)$. There are $l \ge 1$, quantifier-free formulas $\delta_i(\theta)$ of scalar sort (for $i = 1, \dots, l$), scalar terms $\tau_{ij}(\theta)$ (for $i = 1, \dots, l$ and $j = 1, \dots, m_i$), and conjunctions $\varphi_i(w, \theta)$ of congruence inequalities, with moduli among the $\tau_{ij}(\theta)$s such that T^{II} implies

$$\forall\theta\Big[\Big(\bigvee_{i=1}^{l} \delta_i(\theta)\Big) \wedge \bigwedge_{i\ne j} \neg\big(\delta_i(\theta) \wedge \delta_j(\theta)\big) \wedge \bigwedge_{i=1}^{l} \Big\{\delta_i(\theta) \to \wedge_{j=1}^{m_i} \tau_{ij}(\theta) \ne 0 \wedge \forall w(\exists v\, \psi(v, w, \theta) \leftrightarrow \varphi_i(w, \theta))\Big\}\Big].$$

Note that just as Weispfenning's quantifier elimination continues to work in an expansion of $\mathcal{M}$ by unary function symbols corresponding to scalar multiplication by individual rationals ([8], Corollary 3.4),[2] so Theorem 2.2 will continue to work in the expansion of $\mathcal{M}^{\mathrm{II}}$ by unary function symbols f_r ($r \in \mathbb{Q}$), of module sort, if one adjoins to T^{II} the universal closures of

$$f_0(x) = 0 \wedge f_1(x) = x$$
$$f_{c+d}(x) = f_c(x) + f_d(x) \wedge f_{cd}(x) = f_c(f_d(x)) \text{ (for } c, d \in \mathbb{Q}).$$

3 May the Scalars Form a Field?

One might hope to make Theorem 2.2 more symmetric between module elements and scalars by allowing the latter to belong to an ordered field in which a discretely ordered subring is distinguished, relative to whose elements congruences may be defined. A convenient means toward this end would replace T^{II} by $T^{\mathrm{II},+}$, a theory in an expansion $\mathcal{M}^{\mathrm{II},+}$ of $\mathcal{M}^{\mathrm{II}}$ obtained by adding, to the scalar sort, function symbols $\lfloor\cdot\rfloor$ and $^{-1}$ taking scalar terms as arguments and producing scalar terms. To obtain $T^{\mathrm{II},+}$ from T^{II}, adjoin axioms

(14) $0^{-1} = 0 \wedge (\rho \neq 0 \rightarrow \rho \cdot \rho^{-1} = 1)$,

(15) $0 = \lfloor 0\rfloor \wedge 1 = \lfloor 1\rfloor \wedge \lfloor\lfloor\rho\rfloor + \lfloor\sigma\rfloor\rfloor = \lfloor\rho\rfloor + \lfloor\sigma\rfloor \wedge \lfloor -\lfloor\rho\rfloor\rfloor = -\lfloor\rho\rfloor \wedge \lfloor\lfloor\rho\rfloor\cdot\lfloor\sigma\rfloor\rfloor = \lfloor\rho\rfloor \cdot \lfloor\sigma\rfloor \wedge \lfloor\lfloor\rho\rfloor \cdot \lfloor x\rfloor\rfloor = \lfloor\rho\rfloor \cdot \lfloor x\rfloor$,

(16) $0 \leq \rho - \lfloor\rho\rfloor < 1$,

(17) $0 < \lfloor\rho\rfloor \rightarrow 1 \leq \lfloor\rho\rfloor$,

and replace (6) by

(6)′ $g(\rho,\sigma) = g(\sigma,\rho) \wedge \lfloor\rho\rfloor = \gamma(\rho,\sigma)g(\rho,\sigma) \wedge \lfloor\sigma\rfloor = \gamma(\sigma,\rho)g(\rho,\sigma) \wedge 1 = \alpha(\rho,\sigma)\gamma(\rho,\sigma) + \beta(\rho,\sigma)\gamma(\sigma,\rho) \wedge \lfloor g(\rho,\sigma)\rfloor = g(\rho,\sigma) = g(\lfloor\rho\rfloor,\lfloor\sigma\rfloor) \wedge \lfloor\gamma(\rho,\sigma)\rfloor = \gamma(\rho,\sigma) = \gamma(\lfloor\rho\rfloor,\lfloor\sigma\rfloor) \wedge \lfloor\alpha(\rho,\sigma)\rfloor = \alpha(\rho,\sigma) = \alpha(\lfloor\rho\rfloor,\lfloor\sigma\rfloor) \wedge \lfloor\beta(\rho,\sigma)\rfloor = \beta(\rho,\sigma) = \beta(\lfloor\rho\rfloor,\lfloor\sigma\rfloor)$,

(10) by

(10)′ $0 = \lfloor 0\rfloor \wedge 1 = \lfloor 1\rfloor \wedge \lfloor\lfloor x\rfloor + \lfloor y\rfloor\rfloor = \lfloor x\rfloor + \lfloor y\rfloor \wedge \lfloor -\lfloor x\rfloor\rfloor = -\lfloor x\rfloor$,

(11) by

(11)′ $0 < \lfloor\rho\rfloor \rightarrow \exists y(\lfloor x\rfloor \leq \lfloor\rho\rfloor \cdot \lfloor y\rfloor < \lfloor x\rfloor + \lfloor\rho\rfloor \cdot 1)$,

and (12) by

(12)′ $x \equiv_\rho y \leftrightarrow \exists z(x = y + \lfloor\rho\rfloor \cdot \lfloor z\rfloor)$.

The ordered real vector space of reals yields a model $\mathcal{R}^+$ of $T^{\mathrm{II},+}$ in which the ordered group of reals with an integer-part operation forms the module domain and the ordered field of reals with an integer-part operation forms the scalar domain.

If one repeats the argument for Theorem 2.2 in this new context one encounters an obstacle in the reduction to the case of $t = 1$ and formulas $\varphi(w, v) = (\sharp)$ (see Section 2). Because

$$\rho \neq 0 \rightarrow (x \equiv_\sigma y \leftrightarrow \rho \cdot x \equiv_{\rho\sigma} \rho \cdot y)$$

no longer holds for arbitrary ρ, σ, and there is no obvious substitute when ρ or σ is irrational, the manipulations of Section 2 seem insufficient. In fact, an example inspired by Theorem 7.1 of [8] shows that $T^{\mathrm{II},+}$ will not obey the analogue of Theorem 2.2 in which all the relevant formulas are $\mathcal{M}^{\mathrm{II},+}$-formulas and the integer parts of the moduli of the congruences are to be nonzero. For let $\psi(v_1, v_2, w, \theta)$ be

$$2v_1 \equiv_2 0 \wedge 2v_2 \equiv_2 0 \wedge w \leq v_2 \wedge \theta \cdot v_2 = v_1.$$

Suppose that there are quantifier-free formulas $\delta_1(\theta), \ldots, \delta_l(\theta)$ of scalar sort, scalar terms $\tau_{ij}(\theta)$ (for $i = 1, \ldots, l$ and $j = 1, \ldots, m_i$), and conjunctions $\varphi_i(w, \theta)$ of congruence inequalities, with moduli among the $\tau_{ij}(\theta)$s,[3] such that $T^{\mathrm{II},+}$ implies

$$\forall\theta\big[\big(\bigvee_{i=1}^{l}\delta_i(\theta)\big) \wedge \bigwedge_{i\neq j}\neg\big(\delta_i(\theta)\wedge\delta_j(\theta)\big) \wedge \bigwedge_{i=1}^{l}\big\{\delta_i(\theta) \to \wedge_{j=1}^{m_i}\lfloor\tau_{ij}(\theta)\rfloor \neq 0$$
$$\wedge\, \forall w(\exists v_1\exists v_2\psi(v_1, v_2, w, \theta) \leftrightarrow \varphi_i(w, \theta))\big\}\big].$$

Then $T^{\mathrm{II},+}$ also implies $(*) =$

$$\forall\theta\big[\big(\bigvee_{i=1}^{l}\delta_i(\theta)\big) \wedge \bigwedge_{i\neq j}\neg\big(\delta_i(\theta)\wedge\delta_j(\theta)\big) \wedge \bigwedge_{i=1}^{l}\big\{\delta_i(\theta) \to \wedge_{j=1}^{m_i}\lfloor\tau_{ij}(\theta)\rfloor \neq 0$$
$$\wedge\, (\exists v_1\exists v_2\psi(v_1, v_2, 1, \theta) \leftrightarrow \varphi_i(1, \theta))\big\}\big],$$

where $\exists v_1\exists v_2\psi(v_1, v_2, 1, \theta) =$

$$\exists v_1\exists v_2(2v_1 \equiv_2 0 \wedge 2v_2 \equiv_2 0 \wedge 1 \leq v_2 \wedge \theta\cdot v_2 = v_1)$$

defines, in the scalar domain of $\mathcal{R}^+$, the set of rational numbers.

To analyze this example one needs the following lemma.

Lemma 3.1 If $\tau(\theta)$ is a scalar term of $\mathcal{M}^{\mathrm{II},+}$ in one variable θ, then there is a co-countable open set $U \subseteq \mathbb{R}$, on every connected component of which $\tau(\theta)$ defines in $\mathcal{R}^+$ a continuous rational function.

Proof The argument goes by induction on the complexity of $\tau(\theta)$. The terms of complexity zero—$0, 1, \theta$—certainly obey the result, and the induction steps for the ring operations are easy. Suppose $\tau(\theta)$ obeys the result and C is a connected component of U. If $\tau(\theta)$ is identically zero on C then $\tau(\theta)^{-1}$ and $\lfloor\tau(\theta)\rfloor$ also are identically zero on C: so suppose that $\tau(\theta)$ is not identically zero on C. $\tau(\theta)$ has only finitely many zeros $a_1 < \cdots < a_k$ in C, and on each interval $C \cap (-\infty, a_1)$, (a_i, a_{i+1}) $(i \leq i < k)$, and $C \cap (a_k, \infty)$ $\tau(\theta)^{-1}$ defines a continuous rational function. Thus $\tau(\theta)^{-1}$ obeys the result relative to a co-countable open set obtained from U by removing finitely many points from each connected component of U. Let $r : C \to \mathbb{R}$ be the continuous rational function defined by $\tau(\theta)$ on C. Since r is a rational function, $r^{-1}\{n\}$ is finite for each $n \in \mathbb{Z}$; since r is also continuous, $r^{-1}(\mathbb{Z})$ is closed and isolated in C. Thus $V = C - r^{-1}(\mathbb{Z})$ is open and co-countable in C, and on every connected component of V $\lfloor\tau(\theta)\rfloor$ defines a continuous, integer-valued rational function (which therefore is constant). Finally, if δ is any one of g, α, β, γ then since

$$\delta(\lfloor\rho\rfloor, \lfloor\sigma\rfloor) = \delta(\rho, \sigma)$$

by $(6)'$, $\delta(\tau_1(\theta), \tau_2(\theta))$ is constant and integer-valued over any set on which $\lfloor\tau_1(\theta)\rfloor$ and $\lfloor\tau_2(\theta)\rfloor$ are constant and integer-valued. □

Applying Lemma 3.1 to the finitely many scalar terms appearing in $(*)$, one finds a nonempty open interval J on which all of these terms define continuous rational functions. Because each $\delta_i(\theta)$ is a propositional combination of identities and inequalities between these scalar terms, each set $\{\alpha \in J : \mathcal{R}^+ \models \delta_j[\alpha]\}$ is a finite union of points and open intervals; so since

$$J = \coprod_{i=1}^{l} \{\alpha \in J : \mathcal{R}^+ \models \delta_j[\alpha]\},$$

one may without loss of generality assume that $\delta_1(\theta)$ is true at every point of J. Thus each $\lfloor \tau_{1j}(\theta) \rfloor$ has a fixed value $n_j \in \mathbb{Z} - \{0\}$, and for $\alpha \in J$

$$\alpha \text{ is rational iff } \mathcal{R}^+ \models \varphi_1(1, \alpha).$$

$\varphi_1(1, \theta)$ is equivalent over J to a conjunction $\wedge_k \beta_k(\theta) \leq_{1;\theta}^{D_k, H_k} \gamma_k(\theta)$, where the moduli of the Hs are among the n_js and each scalar term appearing in the conjunction defines a continuous rational function on J. Use the definition (13) to replace $\beta_k(\theta) \leq_{1;\theta}^{D_k, H_k} \gamma_k(\theta)$ by the formula

$$\exists x_1 \ldots \exists x_m (\beta_k(\theta) \leq x \leq \gamma_k(\theta) \wedge D_k(x) \wedge H_k(x; 1; \theta)),$$

where $H_k(x; 1; \theta)$ may be written in the form

$$\bigwedge_l \sum_{i=1}^{m} \zeta_{li}(\theta) x_i + \zeta_l(\theta) \cdot 1 \equiv_{m_l} 0,$$

each m_l being one of the n_js and each ζ defining a continuous rational function on J. Fix an irrational $\alpha \in J$ and let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of rationals in J with limit α. Since every α_j obeys $\varphi_1(1, \theta)$, there are corresponding reals $z_1^{(j)}, \ldots, z_m^{(j)}$ for which

$$\beta_k(\alpha_j) \leq z^{(j)} \leq \gamma_k(\alpha_j) \wedge D_k(z^{(j)}) \wedge \bigwedge_l \sum_{i=1}^{m} \zeta_{li}(\alpha_j) z_i^{(j)} + \zeta_l(\alpha_j) \cdot 1 \equiv_{m_l} 0.$$

Since $\alpha_j \to \alpha \in J$ and β_k, γ_k, and the ζs are continuous on J, $\beta_k(\alpha_j) \to \beta_k(\alpha)$, $\gamma_k(\alpha_j) \to \gamma_k(\alpha)$, and each $\zeta(\alpha_j) \to \zeta(\alpha)$; so by selecting a subsequence if necessary one may assume that $\{z^{(j)}\}_{j=1}^{\infty}$ has a limit z. Thus each $\left\{ \sum_{i=1}^{m} \zeta_{li}(\alpha_j) z_i^{(j)} + \zeta_l(\alpha_j) \cdot 1 \right\}_{j=1}^{\infty}$ is a convergent sequence in the closed subset $m_l \mathbb{Z}$ of $\mathbb{R}$, and one concludes that

$$\bigwedge_l \sum_{i=1}^{m} \zeta_{li}(\alpha) z_i + \zeta_l(\alpha) \cdot 1 \equiv_{m_l} 0.$$

Because weak inequalities are preserved under limits, α obeys $\beta_k(\theta) \leq_{1;\theta}^{D_k, H_k} \gamma_k(\theta)$; k being arbitrary, one concludes that α is rational, contrary to hypothesis. So $T^{\mathrm{II},+}$ does not obey the most obvious analogue of Theorem 2.2.

A crucial ingredient in this argument—and in the argument behind Theorem 7.1 of [8]—is that some nonempty open interval of reals is partitioned by the $\delta_i(\theta)$s into finitely many open intervals and points. Though one might block this conclusion by weakening $(6)'$ so that g, α, β, γ are not determined by the integer parts of their arguments, this requirement seems reasonable in a context where one may assume that congruences have integer moduli. While one might respond to the example by suggesting the consideration of formulas $\delta_i(\theta)$ that contain quantifiers, the logical simplicity of $\exists v \, \psi(v, 1, \theta)$ complicates the formulation of nontrivial analogues of Theorem 2.2 that might hold in $\mathcal{R}^+$.

4 Concluding Remarks

The projection result for $\mathbb{Z}$ implies analogues of Farkas's Lemma: see Corollary 4.2 and Corollary 5.2 of [5] and Corollary 7.4 of [4]. The same arguments, using the results of Section 2, will yield mixed-integer analogues of Farkas's Lemma.[4]

Just as one may exploit Fourier-Motzkin elimination to solve linear programs (see [2], Section 4-4), so one might use Theorem 2.1 to solve mixed-integer linear programs when the parameters of the problem are rational. This restriction is included so that the congruence inequalities to which one reduces the given problem are decidable: certainly a condition like $2x \equiv_2 0$ ("x is an integer") is decidable in $\mathbb{Q}$, but not generally in $\mathbb{R}$, and the decidability in general of congruence inequalities over the rationals follows from Weispfenning's proof that the $\mathcal{M}$-theory of the reals is decidable (see his more general Corollary 3.5). Yet the quantifier-elimination method behind Theorem 2.1 is even less efficient than Fourier-Motzkin elimination: while the latter may approximately square the number of inequalities whenever a variable is eliminated ([2], op. cit.), the former exponentiates the number of congruence inequalities whenever a variable is eliminated. So if Fourier-Motzkin elimination is regarded as an inefficient means of solving linear programs ([2], op. cit.), the method behind Theorem 2.1 will be even less practical.

One might hope to learn more about the structure of feasibility sets for mixed-integer programs by studying the structure of finite intersections of generalized half-spaces. The former may be defined, with parameters in $\mathcal{R}'$, by conjunctions of weak inequalities and congruences, and these are special cases of congruence inequalities. While Weispfenning's quantifier-elimination theorem for $\mathcal{M}$-formulas already provides useful structural information—see his Theorem 6.1—the study of a restricted class of formulas allowing the definition of feasibility sets, but not of all $\mathcal{M}$-definable sets, may help to focus attention on properties of feasibility sets not shared by arbitrary $\mathcal{M}$-definable sets.

Notes

1. The square brackets of [8] are here replaced by $\lfloor\cdot\rfloor$.

2. A model-theoretic proof of a similar result appears in the Appendix to [3]. I am grateful to the referee for pointing out Exercise III.4.15 in [7], which clearly sketches an earlier effective quantifier elimination analogous to these results from [3] and [8].

3. As in Section 2 the module-sort symbol $\lfloor\cdot\rfloor$ is not to occur in the φ_is, though now the scalar-sort symbol $\lfloor\cdot\rfloor$ may occur in them.

4. Aschenbrenner [1] establishes a Farkas-type characterization of the mixed-integer solvability of systems of linear equations over the reals.

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