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Dual Gaggle Semantics for Entailment

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Abstract A *sequent calculus* for the positive fragment of entailment together with the Church constants is introduced here. The single cut rule is admissible in this consecution calculus. A *topological dual gaggle semantics* is developed for the logic. The category of the topological structures for the logic with frame morphisms is proven to be the *dual category* of the variety, that is defined by the equations of the algebra of the logic, with homomorphisms. The *duality results* are extended to the logic of entailment that includes a De Morgan negation.

1 Introduction

There is a deeply ingrained association between conjunction and intersection. Indeed, this way of thinking about conjunction goes back to Boole's interpretation of his own algebra in terms of classes. The connection between conjunction and intersection has been reinforced by the possible worlds semantics for various modal logics. (Of course, we mean by "conjunction" a connective that possesses the same properties as conjunction in classical logic does.) Models in which conjunction is interpreted as intersection may be called meet-representations (of the Lindenbaum algebra of the logic) following Birkhoff and Frink [11].

However, sometimes intersection stands for some other operation in a model, perhaps, for technical reasons. Of course, a Boolean algebra is self-dual, and so switching the interpretation of conjunction and disjunction in classical logic is easy. Many important logics are not extensions of classical logic; hence, changing the interpretation of conjunction sometimes leads to unexpected results. An interesting example is furnished by the so-called Kleene logic, that may be thought to formalize the logic of regular expressions (or regular languages). Concatenation is a regular operation on languages (defined from concatenation of strings), which is associative and has an identity element $\{\varepsilon\}$. Another regular operation is $+$ (i.e., the union of regular

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languages), and the third one is the Kleene star. In the absence of a conjunction-like operation, a straightforward move is to replace filters (theories) with ideals (co-theories) in structures for the logic, that is, to "dualize" the usual semantics. Further, $+$ is interpreted as intersection in a model, and the finite binary consequence relation of the logic becomes $\supseteq$ (i.e., the converse of set inclusion). Although it is possible to complete the dualization for the Kleene logic—moreover, in the process exciting new technical results emerge—the resulting semantics is not as smooth and agreeable as, perhaps, expected.[1]

The aim of this paper is to investigate the possibility of building a *dual gaggle* semantics for two logics ($\mathbf{E}^{\boldsymbol{t}}_{+}$ and $\mathbf{E}^{\boldsymbol{t}}$) in the entailment family.[2] The construction of a sound and complete dual gaggle semantics for the positive fragment of the logic of ticket entailment proved to be successful. Therefore, our expectation is that the dual gaggle semantics for $\mathbf{E}^{\boldsymbol{t}}_{+}$ and $\mathbf{E}^{\boldsymbol{t}}$ will be pleasing too—independently of the "modal character" of entailment that is absent in the logic of ticket entailment. (I hope that the reader will agree after reading this paper that the semantics defined here for $\mathbf{E}^{\boldsymbol{t}}_{+}$ and $\mathbf{E}^{\boldsymbol{t}}$ are indeed well-behaved from a logical point of view.)

First, we define a dual gaggle semantics for $\mathbf{E}^{\boldsymbol{t}}_{+}$, the *positive fragment of the logic of entailment* (including the constants $\boldsymbol{t}$, $\boldsymbol{F}$, and $\boldsymbol{T}$), and we also give a topological characterization of the structures. Then in Section 3, we give a new formalization of $\mathbf{E}^{\boldsymbol{t}}_{+}$ in the form of a consecution calculus. Finally, we consider $\mathbf{E}^{\boldsymbol{t}}$, the *logic of entailment* (with the constants $\boldsymbol{t}$, $\boldsymbol{T}$, $\boldsymbol{F}$, and $\boldsymbol{f}$ included), and we define a dual gaggle semantics together with a topology in Section 4. We prove that the variety of $\mathbf{E}^{\boldsymbol{t}}$ (of $\mathbf{E}^{\boldsymbol{t}}_{+}$) algebras with homomorphisms and the class of topological structures for $\mathbf{E}^{\boldsymbol{t}}$ (for $\mathbf{E}^{\boldsymbol{t}}_{+}$) are *dual categories*, and the canonical constructions between them are *functors*. In each section, we add components one by one to provide ample justifications for the components of the semantics. Additionally, we anticipate that the piecemeal presentation yields deep insights (that have been "distilled" from the "technicalities") into the logics themselves. In order to minimize distraction from the development of the ideas, we gather the lemmas and theorems together with brief outlines of their proofs into separate subsections toward the end of the sections. We briefly overview the main results of the paper in Section 5.

2 Positive Entailment

Not all logics contain a conjunction, but some do—together with a disjunction and, perhaps, other connectives. Given the familiar Boolean (classical) experience of dualizing the interpretation of $\wedge$ and $\vee$, our first thought may be to interpret a lonely disjunction through intersection. We pursue this idea, but we add an interpretation for *conjunction as union*. As it shall become clear by the end of this section, the semantics of $\mathbf{E}^{\boldsymbol{t}}_{+}$ defined on this basis is as nice as the usual one.

To explicate the dualization more formally, we start with the assumption that there is a conjunction ($\wedge$) and a disjunction ($\vee$) in the language of a logic and they possess all the properties that they do in classical logic (and which are expressible without negation). The customary interpretation of conjunction and disjunction over some set of situations U is as follows.[3]

(1) $u \vDash \varphi \wedge \psi \quad \Leftrightarrow \quad u \vDash \varphi$ and $u \vDash \psi$,
(2) $u \vDash \varphi \vee \psi \quad \Leftrightarrow \quad u \vDash \varphi$ or $u \vDash \psi$.

The definitions of $\cap$ and $\cup$ make clear that $\wedge \mapsto \cap$ and $\vee \mapsto \cup$—as long as formulas are interpreted into sets of situations. Swapping $\cap$ and $\cup$ may change which formulas are true in some situation in a concrete model; however, the more abstract properties of intersection and union (such as commutativity, associativity etc.) are the same, which yields a certain invariance between the two models. This means that the following interpretations are equally good.

(3) $u \vDash \varphi \wedge \psi \quad \Leftrightarrow \quad u \vDash \varphi$ or $u \vDash \psi$,
(4) $u \vDash \varphi \vee \psi \quad \Leftrightarrow \quad u \vDash \varphi$ and $u \vDash \psi$.

Perhaps a first impression is that (3) and (4) can hardly make sense informally. Oftentimes, situations are thought to be deductively closed, and in particular, there may be situations that comprise exactly what a formula implies (or a little more). If so, a situation has to verify both conjuncts to verify their conjunction—as stated by (1). On the other hand, if situations are thought to include all the formulas that imply a formula (or slightly more), then for a disjunction to be true in a situation both disjuncts should be true—as (4) states—because either disjunct implies the disjunction. (2) and (3) can be phrased in a similar fashion. To emphasize the difference between the two informal ways of thinking about what a situation is, we will call situations *co-situations* when we think of the situations as objects that are compatible with (3) and (4).

If there are further logical connectives, then the "picture" gets more complicated, because intensional operations—such as relevant implication and entailment—are modeled using additional information about the co-situations. More precisely, a *relation* on the set of co-situations is exploited. One of the goals of the present enterprise is to pinpoint the effects the dualization of the interpretation of $\wedge$ and $\vee$ has on the modeling of the intensional connectives in the semantics of the logic of entailment. To elucidate the role of each ingredient, we build the semantics step by step, to start with, for $\mathbf{E}_{+}^{\boldsymbol{t}}$.

Definition 2.1 (Priestley space) The ordered topological space $\mathfrak{T} = \langle U, \leq, \mathfrak{O} \rangle$ is a *Priestley space* if and only if it is compact and totally order disconnected.[4]

A Priestley space gives rise to a *distributive lattice* with the elements being clopen cones (see Priestley [23]). In other words, if formulas containing $\wedge$ and $\vee$ are interpreted by clopen cones, then all the formulas expressing the usual properties of $\wedge$ and $\vee$ become true.[5] Either (1) and (2), or (3) and (4), would be appropriate interpretations for $\wedge$ and $\vee$, but from now on we assume the latter pair of clauses.

There are pairs of theorems in $\mathbf{E}_{+}^{\boldsymbol{t}}$ (indeed, $\aleph_0$ many pairs) such that neither of the two elements implies the other. Thus a subset O of U (the set of co-situations) has to be selected to reflect theoremhood, that (canonically) may be thought to be the principal cone generated by $\{\varphi : \ \vdash \varphi \to \boldsymbol{t}\}$ (where $\vdash$ means that the formula that follows is provable in, i.e., a theorem of $\mathbf{E}_{+}^{\boldsymbol{t}}$). Formally, we postulate that $O \subseteq U$ and also $O \neq \varnothing$; therefore, U is not empty either.

The *provable formulas* are (or provably equivalent to) implications; hence we add a ternary relation R (with tonicity $R{\downarrow}{\uparrow}{\uparrow}$) that allows the distinguished co-situations to be connected to the order relation via (5). ($V^{\uparrow}$ denotes the upset or cone generated by V, given $V \subseteq U$. We follow the usual conventions to omit parentheses and to occasionally insert dots.)

(5) $\exists o \notin O^{\uparrow}.\, Rxyo \quad \Leftrightarrow \quad x \leq y.$

In order to be able to utilize this condition, we introduce the notion of *satisfaction* for implicational formulas. The last three clauses already foretold that the present semantics is considerably different from the usual relational semantics for relevance logics, and (6) reinforces this point.

(6) $z \vDash \varphi \to \psi \quad \Leftrightarrow \quad \exists x, y.\, Rxyz \wedge y \nvDash \varphi \wedge x \vDash \psi.$

If we assume that an *assignment* maps every propositional variable p into an upset of co-situations, then it is straightforward to show that the set of co-situations at which $\varphi \to \psi$ holds is an upset too. (7) ensures that $\boldsymbol{t}$'s interpretation is upward closed too.

(7) $u \vDash \boldsymbol{t} \quad \Leftrightarrow \quad \exists o \in O.\, o \leq u.$

For the sake of comparison, it may be useful to emphasize that the dual character of the whole semantics leaves unchanged the upward closed character of propositions—which is often cast as a "heredity lemma." (Quite obviously, the extensional connectives yield upsets too, as the interested reader might wish to verify.)

To illustrate the role of (5), we outline the proof that $\varphi \to \varphi$ is *valid*, that is, that every co-situation that makes $\varphi \to \varphi$ true is in the upward closure of the distinguished subset O. Let us assume that $o \vDash \varphi \to \varphi$ but $o \notin O^{\uparrow}$. By (6), there are x and y such that $Rxyo$ and $x \vDash \varphi$ without $y \vDash \varphi$ being true. However, $o \notin O^{\uparrow}$ and $Rxyo$ together with (5) imply that $x \leq y$. Then $y \vDash \varphi$, which is a contradiction, and so o has to be an element of $O^{\uparrow}$.

Before we proceed to postulating two other groups of conditions, we show that the pieces of the construction that we outlined so far can be built from $\mathbf{E}^{\boldsymbol{t}}_{+}$. First of all, we define prime co-theories as follows.

(8) x is a *prime co-theory* (i.e., $x \in \mathfrak{P}_{\mathrm{co}}$) iff (i) – (iii) hold, where
(i) $x \neq \varnothing$ and $\exists \varphi.\, \varphi \notin x$,
(ii) $\varphi \in x$ and $\vdash \psi \to \varphi$ imply $\psi \in x$,
(iii) $\varphi \wedge \psi \in x$ implies that either $\varphi \in x$ or $\psi \in x$.

The "smallest" theorem $\boldsymbol{t}$, which implies all the theorems of $\mathbf{E}^{\boldsymbol{t}}_{+}$, may be proved to have the following property: $\vdash \varphi \wedge \psi \,.\!\to \boldsymbol{t}$ only if either $\vdash \varphi \to \boldsymbol{t}$ or $\vdash \psi \to \boldsymbol{t}$. The definition of prime co-theories together with this property of $\mathbf{E}^{\boldsymbol{t}}_{+}$ means that the set of all formulas that provably imply $\boldsymbol{t}$ is a prime co-theory. This justifies our claim above that the set of co-situations O can be thought of to be generated (canonically) by a minimal prime co-theory. We define R on $\mathfrak{P}_{\mathrm{co}}$ by (9).

(9) $Rxyz \quad \Leftrightarrow \quad \forall \varphi, \psi.\, \varphi \notin y \wedge \psi \in x \,.\!\Rightarrow \varphi \to \psi \in z.$

Having this definition of R, we can prove that (5) holds. Let us assume $x \leq y$ (i.e., $x \subseteq y$). As an approximation to a co-theory that we seek to be an instantiation of $\exists o$, let o' be the set $\{\xi\colon \exists \varphi_1, \varphi_2 \notin y\ \exists \psi_1, \psi_2 \in x.\ \vdash \xi \to (\varphi_1 \to \psi_1 \,.\vee.\, \varphi_2 \to \psi_2)\}$. A tacit assumption in (9) is that x, y, and z are prime co-theories. However, if we disregard for a moment this assumption with respect to z, then we can write $Rxyo'$ by (9). If $\boldsymbol{t}$ were an element of o', then the formula in the definition—with $\boldsymbol{t}$ in place of ξ—would be a theorem. $\varphi_1 \wedge \varphi_2 \,.\!\to \varphi_1$ and $\varphi_1 \wedge \varphi_2 .\!\to \varphi_2$ are provable, and $\varphi_1 \wedge \varphi_2 \notin y$, because $\varphi_1 \notin y$ and $\varphi_2 \notin y$ according to the definition of o'. Similarly, $\psi_1 \vee \psi_2 \in x$ by the definition of o', and of course a disjunction is implied by either disjunct. If we denote $\varphi_1 \wedge \varphi_2$ by φ, and $\psi_1 \vee \psi_2$ by ψ, then $\vdash \boldsymbol{t} \to \varphi \to \psi$ has to be provable too. By the definition of prime co-theories, $\varphi \in x$, and therefore, by the starting assumption, $\varphi \in y$. In sum, $\boldsymbol{t}$ is not an element of o'. It is easy to verify that (9) gives the required tonicity to R; in particular, R is monotone in its last argument place. Maximizing o' while preserving the exclusion of $\boldsymbol{t}$, we obtain a prime co-theory o, which is obviously in the relation R to x and y; that is, $Rxyo$ holds (due to $Rxyo'$ and $o' \subseteq o$).

To complete the construction of a model of the logic $\mathbf{E}_{+}^{\boldsymbol{t}}$ we have to define an *assignment* function. We denote this function by η and define it as follows.

(10) $x \vDash_{\eta} p \quad \Leftrightarrow \quad p \in x,$

where p is a propositional variable, $x \in \mathfrak{P}_{\mathrm{co}}$, and $x \vDash_{\eta} p$ is a notation for $x \in \eta(p)$.

We use $x \vDash_{\eta} p$ to stress that η is the restriction to propositional variables of the *valuation* function for all formulas, or equivalently, of the *satisfaction* relation. If η is extended according to (3), (4), (6), and (7), then (11) holds (where $x \in \mathfrak{P}_{\mathrm{co}}$ is assumed as before). Then—as intended—the form of (11) closely resembles that of (10).

(11) $x \vDash \varphi \quad \Leftrightarrow \quad \varphi \in x.$

The demonstration of (11) is by induction, and the steps are obvious or easy except for formulas that have $\to$ as their main connective; hence we sketch this case. From left to right, we start with supposing $z \vDash \varphi \to \psi$; that is, $\exists x, y.\, Rxyz \wedge x \vDash \psi \wedge y \nvDash \varphi$, by (6). The inductive hypothesis and (9) yield $\varphi \to \psi \in z$.

For the right-to-left direction, let the initial assumption be $\varphi \to \psi \in z$. φ generates a theory and ψ generates a co-theory, let us say, υ' and x', respectively. Obviously, we have $\forall \varphi \in \upsilon' \forall \psi \in x'.\, \varphi \to \psi \in z$. By maximizing the pair $\langle \upsilon', x' \rangle$, while keeping its relation to z (expressed by the previous formula), we obtain $\langle \upsilon, x \rangle$ which may be shown in the usual way to be a prime theory and a prime co-theory, respectively. Taking the complement of υ for y, $\varphi \notin y$; that is, by the hypothesis of the induction, $y \nvDash \varphi$. Also, from $\psi \in x$ we get $x \vDash \psi$, and it follows by (6), that $z \vDash \varphi \to \psi$, as we intended to prove.

The *topological characterization* of the structures builds upon Priestley spaces. The definitions of $\wedge$ and $\vee$ given in (3) and (4) suffice for those operations, because the set of clopen upsets is closed under $\cap$ and $\cup$. To guarantee a similar closure property for $\to$, we postulate (12).

(12) $\{\, x \colon \exists y \notin V_1\, \exists z \in V_2.\, Rzyx \,\}$ is a clopen cone, when so are V_1 and V_2.

The consequent could be slightly weakened by omitting the requirement of upward closure, because R was stipulated to have a certain tonicity, in particular, $R\_\_\uparrow$.

The function η and its extension through $\vDash$—which we also denote by η for a moment—is important because $\eta\varphi_1$ is distinct from $\eta\varphi_2$ whenever $\nvdash \varphi_1 \to \varphi_2$ or $\nvdash \varphi_2 \to \varphi_1$. Given a Priestley space with O and R added as above, a function μ may be defined that plays a similar—but dual—role to η.[6] Let $\mu(x)$ be the set of clopen cones V such that $x \in V$. For μ to be "well-behaved," μ has to be a *homeomorphism* and a *relational isomorphism* between the starting topological structure and the canonical structure of the algebra of the clopen cones. (13) ensures that μ fully preserves R.

(13) $\neg Rxyz \quad \Rightarrow \quad \exists V_1, V_2.\, x \in V_1 \wedge y \notin V_2 \wedge z \notin V_2 \to V_1,$

where V_1 and V_2 are clopen cones.

To illustrate the need for (13), we show that $R\mu x \mu y \mu z \Rightarrow Rxyz$. Let us assume $\neg Rxyz$. To establish that $R\mu x \mu y \mu z$ does not obtain, there should exist some clopen upsets V_1 and V_2 such that $V_1 \in \mu x$ but $V_2 \notin \mu y$ and $V_2 \to V_1 \notin \mu z$. Suitable sets are guaranteed to exist by (13). (It may be noted here that given an arbitrary set of co-situations U with $\leq$, O, R and a set of opens, and having the properties of compactness and total order disconnectedness, there is no reason for (13) to hold, unless stipulated.) The other direction, that is, $Rxyz \Rightarrow R\mu x\mu y\mu z$, holds due to (6), (9), and (12) together with the definition of μ, as the reader can easily verify.

The conditions we have postulated for O so far do not give a sufficiently tight characterization of the distinguished co-situations. To ensure that among the clopen cones there is a set that can function as $\boldsymbol{t}$, we need (14), and the interaction of O and μ requires (15) to be postulated.

(14) $O^{\uparrow}$ is a clopen subset.
(15) $O = O^{\uparrow}$.

The constructions described above provide *soundness*, *completeness*, as well as a *topological characterization* of the structures—except that we have not yet taken care of certain features of the entailment operation. Implicational theorems of $\mathbf{E}^{\boldsymbol{t}}_{+}$ include *suffixing*, *contraction*, and *specialized assertion*. (Indeed, these implicational formulas together with detachment suffice as an axiomatization of the implicational fragment of the logic of entailment.) These theorems do not require the addition of new components to the already defined structure. Rather they can be accommodated by stipulating that the accessibility relation satisfies some further conditions. That is, we make R more specific by postulating (16) – (18).

(16) $Rzvy \wedge Rwuz\,. \ \Rightarrow\ \exists x.\, Rwxv \wedge Rxuy.$
(17) $Rzvy \ \Rightarrow\ \exists x.\, Rxvy \wedge Rzvx.$
(18) $\exists o \notin O.\, Ryoy.$

These conditions again look quite unlike the clauses usually stipulated in a semantics that relies on situations. Nonetheless, it is straightforward to verify that (16) – (18) suffice to validate the characteristic implicational axioms of $\mathbf{E}^{\boldsymbol{t}}_{+}$ and that these conditions are true on the canonical structure when the accessibility relation is defined as in (9). It may be interesting to note that instances of *squeeze-type* lemmas involve theories or co-theories not merely according to the dual character of the whole semantics, but additionally determined by the distribution type of the operations.

Given a syntactic presentation of a logic—such as that for $\mathbf{E}^{\boldsymbol{t}}_{+}$ in the next section—it is natural to use η in the proof of the completeness theorem with respect to a class of structures. However, typically—and concretely, for $\mathbf{E}^{\boldsymbol{t}}_{+}$—$\eta$ is not injective; that is, it assigns the same value to distinct formulas in some cases. Often, it is easier to deal with a function that is invertible. Taking the converse relation of η and grouping together all the objects that are in the relation η^{-1} to the same cone of prime co-theories yields the same set of objects (due to completeness) as does forming equivalence classes based on mutual provability between formulas. Just as the connectives may be easily redefined for the new objects, so may η be adjusted in an obvious way too. (We denote the resulting function by η'.)

Typical steps of proving a logic sound and complete with respect to a class of structures include building a *model* from a structure and building a *structure* from (the algebra of) the logic. The topological characterization of structures, on one hand, sharpens the description of the model constructed from the canonical structure of the logic by "pruning" the codomain of η' so that η' becomes onto. On the other hand, when μ is a homeomorphism and a relational isomorphism between a structure and the canonical structure of its algebra, then the class of structures is delineated more precisely than commonly.[7]

Valuations are *homomorphisms*, and functions that preserve operations (and constants, if there are any) are of interest for investigating properties of the objects they link. Similarly, maps between structures contribute to the study of their features. The map m that we are to introduce has a certain similarity to *rp-morphisms* that were introduced by Mares in [21]. (Incidentally, his paper gives an excellent illustration of how the investigation of maps between structures contributes both to the understanding of the structures themselves and to the understanding of the logics for which the structures are defined.) However, our motivation to define a certain kind of maps is independent of considerations of Halldén completeness and rp-morphisms, and it stems from our aim to obtain a functorial duality result. Thus, we proceed to investigate and stipulate properties for functions between structures which guarantee that the duality of objects can be extended to them as well.

Our main interest is in the *logics* and *their semantics* (not in classes of structures), which means that we will not want to vary the homomorphisms. Accordingly, we postulate (19) and (20) to limit the set of permitted frame morphisms.[8] We take m to be a map from $\langle X_1, \leq_1, R_1, O_1, \mathfrak{O}_1\rangle$ into $\langle X_2, \leq_2, R_2, O_2, \mathfrak{O}_2\rangle$ where both topological spaces are of the sort described so far. A *frame morphism* m is stipulated to be *continuous* and *order-preserving* as well as to satisfy (19) – (20).

(19) $R_1xyz \quad \Rightarrow \quad R_2mxmymz.$

(20) $R_2xymz \quad \Rightarrow \quad \exists u, v.\, R_1uvz \wedge x \leq_2 mu \wedge mv \leq_2 y.$

Conditions (19) and (20) may be thought to encode the entailment operation into frame morphisms, so to speak. Indeed, it is easy to verify that these are exactly the conditions that can ensure that the inverse image of a frame morphism (restricted to clopen cones, of course) preserves entailment. It is interesting to note that the order relation is a crucial element in (20).

Lastly, before we state some theorems and lemmas, we should dissipate a potential ambiguity. The zeroary constants $\boldsymbol{F}$ and $\boldsymbol{T}$—which are quite like their classical analogues—are not always included in relevance logics. Classically, every theorem is a notational variant of $\boldsymbol{T}$, but this is not true in relevance logics. Although $\boldsymbol{T}$ (if included) is a theorem of $\mathbf{E}^{\boldsymbol{t}}_{+}$, it does not imply all the theorems; hence it has a reduced significance. However, in order to make our topological representation results to straightforwardly expand previous ones—especially, Priestley's duality theorems for distributive lattices—we assume that $\boldsymbol{F}$ and $\boldsymbol{T}$ are in $\mathbf{E}^{\boldsymbol{t}}_{+}$. The properties of these constants are easy to characterize: $\boldsymbol{F}$ entails every formula, whereas $\boldsymbol{T}$ is entailed by all formulas. The interaction with entailment is summarized by the two formulas $\boldsymbol{T} \to \boldsymbol{F} \to \varphi$ and $\boldsymbol{T} \to \varphi \to \boldsymbol{T}$. In a model, U and $\varnothing$ interpret $\boldsymbol{F}$ and $\boldsymbol{T}$, respectively.

2.1 Pertinent theorems and lemmas

Theorem 2.2 (Soundness) *If φ is a theorem of $\mathbf{E}^{\boldsymbol{t}}_{+}$, then given any structure and valuation, $u \vDash \varphi$* implies *$u \vDash \boldsymbol{t}$.*

Proof The structure of the proof is usual modulo dualization.[9] □

This theorem does not make use of the topological component of a structure. However, the selection of a certain set of subsets of co-situations characterizes the structures more precisely, which is the content of the next lemma.

Theorem 2.3 (From structures to structures) *Every structure is* homeomorphic *and* relationally isomorphic *to a structure consisting of sets.*

Proof The structure of sets is defined as the set of proper nonempty prime ideals (the analogues of prime co-theories) of clopen cones. By the soundness theorem, we know that there is an $\mathbf{E}^{t}_{+}$ algebra that emerges from a structure, and the clauses that refer to the topological component guarantee that the elements in the algebra are clopen cones. From the proof of the next theorem follows that the proper nonempty prime ideals of the Lindenbaum algebra of $\mathbf{E}^{t}_{+}$ (or equivalently, the prime co-theories of $\mathbf{E}^{t}_{+}$) may be shown to satisfy all the conditions that are essential to the proof of soundness. A function that may be shown to be a homeomorphism is μ, which is defined as follows.

$$\mu x = \{\, V : x \in V \text{ and } V \text{ is a clopen cone} \,\}.$$

(μ may be easily proven to be continuous and 1–1, which suffices to show that μ is a homeomorphism because a Priestley space is Hausdorff.) Furthermore, μ and its inverse preserve all three relations of a structure. (Cf. the remarks following (13).) □

Theorem 2.4 (Completeness) *If φ is not a theorem of $\mathbf{E}^{t}_{+}$, then* there is *a structure, a valuation, and a co-situation u such that $u \vDash \varphi$* and *$u \nvDash \boldsymbol{t}$.*

Proof The set of proper nonempty prime co-theories $\mathfrak{P}_{\mathrm{co}}$ forms a structure (without a topology), with inclusion, with O being the set of those $x \in \mathfrak{P}_{\mathrm{co}}$ which contain $\boldsymbol{t}$, and with R defined as $\{\, \langle x, y, z \rangle : \forall \varphi \notin y \, \forall \psi \in x . \, \varphi \to \psi \in z \,\}$. The topology may be added by specifying its base as $\{\, \eta\varphi \cap -\eta\psi : \varphi, \psi \text{ are formulas} \,\}$, where η is the canonical valuation (defined above). The "type-lifted" version of this valuation (which we denoted by η') is an isomorphism between the algebra of $\mathbf{E}^{t}_{+}$ and the algebra of clopen cones of the structure on $\mathfrak{P}_{\mathrm{co}}$. (In other words, the topology delineates the codomain of the canonical valuation function exactly.) □

At this point, we know that it is possible to go back and forth between certain structures and algebras; that is, they are each other's counterpart (or more formally, they are duals). Moreover, in a precise sense (of an isomorphism and of a homeomorphism) the second duals are (essentially) the same as the original algebra or the original structure. This *object level duality* can be extended to functions that connect the objects. Indeed, by postulating (19) and (20) we already anticipated that we intend to consider such an extension, the first step toward which is the next lemma.

Lemma 2.5 *If m is a frame morphism, then m^{-1} is a* homomorphism *between the algebras of clopen cones on the structures. Distinct ms yield distinct m^{-1}s.*

Proof The bulk of the proof consists of a straightforward verification that the inverse image of a frame morphism preserves the operations and constants. □

Lemma 2.6 *If h is a homomorphism between algebras, then h^{-1} is a* frame morphism *between the structures emerging from the algebras. Distinct hs yield distinct h^{-1}s.*

The proof this lemma is quite similar (dually) to the previous proof.

Lemma 2.7 (Harmony of second duals) *η' (μ) are* in harmony *with the second duals of homomorphisms (of frame morphisms) between algebras (between structures).*

Proof The gist of the proof (for η') is to show that η' *commutes* with h and $h^{-1\,-1}$. The proof crucially depends on the definition of η' as well as on the appropriate characterization of the codomain of h^{-1}. Assuming that $[\varphi]$ is φ's equivalence class, $\eta'(h([\varphi])) = h^{-1\,-1}(\eta'([\varphi]))$. Similarly, for μ one can prove that $\mu(m(x)) = m^{-1\,-1}(\mu(x))$. $\square$

Theorem 2.8 (Functorial duality) *The category of* $\mathbf{E}^{\boldsymbol{t}}_{+}$ *algebras with homomorphisms (i.e., maps preserving* $\wedge, \vee, \rightarrow, \boldsymbol{t}, \boldsymbol{T}$*, and* $\boldsymbol{F}$*) is the* dual of the category *of structures for* $\mathbf{E}^{\boldsymbol{t}}_{+}$ *with frame morphisms.*

This theorem is a culmination of the results concerning $\mathbf{E}^{\boldsymbol{t}}_{+}$ and its dual gaggle semantics in this section, and a proof of the theorem may be obtained by combining the proofs of the above lemmas and theorems together with verifying that the "back-and-forth moves" between the categories can be proven to be functors in the sense of category theory. The inverse image constructions explain why one category turns out to be the dual of the other.

3 Formalizations

We formalize $\mathbf{E}^{\boldsymbol{t}}_{+}$ by extending the *sequent calculus* $LE^{\boldsymbol{t}}_{\rightarrow}$.[10] To add the extensional connectives $\wedge$ and $\vee$, we adapt the solution that was introduced by Dunn [14] (see also Dunn [15], §3.10). That is, the consecution calculus for $\mathbf{E}^{\boldsymbol{t}}_{+}$, that we are to define here, contains two structural connectives. The combinatory rules for B' and W in the LC calculi introduced in Dunn and Meyer [18] guide us in shaping the structural rules for the intensional structures. The straightforward parallel between combinatory rules (for proper combinators) and implicational formulas (that are their principal type schemes) may seem to have become obscured in $LE^{\boldsymbol{t}}_{\rightarrow}$ with the $\hat{\boldsymbol{t}}_r\vdash$ rule. However, specialized assertion is the principal type scheme of a combinator, let us say $\mathsf{4}$, with the axiom $\mathsf{4}x \ \triangleright\ x\mathsf{I}$ (where I is the unary identity combinator).[11] The corresponding rule (i.e., the only structural rule beyond $\check{\boldsymbol{t}}_l \vdash$ that is necessary to prove that specialized assertion is a theorem in $LE^{\boldsymbol{t}}_{\rightarrow}$) should be reminiscent of a dual combinatory rule applied in the reverse direction, as it really is.

Bounding a distributive lattice is often considered a trivial move (as in Stone [25]), and adding $\boldsymbol{F}$ and $\boldsymbol{T}$ to the language of a logic is similarly easy. For technical reasons, which we mentioned in the preceding section, we want to formalize positive entailment so that it contains these so-called Church constants. Although adding $\boldsymbol{T}$ is easy, incorporating "absurdity" is a bit more complicated.[12] To hint at one of the problems, we note that $\boldsymbol{F}$ and $\rightarrow$ are expected to interact in a way which can be described by saying that $\rightarrow$ is a 0–1 operation (with $\rightarrow\colon 0, 1 \ \longrightarrow\ 1$) in the algebra of entailment. Since intensional and extensional structures may be embedded into each other repeatedly—moreover, intensional structures cannot be viewed equivalently as "flat" structures—the usual [ ] notation for a single occurrence of a structure in a structure or the use of the structural connective ; is not sufficient here. In other words, a complication results from $(\varphi \rightarrow \psi \rightarrow \xi) \rightarrow \psi \rightarrow \varphi \rightarrow \xi$ not being a theorem of entailment, which means that ; cannot be taken to be a polyadic connective.

We define $LE^{\boldsymbol{t}}_{+}$ to be a *single right-handed* consecution calculus. The logic of entailment does not permit unrestricted permutations and thinnings; thus, to ensure that $\wedge$ and $\vee$ have their "usual" properties, two punctuation marks may be used to build up the antecedent of a consecution. (See [15], §3.10.)

Definition 3.1 (Structures)

(i) If φ is a formula, then φ is a structure.
(ii) If $\alpha_1, \ldots, \alpha_n$ are structures, so is $\alpha_1, \ldots, \alpha_n$.
(iii) If α_1 and α_2 are structures, so is $(\alpha_1; \alpha_2)$.

Informally, (ii) defines structures that are extensional—they are formed by , . Due to the lack of parentheses in (ii), repeated applications of (ii) yield one extensional structure from several.[13] Although an extensional structure containing several ,s conceals the history of its construction (when viewed outside of the context of a derivation), this would not prevent us from defining the notion of an *occurrence of a structure* in a structure unambiguously. However, we do not need to go into the details of the definition of occurrences here; we remark only that a particular occurrence of a structure α in a structure β is indicated by $\beta[\alpha]$, as usual. Now we recall the axiom and the rules of $LE^{\boldsymbol{t}}_{\to}$ from [7], §1.

$$\varphi \vdash \varphi \ \text{id} \qquad\qquad \frac{\alpha[\beta] \vdash \varphi}{\alpha[\boldsymbol{t}; \beta] \vdash \varphi}\ \check{\boldsymbol{t}}_l\vdash$$

$$\frac{\alpha \vdash \varphi \quad \beta[\psi] \vdash \xi}{\beta[\varphi \to \psi; \alpha] \vdash \xi}\ \to\vdash \qquad\qquad \frac{\alpha; \varphi \vdash \psi}{\alpha \vdash \varphi \to \psi}\ \vdash\to$$

$$\frac{\alpha[\gamma; (\beta; \delta)] \vdash \varphi}{\alpha[\beta; \gamma; \delta] \vdash \varphi}\ \mathrm{B}'\vdash \qquad \frac{\alpha[\beta; \gamma; \gamma] \vdash \varphi}{\alpha[\beta; \gamma] \vdash \varphi}\ \mathrm{W}\vdash \qquad \frac{\alpha[\beta; \boldsymbol{t}] \vdash \varphi}{\alpha[\beta] \vdash \varphi}\ \hat{\boldsymbol{t}}_r\vdash$$

Conjunction and *disjunction* are added by the structure-free rules (which share their "contexts").[14]

$$\frac{\alpha[\varphi] \vdash \xi}{\alpha[\varphi \wedge \psi] \vdash \xi}\ \wedge_1\vdash \qquad \frac{\alpha[\varphi] \vdash \xi}{\alpha[\psi \wedge \varphi] \vdash \xi}\ \wedge_2\vdash \qquad \frac{\alpha \vdash \varphi \quad \alpha \vdash \psi}{\alpha \vdash \varphi \wedge \psi}\ \vdash\wedge$$

$$\frac{\alpha[\varphi] \vdash \xi \quad \alpha[\psi] \vdash \xi}{\alpha[\varphi \vee \psi] \vdash \xi}\ \vee\vdash \qquad \frac{\alpha \vdash \varphi}{\alpha \vdash \psi \vee \varphi}\ \vdash\vee_1 \qquad \frac{\alpha \vdash \varphi}{\alpha \vdash \varphi \vee \psi}\ \vdash\vee_2$$

The above rules do not involve the structural connective , , but the following structural rules do, and they are inevitable for the provability of distributivity of $\wedge$ and $\vee$.

$$\frac{\alpha[\beta, \gamma] \vdash \varphi}{\alpha[\gamma, \beta] \vdash \varphi}\ T\vdash \qquad \frac{\alpha[\beta, \beta] \vdash \varphi}{\alpha[\beta] \vdash \varphi}\ M\vdash \qquad \frac{\alpha[\beta] \vdash \varphi}{\alpha[\beta, \gamma] \vdash \varphi}\ K\vdash$$

In combinatory logic (or in the structurally free logics), it is convenient to be able to talk about a term (or a structure) without exhibiting all the details of its shape.[15] An intensional structure that is formed, for example, from $(\alpha; \beta)$ and γ is characterized—to a certain extent—by α, β, and γ. However, $(\gamma; (\alpha; \beta))$ and $((\alpha; \beta); \gamma)$ are, obviously, different structures, and in $LE^{\boldsymbol{t}}_{+}$ there is no rule of permutation that would allow us to disregard the order of the substructures. To indicate that a structure is composed by repeated applications of (i) and (iii) from all of $\alpha_1, \ldots, \alpha_n$, we use the notation $\langle\!\langle\, \alpha_1, \ldots, \alpha_n \,\rangle\!\rangle$. For our purposes it is unimportant to describe a structure as composed of as few substructures as possible; therefore, we assume that $\alpha_1, \ldots, \alpha_n$ is a complete listing of occurrences of some substructures. In other words, each α occurs exactly once in $\langle\!\langle\, \alpha_1, \ldots, \alpha_n \,\rangle\!\rangle$ (and we allow the possibility that some αs have the same shape). In limited contexts, we use a $\langle\!\langle\ \ \rangle\!\rangle$ expression for one fixed structure that may be characterized in this fashion.

The replacement of one of the αs by a $\langle\!\langle\ \ \rangle\!\rangle$ expression preserves the defining feature of the $\langle\!\langle\ \ \rangle\!\rangle$ expressions. That is, if two structures may be described as $\langle\!\langle \alpha_1, \ldots, \alpha_l, \ldots, \alpha_n \rangle\!\rangle$ and $\langle\!\langle \beta_1, \ldots, \beta_m \rangle\!\rangle$, then $[\langle\!\langle \beta_1, \ldots, \beta_m \rangle\!\rangle/\alpha_l]\langle\!\langle \alpha_1, \ldots, \alpha_l, \ldots, \alpha_n \rangle\!\rangle$ is $\langle\!\langle \alpha_1, \ldots, \beta_1, \ldots, \beta_m, \ldots, \alpha_n \rangle\!\rangle$ (in which α_l no longer occurs). Informally, the replacement corresponds to supplanting a leaf in an ordered binary tree by an(other) ordered binary tree. The so obtained tree is binary and ordered, and its leaves comprise the leaves of the first tree save the one replaced, and those of the second tree. By this description we aim to emphasize that combining structures that are completely intensional up to a certain depth yields a like structure, which is useful in the formulation of one of the axioms and in the proof of the cut theorem.

The following two axioms add *absurdity* and *triviality*, and they complete the description of the set of axioms and rules for $LE^{\boldsymbol{t}}_{+}$. (n is a natural number; that is, $\boldsymbol{F} \vdash \varphi$ is an instance of $\boldsymbol{F} \vdash$.)

$$\langle\!\langle \boldsymbol{F}, \alpha_1, \ldots, \alpha_n \rangle\!\rangle \vdash \varphi \quad {\scriptstyle \boldsymbol{F}\vdash} \qquad\qquad \alpha \vdash \boldsymbol{T} \quad {\scriptstyle \vdash\boldsymbol{T}}.$$

To make use of the consecution calculus, the following *cut rule* has to be added and proven eliminable (or it has to be proven admissible).

$$\frac{\alpha \vdash \varphi \qquad \beta[\varphi] \vdash \psi}{\beta[\alpha] \vdash \psi}\ \text{cut}.$$

The above cut rule is sometimes called *single cut* (to distinguish it from other rules such as versions of multiple cut and the mix rule). This is "the cut rule" that was introduced by Gentzen, except that (i) our consecutions are right singular (completely unlike sequents in LK though somewhat like those in LJ), and (ii) our use of more refined structures necessitates the exploitation of the square brackets.

Provability of a consecution is defined as usual; that is, a consecution is provable when there is a derivation of the consecution solely from the axioms. Furthermore, a provable consecution always has a cut-free proof. The proof of this theorem is by *triple induction* (on the degree of the cut formula (δ), and on the rank (ϱ) and the contraction measure (μ) of the cut) for the fragment that does not include the constants. The inclusion of the constants requires an extra lemma to ensure that certain proofs that contain a cut on $\boldsymbol{t}$ can be transformed into cut-free proofs.[16]

It may be interesting to note that $\boldsymbol{t}$ in $LE^{\boldsymbol{t}}_{\to}$ causes fewer complications in the proof of the admissibility of the single cut than it does in $LE^{\boldsymbol{t}}_{+}$ despite $\boldsymbol{t}$'s essential presence in both consecution calculi. $\boldsymbol{t}$ plays a role in the definition of theoremhood: φ is a theorem of $LE^{\boldsymbol{t}}_{+}$ if and only if the consecution $\boldsymbol{t} \vdash \varphi$ is provable. Interestingly, the rule $\hat{\boldsymbol{t}}_r \vdash$, which is distinctive for entailment, does not interfere at all with the inductive proof of the cut theorem, and it does not destroy the subformula property for theorems.

A *Hilbert-style formalization* for $\mathbf{E}^{\boldsymbol{t}}_{+}$ may be obtained by taking $\mathbf{E}_{\to 2}$ (see Anderson and Belnap [1], §8.3.3) and adding to it axioms and a rule for $\wedge$ and $\vee$ (see [1], §27.1). $\boldsymbol{t}$ may be added by the rules $\vdash \varphi$ if and only if $\vdash \boldsymbol{t} \to \varphi$ (where $\vdash$ indicates theoremhood in the Hilbert-style system), together with the axiom $\boldsymbol{t} \to \boldsymbol{t}\,. \to \boldsymbol{t}$.[17] To add the relevantly less interesting constants, the following axioms—already hinted at above—might be considered.

$$\boldsymbol{F} \to \varphi \qquad\quad \varphi \to \boldsymbol{T} \qquad\quad \boldsymbol{T} \to \boldsymbol{F} \to \varphi \qquad\quad \boldsymbol{T} \to \varphi \to \boldsymbol{T}$$

In relevance logics, the genuine connective to combine premises is *fusion* (sometimes called co-tenability), and it may be added to the axiom system described so far by the usual rules: $\vdash \varphi \to \psi \to \xi$ if and only if $\vdash \varphi \circ \psi\,. \to \xi$.

The corresponding consecution rules for $LE^{\boldsymbol{t}}_{+}$ are $\circ\vdash$, $\vdash\circ$, and $\mathrm{b}_{\boldsymbol{t}}\vdash$.

$$\frac{\alpha[\varphi;\psi]\vdash\xi}{\alpha[\varphi\circ\psi]\vdash\xi}\ {\circ\vdash}\qquad\qquad \frac{\alpha\vdash\varphi\quad \beta\vdash\psi}{\alpha;\beta\vdash\varphi\circ\psi}\ {\vdash\circ}\qquad\qquad \frac{\alpha[\boldsymbol{t};\beta;\gamma]\vdash\varphi}{\alpha[\boldsymbol{t};(\beta;\gamma)]\vdash\varphi}\ \mathrm{b}_{\boldsymbol{t}}{\vdash}$$

The logic of entailment is centered around $\to$, and so $\circ$ is not always added. The inclusion of $\circ$ is advantageous sometimes. For instance, the proof of the equivalence of an axiomatic and a sequent calculus becomes more transparent. The expansion of $LE^{\boldsymbol{t}}_{+}$ with the rules $\circ\vdash$ and $\vdash\circ$, however, forces the addition of $\mathrm{b}_{\boldsymbol{t}}\vdash$ as well, which in turn further complicates the proof of the admissibility of the single cut rule. (Two new subinductions on χ and ε seem to be needed, where χ is the height of the derivation of the left premise when $\varrho_r \geq 2$ and the cut is on $\boldsymbol{t}$, whereas ε is its extensional depth.)

Assuming the presence of $\circ$, the following three conditions must be added to the definition of a structure for the logic.

(21) $u \vDash \varphi\circ\psi \quad\Leftrightarrow\quad \forall y,z.\, Ruyz \wedge y \nvDash \psi. \Rightarrow z \vDash \varphi$.

(21) is the definition of $\circ$ for sets of co-situations on a structure (and should be compared to (6) for $\to$). The Hilbert-style rules for fusion make very clear that $\to$ is the right residual of $\circ$ (and of course this is provable in the consecution calculus too). Residuation implies that we may assume that the R in (6) and the R in (21) are one and the same relation, which considerably simplifies the whole semantics. However, for a complete topological characterization of the structures, two further stipulations are needed, when fusion is included.

(22) $\{x : \forall y \notin V_2\, \forall z \notin V_1.\, \neg Rxyz\}$ is a clopen cone, if so are V_1 and V_2.
(23) $R_2mxyz \quad\Rightarrow\quad \exists u,v.\, R_1xuv \wedge mu \leq_2 y \wedge mv \leq_2 z$.

(22) is analogous to (12), whereas (23) parallels (20). If $\circ$ and the corresponding conditions (21) – (23) are added, then all the lemmas and theorems from Section 2.1 are true—with certain applicable modifications. (Cf. Section 4.1.)

3.1 Pertinent theorems and lemmas

Theorem 3.2 (Cut-free proofs) *A provable consecution of $LE^{\boldsymbol{t}}_{+}$ has a* cut-free *proof.*

Proof The proof is by a *triple induction* on δ (the degree of the cut formula), ϱ (the rank of the cut), and μ (the contraction measure of the cut) together with an induction on κ (the height of the proof of the right premise when $\varrho = 2$ and the cut formula is $\boldsymbol{t}$). The presence of $\boldsymbol{t}$ in other relevant consecution calculi (for example, in $LT^{\circ\boldsymbol{t}}_{\to}$ and $LB^{\circ\boldsymbol{t}}_{+}$) leads to an extra induction; however, here the difficulty arises from the interaction between $\boldsymbol{t}$, $\boldsymbol{F}$, and the distributivity of $\wedge$ and $\vee$.[18] □

Theorem 3.3 (Equivalence) *If φ is a* theorem *of the axiomatic system $\mathbf{E}^{\boldsymbol{t}}_{+}$, then $\boldsymbol{t}\vdash\varphi$ is a* provable consecution *in $LE^{\boldsymbol{t}}_{+}$, and vice versa. Therefore, all theorems of $\mathbf{E}^{\boldsymbol{t}}_{+}$ have cut-free proofs in $LE^{\boldsymbol{t}}_{+}$.*

The proof of the equivalence can proceed along the usual lines. It may be interesting to note though that prefixing (that is, a theorem of entailment) has a much shorter proof in the sequent calculus than (the proof I know of) in the axiom system. A reason behind the discrepancy is that if there are no combinators in the consecution calculus (unlike in structurally free logics), then the correspondence between structural rules and proof terms, therefore, the formulas-as-types correspondence, becomes less precise.

4 Treatments of Negation

Negation often causes problems. We consider an extension of the axiomatic formulation with the following three axioms.

$$\varphi \to {\sim}\psi \,.\!\to\, \psi \to {\sim}\varphi \qquad\qquad {\sim}{\sim}\varphi \to \varphi \qquad\qquad \varphi \to {\sim}\varphi \,.\!\to\, {\sim}\varphi$$

The three axioms, respectively, express that $\sim$ is self-residuated; $\sim$ is identified with two operations from a different family; and $\sim$ obeys reductio. In algebraic terms (i.e., in the Lindenbaum algebra of $\mathbf{E}^{\boldsymbol{t}}$) $\sim$ is an involution (without being an identity). To word the same observation differently, $\wedge$, $\vee$, and $\sim$ span a De Morgan lattice, the order of which coincides with the order induced by provable entailments.

De Morgan lattices are elegant because their negation forces the lattice to be self-dual. At the same time, the distributivity of the lattice (that is, the provability of the formula $\varphi \wedge (\psi \vee \xi) \,.\!\to.\, (\varphi \wedge \psi) \vee \xi$) allows a relatively uncomplicated treatment of negation.

To obtain a structure on which negation is definable for certain propositions, we add a new *binary relation* R' onto the set of co-situations. (Formally, $R' \subseteq U \times U$ and R' is monotone in its second argument place.) (24) defines negation on propositions, which are sets of co-situations.[19]

(24) $u \vDash {\sim}\varphi \;\;\Leftrightarrow\;\; \exists x.\, R'xu \wedge x \nvDash \varphi.$

Due to the postulated tonicity of R', if $u \vDash {\sim}\varphi$ and $u \leq x$, then also $x \vDash {\sim}\varphi$. The addition of clauses (25) – (27) ensures that negation defined on propositions satisfies the three new axioms.

(25) $R'xy \;\Rightarrow\; \exists z.\, R'zx \wedge z \leq y.$
(26) $\exists y.\, R'yx \wedge \forall z.\, R'zy \Rightarrow x \leq z.$
(27) $R'xz \wedge x \leq y \,.\!\Rightarrow \exists w, v.\, Rwvy \wedge v \leq z \wedge x \leq w.$

To guarantee that the set of clopen cones is closed under the negation operation, we stipulate (28). Further, (29) suffices for the extension of μ (to structures that include the relation R') to remain well-behaved.

(28) ${\sim}V_1$ is clopen whenever V_1 is a clopen cone.
(29) $\neg R'xy \;\Rightarrow\; \exists V_1.\, x \notin V_1 \wedge y \notin {\sim}V_1$, where V_1 is a clopen cone.

The structure defined from $\mathbf{E}^{\boldsymbol{t}}$ (similarly, as before) may be proven to satisfy the newly added conditions once we specify R' as $\{\, \langle x, y\rangle : \forall\varphi \notin x.\, {\sim}\varphi \in y \,\}$.

If a valuation η is defined as in (10), then with the above definitions it may be shown to be a homomorphism for negation. Although the elements of a proposition are prime co-theories, a natural proof proceeds via maximizing a theory. (This step is an analogue of the "primeness lemma" for a unary antitone operation.)

To ensure that the duality between the objects of the categories can be extended to *full duality*, the frame morphisms' interplay with the relation R' has to be circumscribed—as by (30) and (31).

(30) $R'_1xy \;\Rightarrow\; R'_2mxmy$.
(31) $R'_2xmy \;\Rightarrow\; \exists z.\, R'_1zy \wedge mz \leq_2 x$.

Given these additions, *functorial duality* can be proven.

For the sake of comparison, it should be noted that the modeling of negation above is not the "usual" one. In semantics of relevance logics, negation is typically modeled from a function on structures which is often denoted by * (see [15], §4.4 and [2], §48.5). * reflects that $\sim$ is perceived as a dual isomorphism. $R'xy$ has a straightforward translation in terms of * and $\leq$; furthermore, all the resulting conditions are true in structures with *. However, the converse of the translation seems incapable of expressing all the usual conditions that * is postulated to satisfy, and in this sense, R' comes with weaker assumptions than * does.[20]

A yet another way to model negation that promptly comes to mind is derived from the other distribution type of $\sim$, namely, $\sim\colon \wedge \longrightarrow \vee$. Using *dualized gaggle theory*, the fact that $\sim$ distributes into $\vee$ leads to its modeling via a universally quantified clause on propositions. This would necessitate replacing not only (24), but also (25) – (27) and (29). Lastly, the two conditions (30) and (31) should have been replaced then by $\neg R''_1xy \;\Rightarrow\; \neg R''_2mxmy$ and $\neg R''_2xmy \;\Rightarrow\; \exists z.\, \neg R''_1zy \wedge x \leq_2 mz$, where R'' is a new binary relation (in place of R').

Another possibility is to "strengthen" either of the primed relations into a function. A deep insight that we wish to reiterate is that $\{\, \varphi : {\sim}\varphi \notin x \,\}$ is not only a prime co-theory of $\mathbf{E^t}$ (when so is x), but in general, distinct from x—unlike it would be in a classical context. It is easy to see that if R' and R'' are "functionalized" into r, then both definitions of negation turn out to be the same: $u \vDash {\sim}\varphi \;\Leftrightarrow\; ru \nvDash \varphi$.

4.1 Pertinent theorems and lemmas

We state the analogues of theorems from Section 2.1 and briefly remark on the augmentations of the previous proofs.[21]

Theorem 4.1 (Soundness) *If φ is a theorem of $\mathbf{E^t}$, then given any structure and valuation, $u \vDash \varphi$* implies *$u \vDash \boldsymbol{t}$.*

The proof requires adding to the previous proof the cases for the $\circ$ rules and the $\sim$ axioms.

Theorem 4.2 (From structures to structures) *Every structure is* homeomorphic *and* relationally isomorphic *to a structure consisting of sets.*

The only necessary addition to the proof of Theorem 2.3 is that R' is preserved in both directions.

Theorem 4.3 (Completeness) *If φ is not a theorem of $\mathbf{E^t}$, then* there is *a structure, a valuation, and a co-situation such that $u \vDash \varphi$* and *$u \nvDash \boldsymbol{t}$.*

The case of fusion is unproblematic, and we already commented on some of the details in proving that η is a homomorphism for $\sim$.

Lemma 4.4 *If m is a frame morphism, then m^{-1} is a* homomorphism *between the algebras of clopen cones on the structures. Distinct ms yield distinct m^{-1}s.*

The new operations are preserved by inverse images due to the stipulations (19), (23), and (21), as well as (30), (31), and (24).

Lemma 4.5 *If h is a homomorphism between algebras, then h^{-1} is a* frame morphism *between the structures emerging from the algebras. Distinct hs yield distinct h^{-1}s.*

The conditions involving m that were just mentioned have to be proven to hold on the top of the previous proof. However, the homomorphism h preserves the new operations, and so no problems arise.

Lemma 4.6 (Harmony of second duals) *η' (μ) are* in harmony *with the second duals of homomorphisms (of frame morphisms) between algebras (between structures).*

The proof of this lemma is exactly as that of Lemma 2.7. The uniformity of the proofs reflects a certain universality of the lemma itself, which underscores that η' and μ are significant objects. On the other hand, it should be pointed out that now η' (μ) is an isomorphism (a homeomorphism) between algebras (between structures) with more stuff in it, so to speak.

Theorem 4.7 (Functorial duality) *The category of $\mathbf{E}^{\boldsymbol{t}}$ algebras with homomorphisms (i.e., maps preserving $\wedge, \vee, \rightarrow, \circ, \sim, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{F}$, and $\boldsymbol{f}$) is the* dual of the category *of structures for entailment with frame morphisms.*

The proof of this theorem is similar to the proof of Theorem 2.8.

5 Conclusions

This paper investigated the possibility of defining a semantics for the logic of entailment and its positive fragment based on co-situations. The functorial duality theorems we proved show that these semantics are as good from a logical point of view as the relational semantics in which situations are taken to be the points. The use of a topological characterization of structures forces the inclusion of the Church constants that are normally less important for relevance logics. We defined new consecution calculi for the positive fragment and for the positive fragment with fusion, in both of which these constants are included. The cut theorem is true for these consecution calculi.

Notes

1. A semantics based on ideals as well as two other semantics for the Kleene logic (and the closely related action logic) may be found in Bimbó and Dunn [9].

2. *Gaggle theory* was invented by Dunn [16]. A *dual gaggle semantics* for $\mathbf{T}_{+}^{\circ\boldsymbol{t}}$ is defined in Bimbó [8], §3.6.

3. $\varphi, \psi, \xi, \ldots$ range over formulas whereas $u, \upsilon, w, x, y, z, \ldots$ denote situations or co-situations. $u \vDash \varphi$ indicates that φ is true in the (co-)situation u.

4. Basic topological notions are defined in various places, for example, in Dunn and Hardegree [17], Ch. 13; Davey and Priestley [13], Appendix A; Clark and Davey [12], Appendix B; and Bimbó and Dunn [10], Ch. 9.

5. This claim is true, but strictly speaking, it cannot be proven until an interpretation for entailment is added. $\rightarrow$ is essential to (i.e., the main connective in) the formulas that turn into (in)equations in the Lindenbaum algebras of the logics $\mathbf{E}_{+}^{\boldsymbol{t}}$ and $\mathbf{E}^{\boldsymbol{t}}$.

6. We assume that it is understood that the definition of the base of a topology from a distributive lattice remains the same in the presence of intensional operations.

7. Categorically speaking, η' and μ can be understood as the functions that are suitable to prove *object duality* for a certain class of topological spaces and a certain class of algebras. (See Theorem 2.8 below.) An exposition of category theory is, for example, Goldblatt [19].

8. The careful reader might notice that (19) and (20) look quite similar to (2) and (3) in Definition 4 in [21]—despite the dual character of our semantics. An important difference is though that (20) cannot be strengthened by replacing $\leq_2$ with $=$, unless we restrict the class of homomorphisms too—the desirability of which we have already excluded for our purposes.

9. Here as well as below we omit most of the details of the proofs and provide only a sketch of the steps or some key definitions. (Notice that parts of some proofs were detailed earlier in this section.) Proofs of topological duality theorems—for various algebraic structures—may be found, for example, in [13] (for distributive lattices), in Urquhart [26] (for lattices), in Bimbó [6] (for ortholattices and De Morgan lattices), and in [10], Ch. 9 (for a variety of logics, or more accurately, for various classes of algebras that are connected to nonclassical logics via their Lindenbaum algebras). In the rest of this section (unless we specify otherwise), by "structures" we mean the structures that have been defined so far, and we call "algebras" or "$\mathbf{E}_+^{\boldsymbol{t}}$ algebras" the algebras that satisfy those equations that characterize the algebra of $\mathbf{E}_+^{\boldsymbol{t}}$.

10. This consecution calculus was introduced in Bimbó [7], §1. A consecution calculus for $\mathbf{E}_\to$ was enunciated in Kripke [20], and negation was added in Belnap and Wallace [4]. Merge-style consecution calculi for various relevance logics are described in [1], §7.3.

11. Some well-known implicational theorems of $\mathbf{E}_+^{\boldsymbol{t}}$ are principal type schemes of other improper combinators. For example, *restricted permutation* is so connected to $\mathsf{2}$ that has axiom $\mathsf{2}xyz \rhd xz(\mathsf{B}'\mathsf{I}y)$—see [8], §4.

12. As far as I know, no sequent calculus has been introduced previously for the positive fragment of any of the standard relevance logics with the constants $\boldsymbol{F}$ and $\boldsymbol{T}$ included.

13. This is quite like Gentzen's treatment of antecedents and succedents of sequents as strings (or sequences) of formulas. We use the notational convention of omitting parentheses from left-associated intensional structures.

14. Cf. Belnap [3] and Negri and von Plato [22].

15. A notation that is useful in specifying structural rules (in a uniform way) that correspond to proper (dual) combinators in structurally free logics was introduced immediately following Definition 2.3 in Bimbó [5]. However, we cannot utilize that notation now, because the presence of $\boldsymbol{F}$ in axiom $\boldsymbol{F} \vdash$ is a must.

16. This lemma has some resemblance to Lemma 4.7 in [5], but here the induction is on κ which is the *height* of the derivation of the right premise when $\varrho = 2$. Lemma 4.7 takes care of the elimination of the cut rule when the cut formula is a fixed-point combinator. Truth can be dealt with more or less in a like fashion because of the limited ways a

fixed-point combinator or $\boldsymbol{t}$ can be introduced. $\boldsymbol{t}$ has been included (with the cut theorem remaining true) into consecution formulations of some of the stronger relevance logics in [8], §3.2.

17. See also the overview of axiomatic systems in Anderson, Belnap, and Dunn [2], §R2.

18. For proofs of cut theorems for consecution calculi that require a somewhat similar induction on χ, see [5], [8], [7], and [10], Ch. 2.

19. An altogether different modeling of De Morgan negation ($\sim$) using a binary relation is given in [6], §4.

20. The usual semantics for relevance logics contains a function that is the central element of the definition of negation on sets of situations, that incorporates a well-known representation of De Morgan lattices. See [15], §4, Routley and Meyer [24], and for a topological characterization Urquhart [27].

21. The terms "structure" and "algebra" now refer to a structure and an algebra obtained by the additions in the last two sections.

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