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Weakening of Intuitionistic Negation for Many-valued Paraconsistent da Costa System

Zoran Majkić

Abstract In this paper we propose substructural propositional logic obtained by da Costa weakening of the intuitionistic negation. We show that the positive fragment of the da Costa system is distributive lattice logic, and we apply a kind of da Costa weakening of negation, by preserving, differently from da Costa, its fundamental properties: antitonicity, inversion, and additivity for distributive lattices. The other stronger paraconsistent logic with constructive negation is obtained by adding an axiom for multiplicative property of weak negation. After that, we define Kripke-style semantics based on possible worlds and derive from it many-valued semantics based on truth-functional valuations for these two paraconsistent logics. Finally, we demonstrate that this model-theoretic inference system is adequate—sound and complete with respect to the axiomatic da Costa-like systems for these two logics.

1 Introduction

Paraconsistent logics are those logics which reject the classical identification of contradictoriness and triviality (the fact that such theory entails all possible consequences). Thus, paraconsistency is the study of contradictory yet nontrivial theories. The big challenge for paraconsistent logics is to avoid allowing contradictory theories to explode and derive anything else and still to reserve a respectable logic, that is, a logic capable of drawing reasonable conclusions from contradictory theories.

It should perhaps be mentioned that Tarski himself considered the possibility of working with inconsistent theories while weakening classical logic so as to avoid triviality. He was not, however, inclined to consider as acceptable any theory that

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contained a contradiction. Such a rigid position has been criticized by other logicians who were more open to normal attitudes of mathematicians concerning contradictions, and they contributed to developing more robust, with respect to classic logic, paraconsistent systems.

There are different approaches to paraconsistent logics. The da Costa approach [9] is to maintain positive fragments of classic (or more appropriate, of intuitionistic) logic and to use weaker forms for non-truth-functional negation. Another trend comes from the relevance logics [1] where the focus is on implication rather than on negation. Adaptive logics ([3], [4]) are also interesting: they are not so concerned about proving consistency, but assume it instead from the very start as some kind of default. The most recent abstract consideration of paraconsistency occurs in LFI (Logic of Formal Inconsistency) systems [6], in particular, its many-valued logics [18] based on complete distributive lattices of algebraic truth-values.

Thus, from my point of view, there are two principal approaches to paraconsistent logic. The first is the nonconstructive approach based on abstract logic (as LFI [6]), where logic connectives and their particular semantics are not considered. The second is the constructive approach and is divided into two parts: axiomatic proof-theoretic (cases of da Costa [9] and [1], [3], [4]), and many-valued (case [18]) model theoretic based on truth-functional valuations (that is, it satisfies the truth-compositionality principle). The best case is when we obtain both proof- and model-theoretic definitions which are mutually sound and complete.

For the axiomatic da Costa system which is presented below, it has been proved [2] that none of these logical calculi is characterizable by *finite* matrices. Therefore, any many-valued semantics used for it must be done by means of a many-valued system with an infinite number of algebraic truth values. (It has to be based on an infinite complete distributive lattice as will be shown in this paper.)

One of the main founders with Jaskowski [16], da Costa built his propositional paraconsistent system C_ω in [9] by weakening the logic negation operator $\neg$ in order to avoid the explosive inconsistency ([6], [8]) of the classic propositional logic, where the ex falso quodlibet proof rule $\frac{A,\ \neg A}{B}$ is valid. In fact, in order to avoid this classic logic rule, he changed the semantics for the negation operator, so that

NdC1 in these calculi the principle of noncontradiction, in the form $\neg(A \wedge \neg A)$, should not be a generally valid schema, but if it does hold for formula A, it is a well-behaved formula and is denoted by A°;

NdC2 from two contradictory formulas, A and $\neg A$, it would not in general be possible to deduce an arbitrary formula B; that is, it does not hold the falso quodlibet proof rule $\frac{A,\ \neg A}{B}$;

NdC3 it should be simple to extend these calculi to corresponding predicate calculi (with or without equality);

NdC4 they should contain most parts of the schemata and rules of classical propositional calculus which do not interfere with the first conditions.

In fact, this paraconsistent propositional logic is made up of the unique modus ponens inferential rule (MP), $A, A \Rightarrow B \vdash B$, and two axiom subsets. The first one is for the positive propositional logic (without negation), composed of the following eight axioms, borrowed from the classic propositional logic of the Kleene L_4 system and also from the more general propositional *intuitionistic* system (these two systems differ only regarding axioms with the negation operator) which uses three binary connectives, $\wedge$ for conjunction, $\vee$ for disjunction, and $\Rightarrow$ for implication:

(PLA) Positive Logic Axioms

(1) $A \Rightarrow (B \Rightarrow A)$,
(2) $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C))$,
(3) $A \Rightarrow (B \Rightarrow (A \wedge B))$,
(4) $(A \wedge B) \Rightarrow A$,
(5) $(A \wedge B) \Rightarrow B$,
(6) $A \Rightarrow (A \vee B)$,
(7) $B \Rightarrow (A \vee B)$,
(8) $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C))$.

And change the original axioms for negation of the classic propositional logic by defining semantics of negation by the following subset of axioms:

(NLA) Logic Axioms for Negation

(9) $A \vee \neg A$,
(10) $\neg\neg A \Rightarrow A$,
(11) $B^{(n)} \Rightarrow ((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$, (Reductio relativization axiom)
(12) $(A^{(n)} \wedge B^{(n)}) \Rightarrow ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \Rightarrow B)^{(n)})$,

where $B^{(0)} = B$ and, recursively, $B^{(n+1)} = (B^{(n)})^{\circ}$ for $1 \leq n \leq \omega$, and B° abbreviate the formula $\neg(B \wedge \neg B)$.

It is easy to see that axiom (11) relativizes the classic reductio axiom $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$ (which is equivalent to the contraposition axiom $(A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A)$ and the trivialization axiom $\neg(A \Rightarrow A) \Rightarrow B$), *only* for propositions B such that $B^{(n)}$ is valid, and in this way avoids the validity of the classic ex falso quodlibet proof rule. It provides a qualified form of reductio, helping to prevent general validity of $B^{(n)}$ in the paraconsistent logic C_n. Axiom (12) regulates only the propagation of n-consistency. It is easy to verify that n-consistency also propagates through negation; that is, $A^{(n)} \Rightarrow (\neg A)^{(n)}$ is provable in C_n. So for any fixed n (from 0 to ω) we obtain a particular da Costa paraconsistent logic C_n.

One may regard C_ω as a kind of syntactic limit [7] of the calculi in the hierarchy. Each C_n is strictly weaker than any of its predecessors; that is, denoting by Th(S) the set of theorems of calculus S, we have

$$\mathrm{Th}(CPL) \supset \mathrm{Th}(C_1) \supset \cdots \supset \mathrm{Th}(C_n) \supset \cdots \supset \mathrm{Th}(C_\omega).$$

Thus, we are fundamentally interested in the C_1 system which is a paraconsistent logic closer to the CPL (classic propositional logic); that is, C_1 is the paraconsistent logic of da Costa's hierarchy obtained by minimal change of CPL.

For this da Costa calculi are not given any truth-compositional model theoretic semantics. If we consider the semantics based on the classic 2-valued complete distributive lattice $(\mathbf{2}, \leq)$ with the set $\mathbf{2} = \{0, 1\}$ of truth values, then the da Costa system can be represented as a kind of *intensional* logic (similar, for example, to intuitionistic logic). But it is still not given any Kripke semantics based on an (infinite) set of possible worlds $\mathcal{W}$ and accessibility relations for its modal operators for implication and weakened negation. Kripke semantics for intensional logic is still compositional but *relative* to possible worlds: the satisfaction of a given formula in a given world

$w \in \mathcal{W}$ is defined by the satisfaction of its principal subformulas in other possible worlds mediated by the defined binary accessibility relation of a Kripke frame.

The *non-truth-functional* bivaluations (mappings from the set of well-formed formulas of C_n into the set **2**) used in [10] and [17] induce the decision procedure for C_n known as quasi matrices instead. In this method, a negated formula within truth-tables must branch: if A takes the value 0, then $\neg A$ takes the value 1 (as usual), but if A takes the value 1, then $\neg A$ can take either the value 0 or the value 1; both possibilities must be considered, as well as the other axioms governing the bivaluations.

Consequently, the da Costa system still needs a kind of (relative) compositional model-theoretic semantics. In this paper we will explain some weak properties of its proposed weakening for a negation operator, so we will not address this problem of compositional model-theoretic semantics for C_n system. Instead, we will do it for the more appropriate da Costa weakening of negations, where fundamental negation properties such as antitonicity and truth inversion are preserved.

The plan of this paper is the following. After a short introduction to complete lattices and modal truth-functional algebraic logics based on Galois connections for modal operators in Section 2, we will define an algebra for the positive fragment of the da Costa System and show that it is distributive lattice logic. In Section 3 we will define the new weakening of negation for such distributive lattice logic, which, differently from da Costa negation, preserves antitonicity and additivity properties. We will define also another stronger paraconsistent da Costa logic, where weak negation is constructive, that is, self-adjoint for distributive lattice. In Section 4 we will define the Kripke possible world semantics for these two paraconsistent logics and based on it, the many-valued semantics based on functional hereditary distributive lattice of algebraic truth-values. Finally, in Section 5 we will show that this many-valued (and Kripke) semantics, based on model-theoretic entailment, is adequate, that is, sound and complete with respect to the proof-theoretic da Costa axiomatic systems of these two paraconsistent logics.

1.1 Introduction into lattice algebras and their extensions Posets and lattices (posets such that for all elements x and y, the set $\{x, y\}$ has both a join (lub—least upper bound) and a meet (glb—greatest lower bound)) with a partial order $\leq$ play an important role in what follows. A *bounded* lattice has a greatest (top) and least (bottom) element, denoted, by convention, by 1 and 0. Finite meets in a poset will be written as $1, \wedge$, and finite joins as $0, \vee$. A lattice (poset) X is *complete* if each (also infinite) subset $S \subseteq X$ (or $S \in \mathcal{P}(X)$ where $\mathcal{P}$ is the symbol for powerset and $\varnothing \in \mathcal{P}(X)$ denotes the empty set) has the least upper bound (supremum) denoted by $\bigvee S \in X$ (when S has only two elements the supremum corresponds to the join operator $\vee$). Each finite bounded lattice is a complete lattice. Each subset S has the greatest lower bound (infimum) denoted by $\bigwedge S \in X$, given as $\bigvee\{x \in X \mid \forall y \in S.x \leq y\}$. The complete lattice is bounded and has the bottom element, $0 = \bigwedge X \in X$, and the top element, $1 = \bigvee X \in X$.

A function $l : X \to Y$ between posets X, Y is *monotone* if $x \leq x'$ implies $l(x) \leq l(x')$ for all $x, x' \in X$. The function $l : X \to Y$ is said to have a right (or upper) adjoint if there is a function $r : Y \to X$ in the reverse direction such that $l(x) \leq y$ if and only if $x \leq r(y)$ for all $x \in X, y \in Y$. Such a situation forms a Galois connection and will often be denoted by $l \dashv r$. Then l is called the left (or lower) adjoint of r. If X, Y are complete lattices (posets) then $l : X \to Y$ has a right adjoint if and only if l preserves all joins. (It is *additive*; that is, $l(x \vee y) = l(x) \vee l(y)$ and $l(0_X) = 0_Y$, where $0_X, 0_Y$ are bottom elements in complete lattices X and Y, respectively.) The right adjoint is then $r(y) = \bigvee\{z \in X \mid l(z) \leq y\}$. Similarly, a monotone function $r : Y \to X$ is a right adjoint (it is *multiplicative*, i.e., has a left adjoint) if and only if r preserves all meets; the left adjoint is then $l(x) = \bigwedge\{z \in Y \mid x \leq r(z)\}$.

Each monotone function $l : X \to Y$ on a complete lattice (poset) X has both a *least* fixed point $\mu l \in X$ and *greatest* fixed point $\nu l \in X$. These can be described explicitly as

$$\mu l = \bigwedge\{x \in X \mid l(x) \leq x\} \text{ and } \nu l = \bigvee\{x \in X \mid x \leq l(x)\}.$$

In what follows we denote by $y < x$ if and only if ($y \leq x$ and not $x \leq y$), and we denote by $x \bowtie y$ two unrelated elements in X (so that not ($x \leq y$ or $y \leq x$)). An element in a lattice $x \in X$ is a *join-irreducible* element if and only if $x = a \vee b$ implies $x = a$ or $x = b$ for any $a, b \in X$. An element in a lattice $x \in X$ is an *atom* if and only if $x > 0$ and $\nexists y.(x > y > 0)$.

Lower set (down closed) is any subset Y of a given poset $(X, \leq)$ such that, for all elements x and y, if $x \leq y$ and $y \in Y$, then $x \in Y$. A *Heyting algebra* is a bounded lattice X with finite meets and joins such that for each element $x \in X$, the function $(\_) \wedge x : X \to X$ has a right adjoint $x \rightharpoonup (\_)$, also called an algebraic implication. An equivalent definition can be given by considering a bounded lattice such that for all x and y in X there is a greatest element z in X, denoted by $x \rightharpoonup y$, such that $z \wedge x \leq y$; that is, $x \rightharpoonup y = \bigvee\{z \in X \mid z \wedge x \leq y\}$ (relative pseudocomplement). In Heyting algebra we can define negation $\neg x$ as a pseudocomplement $x \rightharpoonup 0$. Then $x \leq \neg\neg x$. A complete Heyting algebra is a Heyting algebra which is complete as a poset. A complete lattice is thus a complete Heyting algebra if and only if the following *distributivity* $x \wedge \left(\bigvee S\right) = \bigvee_{y \in S}(x \wedge y)$ holds.

The negation and implication operators can be represented as monotone functions,

$$\neg : X \to X^{OP} \text{ and } \Rightarrow : X \times X^{OP} \to X^{OP},$$

where X^{OP} is the lattice with inverse partial ordering and $\wedge^{OP} = \vee$, $\vee^{OP} = \wedge$.

The smallest complete distributive lattice is denoted by $\mathbf{2} = \{0, 1\}$ with classic two values, false and true, respectively. It is also complemented Heyting algebra; consequently, it is Boolean. A *Galois algebra* is a complete Heyting algebra B both with a "nexttime" monotone function from B to B that preserves all meets (i.e., right adjoint). Such Galois algebras are often called Heyting algebras with (unary) modal operators.

2 Algebra for Positive Fragment of da Costa System

It was previously mentioned that the set PLA of positive axioms of the da Costa system is equal to the positive intuitionistic (thus also classic) fragment of the propositional logic with connectives $\wedge$, $\vee$, and $\Rightarrow$, for conjunction, disjunction, and implication, whose algebraic version is defined by $x \rightharpoonup y = \bigvee\{z \in X \mid z \wedge x \leq y\}$ (relative pseudocomplement). Consequently, we obtain positive Heyting algebra (without negation) which is a complete distributive lattice $(X, \leq)$, where meet and join operators act as algebraic conjunction and disjunction, with the sublattice $(\mathbf{2}, \leq)$ where $\mathbf{2} = \{0, 1\}$ is the set of classic logic values, which are the bottom and the top values in X, respectively. Notice that in the case of 2-valued lattice this Heyting algebra becomes Boolean algebra with classic 2-valued implication. From this point of view, other axioms in the da Costa system are used only to define the weak-negation $\neg$, different from the pseudocomplement which is used in full Heyting algebras.

Let $\mathcal{L}$ be a propositional logic language obtained as free algebra from connectives in Σ of an algebra based on the complete lattice $(X, \leq)$ of algebraic truth-values (for example, meet, join, and implication $\{\wedge, \vee, \Rightarrow\} \subseteq \Sigma$ are binary operators, negation $\neg \in \Sigma$ and other modal operators are unary operators, while each $x \in X \subseteq \Sigma$ is a constant (nullary operator)) and on a set Var of propositional variables (letters) denoted by $p, r, q, \ldots$. We will use letters $A, B, \ldots$ for formulas of $\mathcal{L}$. We define a (many-valued) *valuation* v as a mapping $v : \mathcal{L} \to X$ (notice that $X \subseteq \mathcal{L}$ are the constants of this language and we will use the same symbols as those used for elements of the lattice X), which is a homomorphism (for example, for any $p, q \in$ Var, $v(p \odot q) = v(p) \odot v(q)$, $\odot \in \{\wedge, \vee, \Rightarrow\}$, and $v(\neg p) = \neg v(p)$, where $\wedge, \vee, \Rightarrow, \neg$ are conjunction, disjunction, implication, and negation, respectively) and acts as the identity for elements in X; that is, for any $x \in X$, $v(x) = x$.

Given a propositional logic language $\mathcal{L}$ (set of logic formulas), we say that $\Vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ defines a (Tarskian) *consequence relation* for $\mathcal{L}$ if the following clauses hold, for any propositional formula A and B, and subsets Γ, Θ of $\mathcal{L}$ called also *theories* (formulas and commas at the left-hand side of $\Vdash$ denote, as usual, sets and unions of sets of formulas):

(1) (reflexivity) if $A \in \Gamma$ then $\Gamma \Vdash A$;
(2) (monotonicity) if $\Gamma \Vdash A$ and $\Gamma \subseteq \Theta$, then $\Theta \Vdash A$;
(3) (cut) if $\Gamma \Vdash A$ and $\Theta, A \Vdash B$, then $\Gamma, \Theta \Vdash B$;
(4) (finiteness) if $\Gamma \Vdash A$ then there is a finite $\Theta \subseteq \Gamma$ such that $\Theta \Vdash A$;
(5) for any homomorphism σ from $\mathcal{L}$ into itself (i.e., substitution), if $\Gamma \Vdash A$, then $\sigma[\Gamma] \Vdash \sigma(A)$, that is, $\{\sigma(B) \mid B \in \Gamma\} \Vdash \sigma(A)$.

We denote by $C : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ the closure operator with $C(\Gamma) =_{\text{def}} \{A \in \mathcal{L} \mid \Gamma \Vdash A\}$ having the following properties: $\Gamma \subseteq C(\Gamma)$ (from reflexivity (1)), $\Gamma \subseteq \Gamma_1$ implies $C(\Gamma) \subseteq C(\Gamma_1)$ (from (5)), and $C(C(\Gamma)) = C(\Gamma)$. Thus we obtain that

(6) $\Gamma \Vdash A$ iff $A \in C(\Gamma)$.

Any theory $\Gamma \subseteq \mathcal{L}$ is called a *closed* theory if and only if $\Gamma = C(\Gamma)$. This closure property corresponds to the fact that $\Gamma \Vdash A$ if and only if $A \in \Gamma$.

If $\Gamma \Vdash A$ for all Γ, we will say that A is a *thesis* of this logic. The sequent calculus was developed by Gentzen [14] inspired by some ideas of Herz [15]. Given a propositional logic language $\mathcal{L}$ (set of logic formulas) a *binary sequent* is a consequence pair of formulas $s = (A; B) \in \mathcal{L} \times \mathcal{L}$, denoted also by $A \vdash B$. A Gentzen system, denoted as a pair $\mathcal{G} = \langle \mathbb{L}, \Vdash \rangle$ where $\Vdash$ is a finitary consequence relation on a set of sequents in $\mathbb{L} \subseteq \mathcal{L} \times \mathcal{L}$, is said to be *normal* if it satisfies the five conditions above. Now we are ready to define the following lattice-based consequence binary relation $\vdash \subseteq \mathcal{L} \times \mathcal{L}$ between formulas (analog to the binary consequence system from [12] for the distributive lattice logic DLL), where each consequence pair $A \vdash B$ is a *sequent* also.

Definition 2.1 ([12]) The Gentzen-like system $\mathcal{G}$ of the DLL (distributive lattice logic) $\mathcal{L}$ contains the following axioms (sequents) and rules.

Axioms $\mathcal{G}$ contains the following sequents.

1a.	$A \vdash A$	(reflexive)
2a.	$A \wedge B \vdash A,\ A \wedge B \vdash B$	(product projections: axioms for meet)
3a.	$A \vdash A \vee B,\ B \vdash A \vee B$	(coproduct injections: axioms for join)
4a.	$A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$	(distributivity axiom)

Inference Rules $\mathcal{G}$ is closed under the following inference rules.

1r.	$\frac{A \vdash B,\ B \vdash C}{A \vdash C}$	(cut/transitivity rule)
2r.	$\frac{A \vdash B,\ A \vdash C}{A \vdash B \wedge C},\ \frac{A \vdash B,\ C \vdash B}{A \vee C \vdash B}$	(lower/upper lattice bound rules)

It is easy to verify that the binary relation $\vdash$ corresponds to the distributive lattice ordering $\leq$; thus, for a given lattice of truth values in $(X, \leq)$, we can also take the set of these lattice axioms,

(a) top/bottom axioms, $A \vdash 1,\ 0 \vdash A$,
(b) the set of sequents which define the poset of the lattice of truth values $(X, \leq)$: for any two $x, y \in X$, if $x \leq y$ then $x \vdash y \in \mathcal{G}$

(if we consider the algebraic values in X as nullary logic constants such that for a given $x \in X$ we have that for every valuation $v(x) = x$).

Notice that the modus ponens rule of propositional logic $\frac{B,\ B \Rightarrow C}{C}$, with the sequent form $\frac{1 \vdash B,\ B \vdash C}{1 \vdash C}$, is a particular case of the transitive rule (1r) when A is equal to 1.

So, for example, we may specify by this Gentzen sequent system $\mathcal{G}$ that a formula A has an algebraic truth value $x \in X$ by means of two true sequents, $A \vdash x$ and $x \vdash A$, and specify the truth of this formula by $1 \vdash A$ (where $1 \in X$ is the top algebraic value in a distributive complete lattice $(X, \leq)$).

It is easy to verify that any *valuation* $v : \mathcal{L} \to X$ (notice that $X \subseteq \mathcal{L}$ are the constants of this language and we will use the same symbols as those used for elements of the lattice X) for this positive fragment of C_ω is truth-functional; that is, it is a homomorphism (for example, for any $p, q \in \mathrm{Var}$, $v(p \odot q) = v(p) \odot v(q), \odot \in \{\wedge, \vee, \Rightarrow\}$).

Example 2.2 The smallest distributive complete lattice is the classic 2-valued logic where $X = \mathbf{2}$. The *infinite* distributive complete lattice logic is, for example, *fuzzy logic*, where $X = [0, 1]$ is the closed interval of reals between 0 and 1, where the algebraic versions for logic connectives $\wedge$ and $\vee$ are $\min, \max : X \times X \to X$ operations, while the implication is defined by relative pseudocomplement; that is, $x \rightharpoonup y = 1$ if $x \leq y$, y otherwise.

Another infinite example is the DCDL defined in what follows, when the set of "possible worlds" $\mathcal{W}$ used for Kripke semantics of intensional logics is infinite.

Particularly important is the distributive powerset lattice when $X = \mathcal{P}(\mathcal{W})$ for a given (finite or infinite set $\mathcal{W}$). It is an example of the positive fragment (without negation) of Heyting algebra $(\mathcal{P}(\mathcal{W}), \subseteq, \bigcap, \bigcup, \rightharpoonup)$, where meet and join operators are set intersection and set union while $\rightharpoonup$ is a relative pseudocomplement for sets. Notice that $(\mathcal{P}(\mathcal{W}), \subseteq)$ is a complete distributive lattice also when the set $\mathcal{W}$ is *infinite*; thus, for any two $S, S' \subseteq \mathcal{W}$, the implication $S \rightharpoonup S' = \bigcup\{Z \mid Z \bigcap S \subseteq S'\}$ is well defined also when the set $\{Z \mid Z \bigcap S \subseteq S'\}$ is infinite. But instead of this set-based complete distributive lattice we will use a function-based lattice which is isomorphic to it.

Definition 2.3 (Functional Complete Distributive Lattice—FCDL) The lattice $(\mathbf{2}^{\mathcal{W}}, \leq, \wedge^a, \vee^a)$ is a complete distributive lattice with elements $f \in \mathbf{2}^{\mathcal{W}}$ that are functions $f : \mathcal{W} \to \mathbf{2}$, with the isomorphism

$$\mathrm{ch} : \big(\mathcal{P}(\mathcal{W}), \subseteq, \bigcap, \bigcup\big) \simeq \big(\mathbf{2}^{\mathcal{W}}, \leq, \wedge^a, \vee^a\big)$$

such that, for any subset $S \subseteq \mathcal{W}$, $\mathrm{ch}(S) = f$ is a characteristic function for S; that is, $S = \mathrm{ch}^{-1}(f) = \{x \in \mathcal{W} \mid f(x) = 1\}$, where ch^{-1} is the inverse of ch, and for any two $f, f' \in \mathbf{2}^{\mathcal{W}}$, we have $f \leq f'$ if and only if $\mathrm{ch}^{-1}(f) \subseteq \mathrm{ch}^{-1}(f')$.

So that ($\circ$ is a composition of functions), $\wedge^a = \mathrm{ch} \circ \bigcap \circ (\mathrm{ch}^{-1} \times \mathrm{ch}^{-1}) : \mathbf{2}^{\mathcal{W}} \times \mathbf{2}^{\mathcal{W}} \to \mathbf{2}^{\mathcal{W}}$ and $\vee^a = \mathrm{ch} \circ \bigcup \circ (\mathrm{ch}^{-1} \times \mathrm{ch}^{-1}) : \mathbf{2}^{\mathcal{W}} \times \mathbf{2}^{\mathcal{W}} \to \mathbf{2}^{\mathcal{W}}$ are meet and join algebraic operators in this complete distributive lattice.

Notice that complete lattices are very important when the lattice is infinite (remember that each finite lattice is complete), when such infinite distributive lattices have to be used for C_n (which cannot have a finite lattice matrix [2]). Such complete distributive lattices have the following Galois connection between conjunction and implication.

Proposition 2.4 *In each complete distributive lattice* $(X, \leq)$ *we have the Galois connection* $\_ \wedge y \dashv y \Rightarrow \_$ *for any* $y \in X$, *where* $\Rightarrow$ *is defined by* $y \Rightarrow z = \bigvee\{x \mid x \wedge y \leq z\}$. *For such lattices we can obtain the simple binary sequent calculi, where structural connective "comma" in the left-hand of a sequent can be replaced by the logic conjunction* $\wedge$ *connective.*

Proof It is a well-known result in the literature. Consequently, Definition 2.1 is adequate also for *infinite complete* distributive latices, as, for example, functional lattice FCDL in Definition 2.3, but also for the following.

Example 2.5 Another case of complete distributive lattices, where the set $\mathcal{W}$ is a poset, are the sublattices of hereditary sets used in Kripke semantics for intuitionistic propositional logic.

Definition 2.6 (Functional Hereditary Sublattice of FCDL—FHL) Let $(\mathcal{W}, \leq)$ be a poset. A subset $S \subseteq \mathcal{W}$ is said to be hereditary if $x \in S$ and $x \leq x'$ implies $x' \in S$. We denote by $\mathcal{H}_W$ the subset of all hereditary subsets of $\mathcal{P}(\mathcal{W})$ so that $\big(\mathcal{H}_W, \subseteq, \bigcap, \bigcup\big)$ is a sublattice of the powerset lattice $\big(\mathcal{P}(\mathcal{W}), \subseteq, \bigcap, \bigcup\big)$.

Then $(\mathcal{F}_W, \leq, \wedge^a, \vee^a)$ is the functional hereditary complete distributive sublattice (FHL) of $(\mathbf{2}^{\mathcal{W}}, \leq, \wedge^a, \vee^a)$, where $\mathcal{F}_W = \{\mathrm{ch}(S) \mid S \in \mathcal{H}_W\} \subseteq \mathbf{2}^{\mathcal{W}}$.

We define also the algebraic implication operator $\Rightarrow^a$ for FHL by

$$\Rightarrow^a = \mathrm{ch} \circ \rightharpoonup \circ (\mathrm{ch}^{-1} \times \mathrm{ch}^{-1}) : \mathcal{F}_W \times \mathcal{F}_W \to \mathcal{F}_W,$$

where $\rightharpoonup$ is the relative pseudocomplement for sets given by $S \rightharpoonup S' = \bigcup\{Z \in \mathcal{H}_W \mid Z \bigcap S \subseteq S'\}$.

The hereditary sets are closed under set intersection and union, thus also under a relative pseudocomplement operator $\rightharpoonup$, which is expressed by using set union and intersection. As a result of this the positive fragment of Heyting algebra $(\mathcal{F}_W, \leq, \wedge^a, \vee^a, \Rightarrow^a)$ is well defined (closed under algebraic operations) with the isomorphism

$$\mathrm{ch} : \big(\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightharpoonup\big) \simeq (\mathcal{F}_W, \leq, \wedge^a, \vee^a, \Rightarrow^a).$$

It is easy to verify that FHL is also a complete distributive lattice where the bottom element $0 : \mathcal{W} \to \mathbf{2}$ is a function such that $\mathrm{ch}^{-1}(0) = \varnothing$ is the empty set, while the top element $1 : \mathcal{W} \to \mathbf{2}$ is a function such that $\mathrm{ch}^{-1}(1) = \mathcal{W}$. Given a propositional logic $\mathcal{L}$, then a homomorphism $v : \mathcal{L} \to X$, where $X = \mathcal{F}_W \subseteq \mathbf{2}^{\mathcal{W}}$, is a truth-functional hereditary valuation for such a *many-valued* logic with connectives $\wedge$, $\vee$, and $\Rightarrow$, whose algebraic counterparts are many-valued algebraic operators $\wedge^a$, $\vee^a$, and $\Rightarrow^a$. □

The Gentzen system in Definition 2.1 is a normal logic, thus monotonic and transitive; therefore, the Deduction Metatheorem holds.

Proposition 2.7 *For any two propositional formulas A and B in the* PLA *fragment of da Costa system, we have*

$$A \vdash B \quad \textit{iff} \quad 1 \vdash (A \Rightarrow B).$$

Notice that we use $1 \vdash (A \Rightarrow B)$ instead of $\vdash (A \Rightarrow B)$ in order to have the binary relation $\vdash$ and so to maintain the equivalence between $\vdash$ and lattice ordering $\leq$ between algebraic truth values.

Proof It is familiar and straightforward to show that $A \vdash B$ implies $1 \vdash (A \Rightarrow B)$ holds for any logic containing axioms (1) and (2) of PLA as provable schemas and having only MP as a primitive rule. Vice versa, by monotonicity and transitivity and MP, we obtain also that $1 \vdash (A \Rightarrow B)$ implies $A \vdash B$. □

Now we are able to demonstrate the following property of the PLA fragment of da Costa system C_ω, which is equal to the positive fragment of intuitionistic (and classic) logic. We know that Boolean algebra (distributive lattice with complements, where $\neg(A \wedge \neg A)$ and $A \vee \neg A$ are theorems) is the algebraic counterpart of CPL (classic propositional logic), whereas Heyting algebra (where $A \vee \neg A$ does not hold) is the algebraic counterpart for intuitionistic propositional logic. The question which we consider here is what is the algebraic counterpart for the PLA (positive fragment common to both of these two propositional logics). The answer is as follows.

Proposition 2.8 *The* PLA *fragment of the da Costa system C_ω is equal to the positive fragment of classic and intuitionistic propositional logic and corresponds to distributive lattice algebra. Thus, it is the normal distributive lattice logic (*DLL*) given by Definition 2.1.*

Proof This is proved in the usual way. Notice that, for example, Dummett's law $A \vee (A \Rightarrow B)$ holds only for distributive *complemented* lattices (i.e., Boolean algebras) but not for any other many-valued distributive lattice. There are *many-valued classic* logics (which satisfy all axioms of the classic logic), that is, many-valued Boolean algebras, as, for example, Belnap's four valued logic, which is isomorphic to the Cartesian product distributive lattice $(\mathbf{2} \times \mathbf{2}, \preceq)$, where partial order is defined by $(x, y) \preceq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. The $\wedge$ and $\vee$ correspond to meet and join lattice operators, with the *classic negation* defined by $\neg(x, y) = (\neg x, \neg y)$ and classic material implication by $(x, y) \Rightarrow (x', y') = \neg(x, y) \vee (x', y')$. For such complemented distributive lattices we have that Dummett's law $A \vee (A \Rightarrow B) = A \vee \neg A \vee B$ is a theorem, because $A \vee \neg A$ holds in these complemented distributive lattices.

But the da Costa system, that is, PLA, *is not a complemented* distributive lattice (does not hold that $A \Rightarrow B = \neg A \vee B$ as in classic logic), so that $A \vee \neg A$ holds (axiom 9) but Dummett's law does not hold in any da Costa system C_n,

$1 \le n \le \omega$. It holds only for the particular case of the 2-valued distributive lattice, that is, when $X = \mathbf{2}$, which is *necessarily* complemented (with *uniquely* defined negation $\neg : \mathbf{2} \to \mathbf{2}^{OP}$ which inverts 0 and 1), but such a minimal complemented distributive lattice cannot be a truth-functional matrix for any da Costa system. □

3 Weak Negation for Distributive Lattices

The da Costa extension of PLA with weak negation $\neg$ is intended to preserve the positive intuitionistic logic (in the special case in which this lattice is **2**, the PLA is a fragment of classic logic) so that other axioms are dedicated to define the weak negation. The classic propositional logic (CPL) is defined by the set of positive axioms in PLA with the following axioms concerning the negation operator $\neg$.

Negative Classic Logic Axioms—NCLA

(9) $A \vee \neg A$ (excluded middle axiom)

(10_c) $A \Rightarrow (\neg A \Rightarrow B)$ (ex falso quodlibet axiom)

(11_c) $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$ (reductio axiom)

Notice that the propositional *intuitionistic* logic is equal to classic logic without the excluded middle axiom (9). The axiom (10_c), by using the Deduction Metatheorem (Proposition 2.7) obtained by PLA, corresponds to the ex falso quodlibet rule $\frac{A,\ \neg A}{B}$, while the reductio axiom (11_c) to the rule $\frac{A \Rightarrow B,\ A \Rightarrow \neg B}{\neg A}$, or in binary-sequent form to the Gentzen-like rule $\frac{A \vdash B,\ A \vdash \neg B}{1 \vdash \neg A}$. Thus, the Gentzen-like CPL is a system composed by the sequent system given for DLL in Definition 2.1, extended by the following.

(9_g) $1 \vdash A \vee \neg A$ (excluded middle axiom)

(10_g) $\frac{1 \vdash A,\ 1 \vdash \neg A}{1 \vdash B}$ (ex falso quodlibet rule)

(11_g) $\frac{A \vdash B,\ A \vdash \neg B}{1 \vdash \neg A}$ (reduct rule)

Da Costa replaced the reductio axiom (11_c) by its relativization axiom (11), dropped the ex falso quodlibet axiom (10_c) in order to obtain a nonexplosive inconsistent logic, and replaced it by the axiom (10) $\neg\neg A \Rightarrow A$ as a way of rendering the negation of his calculi a bit stronger, using as argument the intended duality with the logics arising from the formalization of intuitionistic logic in which only the converse, that is, the formula $A \Rightarrow \neg\neg A$, is valid.

Our choice will be different from his, because from the precedent results we have seen that PLA is a positive intuitionistic logic fragment; thus it will be natural to use a kind of weakening for the intuitionistic negation (which drops the excluded middle axiom (9)). It is not possible to use directly intuitionistic negation because in that case we are not able to realize da Costa's relativization of the reductio axiom (11_c). This is because in pure intuitionistic logic (PLA plus two negation axioms (10_c) and (11_c)) the formula B° is valid for any B.

But before we start to define the intuitionistic version of da Costa negation weakening, let us consider some other reasons why the original da Costa negation is an inadequate semantics for negation, considering the distributive lattice ordering determined by PLA fragment of his logic. In fact, with respect to the lattice $(X, \le)$ the first two axioms for negation, $A \vee \neg A$ and $\neg\neg A \Rightarrow A$, become (from Proposition 2.8) $1 \vdash A \vee \neg A$ and $\neg\neg A \vdash A$, that is, $1 \le A \vee \neg A$ and $\neg\neg A \le A$, for the lattice ordering $\leq$. So that for the bottom logic value $0 \in \mathbf{2} \subseteq X$, we obtain $1 \leq 0 \vee \neg 0 = \neg 0$, that is, $\neg 0 = 1$, and for the top logic value, $1 \in \mathbf{2} \subseteq X$, we have that $\neg\neg 0 \leq 0$, that is, (from $\neg 0 = 1$) we obtain $\neg 1 = 0$. Consequently, for every negation operator, the top and bottom values are inverted by it. But for (finite or infinite) lattice-based logic the negation has to satisfy also the antitonicity; that is, it must be a monotonic mapping $\neg : X \to X^{OP}$, or, equivalently, an antitonic mapping $\neg : X \to X$ so that for any two logic formulas A, B must hold that, $A \leq B$ implies $\neg B \leq \neg A$, or, equivalently, $1 \vdash (A \Rightarrow B)$ implies $1 \vdash (\neg B \Rightarrow \neg A)$, and corresponding *antitonicity axiom* $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$. But it does not hold in da Costa weakening of negation.

Proposition 3.1 *The antitonicity* $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$ *and the contraposition* $(A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A)$ *for negation operator* $\neg$ *do not hold in* C_n.

Proof The reductio axiom (a) $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$ of classic propositional logic (CPL) is independent of other axioms in propositional logic. Thus it cannot be derived from the first ten axioms of the da Costa system which are a subset of axioms for CPL. In classic propositional logic, in order to derive theorem $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$, we need axiom (a) (take, for example, a bivaluation which does not satisfy contraposition in C_1, for A and B atomic formulas). From the fact that axiom (a) is relativized to B° in axiom 11 of da Costa, it means that the antitonicity of negation cannot be derived from his axiom system.

Suppose that the contraposition (b) $(A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A)$ holds in C_n, then

1.	$A \Rightarrow B$	IP
2.	$A \Rightarrow \neg B$	IP
3.	$(A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A)$	Ax.b
4.	$B \Rightarrow \neg A$	2, 3, MP
5.	$\neg A$	1, 4, transitivity.

Thus we obtain $A \Rightarrow B, A \Rightarrow \neg B \Vdash \neg A$; so from the Deduction Metatheorem, $A \Rightarrow B \Vdash (A \Rightarrow \neg B) \Rightarrow \neg A$. Consequently, $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$, which is a contradiction because this does not hold in da Costa (it is relativized to B° in axiom 11 of da Costa system). □

It means that if we want to obtain semantically correct weakening of negation we need to add the antitonicity axiom to the da Costa system. Thus, together with the fact that PLA corresponds to general many-valued logic based on the complete distributive lattice, that is, on intuitionistic positive logic, in what follows we will change the two negation axioms of C_n, 9 and 10, with the following set of axioms, in order to obtain the Kripke semantics for such a modified da Costa system, denoted by Z_n.

Definition 3.2 (Z_n system) In order to define Z_n we use the da Costa axioms weakening of intuitionistic negation by relativization of the unique remaining reductio axiom (after dropping also the ex falso quodlibet intuitionistic axiom),

(11) $B^{(n)} \Rightarrow ((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$ (Reductio relativization axiom)
(12) $(A^{(n)} \wedge B^{(n)}) \Rightarrow ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \Rightarrow B)^{(n)})$

with the following set of new axioms,

(9b)	$(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$	(antitonicity)
(10b)	$1 \Rightarrow \neg 0$, $\neg 1 \Rightarrow 0$	(inversion axioms)
(11b)	$A \Rightarrow 1$, $0 \Rightarrow A$	(top/bottom axioms)
(12b)	$(\neg A \wedge \neg B) \Rightarrow \neg(A \vee B)$	(additive modal negation axiom)

where 0 and 1 are considered as contradiction and tautology nullary logic operators (constants) in this propositional logic; that is, for every valuation v, $v(1) = 1$ and $v(0) = 0$ (the symbol on the left side is logic constant, while on the right is logic value). We denote, by CZ_n, the *constructive* Z_n system by adding the axiom

(13b) $\neg(A \wedge B) \Rightarrow (\neg A \vee \neg B)$ (multiplicative modal negation axiom).

Notice that axiom (11b) corresponds to $A \leq 1$, $0 \leq A$, while axiom (12b) expresses the inverting of top and bottom elements; that is, from the axiom $1 \Rightarrow \neg 0$, that is, $1 \leq \neg 0$, we obtain $\neg 0 = 1$, because 1 is the top lattice value. (We denote an equivalence $A = B$ if and only if $A \vdash B$ and $B \vdash A$). Analogously, we obtain $\neg 1 = 0$.

Proposition 3.3 *From the Z_n system without axiom (12b) we obtain the following theorems:*

$$\neg(A \vee B) \Rightarrow (\neg A \wedge \neg B)$$

and

$$(\neg A \vee \neg B) \Rightarrow \neg(A \wedge B).$$

Thus, in Z_n, the equivalence $\neg(A \vee B) = (\neg A \wedge \neg B)$ holds, while in CZ_n, the equivalence $\neg(A \wedge B) = (\neg A \vee \neg B)$ also holds.

Proof We have

1.	$B \Rightarrow (A \vee B)$	Ax.7
2.	$(B \Rightarrow (A \vee B)) \Rightarrow (\neg(A \vee B) \Rightarrow \neg B)$	Ax.9b (substitution $A \mapsto B$ and $B \mapsto A \vee B$)
3.	$\neg(A \vee B) \Rightarrow \neg B$	1, 2, MP
4.	$A \Rightarrow (A \vee B)$	Ax.6
5.	$(A \Rightarrow (A \vee B)) \Rightarrow (\neg(A \vee B) \Rightarrow \neg A)$	Ax.9b (substitution and $B \mapsto A \vee B$)
6.	$\neg(A \vee B) \Rightarrow \neg A$	4, 5, MP
7.	$\neg(A \vee B) \Rightarrow (\neg A \wedge \neg B)$	4, 6, deduction in case 6 of Proposition 2.7

Thus we obtain from the Deduction Metatheorem that $\neg(A \vee B) \vdash (\neg A \wedge \neg B)$, and from the axiom (12b) $(\neg A \wedge \neg B) \vdash \neg(A \vee B)$. Also, thus, we obtain the homomorphism equivalence $\neg(A \vee B) = (\neg A \wedge \neg B)$ in Z_n. So we obtain a homomorphic property of negation operator for join semilattice in Z_n, $\neg : (X, \leq, \vee) \rightarrow (X, \leq, \vee)^{OP}$, where $\vee^{OP}$ corresponds to the meet operator $\wedge$ in a distributive lattice $(X, \leq, \wedge, \vee)$, so that $\neg(A \vee B) = \neg A \vee^{OP} \neg B = \neg A \wedge \neg B$ (and also $\neg 0 = 0^{OP} = 1$). Analogously we obtain

1.	$A \wedge B \Rightarrow B$	Ax.5
2.	$((A \wedge B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \wedge B))$	Ax.9b (substitution $A \mapsto A \vee B$)

3.	$\neg B \Rightarrow \neg(A \wedge B)$	1, 2, MP
4.	$A \wedge B \Rightarrow A$	Ax.4
5.	$((A \wedge B) \Rightarrow A) \Rightarrow (\neg A \Rightarrow \neg(A \wedge B))$	Ax.9b (substitution $A \mapsto A \vee B$ and $B \mapsto A$)
6.	$\neg A \Rightarrow \neg(A \wedge B)$	4, 5, MP
7.	$(\neg A \vee \neg B) \Rightarrow \neg(A \wedge B)$	4, 6, deduction in case 6 of Proposition 2.7.

Thus we obtain from the Deduction Metatheorem that $(\neg A \vee \neg B) \vdash \neg(A \wedge B)$, and from axiom (13) $\neg(A \wedge B) \Rightarrow (\neg A \vee \neg B)$. Also we obtain the homomorphism equivalence $\neg(A \wedge B) = (\neg A \vee \neg B)$ in CZ_n.

So we obtain a homomorphic property of negation operator for a distributive lattice in CZ_n, $\neg : (X, \leq, \wedge, \vee) \to (X, \leq, \wedge, \vee)^{OP}$, where $\wedge^{OP}$ corresponds to the meet operator $\vee$ so that $\neg(A \wedge B) = \neg A \wedge^{OP} \neg B = \neg A \vee \neg B$ (and also $\neg 1 = 1^{OP} = 0$) corresponds to the multiplicativity of $\neg$; thus in CZ_n the negation is selfadjoint (both additive and multiplicative). □

Remark 3.4 In this way we obtained that the monotone weak-negation operator on the distributive lattice $\neg : (X, \leq, \vee) \to (X, \leq, \vee)^{OP}$ is an additive algebraic operator for Z_n and can be defined as a selfadjoint modal operator in CZ_n for a distributive lattice with the Galois connection (see preliminaries), $\neg B \leq^{OP} A$ if and only if $B \leq \neg A$; that is (from $\leq^{OP}$ equal to $\geq$), $A \vdash \neg B$ if and only if $B \vdash \neg A$, or from the Deduction Metatheorem, in sequents $1 \vdash (A \Rightarrow \neg B)$ if and only if $1 \vdash (B \Rightarrow \neg A)$ so that in CZ_n the *contraposition* axiom $(A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A)$ holds, as in classic propositional logic. In CZ_n the following also holds:

1.	$(\neg B \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg\neg B)$	contraposition (with substitution $A \mapsto \neg B$)
2.	$(\neg B \Rightarrow \neg B)$	Ax.1
3.	$B \Rightarrow \neg\neg B$	1, 2, MP, (that is, $B \leq \neg\neg B$).

That is, we obtained the inverted version of the replaced da Costa axiom (10) as in *constructive* logics (that is, the reason that we consider CZ_n a *constructive* Z_n), where we are not able to deduce B from the fact that the negation of B cannot be proved (for example, in intuitionistic logic). Thus in the ZC_ω system, antitonicity and contraposition and constructive negation $A \Rightarrow \neg\neg A$ all hold, which is not case with the da Costa system C_ω.

4 Kripke Semantics for the Paraconsistent Logic Z_n

Now we will introduce a hierarchy of negation operators for many-valued logics based on complete lattices of truth values $(X, \leq)$ with respect to their homomorphic properties. The negation with the lowest requirements (which inverts the truth ordering of the lattice of truth values and is able to produce the falsity and truth, that is, the bottom and top elements of the lattice) denominated “general” negation can be defined in any complete lattice (see the example below).

Definition 4.1 ([18] Hierarchy of Negation Operators) Let $(X, \leq, \wedge, \vee)$ be a complete lattice. Then we define the following hierarchy of negation operators on it.

1. A *general* negation is a monotone mapping between posets ($\leq^{OP}$ is inverse of $\leq$),

$$\neg : (X, \leq) \to (X, \leq)^{OP} \text{ such that } \{0, 1\} \subseteq \{y = \neg x \mid x \in X\}.$$

2. A *split* negation is a general negation extended into join-semilattice homomorphism,

$$\neg : (X, \leq, \vee) \rightarrow (X, \leq, \vee)^{OP},$$

with

$$(X, \leq, \vee)^{OP} = (X, \leq^{OP}, \vee^{OP}), \vee^{OP} = \wedge.$$

3. A *constructive* negation is a general negation extended into full lattice homomorphism,

$$\neg : (X, \leq, \wedge, \vee) \rightarrow (X, \leq, \wedge, \vee)^{OP},$$

with

$$(X, \leq, \wedge, \vee)^{OP} = (X, \leq^{OP}, \wedge^{OP}, \vee^{OP}), \text{ and } \wedge^{OP} = \vee.$$

4. A *De Morgan* negation is a constructive negation when the lattice homomorphism is an involution ($\neg\neg x = x$).

The names given to these different kinds of negations follow from the fact that a split negation introduces the second right adjoint negation, that a constructive negation satisfies the constructive requirement (as in Heyting algebras) $\neg\neg x \geq x$, while De Morgan negation satisfies well-known De Morgan laws.

Lemma 4.2 ([18] Negation properties) *Let $(X, \leq)$ be a complete lattice. Then the following properties for negation operators hold, for any $x, y \in X$:*

1. *for general negation,*

$$\neg(x \vee y) \leq \neg x \wedge \neg y, \neg(x \wedge y) \geq \neg x \vee \neg y, \text{ with } \neg 0 = 1, \neg 1 = 0;$$

2. *for split negation,*

$$\neg(x \vee y) = \neg x \wedge \neg y, \neg(x \wedge y) \geq \neg x \vee \neg y;$$

it is an additive modal operator with right adjoint (multiplicative) negation $\sim: (X, \leq)^{OP} \rightarrow (X, \leq)$, and Galois connection $\neg x \leq^{OP} y$ if and only if $x \leq \sim y$ such that $x \leq \sim \neg x$ and $x \leq \neg \sim x$;

3. *for constructive negation,*

$$\neg(x \vee y) = \neg x \wedge \neg y, \neg(x \wedge y) = \neg x \vee \neg y;$$

it is a self-adjoint operator, $\neg = \sim$, with $x \leq \neg\neg x$ satisfying proto *De Morgan inequalities $\neg(\neg x \vee \neg y) \geq x \wedge y$ and $\neg(\neg x \wedge \neg y) \geq x \vee y$;*

4. *for De Morgan negation,*

$$\neg\neg x = x;$$

it satisfies also De Morgan laws $\neg(\neg x \vee \neg y) = x \wedge y$ and $\neg(\neg x \wedge \neg y) = x \vee y$, and is contrapositive; that is, $x \leq y$ if and only if $\neg x \geq \neg y$.

Proof The proof can be found in [18]. □

Remark 4.3 We will see that the system Z_n without axiom (12b) corresponds to a particular case of *general* negation, that the whole system Z_n corresponds to a particular case of *split* negation, while the system CZ_n corresponds to a particular case of *constructive* negation.

Generally lattices arise concretely as substructures of closure systems (intersection systems) where a closure system is a family $\mathcal{F}(\mathcal{W})$ of subsets of a set $\mathcal{W}$ such that $\mathcal{W} \in \mathcal{F}(\mathcal{W})$ and if $A_i \in \mathcal{F}(\mathcal{W}), i \in I$, then $\bigcap_{i\in I} A_i \in \mathcal{F}(\mathcal{W})$. Closure operators Γ are canonically obtained by composition of the two maps of Galois connection. The Galois connections can be obtained from any binary relation based on a set $\mathcal{W}$ [5] (Birkhoff *polarity*) in a canonical way: If $(\mathcal{W}, \mathcal{R})$ is a set with a particular relation based on a set $\mathcal{W}$, $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$, with mappings $\lambda : \mathcal{P}(\mathcal{W}) \to \mathcal{P}(\mathcal{W})^{OP}, \varrho : \mathcal{P}(\mathcal{W})^{OP} \to \mathcal{P}(\mathcal{W})$ such that for subsets $U, V \in \mathcal{P}(\mathcal{W})$,

$$\lambda U = \{w \in \mathcal{W} \mid \forall u \in U.((u, w) \in \mathcal{R})\}, \rho V = \{w \in \mathcal{W} \mid \forall v \in V.((w, v) \in \mathcal{R})\},$$

where $(\mathcal{P}(\mathcal{W}), \subseteq)$ is the *powerset poset* complete distributive lattice with bottom element empty set $\varnothing$ and top element $\mathcal{W}$, and $\mathcal{P}(\mathcal{W})^{OP}$ its dual (with $\subseteq^{OP}$ inverse of $\subseteq$), then we have the induced Galois connection $\lambda \dashv \rho$; that is, $\lambda U \subseteq^{OP} V$ if and only if $U \subseteq \rho V$.

It is easy to verify that λ and ρ are two antitonic set-based operators which invert empty set $\varnothing$ and $\mathcal{W}$, thus can be used as set-based negation operators. The negation as modal operator has a long history [11]. The following lemma is useful for connecting these set-based operators with the operation of negation in complete lattices. But instead of compatibility relation C as in [19] we will use its complement, that is, the incompatibility relation $\mathcal{R} = \mathcal{W} \times \mathcal{W} - C$.

Lemma 4.4 (Incompatibility relation) *Let $(\mathcal{W}, \leq)$ be a poset. Then we can use the binary relation $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ as an* incompatibility *relation for set-based negation operators λ and ρ with the following properties. For any $U, V \subseteq \mathcal{W}$,*

1. *$\lambda(U \bigcup V) = \lambda U \bigcup^{OP} \lambda V = \lambda U \bigcap \lambda V$, with $\lambda\varnothing = \varnothing^{OP} = \mathcal{W}$ (additivity);*
2. *$\rho(U \bigcap^{OP} V) = \rho(U \bigcup V) = \rho U \bigcap \rho V$, with $\rho \mathcal{W}^{OP} = \rho\varnothing = \mathcal{W}$ (multiplicativity);*
3. *while*

$$\lambda(U \bigcap V) \supseteq \lambda U \bigcup \lambda V,$$
$$\rho(U \bigcap V) \supseteq \rho U \bigcup \rho V,$$

and

$$\lambda\rho V \supseteq V, \rho\lambda U \supseteq U.$$

We denote by $\Re$ the class of such binary incompatibility relations $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ which are also hereditary*; that is,*

4. *if $(u, w) \in \mathcal{R}$ and $(u, w) \preceq (u', w')$ then $(u', w') \in \mathcal{R}$, where $(u, w) \preceq (u', w')$ if and only if $u \leq u'$ and $w \leq w'$; then, for a hereditary incompatibility relation $\mathcal{R}$ we obtain the additive modal operator $\neg^a = \mathrm{ch} \circ \lambda \circ \mathrm{ch}^{-1} : (\mathcal{F}_W, \leq) \to (\mathcal{F}_W, \leq)^{OP}$, where $\mathcal{F}_W$ is a complete distributive lattice* FHL *in Definition 2.6, with the following isomorphism:*
5. *$\mathrm{ch} : (\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightharpoonup, \lambda) \simeq (\mathcal{F}_W, \leq, \wedge^a, \vee^a, \Rightarrow^a, \neg^a)$, between these two Paraconsistent Heyting (P-Heyting) algebras, where the pseudocomplement negation is replaced by modal paraconsistent negation.*

Proof The additivity of λ and multiplicativity of ρ are standard results from Birkhoff polarity. Let us show that λ is closed under hereditary sets in $\mathcal{H}_W$ given in Definition 2.6. In fact, given a hereditary set $S \in \mathcal{H}_W$, if $x \in \lambda(S)$ and $x \leq x'$, then

$$\forall u \in U.((u, x) \in \mathcal{R}) \text{ and } x \leq x', \text{ so } \forall u \in U.((u, x) \in R) \text{ and } (u, x) \preceq (u, x'),$$

and from point 4 we obtain $\forall u \in U.((u, x') \in \mathcal{R})$; that is, $x' \in \lambda(S)$. Consequently, we have

$$\lambda : (\mathcal{H}_W, \leq) \to (\mathcal{H}_W, \leq)^{OP},$$

and

$$\neg^a = \mathrm{ch} \circ \lambda \circ \mathrm{ch}^{-1} : (\mathcal{F}_W, \leq) \to (\mathcal{F}_W, \leq)^{OP}.$$

Isomorphism 5 is only the algebraic extension of the isomorphism of the positive fragment of Heyting algebras as defined in Definition 2.6 by the modal negation operator obtained from the hereditary incompatibility relation and Birkhoff polarity additive operator λ. □

It is easy to see that, for any given hereditary incompatibility relation $\mathcal{R} \in \Re$, the additive algebraic operator $\neg^a$ can be used as the split negation for Z_n (or constructive negation, when λ is selfadjoint, that is, $\lambda = \rho$, for CZ_n) in Definition 4.1. We obtained the P-Heyting algebra $(\mathcal{F}_W, \leq, \wedge^a, \vee^a, \Rightarrow^a, \neg^a)$ by extending the positive fragment of Heyting algebra of FHL in Definition 2.6 by this new algebraic weakened paraconsistent modal negation $\neg^a$.

Now we have seen that the additive negation in Z_n has to satisfy the axioms (11) and (12) in Definition 3.2 also so that the set of hereditary incompatibility relations $\Re_{Z_n}$ for weakened negation in Z_n is a subset of hereditary incompatibility relations, that is, $\Re_{Z_n} \subset \Re$. For the constructive negation used in CZ_n we have that $\lambda = \rho$; that is, the hereditary incompatibility relation is a *symmetric* relation in $\Re_{Z_n}$.

In order to be able to use this semantics for weakened negation in Z_n we have only to prove that there exists the distributive lattice $(X, \leq)$ such that intensional (modal) negation in Z_n can be represented as many-valued truth-functional split negation. In order to obtain this result we will first define the intensional Kripke-like semantics for paraconsistent propositional logic Z_n and then from it demonstrate that there is at least an infinite distributive lattice such that Z_n can be represented as many-valued truth-functional propositional logic. The Kripke semantics for Z_n and CZ_n logic can be defined as modified Kripke semantics for intuitionistic positive fragment (correspondent to PLA positive fragment of Z_n and CZ_n) with weakened paraconsistent da Costa negation instead of intuitionistic negation (pseudocomplement in Heyting algebras).

Definition 4.5 We define the Kripke model $\mathcal{M} = (\mathcal{W}, \leq, \mathcal{R}, V)$ where $(\mathcal{W}, \leq)$ is a poset, $\mathcal{R} \in \Re_{Z_n}$ is an incompatibility binary accessibility relation for weakened paraconsistent da Costa negation, and a mapping $V : \mathrm{Var} \times \mathcal{W} \to \mathbf{2}$ such that for any propositional letter $p \in \mathrm{Var}$, if $w \leq w'$ then $V(p, w) \leq V(p, w')$, with $\mathbf{2} \subset \mathrm{Var}$ such that $\forall w.(V(0, w) = 0$ and $V(1, w) = 1)$. Then, for any world $w \in \mathcal{W}$ we define the satisfaction relation for any propositional formula A, denoted by $\mathcal{M} \models_w A$, as follows:

1. $\mathcal{M} \models_w p$ iff $V(p, w) = 1$, for any $p \in \mathrm{Var}$;
2. $\mathcal{M} \models_w A \wedge B$ iff $\mathcal{M} \models_w A$ and $\mathcal{M} \models_w B$;

3. $\mathcal{M} \models_w A \vee B$ iff $\mathcal{M} \models_w A$ or $\mathcal{M} \models_w B$;
4. $\mathcal{M} \models_w A \Rightarrow B$ iff $\forall y((y \leq w$ and $\mathcal{M} \models_w A)$ implies $\mathcal{M} \models_y B)$;
5. $\mathcal{M} \models_w \neg A$ iff $\forall y(\mathcal{M} \models_y A$ implies $(y, w) \in \mathcal{R})$.

It is easy to see that points from 1 to 4 are identical regarding the satisfaction relation of intuitionistic propositional logic. Point 5 defines this relation for the new modal paraconsistent weakened negation.

Let $v = [V] : \text{Var} \to \mathbf{2}^{\mathcal{W}}$ be the mapping obtained by currying (λ-abstraction) of the function V (where [_] is λ-abstraction operator) such that for any $p \in \text{Var}$, $w \in \mathcal{W}$, $v(p)(w) = V(p, w)$. Then for each function $v(p) : \mathcal{W} \to \mathbf{2}$ we obtain that it is hereditary; that is, from Definition 4.5, if $w \leq w'$ then $v(p)(w) \leq v(p)(w')$. That is, for any $p \in \text{Var}$, we have that $v(p) \in \mathcal{F}_W$, and consequently, v is a many-valued propositional valuation with a set of algebraic values equal to the complete distributive lattice $\mathcal{F}_W$, that is, $v : \text{Var} \to \mathcal{F}_W$. Let us show that v can be homomorphically extended to all formulas in Z_n so that it is a truth-functional many-valued valuation for the paraconsistent logic Z_n. We denote by $\mathbb{V}_m \subset \mathcal{F}_W^{Z_n}$ the set of all homomorphic *many-valued valuations*.

Corollary 4.6 *For any Kripke model $\mathcal{M} = (\mathcal{W}, \leq, \mathcal{R}, V)$ of the paraconsistent propositional logic Z_n, given by Definition 4.5, we obtain the many-valued truth-functional valuation as a homomorphism $v = [V] : Z_n \to \mathcal{F}_W$ between free generated algebra of formulas in Z_n (by the carrier set* Var *and logic connectives in $\Sigma = \{\wedge, \vee, \Rightarrow, \neg\}$) and the P-Heyting algebra $(\mathcal{F}_W, \leq, \wedge^a, \vee^a, \Rightarrow^a, \neg^a)$. Thus, for any given frame $(\mathcal{W}, \leq, \mathcal{R})$, we have the bijective correspondence between Kripke valuations $V \in \mathbf{2}^{\text{Var} \times \mathcal{W}}$ for Z_n (where the set of propositional letters* Var $\subseteq Z_n$ *is a subset of atomic formulas in Z_n) and many-valued truth-functional valuations $v \in \mathbb{V}_m \subset \mathcal{F}_W^{Z_n}$, given by the currying operator* [_] : $\mathbf{2}^{\text{Var} \times \mathcal{W}} \to \mathbb{V}_m$.

Proof Let us denote by $\|A\|$ for a given formula A the set of worlds where A is satisfied; that is, $\|A\| = \{w \in \mathcal{W} \mid \mathcal{M} \models_w A\}$. Then by structural induction we have the following cases.

1 $A = p$ is a propositional letter: Thus,

$$\|p\| = \{w \in \mathcal{W} \mid V(p, w) = 1\} = \{w \in \mathcal{W} \mid v(p)(w) = 1\} = \text{ch}^{-1}(v(p))$$

which is a hereditary set because $v(p) \in \mathfrak{F}_W$.

2 $A = A_1 \wedge A_2$: Then $\|A\| = \|A_1 \wedge A_2\| =$ (from point 1 in Definition 4.5) $= \|A_1\| \bigcap \|A_2\|$, which is a hereditary set, because from inductive hypothesis both $\|A_1\|$ and $\|A_2\|$ are hereditary, and their intersection is a hereditary from Lemma 4.4.

3 $A = A_1 \vee A_2$: Then $\|A\| = \|A_1 \vee A_2\| =$ (from point 2 in Definition 4.5) $= \|A_1\| \bigcup \|A_2\|$, which is a hereditary set, because from inductive hypothesis both $\|A_1\|$ and $\|A_2\|$ are hereditary, and their union is hereditary from Lemma 4.4.

4 $A = A_1 \Rightarrow A_2$: Then $\|A\| = \|A_1 \Rightarrow A_2\| =$ (from point 3 in Definition 4.5 as for intuitionistic logic) $= \|A_1\| \rightharpoonup \|A_2\|$, which is a hereditary set, because from inductive hypothesis both $\|A_1\|$ and $\|A_2\|$ are hereditary, and their relative pseudo-complement is hereditary from Lemma 4.4.

5 $A = \neg A_1$: Then $\|A\| = \|\neg A_1\| =$ (from point 1 in Definition 4.5) $= \{w \in \mathcal{W} \mid \forall y(\mathcal{M} \models_y A_1$ implies $(y,w) \in \mathcal{R})\} = \{w \in \mathcal{W} \mid \forall y(y \in \|A_1\|$ implies $(y,w) \in \mathcal{R})\} = \{w \in \mathcal{W} \mid \forall y \in \|A_1\|.(y,w) \in \mathcal{R})\} = \lambda(\|A_1\|)$, which is a hereditary set, because from inductive hypothesis $\|A_1\|$ is hereditary, and, consequently, from Lemma 4.4, the set $\lambda(\|A_1\|)$ is hereditary also. Consequently, we obtained the homomorphism $\|\_\| : (Z_n, \wedge, \vee, \Rightarrow, \neg) \to (\mathcal{H}_W, \bigcap, \bigcup, \rightharpoonup, \lambda)$ such that $\|A \odot B\| = \|A\| \odot^s \|B\|$, where $\odot \in \{\wedge, \vee, \Rightarrow\}$ and $\wedge^s = \bigcap, \vee^s = \bigcup, \Rightarrow^s = \rightharpoonup$ and $\|A\| = \lambda(\|A_1\|)$.

Consequently, from the homomorphism ch, given in Definition 2.6 of Example 2, we obtain, by composition of these two homomorphisms, the many-valued truth-functional homomorphism (a valuation for Z_n),

$$v = \mathrm{ch} \circ \|\_\| : (Z_n, \wedge, \vee, \Rightarrow, \neg) \to (\mathcal{F}_W, \wedge^a, \vee^a, \Rightarrow^a, \neg^a). \qquad \square$$

From the bijection between Kripke valuations and many-valued valuations of Z_n, we have that each *valid* formula A in a Kripke model $\mathcal{M}$ (with valuation V), that is, with $\|A\| = \mathcal{W}$, has the top value $1 = f_1 : \mathcal{W} \to \mathbf{2}$ in the complete distributive lattice FHL $\mathcal{F}_W$ (such that $\forall w \in \mathcal{W}. f_1(w) = 1$), and vice versa.

Now that we have demonstrated that there is a complete distributive lattice FHL for which the paraconsistent propositional logic Z_n is a truth-functional many-valued logic for this set of algebraic truth values, we are able to define the entailment for it.

5 Sound and Complete Truth-Functional Semantics for Z_n

We have shown how we are able to define the truth-functional many-valued semantics for paraconsistent logics Z_n and CZ_n based on Kripke models in Definition 4.5. Let us show that this Kripke (or many-valued) semantics is sound and complete for deductive-theoretic-based systems of Z_n and CZ_n. In order to do this we will first define the following 2-valued binary Gentzen-like system for Z_n and CZ_n based on the complete distributive lattice FHL of algebraic truth values in $\mathcal{F}_W$, defined in Definition 2.6.

Definition 5.1 The Gentzen-like system $\mathcal{G}_{Z_n} = \langle \mathbb{L}, \Vdash \rangle$, with a set of axioms $\mathbb{L}$ and Tarskian consequence relation $\Vdash$, of the paraconsistent system Z_n in Definition 3.2, considered as many-valued logic based on a distributive lattice $(\mathcal{F}_W, \leq)$ of algebraic truth-values, contains the sequent system of distributive lattice given by Definition 2.1 (four axioms and two inference rules) and the following axioms and rules.

Axioms

5a. $1 \vdash \neg 0$ and $\neg 1 \vdash 0$ (inversion axioms)
6a. $A \vdash 1, \; 0 \vdash A$ (top/bottom axioms)
7a. $(\neg A \wedge \neg B) \vdash \neg(A \vee B)$ (additive modal negation axiom)

Inference Rules

3r. $\frac{A \vdash B}{\neg B \vdash \neg A}$ (antitonicity rule)

4r. $\frac{1 \vdash B^{(n)}, \; A \vdash B, \; A \vdash \neg B}{1 \vdash \neg A}$ (relativized reduction rule)

5r. $\frac{1 \vdash (A^{(n)} \wedge B^{(n)})}{1 \vdash ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \Rightarrow B)^{(n)})}$ (propagation rule)

The Gentzen-like system $\mathcal{G}_{CZ_n}$ of the constructive paraconsistent system CZ_n contains also the axiom,

8a. $\neg(A \wedge B) \vdash (\neg A \vee \neg B)$ (multiplicative modal negation axiom).

In what follows we will denote by $\mathcal{G}^*_{\mathcal{L}}$ the extension by the set of *constant axioms* (they are not axiom schemas) of the Gentzen-like systems $\mathcal{G}_{\mathcal{L}}$ which define the poset of the lattice of truth values $(\mathcal{F}_W, \leq)$: for any two constants $x, y \in \mathcal{F}_W - \{0, 1\}$, if $x \leq y$ then $x \vdash y \in \mathbb{L}$. We will denote by $\Vdash^*$ the entailment relation for this extended system.

Notice that $\mathcal{G}_{\mathcal{L}}$, where $\mathcal{L} \in \{Z_n, CZ_n\}$, are Gentzen-like systems equivalent to propositional logics Z_n and CZ_n in Definition 3.2. The extensions $\mathcal{G}^*_{\mathcal{L}}$ are strictly stronger (monotonic extensions) and correspond to the modified paraconsistent logics *extended* by the set of constants in a given lattice $\mathcal{F}_W$. Thus, for a given set of hypothesis Γ, we have from the monotonic property that $\Gamma \Vdash s$ implies $\Gamma \Vdash^* s$, but not vice versa. But in what follows we will see that for any sequent of the form $s = (1 \vdash A)$ we have that $\Gamma \Vdash^* s$ implies $\Gamma \Vdash s$ holds also; that is, $\Gamma \Vdash s$ if and only if $\Gamma \Vdash^* s$.

Thus from the algebraic point of view, the paraconsistent systems Z_n and CZ_n are distributive lattice with a negative modal operator $\neg$, weakened by the rules (4r) and (5r) in order to obtain the paraconsistent logic. We can therefore use the Dunn's gaggle theory (the $\mathcal{G}_{Z_n}$ system without rules (4r) and (5r) is equal to the logical system K_- of Dunn [13]).

Definition 5.2 (Truth-preserving entailment in FLA for Z_n and CZ_n) For any two formulas $A, B \in \mathcal{L}$, where $\mathcal{L} \in \{Z_n, CZ_n\}$, the truth-preserving consequence pair (sequent), denoted by $A \vdash B$, is satisfied by a given Kripke valuation $V : \mathrm{Var} \times \mathcal{W} \to \mathbf{2}$, that is, by a many-valued valuation $v = [V] : \mathcal{L} \to (\mathcal{F}_W, \leq)$ if and only if $v(A) \leq v(B)$. This sequent is a tautology if it is satisfied by all valuations, that is, when $\forall v \in \mathbb{V}_m(v(A) \leq v(B))$, or, equivalently from Corollary 4.6, by all Kripke valuations in Definition 4.5, that is, $\forall V \in \mathbf{2}^{\mathrm{Var}\times\mathcal{W}}(\|A\| \subseteq \|B\|)$.

For a normal Gentzen-like sequent system $\mathcal{G}_{\mathcal{L}}$ in Definition 5.1 of the many-valued logic $\mathcal{L} \in \{Z_n, CZ_n\}$, we state that a many-valued valuation v is its *model* if it satisfies all sequents in $\mathcal{G}_{\mathcal{L}}$. The set of all models of a given set of sequents (theory) Γ is denoted by

$$\mathrm{Mod}_{\mathcal{K}}(\Gamma) =_{\mathrm{def}} \{V \in \mathbf{2}^{\mathrm{Var}\times\mathcal{W}} \mid \forall (A \vdash B) \in \Gamma.(\|A\| \subseteq \|B\|)\},$$

or equivalently,

$$\mathrm{Mod}_{\Gamma} =_{\mathrm{def}} \{v = [V] \mid V \in \mathrm{Mod}_{\mathcal{K}}(\Gamma)\} \subseteq \mathbb{V}_m \subset \mathcal{F}_W^{\mathcal{L}}.$$

Proposition 5.3 (Soundness) *All the axioms of the Gentzen-like sequent system $\mathcal{G}^*_{\mathcal{L}}$ in Definition 5.1 of the many-valued logic $\mathcal{L} \in \{Z_n, CZ_n\}$, based on complete distributive* FLA *lattice $(\mathcal{F}_W, \leq)$ of algebraic truth values, are the tautologies, and all its rules are sound for model satisfiability and preserve tautologies.*

Proof It is straightforward to check that all axioms in a Gentzen-like system in Definition 5.1 are tautologies (all constant sequents specify the poset of the complete lattice $(\mathcal{F}_W, \leq)$, thus are tautologies). It is also straightforward to check that all rules preserve tautologies. Moreover, if all premises of any rule in $\mathcal{G}_{\mathcal{L}}$, $\mathcal{L} \in \{Z_n, CZ_n\}$ are satisfied by the given many-valued valuation $v : \mathcal{L} \to \mathcal{F}_W$, then also the deduced sequent of the rule is satisfied by the same valuation; that is, the rules are sound for model satisfiability. □

Thus we are now able to introduce the many-valued valuation-based (i.e., model-theoretic) semantics for paraconsistent propositional many-valued logics Z_n and CZ_n.

Definition 5.4 A *many-valued* model-theoretic semantics of a given many-valued logic $\mathcal{L}$, where $\mathcal{L} \in \{Z_n, CZ_n\}$, extended by the set of propositional constants (truth values) in $\mathcal{F}_W$, with a Gentzen system $\mathcal{G}^*_{\mathcal{L}} = \langle \mathbb{L}, \Vdash^* \rangle$ in Definition 5.1, is the semantic deducibility relation $\models_m$, defined for any $\Gamma = \{s_i = (A_i \vdash B_i) \mid i \in I\}$ and sequent $s = (A \vdash B) \in \mathbb{L} \subseteq \mathcal{L} \times \mathcal{L}$ by "$\Gamma \models_m s$ if and only if all many-valued models of Γ are the models of s"; that is,

$$\begin{aligned}\Gamma \models_m s \text{ iff } &\forall v \in \mathbb{V}_m(\forall (A_i \vdash B_i) \in \Gamma(v(A_i) \le v(B_i)) \text{ implies } v(A) \le v(B))\\ \text{iff } &\forall v \in \mathrm{Mod}_\Gamma(\forall (A_i \vdash B_i) \in \Gamma(v(A_i) \le v(B_i)) \text{ implies } v(A) \le v(B))\\ \text{iff } &\forall v \in \mathrm{Mod}_\Gamma(v(A) \le v(B)).\end{aligned}$$

This model-theoretic entailment $\models_m$ for $\mathcal{L} \in \{Z_n, CZ_n\}$ is, from Corollary 4.6, bijectively correspondent to the Kripke semantics for $\mathcal{L}$ given in Definition 4.5; that is,

$$\Gamma \models_m (A \vdash B) \text{ iff } \forall V \in \mathrm{Mod}_{\mathcal{K}}(\Gamma)(\|A\| \subseteq \|B\|).$$

It is easy to verify that the Gentzen-like system $\mathcal{G}^*_{\mathcal{L}} = \langle \mathbb{L}, \Vdash^* \rangle$ is a *normal* logic.

Theorem 5.5 *The many-valued model theoretic semantics is an adequate semantics for many-valued logic $\mathcal{L}$, where $\mathcal{L} \in \{Z_n, CZ_n\}$, extended by the set of propositional constants (truth values) in $\mathcal{F}_W$ and specified by a Gentzen-like logic system $\mathcal{G}^*_{\mathcal{L}} = \langle \mathbb{L}, \Vdash^* \rangle$ in Definition 5.1; that is, it is sound and complete. Consequently, $\Gamma \models_m s$ if and only if $\Gamma \Vdash^* s$.*

Proof Let us prove that, for any many-valued model $v \in \mathrm{Mod}_\Gamma$, the obtained sequent bivaluation $\beta = \mathrm{eq} \circ \langle \pi_1, \wedge \rangle \circ (v \times v) : \mathcal{L} \times \mathcal{L} \to \mathbf{2}$ is the characteristic function of the closed theory $\Gamma_v = C(T)$ with $T = \{A \vdash x, x \vdash A \mid A \in \mathcal{L}, x = v(A)\}$, where, for $X = \mathcal{F}_W$, $\pi_1 : X \times X \to X$ is the first projection, $\mathrm{eq} : X \times X \to \mathbf{2} \subseteq X$ is the equality characteristic function such that $\mathrm{eq}(x, y) = 1$ if $x = y$.

From the definition of β we have that

$$\begin{aligned}\beta(A \vdash B) &= \beta(A; B) = \mathrm{eq} \circ \langle \pi_1, \wedge \rangle \circ (v \times v)(A; B)\\ &= \mathrm{eq} \circ \langle \pi_1, \wedge \rangle (v(A), v(B)) = \mathrm{eq}\langle \pi_1(v(A), v(B)), \wedge(v(A), v(B))\rangle\\ &= \mathrm{eq}(v(A), \wedge(v(A), v(B))) = \mathrm{eq}(v(A), v(A) \wedge v(B)).\end{aligned}$$

Thus $\beta(A \vdash B) = 1$ if and only if $v(A) \le v(B)$, that is, when this sequent is satisfied by v.

1. Let us show that for any sequent s, $s \in \Gamma_v$ implies $\beta(s) = 1$: First, for any sequent $s \in T$, it is of the form $A \vdash x$ or $x \vdash A$, where $x = v(A)$ so that they are satisfied by v (it holds that $v(A) \le v(A)$ in both cases). Consequently, *all* sequents in T are satisfied by v. By means of Proposition 5.3 we have that all inference rules in $\mathcal{G}^*_{\mathcal{L}}$ are sound with respect to the model satisfiability. Thus for any deduction $T \Vdash^* s$ (i.e., $s \in \Gamma_v$) where all sequents in premises are satisfied by the many-valued valuation (model) v, also the deduced sequent

$s = (A \vdash B)$ must be satisfied. That is, it must hold $v(A) \leq v(B)$; that is, $\beta(s) = 1$.

2. Let us show that for any sequent s, $\beta(s) = 1$ implies $s \in \Gamma_v$: For *any* sequent $s = (A \vdash B) \in \mathcal{L} \times \mathcal{L}$ if $\beta(s) = 1$ then $x = v(A) \leq v(B) = y$ (i.e., s is satisfied by v). From the definition of T, we have that $A \vdash x, y \vdash B \in T$, and from $x \leq y$ we have $x \vdash y \in \mathbb{L}$ (where $\mathbb{L}$ are axioms (sequents) in $\mathcal{G}^*_{\mathcal{L}}$, with $\{x \vdash y \mid x, y \in X, x \leq y\} \subseteq \mathbb{L}$, thus satisfied by every valuation) by the transitivity rule, from $A \vdash x, x \vdash y, y \vdash B$, we obtain that $T \Vdash^* (A \vdash B)$; that is, $s = (A \vdash B) \in C(T) = \Gamma_v$.

So from (1) and (2) we obtain that $\beta(s) = 1$ if and only if $s \in \Gamma_v$; that is, the sequent bivaluation β is the characteristic function of the closed set. Consequently, any many-valued *model* v of this many-valued logic $\mathcal{L}$ corresponds to the *closed* bivaluation β which is a characteristic function of a closed theory of sequents: we define the set of all closed bivaluations obtained from the set of many-valued models $v \in \mathrm{Mod}_\Gamma$: $\mathrm{Biv}_\Gamma = \{\Gamma_v \mid v \in \mathrm{Mod}_\Gamma\}$. From the fact that Γ is satisfied by every $v \in \mathrm{Mod}_\Gamma$ we have that for every $\Gamma_v \in \mathrm{Biv}_\Gamma$ we have that $\Gamma \subseteq \Gamma_v$ so that $C(\Gamma) = \bigcap \mathrm{Biv}_\Gamma$ (intersection of closed sets is a closed set also). Thus, for $s = (A \vdash B)$,

$$
\begin{aligned}
\Gamma \models_m s \text{ iff } & \forall v \in \mathrm{Mod}_\Gamma(\forall (A_i \vdash B_i) \in \Gamma(v(A_i) \leq v(B_i)) \text{ implies } v(A) \leq v(B)) \\
\text{iff } & \forall v \in \mathrm{Mod}_\Gamma(\forall (A_i \vdash B_i) \in \Gamma(\beta(A_i \vdash B_i) = 1) \text{ implies } \beta(A \vdash B) = 1) \\
\text{iff } & \forall v \in \mathrm{Mod}_\Gamma(\forall (A_i \vdash B_i) \in \Gamma((A_i \vdash B_i) \in \Gamma_v) \text{ implies } s \in \Gamma_v) \\
\text{iff } & \forall \Gamma_v \in \mathrm{Biv}_\Gamma(\Gamma \subseteq \Gamma_v \text{ implies } s \in \Gamma_v) \\
\text{iff } & \forall \Gamma_v \in \mathrm{Biv}_\Gamma(s \in \Gamma_v), \text{ because } \Gamma \subseteq \Gamma_v \text{ for each } \Gamma_v \in \mathrm{Biv}_\Gamma \\
\text{iff } & s \in \bigcap \mathrm{Biv}_\Gamma = C(\Gamma), \text{ that is, iff } \Gamma \Vdash^* s.
\end{aligned}
$$

□

Now we have the following corollary that demonstrates that the many-valued semantics for paraconsistent logic Z_n, obtained from its Kripke semantics, is sound and complete.

Corollary 5.6 *The Kripke semantics and derived many-valued model theoretic semantics are adequate semantics for paraconsistent logic $\mathcal{L}$, where $\mathcal{L} \in \{Z_n, CZ_n\}$. That is, for any deduced formula A from a given set of hypotheses Γ in $\mathcal{L}$, that is, when $\Gamma \Vdash (1 \vdash A)$, then $\Gamma \models_m (1 \vdash A)$ so that the formula A is valid in the Kripke frame in Definition 4.5, and vice versa.*

Proof If $\Gamma \Vdash (1 \vdash A)$ then, from monotonicity, $\Gamma \Vdash^* (1 \vdash A)$, and from Theorem 5.5, $\Gamma \models_m (1 \vdash A)$. Vice versa, if $\Gamma \models_m (1 \vdash A)$ then from Theorem 5.5, $\Gamma \Vdash^* (1 \vdash A)$. Let us show that $\Gamma \Vdash^* (1 \vdash A)$ implies $\Gamma \Vdash (1 \vdash A)$ also. We have to show that in order to deduce the sequents of the form $1 \vdash A$ we do not need the constant sequent axioms. By recursive induction, all sequents in Γ are of the form $1 \vdash A$. Also all axiom schemas (different from 5a) of this form are reducible to the sequent $1 \vdash 1$, while the axiom schema 5a reduces to $1 \vdash \neg 0$. From the Deduction Metatheorem instead, each axiom schema in $\mathcal{L}$, $A \vdash B$ can be transformed into correspondent axiom schema $1 \vdash A \Rightarrow B$.

Let us suppose that we make the step by step deductions from Γ by using the inference rules. Suppose that up to the current nth step all deduced formulas are obtained without using constant axioms. Then in the next step, a new deduced formula of the form $1 \vdash A$ can be obtained from inference rules in one of the possible cases:

Case 1 from rule (1r) $\frac{A \vdash B,\ B \vdash C}{A \vdash C}$

In order to deduce $1 \vdash C$ we need previously to have deduced sequents $1 \vdash B$ and $B \vdash C$ (from inductive hypothesis without constant axioms). Thus $1 \vdash C$ is also deduced without constant axioms.

Case 2 from rule (2r) $\frac{A \vdash B,\ A \vdash C}{A \vdash B \wedge C}$

In order to deduce $1 \vdash B \wedge C$ we need previously to have deduced sequents $1 \vdash B$ and $1 \vdash C$ (from inductive hypothesis without constant axioms). Thus $1 \vdash B \wedge C$ is also deduced without constant axioms.

from another rule $\frac{A \vdash B,\ C \vdash B}{A \vee C \vdash B}$

In order to deduce $1 \vdash B$ we need sequents $1 \vdash A \vee C$ (so both with axiom 6a, $A \vee C \vdash 1$, we have that $1 = A \vee C$) and $A \vdash 1$ (axiom 6a), and $C \vdash 1$ (axiom 6a), deduced previously (from hypothesis) without constant axioms. Thus $1 \vdash B$ is also deduced without constant axioms.

Case 3 from rule (3r) $\frac{A \vdash B}{\neg B \vdash \neg A}$

In order to deduce $1 \vdash \neg A$ we need previously to have deduced sequent $1 \vdash B$ (from inductive hypothesis without constant axioms). Thus $1 \vdash \neg A$ is also deduced without constant axioms.

Case 4 from rule (4r) $\frac{1 \vdash B^{(n)},\ A \vdash B,\ A \vdash \neg B}{1 \vdash \neg A}$

In order to deduce $1 \vdash \neg A$ we need previously to have deduced sequents $1 \vdash B^{(n)}$, and $A \vdash B,\ A \vdash\ \neg B$ (from inductive hypothesis without constant axioms). Thus $1 \vdash \neg A$ is also deduced without constant axioms.

Case 5 from rule (5r) $\frac{1 \vdash (A^{(n)} \wedge B^{(n)})}{1 \vdash ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \Rightarrow B)^{(n)})}$

In order to deduce $1 \vdash ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \Rightarrow B)^{(n)})$ we need sequent $1 \vdash (A^{(n)} \wedge B^{(n)})$, deduced previously (from inductive hypothesis) without constant axioms. Thus $1 \vdash ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \Rightarrow B)^{(n)})$ is also deduced without constant axioms. □

6 Conclusion

In this paper we have developed a new weakening of negation based on the da Costa method but by preserving fundamental properties of negation as antitonicity, inversion of top/bottom truth values and additivity, with respect to the distributive lattice logic represented by the positive fragment of propositional logic: this positive fragment determines the semantics for logic conjunction, disjunction, and implication by meet, join, and relative pseudocomplement of this complete lattice. Moreover, if we preserve also the multiplicative property for this weak negation, we obtain *constructive* paraconsistent negation which satisfies also the contraposition law for negation; such constructive negation is paraconsistent weakening of intuitionistic negation.

We defined the Kripke-style semantics for these two paraconsistent negations with modal negation and show that it is a conservative extension of the positive fragment of Kripke semantics for intuitionistic propositional logic, where only the satisfaction for negation operator is changed by adopting an incompatibility accessibility relation for this modal operator which comes from Birkhoff polarity theory based on a Galois connection for negation operator. After that we derived the many-valued semantics for this logic based on truth-functional valuations and have shown that this model-theoretic semantics for obtained substructural paraconsistent logics is sound and complete.

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ETF
Applied Mathematics Department
University of Belgrade
SERBIA
majkic@etf.bg.ac.yu
http://www.geocities.com/zoran_it/