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Models Omitting Given Complete Types

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Abstract We consider a problem of constructing a model that omits given complete types. We present two results. The first one is related to the Lopez-Escobar theorem and the second one is a version of Morley's omitting types theorem.

1 Introduction

The well-known omitting types theorem for a countable theory T states that if a type $p(x)$ is nonprincipal then there is a model that omits $p(x)$. This result can be easily generalized to the case of countably many types:

(*) Let $S = \{p_i(x)\}_{i\in\omega}$ be a set of types such that each $p_i(x)$ is nonprincipal. Then there is a model that omits all $p_i(x)$s.

Shelah showed that if T is complete, $\kappa < 2^\omega$, and $\{p_i(x)\}_{i<\kappa}$ is a set of complete nonprincipal types, then there is a model that omits all $p_i(x)$s. In general, some set theoretical axioms are necessary if we want to generalize (*) to the case when S is not countable and p_is are not necessarily complete. (See [3] and [2]).

There is another kind of omitting types theorem due to Morley. Morley's omitting types theorem states that if for each $i < \omega_1$ there is a model M_i of cardinality $\beth_i$ omitting given type $p(x)$ then there is an arbitrarily large model omitting $p(x)$. Again this theorem can be generalized to the case of countably many types without major changes in the proof.

From the Lopez-Escobar theorem [1] for $\mathcal{L}_{\omega_1,\omega}$ we can deduce the following: Let T be a theory formulated in a countable language with the binary relation $<$ expressing a linear order. Let $p(x)$ be a type. Suppose that T has a model omitting $p(x)$ such that the order type of $<$ is ω_1. Then there is a model omitting $p(x)$ such that $<$ is non-well-ordered. Morley's omitting types theorem has a strong relation with the Lopez-Escobar theorem. In fact, Morley's theorem can be proven via the Lopez-Escobar theorem. This can be explained well using Hanf numbers $\mu(\lambda,\kappa)$ and $\delta(\lambda,\kappa)$ introduced by Shelah (see Preliminaries, Section 2).

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In this paper, we are interested in omitting complete types and prove two theorems. The first one is related to the Lopez-Escobar theorem and the second one is a version of Morley's omitting types theorem.

Theorem 1.1 *Let $\mathcal{L}$ be a countable language having a binary relation $<$. Let T be a complete $\mathcal{L}$-theory and R a set of complete types with $|R| < 2^\omega$. Suppose that there is a model $M \models T$ omitting R and with the order type ω_1. Then there is a model $N \models T$ omitting R such that N has an infinite descending sequence with respect to $<^N$.*

Theorem 1.2 *Let T be a countable complete theory and R a set of complete types with $|R| < 2^\omega$. Suppose that for each $i < \omega_1$, there is a model $M_i \models T$ with the following properties:*

1. *$|M_i| \geq \beth_i$,*
2. *M_i omits each member of R.*

Then for each κ there is a model M omitting R and with $|M| \geq \kappa$.

2 Preliminaries

We recall some basic definitions and facts necessary for understanding our results. Throughout $\mathcal{L}$ is a countable language and T is an $\mathcal{L}$-theory. A model means a model of T. A formula means an $\mathcal{L}$-formula. Formulas are denoted by φ, ψ and so on. Variables are denoted by x, y and so on. A finite tuple of variables are denoted by $\bar{x}$, $\bar{y}$ and so on. A set $p(x)$ of formulas (with the free variable x) is called a type if $p(x)$ is consistent with T. A complete type is a type $p(x)$ with the property that if $\varphi(x)$ is a formula then either $\varphi(x)$ or $\neg\varphi(x)$ belongs to $p(x)$. Let M be a model and $\bar{a} \in M$. The type $\text{tp}(\bar{a})$ of $\bar{a}$ is the set $\{\varphi(\bar{x}) : M \models \varphi(\bar{a})\}$. M is said to realize $p(\bar{x})$ if there is $\bar{a} \in M$ with $p(\bar{x}) \subset \text{tp}(\bar{a})$. Otherwise, M is said to omit $p(x)$. In this paper, we say that M omits a set R of types if M omits every member of R.

Let $\varphi(x, \bar{y})$ be a formula and $F(\bar{y})$ a function symbol. F is called a Skolem function (for φ) if T proves $\forall\bar{y}(\exists x\varphi(x, \bar{y}) \to \varphi(F(\bar{y}), \bar{y}))$. If every formula has a Skolem function, then we say that T has built-in Skolem functions. A simple argument shows that every T can be extended to a theory T^* (formulated in an extended language) having built-in Skolem functions. Suppose that T has built-in Skolem functions. If M is a model and A is a subset of M, then the Skolem closure (closure of A by Skolem functions) of A is an elementary submodel of M.

Now we recall Shelah's μ and λ (Hanf numbers of omitting types). Since we are only interested in countable languages, we write those definitions by omitting unnecessary parameters.

Definition 2.1 Let κ be a cardinal.

1. $\mu(\kappa)$ is the first cardinal μ with the following property: if $\mathcal{L}$ is countable, T is an $\mathcal{L}$-theory, R is a set of types with $|R| \leq \kappa$, and for every $\chi < \mu$ there is a model of cardinality $\geq \chi$ omitting R, then there is an arbitrarily large model omitting R.
2. $\delta(\kappa)$ is the first cardinal δ with the following property: if $\mathcal{L}$ is a countable language with the binary relation $<$, T is an $\mathcal{L}$-theory, R is a set of types with $|R| \leq \kappa$, and there is a model omitting R whose order type (with respect to $<$) is $\geq \delta$, then there is a model omitting R such that $<$ is not well-ordered.

There is a relation between μ and δ (see Theorem 5.4 in [3]).

Fact 2.2 $\mu(\kappa) = \beth_{\delta(\kappa)}$.

Recall that $\beth_i$ is the beth function defined by $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and $\beth_\delta = \sup_{i<\delta} \beth_i$ for limit δ. The Lopez-Escobar theorem yields $\delta(1) = \omega_1$. So from the above fact, we have $\mu(1) = \beth_{\omega_1}$. This equation proves Morley's omitting types theorem. For proving Fact 2.2, we need to work on incomplete types. So Theorem 1.1 and Fact 2.2 do not directly yield Theorem 1.2. Since our two theorems can be proven using similar arguments, the author believes that there is a version of Fact 2.2 by which Theorem 1.2 can be easily deduced from Theorem 1.1. However, the author has not succeeded in proving Theorem 1.2 as a direct corollary to Theorem 1.1. So we give separate proofs to each theorem.

3 Proof of Theorem 1.1

We assume T has built-in Skolem functions. We prepare a countable set $X = \{x_i : i \in \omega\}$ of variables. Let $\{t_i : i \in \omega\}$ be an enumeration of all the $\mathcal{L}$-terms whose variables belong to X. We may assume that the variables of t_n are contained in $\bar{x}_n = x_0, \dots, x_{n-1}$. So we may assume $t_n = t_n(\bar{x}_n)$ by adding dummy variables. For $\bar{a} = (a_0, \dots, a_m)$ with $m \geq n-1$, $t_n(\bar{a})$ means $t_n(a_0, \dots, a_{n-1})$. By $otp(<^M) = \omega_1$, we assume the universe of M is ω_1 and $M|\{<\} = (\omega_1, <)$.

Definition 3.1 Let $m \in \omega$. An m-sequence is a sequence of length m. Let S be an uncountable set of descending m-sequences of elements in M. We introduce the following terminologies.

1. S is uniformly uncountable if for every $i < \omega_1$ $S_{>i} = \{(a_0, \dots, a_{m-1}) \in S : a_{m-1} > i\}$ is uncountable.
2. Let $n \leq m$. We will say that S is t_n-uniform if
   (1) S is uniformly uncountable, and
   (2) whenever $\bar{a}, \bar{b} \in S$ then $\mathrm{tp}(t_n(\bar{a})) = \mathrm{tp}(t_n(\bar{b}))$.
   We also say that S is essentially t_n-uniform if S has a t_n-uniform subset S_0.

Lemma 3.2 *Let S be uniformly uncountable. For each $i < \omega$, let T_i be a subset of S such that T_i is not uniformly uncountable. Then $S \setminus \bigcup_{i\in\omega} T_i$ is uniformly uncountable.*

Proof For each $i < \omega$, choose $j_i < \omega_1$ such that $(T_i)_{>j_i}$ is countable. Let $j^* = \sup_{i<\omega} j_i$. Then every $(T_i)_{>j^*}$ is countable. Let $k > j^*$. Then

$$(S \setminus \bigcup_{i\in\omega} T_i)_{>k} \supset (S_{>k} \setminus \bigcup_{i\in\omega} (T_i)_{>j^*}).$$

So $(S \setminus \bigcup_{i\in\omega} T_i)_{>k}$ is an uncountable set. So $S \setminus \bigcup_{i\in\omega} T_i$ is uniformly uncountable. □

Definition 3.3 Let $m_0 < m_1 \in \omega$. For $i = 0, 1$, let S_i be a set of descending m_i-sequences. We write $S_0 > S_1$ if whenever $(a_0, \dots, a_{m_1-1}) \in S_1$ then $(a_0, \dots, a_{m_0-1}) \in S_0$.

Lemma 3.4 *Let $m_0 < m_1 \in \omega$. Let S_0 be a uniformly uncountable set of descending m_0-sequences. Then there is a uniformly uncountable set S_1 of descending m_1-sequences with $S_0 > S_1$.*

Proof We may assume $m_1 = m_0 + 1$. For each $i \in \omega_1$, we can choose $\bar{a}_i = (a_0, \dots, a_{m_0-1}) \in S_0$ such that $a_{m_0-1} > i$. Let S_1 be the set of all $(\bar{a}_i, i)$s. Then S_1 clearly satisfies our requirements. □

Lemma 3.5 *Let $n \leq m$. Let S be a uniformly uncountable set of descending m sequences of elements in M. Then one of the following holds:*

1. *S is essentially t_n-uniform, or*
2. *for any uniformly uncountable subset S_0 of S, there is a formula $\varphi(x)$ such that both S_0^{φ,t_n} and $S_0^{\neg\varphi,t_n}$ are uniformly uncountable, where $S_0^{\varphi,t_n} = \{\bar{a} \in S_0 : M \models \varphi(t_n(\bar{a}))\}$ and $S_0^{\neg\varphi,t_n} = \{\bar{a} \in S_0 : M \models \neg\varphi(t_n(\bar{a}))\}$.*

Proof Suppose that (2) is not the case for S_0. Then, for each formula φ, define T_φ by

$$T_\varphi = \begin{cases} S_0^{\varphi,t_n} & \text{if } S_0^{\varphi,t_n} \text{ is countable,} \\ S_0^{\neg\varphi,t_n} & \text{otherwise.} \end{cases}$$

Since T_φ is not uniformly uncountable, $S_1 = S_0 \setminus \bigcup_{\varphi \in L} T_\varphi$ is uniformly uncountable by Lemma 3.2. Moreover, if $\bar{a}, \bar{b} \in S_1$, then we have $M \models \varphi(t_n(\bar{a})) \leftrightarrow \varphi(t_n(\bar{b}))$ for every φ. This shows that S_1 is t_n-uniform. □

Using Lemma 3.5 we can also show the following.

Lemma 3.6 *Let $n \leq m$ and S a uniformly uncountable set of m-sequences. Suppose that S is not essentially t_n-uniform. Then, for any uniformly uncountable S_0, $S_1 < S$, there are uniformly uncountable subsets $S_0' \subset S_0$ and $S_1' \subset S_1$, and a formula $\varphi(x)$, such that $(S_0')^{\varphi,t_n} = S_0'$ and $(S_1')^{\neg\varphi,t_n} = S_1'$.*

Proof By Lemma 3.5, there are formulas $\theta_i(x)$ $(i = 0, 1)$ such that $\theta_i(x)$ divides S_i into two uniformly uncountable sets. If $\theta_0(x)$ divides S_1 into two uniformly uncountable sets, then we are done. So we can assume $S_1^{\theta_0,t_n} = S_1$. For the same reason, we can assume $S_0^{\theta_1,t_n} = S_0$. Now let $S_0' = S_0^{\theta_0,t_n}$ and $S_1' = S_1^{\neg\theta_1,t_n}$. Let $\varphi(x)$ be the formula $\theta_0(x) \leftrightarrow \theta_1(x)$. Then $\varphi(x)$ has the required property. □

We put $S_{\langle\rangle} = \omega_1$. Now using Lemma 3.6 we can inductively choose sets S_η $(\eta \in 2^{<\omega})$ and formulas $\varphi_{\eta,n}(x)$ $(\eta \in 2^{<\omega},\ n \leq \mathrm{lh}(\eta))$ with the following properties:

1. S_η is a set of descending $\mathrm{lh}(\eta)$-sequences;
2. S_η is uniformly uncountable;
3. $S_\eta > S_{\eta'}$, if η' extends η;
4. if S_η is essentially t_n-uniform and $n = \mathrm{lh}(\eta)$ then S_η is t_n-uniform;
5. for each η and for each $n \leq \mathrm{lh}(\eta)$ such that $S_{\eta|n}$ is not t_n-uniform, we have $(S_{\eta\frown 0})^{\varphi_\eta,t_n} = S_{\eta\frown 0}$ and $(S_{\eta\frown 1})^{\neg\varphi_\eta,t_n} = S_{\eta\frown 1}$.

Let K_{u} be the set of all ηs such that S_η is $t_{\mathrm{lh}(\eta)}$-uniform. Let $K_{\mathrm{nu}} = 2^{<\omega} \setminus K_{\mathrm{u}}$. For $\eta \in K_{\mathrm{u}}$, let $p_\eta(\bar{x}_n)$ be the type $\mathrm{tp}(t_n(\bar{a}))$, where $\bar{a} \in S_\eta$ and $n = \mathrm{lh}(\eta)$. p_η does not depend on the choice of $\bar{a}$ by the t_n-uniformity. p_η is a type realized in M, whence $p_\eta \notin R$. For an infinite path $\mu \in 2^\omega$, let $\Gamma_\mu((x_i)_{i\in\omega})$ be the following set of formulas:

$$\begin{aligned}\{x_i > x_{i+1} : i \in \omega\} \;&\cup\; \{\varphi_{\mu|n,m}(t_m(\bar{x}_m)) : m \le n,\ \mu|m \in K_{\mathrm{nu}},\ \mu(n) = 0\}\\ &\cup\; \{\neg\varphi_{\mu|n,m}(t_m(\bar{x}_m)) : m \le n,\ \mu|m \in K_{\mathrm{nu}},\ \mu(n) = 1\}\\ &\cup\; \textstyle\bigcup_{\mu|n\in K_{\mathrm{u}}} p_{\mu|n}(\bar{x}_n).\end{aligned}$$

The following claim is easily proven, using the $S_{\mu|n}$s.

Claim 3.7 *Γ_μ is consistent.*

For each $\mu \in 2^\omega$, (in a big model) choose a realization I_μ of Γ_μ. For $\eta \in K_{\mathrm{nu}}$, let

$$X_\eta = \{\mu \in 2^\omega : \eta < \mu,\ \mathrm{tp}(t_n(I_\mu)) \in R\},$$

where $n = \mathrm{lh}(\eta)$.

Claim 3.8 *If $\eta \in K_{\mathrm{nu}}$, then $|X_\eta| \le |R|$.*

Otherwise, there are two different infinite paths μ and μ' both extending η such that $\mathrm{tp}(t_n(I_\mu)) = \mathrm{tp}(t_n(I_{\mu'}))$, where $n = \mathrm{lh}(\eta)$. But this is impossible, since these two types are distinguished by a formula of the form $\varphi_{\eta',n}$, where η' is the maximum common initial segment of μ and μ'.

By Claim 3.8, we can choose $\mu \in 2^\omega \setminus \bigcup_{\eta\in K_{\mathrm{nu}}} X_\eta$. Let $N \supset I_\mu$ be the Skolem closure of I_μ. Since $\{t_n\}_{n\in\omega}$ is an enumeration of all the Skolem terms, we have $N = \{t_n(\bar{b}_n) : n \in \omega\}$, where $\bar{b}_n$ are the first n elements of I_μ.

If $\mu|n \in K_{\mathrm{u}}$, then we have $\mathrm{tp}(t_n(\bar{b}_n)) = p_{\mu|n} \notin R$. While in the opposite case ($\mu|n \in K_{\mathrm{nu}}$), by our choice of μ, we also have $\mathrm{tp}(t_n(\bar{b}_n)) \notin R$.

4 Proof of Theorem 1.2

We assume that T has built-in Skolem functions. We work in a big model of T. As in Section 3, $\{t_n(\bar{x}_n) : n \in \omega\}$ is an enumeration of all the terms. Let $\{I_i : i \in X\}$ be a set of infinite n-indiscernible sequences, where X is an uncountable set. (Recall that $I = \{a_i : i < \alpha\}$ is called an n-indiscernible sequence if whenever $i_0 < \cdots <_{n-1} < \alpha$, $j_0 < \cdots < j_{n-1} < \alpha$, then $\mathrm{tp}(a_{i_0}, \ldots, a_{i_{n-1}}) = \mathrm{tp}(a_{j_0}, \ldots, a_{j_{n-1}})$.) In this section, we will say that the set $\{I_i : i \in X\}$ is t_n-uniform if the following condition holds:

(*) If $i_0, i_1 \in X$ and $\bar{a}_j$ is an n-tuple (ordered increasingly) from I_{i_j} $(j = 0, 1)$, then $\mathrm{tp}(t_n(\bar{a}_0)) = \mathrm{tp}(t_n(\bar{a}_1))$.

If there is an uncountable subset Y of X such that $\{I_i : i \in Y\}$ is t_n-uniform, then we will say that $\{I_i : i \in X\}$ is essentially t_n-uniform. The following two claims can be proven by the same argument as those in Lemmas 3.5 and 3.6.

Claim 4.1 *Let $\{I_i : i \in X\}$ be a set of n-indiscernible sequences with $|X| = \omega_1$. Then one of the following cases holds:*

1. *$\{I_i : i \in X\}$ is essentially t_n-uniform;*
2. *there is a formula $\varphi(x)$ such that both*

$$X^{\varphi,t_n} = \{i \in X : \varphi(x) \in \mathrm{tp}(t_n(a_{i,0}, \ldots, a_{i,n-1}))\}$$

and

$$X^{\neg\varphi,t_n} = \{i \in X : \neg\varphi(x) \in \mathrm{tp}(t_n(a_{i,0}, \ldots, a_{i,n-1}))\}$$

are uncountable, where $a_{i,0}, \ldots, a_{i,n-1}$ is the beginning part of I_i.

Claim 4.2 *Let $\{I_i : i \in X\}$ be a set of n-indiscernible sequences with $|X| = \omega_1$. Suppose that $\{I_i : i \in X\}$ is not essentially t_k-uniform, where $k \leq n$. Then for any uncountable subsets $X_i \subset X$ $(i = 0, 1)$, we can find uncountable sets $X'_i \subset X_i$ $(i = 0, 1)$ and $\varphi(x)$ such that $X'_0 \subset X^{\varphi, t_k}$ and $X'_1 \subset X^{\neg\varphi, t_k}$.*

We put $X_\varnothing = \omega_1$, and for each $i \in X_\varnothing$ we fix a sequence $I_\varnothing(i)$ enumerating the universe M_i. Using the Erdös-Rado theorem and Claim 4.2, for $\eta \in 2^{<\omega}$, we can inductively choose $X_\eta \subset \omega_1$, $\{I_\eta(i) : i \in X_\eta\}$ and formulas $\varphi_{\eta,k}$ with the following properties.

1. If $\eta < \nu$, then
   (a) X_ν is an uncountable subset of X_η;
   (b) $I_\nu(i)$ is a subsequence of $I_\eta(i)$ for each $i \in X_\nu$.
2. $|I_\eta(i)| < |I_\eta(j)|$ for all $i, j \in X_\eta$ with $i < j$,
   and $\sup\{|I_\eta(i)| : i \in X_\eta\} = \beth_{\omega_1}$.
3. If $\eta \in 2^n$ then
   (a) each $I_\eta(i)$ is an infinite n-indiscernible sequence;
   (b) $\{I_\eta(i) : i \in X_\eta\}$ is essentially t_n-uniform $\Rightarrow$ it is t_n-uniform.
4. If $\eta \in 2^n$ and $k \leq n$ then
$$\{I_i : i \in X_\eta\} \text{ is not } t_k - \text{uniform } \Rightarrow X_{\eta\frown 0} \subset (X_\eta)^{(\varphi_{\eta,k}),t_k} \text{ and}$$
$$X_{\eta\frown 1} \subset (X_\eta)^{(\neg\varphi_{\eta,k}),t_k}.$$

For $\eta \in 2^n$ such that $\{I_i : i \in X_\eta\}$ is t_n-uniform, let
$$p_\eta(x) = \text{tp}(t_n(a_{i_0}, \dots, a_{i_{n-1}})),$$
where $a_{i_1}, \dots, a_{i_{n-1}}$ is the beginning part of I_i. By the t_n-uniformity, this definition is well defined. p_η is a type realized in M_i, so p_η does not belong to R.

Although it is not used in our proof, we remark the following.

Remark Suppose that there is an infinite path $\nu \in 2^\omega$ such that for each $n \in \omega$ $\{I_{\nu|n}(i) : i \in X_{\nu|n}\}$ is t_n-uniform. Then we can easily find an infinite indiscernible sequence I whose Skolem closure only realizes types in $\{p_{\nu|n} : n \in \omega\}$. This can be shown as below. Let $\Gamma(\{x_i\}_{i\in\omega})$ be the following set of formulas.
$$\{\{x_i\}_{i\in\omega} \text{ is indiscernible}\} \cup \bigcup_{n\in\omega} p_{\nu|n}(t_n(\bar{x}_n)).$$
Clearly, it is consistent. Choose a realization I of Γ. Let M be the Skolem closure of I. Then each element of M has the form $t_n(\bar{a})$, where $\bar{a}$ is an n-tuple from I. So, by the indiscernibility, $t_n(\bar{a})$ realizes $p_{\eta|n}$, which is not a member of R. By a compactness argument, the cardinality of I can be chosen arbitrarily large.

For $\nu \in 2^\omega$, we define the following:

1. K_ν is the set of all $n \in \omega$ such that $\{I_{\nu|n}(i) : i \in X_{\nu|n}\}$ is not t_n-uniform;
2. for $n \in K_\nu$, let $\Gamma^n_\nu(x)$ be the set
$$\bigcup_{n\leq m\in\omega} \{\varphi_{\nu|m,n}(x) : \nu(m) = 0\} \cup \bigcup_{n\leq m\in\omega} \{\neg\varphi_{\nu|m,n}(x) : \nu(n) = 1\};$$
   (Recall that $\varphi_{\eta,n}(x)$ was a formula dividing X_η into two uncountable sets.)
3. finally let $\Delta_\nu(\{x_i\}_{i<\kappa})$ be the set
$$\{\{x_i\}_{i<\kappa} \text{ is indiscernible}\} \cup \bigcup_{n\in K_\nu} \Gamma^n_\nu(t_n(\bar{x}_n)) \cup \bigcup_{n\notin K_\nu} p_{\nu|n}(t_n(\bar{x}_n)).$$

Claim 4.3 *$\Delta_\nu(\{x_i\}_{i<\kappa})$ is finitely satisfiable.*

Let $n^* \in \omega$. Choose $i \in X_{\nu|n^*}$ arbitrarily, and let $I = I_{\nu|n^*}(i)$. Clearly, I is an n^*-indiscernible sequence. Let $n < n^*$. First assume $n \notin K_\nu$. The first n-elements of I clearly realize the type $p_{\nu|n}(t_n(\bar{x}))$. Then assume $n \in K_\nu$ and let $n \leq m < n^*$. Suppose $\nu(m) = 0$. Then $X_{\nu|n^*} \subset X_{\nu|m^\frown 0} \subset (X_{\nu|m})^{\varphi_{\nu|m}, t_m}$. So if $\bar{a}$ is the first n-element of I, then it satisfies $\varphi_{\nu|m,n}(t_n(\bar{x}_n))$. For the same reason, if $\nu(m) = 1$, $\bar{a}$ satisfies $\neg\varphi_{\nu|m,n}(t_n(\bar{x}_n))$. The above argument shows that $\Delta_\nu(\{x_i\}_{i<\kappa})$ is finitely satisfiable.

Claim 4.4 *Let $\eta \neq \eta'$ be two infinite paths. If $\eta|n = \eta'|n$ and $n \in K_\eta \cap K_{\eta'}$, then $\Gamma^n_\eta(x)$ and $\Gamma^n_{\eta'}(x)$ are contradictory.*

Choose the largest $m \geq n$ with $\eta|m = \eta'|m$. We can assume $\eta(m) = 0$ and $\eta'(m) = 1$. Then $\Gamma^n_\eta(x)$ contains $\psi_{\eta|m,n}(x)$, while $\Gamma^n_{\eta'}(x)$ contains $\neg\psi_{\eta|m,n}(x)$.

Claim 4.5 *There is a model N such that*

1. *N omits each member of R;*
2. *the domain of N is the Skolem hull of an infinite indiscernible sequence.*

For each $\nu \in 2^\omega$, choose $J_\nu = (a_{\nu,i})_{i<\kappa}$ realizing Δ_η. Let $\mathcal{K} = \{\nu|n : \nu \in 2^\omega, n \in K_\nu\}$. For $\eta \in \mathcal{K}$, let S_η be the set

$$\{\nu \in 2^\omega : \eta < \nu,\ \mathrm{tp}(t_n(a_{\nu,0}, \ldots, a_{\nu,n-1})) \in R\},$$

where $n = \mathrm{len}(\eta)$. By Claim 4.4 and our assumption that $|R| < 2^\omega$, we have $|S_\eta| \leq |R| < 2^\omega$. So we can choose $\nu \in 2^\omega \setminus \bigcup_{\eta\in\mathcal{K}} S_\eta$.

J_ν is an infinite indiscernible sequence. By our choice of ν, if $\nu|n \in \mathcal{K}$, then $t_n(a_{\nu|n,0}, \ldots, a_{\nu|n,n-1})$ does not realize R. If $\nu|n \notin \mathcal{K}$, then $t_n(a_{\nu|n,0}, \ldots, a_{\nu|n,n-1})$ realizes $p_{\nu|n}$, which is not a member of R. Let N be the Skolem hull of J_ν. By the indiscernibility (and the fact that t_ks enumerate all the Skolem terms), N does not realize a member of R.

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