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A Note on Logics of Ignorance and Borders

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Abstract We present and show topological completeness for LB, the logic of the topological border. LB is also a logic of epistemic ignorance. Also, we present and show completeness for LUT, the logic of unknown truths. A simple topological completeness proof for **S4** is also presented using a T_1 space.

1 Introduction

To say an agent is ignorant of the truth or falsity of proposition A, we mean the agent doesn't know A and doesn't know the negation of A. We present an axiom system intended to capture this notion of ignorance. The axiom system is LB, as it is also the logic of the topological border, and a topological model is used to show completeness. We also present and show completeness for the logic LUT, describing the logic of true statements which the agent is ignorant of.

The logic **S4** is often taken as a logic of knowledge. Also, **S4** is topologically complete when the box is interpreted as the interior (McKinsey and Tarski [7]). To express that an agent is ignorant of the truth or falsity of A, we may say, "At world w, $\neg KA \wedge \neg K \neg A$ is true" (where KA means the agent knows that A). Similarly, if we want to say a point w is a member of the border of the set A, we would write "$w \in -I(A) \cap -I(-A)$" (where $I(A)$ is the interior of A). Since knowledge and the interior both obey **S4**, the structural resemblance between ignorance and the border is clear, and LB describes both. Considering LUT, to express that a statement is true but unknown, we may say $A \wedge \neg KA$ is true at world w.

In van der Hoek and Lomuscio [5] a logic of ignorance was introduced. A relational model was used to show completeness and they imposed no restrictions on the relation (corresponding to the modal logic K); a logic of ignorance for when the relation is transitive was also worked out. Here we show topological completeness for a logic of ignorance corresponding to **S4** (in relational terms, this corresponds to a logic of ignorance where the relation is reflexive and transitive).

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Apart from the motivation given in [5], our own is mainly philosophical. We are interested in exploring connections between epistemology and topology, as has previously been done in van Benthem and Sarenac [2], Dabrowski, Moss, and Parikh [3], Georgatos [4], Kelly [6], Steinsvold [8], and Vickers [9]. For relations between logic and topology in general, see Aiello, Pratt-Hartmann, and van Benthem [1].

Following the philosophical interest in T_1 spaces discussed in [8], we also include a completeness proof using a T_1 space for **S4**. The proof is a simple variation of the completeness proof for **S4** presented in the first chapter of [1]. The completeness proofs for LB and LUT, both nonnormal modal logics, are structurally similar to the aforementioned proof as well.

2 Topological Definitions

Definition 2.1 Let W be a nonempty set. Let τ be a subset of the power set of W such that

1. τ is closed under finite intersection,
2. τ is closed under union,
3. $W \in \tau$,
4. $\varnothing \in \tau$.

τ is a *topology* on W.

Definition 2.2 Where τ is a topology on W, members of W are *points* or *worlds*, and $\langle W, \tau \rangle$ is a *topological space*. The members of the topology τ are *open sets*, or *opens*. We will use O and U as variables for open sets. A set is *closed* if and only if the complement is open. The *interior of* A, $I(A)$, is the union of all open subsets of A. The *closure of* A, $C(A)$, is the intersection of all closed supersets of A. The *border of* A, $b(A)$, is $C(A) \cap C(-A)$.

The interior operator and closure operator are duals, $I(A) = -C(-A)$. A point x is a member of $I(A)$ if and only if there is some $O \subseteq A$ and $x \in O$. A point x is in $C(A)$ if and only if, for all O, if $x \in O$, then $O \cap A$ is nonempty.

Definition 2.3 A topological space is T_1 if and only if every cofinite set is open.

Definition 2.4 Let τ be a topology and let $\mathfrak{B} \subseteq \tau$. $\mathfrak{B}$ is a *base* for the topology τ if and only if every member of τ can be formed using unions of members of $\mathfrak{B}$. The members of $\mathfrak{B}$ are called *basic open sets*. If $\mathfrak{B}$ is a base for τ, τ is *generated* by $\mathfrak{B}$.

The following useful theorem is a common result; we omit proof.

Proposition 2.5 *Let $\mathfrak{B}$ be a subset of the power set of some set W. $\mathfrak{B}$ is a base for a topology on W if and only if*

1. $\bigcup\{Y | Y \in \mathfrak{B}\} = W$, *and*
2. *if X and Y are members of $\mathfrak{B}$ and $w \in X \cap Y$, then there is some $Z \in \mathfrak{B}$ and $w \in Z \subseteq X \cap Y$.*

3 Topological Semantics

Definition 3.1 Given a topological space $\langle W, \tau \rangle$, we define a *valuation* V as a function from atomic sentences to subsets of W. Intuitively, if $b \in V(p)$ then p is true at the world b. A *topological model* $M = \langle W, \tau, V \rangle$ is a topological space with a valuation.

Definition 3.2 (Truth in a topological model M at a point w)

$$\begin{array}{rll} M, w \models p & \text{iff} & w \in V(p) \\ M, w \models \bot & \text{iff} & 0 = 1 \\ M, w \models A \rightarrow B & \text{iff} & M, w \models A \text{ then } M, w \models B \\ M, w \models KA & \text{iff} & (\exists O)[w \in O \text{ and } (\forall x \in O)(M, x \models A)] \\ M, w \models \Box A & \text{iff} & (\exists O)[w \in O \text{ and } ((\forall x \in O)(M, x \models A) \\ & & \qquad\qquad\qquad \text{or } (\forall x \in O)(M, x \not\models A))] \\ M, w \models \blacksquare A & \text{iff} & M, w \models A \text{ and } (\forall O)(\text{ if } w \in O \text{ then } (\exists y \in O)(M, y \not\models A)) \end{array}$$

$\neg A$ is defined as $A \rightarrow \bot$, and $\top$ is defined as $\neg\bot$, so it is always true.

K is the box for **S4**, representing knowledge. $\Box A$ is the dual of ignorance (the agent knows whether or not A ($KA \vee K\neg A$)). This is the box for the logic LB, the logic of borders.

Definition 3.3 Given a topological model $M = \langle W, \tau, V \rangle$ and a formula A,

$$[A]^M = \{x \in W | M, x \models A\}.$$

To be sure that we are capturing the right operator, b, note that

$$M, w \models \Diamond A \text{ if and only if } w \in C([A]^M) \text{ and } w \in C([\neg A]^M).$$

The black square $\blacksquare$ is the box for LUT, the logic of unknown truths. Thus $\blacksquare A$ is intended to represent $A \wedge \neg KA$. Epistemically, we may think of LUT as a more specific logic of ignorance. For if $A \wedge \neg KA$ is true, then (since $A \rightarrow \neg K \neg A$), $\neg KA \wedge \neg K \neg A$. In other words, if $\blacksquare A$ is true at a point, then $\Diamond A$ will be true as well. Topologically, we may think of LUT as the logic of a type of border point.

Definition 3.4 A sentence φ is *valid in a topological model* if and only if φ is true at every point in the model. A sentence φ is *valid in a topological space* if and only if φ is valid in every topological model based on the space.

Each of our boxes can define the other. Thus all of the following sentences are valid:

$$\begin{array}{rcl} (\Box A \wedge A) & \leftrightarrow & KA, \\ (KA \vee K\neg A) & \leftrightarrow & \Box A, \\ \blacksquare A & \leftrightarrow & (A \wedge \neg KA), \\ (\blacklozenge \neg A \wedge A) & \leftrightarrow & KA, \\ (A \wedge \neg(\Box A \wedge A)) & \leftrightarrow & \blacksquare A, \\ [(\blacklozenge \neg A \wedge A) \vee (\blacklozenge A \wedge \neg A)] & \leftrightarrow & \Box A. \end{array}$$

4 The Logic S4

The following is a nonstandard, though equivalent, variant of **S4**:

Name	Axiom Scheme
N	$K\top \leftrightarrow \top$
T	$KA \rightarrow A$
K	$(KA \wedge KB) \leftrightarrow K(A \wedge B)$

along with the following inference rule, R4,

$$\text{from } \mathbf{S4} \vdash KA \rightarrow B, \text{ infer } \mathbf{S4} \vdash KA \rightarrow KB,$$

with MP, substitution of equivalents, and all tautologies and their modal instances.

The 4 axiom (or the KK thesis), $KA \rightarrow KKA$, is an immediate consequence of R4. For **S4** $\vdash KA \rightarrow KA$ (tautology) and applying R4 gives us **S4** $\vdash KA \rightarrow KKA$. (Furthermore, R4 is a valid rule of inference in the logic **K4**). We leave topological soundness to the reader.

4.1 Completeness for S4

Let W be the maximally consistent sets of **S4**.

Definition 4.1 For any formula A, $\widehat{A} = \{x \in W | A \in x\}$.

Let $\mathfrak{B} = \{\, \widehat{KA} - F \mid$ where A is a formula and F is a finite subset of $W\,\}$.

Proposition 4.2 *$\mathfrak{B}$ is a base for a topology on W.*

Proof We show this with Proposition 2.5 in mind. Where F is the empty set, $\widehat{K\top} - F = W$, and so $W \in \mathfrak{B}$, satisfying the first condition of Proposition 2.5. To see that the second condition of Proposition 2.5 holds, assume $\widehat{KA} - F_1$ and $\widehat{KB} - F_2$ are members of $\mathfrak{B}$. We show that the intersection of the two sets are also members of $\mathfrak{B}$. For $(\widehat{KA} - F_1) \cap (\widehat{KB} - F_2) = \widehat{KA \wedge KB} - (F_1 \cup F_2)$. By axiom K, $\widehat{KA \wedge KB} = K\widehat{(A \wedge B)}$, so $(\widehat{KA} - F_1) \cap (\widehat{KB} - F_2) = K\widehat{(A \wedge B)} - (F_1 \cup F_2)$. Since $A \wedge B$ is a formula and $F_1 \cup F_2$ is a finite set, $K\widehat{(A \wedge B)} - (F_1 \cup F_2)$ is in $\mathfrak{B}$. $\square$

Let τ be the topology generated by $\mathfrak{B}$. For all p, let $x \in V(p)$ if and only if $p \in x$. This gives a topological model, $M = \langle W, \tau, V\rangle$.

Proposition 4.3 *For all formulas A, $(\forall w \in W)(M, w \models A$ if and only if $A \in w)$.*

Proof The nonmodal cases are straightforward. Assume $w \in \widehat{KA}$. By axiom T, $\widehat{KA} \subseteq \widehat{A}$. By induction hypothesis (hereafter, IH), A is true at all points in $\widehat{KA}$. Where F is empty, $\widehat{KA} - F = \widehat{KA}$; thus $\widehat{KA}$ is open, so $M, w \models KA$.

Assume $M, w \models KA$. Thus for some open set O, $w \in O$ and A is true at all points in O. Since O is formed using the basic open sets from $\mathfrak{B}$, there must be some $\widehat{KB} - F \in \mathfrak{B}$, $w \in \widehat{KB} - F$, and $\widehat{KB} - F \subseteq [A]^M$. By IH, $\widehat{KB} - F \subseteq \widehat{A}$, which implies that $\widehat{KB \wedge \neg A} \subseteq F$. But then F must be empty, and $KB \wedge \neg A$ inconsistent (for all consistent formulas C, $\widehat{C}$ is infinite). Thus **S4** $\vdash KB \rightarrow A$. By the inference rule R4, **S4** $\vdash KB \rightarrow KA$. Thus, $KA \in w$. $\square$

By the construction of $\mathfrak{B}$, all cofinite sets are in τ (for all finite F, $\widehat{K\top} - F \in \mathfrak{B}$), and thus our space is T_1. As mentioned, our model is a simple variation of the model used in Chapter 1 of [1] to show completeness for **S4**. There, the set of all $\widehat{KA}$ is used as a base.

5 LB: The Logic of Borders

The axiom system for LB is as follows:

Name	Axiom Scheme
N	$\Box\top \leftrightarrow \top$
Z	$\Box A \leftrightarrow \Box\neg A$
R	$(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

along with the following inference rule, WM,

$$\text{from LB} \vdash (\Box A \wedge A) \rightarrow B, \text{ infer LB} \vdash (\Box A \wedge A) \rightarrow (\Box B \wedge B),$$

with MP, substitution of equivalents, and all tautologies and their modal instances.

The following theorem will be useful in proving completeness for LB.

Proposition 5.1 LB $\vdash (\Box A \wedge \Box B \wedge A \wedge B) \leftrightarrow (\Box(A \wedge B) \wedge A \wedge B)$.

Proof

1.	LB $\vdash (A \wedge B) \rightarrow B$	tautology
2.	LB $\vdash (\Box(A \wedge B) \wedge A \wedge B) \rightarrow B$	1, classical logic
3.	LB $\vdash (\Box(A \wedge B) \wedge A \wedge B) \rightarrow (\Box B \wedge B)$	using WM on line 2
4.	LB $\vdash (A \wedge B) \rightarrow A$	tautology
5.	LB $\vdash (\Box(A \wedge B) \wedge A \wedge B) \rightarrow A$	4, classical logic
6.	LB $\vdash (\Box(A \wedge B) \wedge A \wedge B) \rightarrow (\Box A \wedge A)$	using WM on line 5
7.	LB $\vdash (\Box(A \wedge B) \wedge A \wedge B) \rightarrow (\Box A \wedge \Box B \wedge A \wedge B)$	3, 6 using classical logic.

Conversely,

8.	LB $\vdash (\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$	axiom R
9.	LB $\vdash (\Box A \wedge \Box B \wedge A \wedge B) \rightarrow (\Box(A \wedge B) \wedge A \wedge B)$	8, classical logic
10.	LB $\vdash (\Box A \wedge \Box B \wedge A \wedge B) \leftrightarrow (\Box(A \wedge B) \wedge A \wedge B)$	7 and 9.

□

We leave soundness to the reader and move to completeness.

5.1 Completeness Let W be the set of all maximally consistent sets of LB. Let

$$\mathfrak{B} = \{\widehat{\Box A \wedge A} \,|\, A \text{ is a formula}\}.$$

Proposition 5.2 *$\mathfrak{B}$ is a base for a topology on W.*

Proof We show this with Proposition 2.5 in mind. By axiom N, $W = \widehat{\Box\top \wedge \top}$, and $\widehat{\Box\top \wedge \top} \in \mathfrak{B}$. Secondly, assume $\widehat{\Box A \wedge A}$ and $\widehat{\Box B \wedge B}$ are in $\mathfrak{B}$. Now, by Proposition 5.1, LB $\vdash (\Box A \wedge A \wedge \Box B \wedge B) \leftrightarrow (\Box(A \wedge B) \wedge A \wedge B)$. Thus, since $A \wedge B$ is a formula, $\widehat{\Box A \wedge A} \cap \widehat{\Box B \wedge B}$ is in $\mathfrak{B}$. □

Let $M = \langle W, \tau, V\rangle$, where $\mathfrak{B}$ generates τ and $x \in V(p)$ if and only if $x \in \widehat{p}$.

Proposition 5.3 *For all formulas A, $(\forall w \in W)(M, w \models A$ if and only if $A \in w)$.*

Proof The nonmodal cases are straightforward. Assume $\Box A \in w$. Either $A \in w$ or not. Assume $A \in w$. Then there is an open set, $\widehat{\Box A \wedge A}$, w is a member of the open set, and $\widehat{\Box A \wedge A} \subseteq \widehat{A}$. By IH, for all x in $\widehat{\Box A \wedge A}$, $M, x \models A$. Thus $M, w \models \Box A$. Assume $A \notin W$, so $\neg A \in w$. Now, by axiom Z, $\widehat{\Box A} = \widehat{\Box\neg A}$, so w is in the basic open set $\widehat{\Box\neg A \wedge \neg A}$, and $\widehat{\Box\neg A \wedge \neg A} \subseteq \widehat{\neg A}$. By IH, for all $x \in \widehat{\Box\neg A \wedge \neg A}$, $M, x \models \neg A$. By definition of truth in a model, $M, w \models \Box A$.

Assume $M, w \models \Box A$. Thus there is some O, $w \in O$ and either $O \subseteq [A]^M$ or $O \subseteq [\neg A]^M$. Assume $O \subseteq [A]^M$. Since O is formed using the basic open sets in $\mathfrak{B}$, there must be some $\widehat{\Box B \wedge B}$, $w \in \widehat{\Box B \wedge B}$, and $\widehat{\Box B \wedge B} \subseteq [A]^M$. By IH, $\widehat{\Box B \wedge B} \subseteq \widehat{A}$, and so LB $\vdash (\Box B \wedge B) \to A$. By WM, LB $\vdash (\Box B \wedge B) \to (\Box A \wedge A)$. Thus $\Box A \in w$. Assume $O \subseteq [\neg A]^M$; so for some basic open set $\widehat{\Box B \wedge B}$, $w \in \widehat{\Box B \wedge B} \subseteq [\neg A]^M$. By IH, LB $\vdash (\Box B \wedge B) \to \neg A$. By WM, LB $\vdash (\Box B \wedge B) \to (\Box\neg A \wedge \neg A)$. Thus $\Box\neg A \in w$. By axiom Z, $\Box A \in w$. □

6 LUT: The Logic of Unknown Truths

The axiom system for LUT is

Name	Axiom Scheme
T	$A \to \blacklozenge A$
U	$\blacklozenge\bot \leftrightarrow \top$
V	$(\blacklozenge A \wedge \blacklozenge B) \to \blacklozenge(A \vee B)$,

along with the following inference rule, SM,

$$\text{from LUT} \vdash (\blacklozenge\neg A \wedge A) \to B, \text{ infer LUT} \vdash (\blacklozenge\neg A \wedge A) \to (\blacklozenge\neg B \wedge B),$$

with MP, substitution of equivalents, all tautologies and their modal instances.

Proposition 6.1 LUT $\vdash (\blacklozenge\neg(A \wedge B) \wedge A \wedge B) \leftrightarrow (\blacklozenge\neg A \wedge A \wedge \blacklozenge\neg B \wedge B)$.

Proof

1.	LUT $\vdash (\blacklozenge\neg A \wedge \blacklozenge\neg B) \to \blacklozenge(\neg A \vee \neg B)$	instance of axiom V
2.	LUT $\vdash (\blacklozenge\neg A \wedge \blacklozenge\neg B) \to \blacklozenge\neg(A \wedge B)$	from 1, De Morgan's
3.	LUT $\vdash (\blacklozenge\neg A \wedge A \wedge \blacklozenge\neg B \wedge A) \to (\blacklozenge\neg(A \wedge B) \wedge A \wedge B)$	from 2, classical logic
4.	LUT $\vdash (A \wedge B) \to A$	tautology
5.	LUT $\vdash (\blacklozenge\neg(A \wedge B) \wedge A \wedge B) \to A$	from 4, classical logic
6	LUT $\vdash (\blacklozenge\neg(A \wedge B) \wedge A \wedge B) \to (\blacklozenge\neg A \wedge A)$	from 5, using SM
7.	LUT $\vdash (A \wedge B) \to B$	tautology
8.	LUT $\vdash (\blacklozenge\neg(A \wedge B) \wedge A \wedge B) \to B$	from 7, classical logic
9.	LUT $\vdash (\blacklozenge\neg(A \wedge B) \wedge A \wedge B) \to (\blacklozenge\neg B \wedge B)$	from 8, using SM
10.	LUT $\vdash (\blacklozenge\neg(A \wedge B) \wedge A \wedge B) \to (\blacklozenge\neg A \wedge A \wedge \blacklozenge\neg B \wedge B)$	from 6 and 9
11.	LUT $\vdash (\blacklozenge\neg(A \wedge B) \wedge A \wedge B) \leftrightarrow (\blacklozenge\neg A \wedge A \wedge \blacklozenge\neg B \wedge B)$	from 3 and 10

□

We leave soundness for the reader and move to completeness.

6.1 Completeness Let W be the set of all maximally consistent sets of LUT. Let

$$\mathfrak{B} = \{\widehat{\blacklozenge\neg A \wedge A} | A \text{ is a formula}\}.$$

Proposition 6.2 *$\mathfrak{B}$ is a base for a topology on W.*

Proof By axiom U, $W = \widehat{\blacklozenge\bot \wedge \top}$, so the first clause of Proposition 2.5 is satisfied. Proposition 6.1 satisfies the second clause of Proposition 2.5. $\square$

Let $M = \langle W, \tau, V\rangle$, where $\mathfrak{B}$ generates τ and $x \in V(p)$ if and only if $x \in \widehat{p}$.

Proposition 6.3 *For all formulas A, $(\forall w \in W)(M, w \models A$ if and only if $A \in w)$.*

Proof The nonmodal cases are straightforward. Assume $M, w \models \blacksquare A$. So $w \in [A]^M \cap C([\neg A]^M)$. By IH, $w \in \widehat{A} \cap C(\widehat{\neg A})$. If $\blacksquare A \notin w$, then $w \in \widehat{\blacklozenge\neg A \wedge A}$, which is a basic open set. Thus $\widehat{\neg A} \cap \widehat{\blacklozenge\neg A \wedge A}$ is nonempty, contradiction. So, $\blacksquare A \in w$.

Conversely, assume $M, w \not\models \blacksquare A$. Then $w \in [\neg A]^M \cup I([A]^M)$. Assume $w \in [\neg A]^M$. By IH, $A \notin w$; thus $\neg A \in w$. By axiom T, $\blacklozenge\neg A \in w$; thus $w \notin \blacksquare A$. Otherwise, assume $w \in I([A]^M)$. Thus there is some O, $w \in O$ and $O \subseteq [A]^M$. By IH, $O \subseteq \widehat{A}$. There must be some $\widehat{\blacklozenge\neg B \wedge B} \in \mathfrak{B}$, where $w \in \widehat{\blacklozenge\neg B \wedge B} \subseteq \widehat{A}$. Thus LUT $\vdash (\blacklozenge\neg B \wedge B) \rightarrow A$, and using the inference rule SM we get LUT $\vdash (\blacklozenge\neg B \wedge B) \rightarrow (\blacklozenge\neg A \wedge A)$. Thus $\blacklozenge\neg A \in w$ and $w \notin \blacksquare A$. $\square$

If we change the base for LUT (LB) to the set of all $\widehat{\blacklozenge\neg A \wedge A} - F$ ($\widehat{\Box A \wedge A} - F$) where F is a finite, we have a T_1 space and completeness still goes through. Both truth lemmas need the fact that for all consistent formulas C, $\widehat{C}$ is infinite.

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