Automorphisms of Countable
Short Recursively Saturated Models of PA

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Abstract   A model of Peano Arithmetic is short recursively saturated if it realizes all its bounded finitely realized recursive types. Short recursively saturated models of PA are exactly the elementary initial segments of recursively saturated models of PA. In this paper, we survey and prove results on short recursively saturated models of PA and their automorphisms. In particular, we investigate a certain subgroup of the automorphism group of such models. This subgroup, denoted $G|_{M(a)}$, contains all the automorphisms of a countable short recursively saturated model of PA which can be extended to an automorphism of the countable recursively saturated elementary end extension of the model.

1   Introduction

The notion of recursive saturation was introduced in the 1970s by Barwise and Schlipf [1] and independently by Ressayre [19]. Every complete consistent theory in a finite language has countable recursively saturated models. Moreover, every theory in a finite language extending PA has continuum many nonisomorphic countable recursively saturated models. Countable recursively saturated models of arithmetic have continuum many elementary initial segments. All but countably many are recursively saturated. This paper explores the countably many elementary initial segments which are not recursively saturated. All these segments have a weaker property than recursive saturation, namely, short recursive saturation.

A model is short recursively saturated if it realizes all finitely realized *bounded* recursive types with finitely many parameters. It had been shown (see Smoryński [23]) that, unlike countable recursively saturated models of arithmetic which are uniquely determined, up to isomorphism, by their complete theory and standard system, there are countably many nonisomorphic short recursively saturated models that have the same theory as well as the same standard system.

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Short recursively saturated models of PA were studied in the late 1970s and early 1980s, in particular, in Kossak [10], Kotlarski [14], Kotlarski [15], Lesan [17], and Smoryński [23]. Since then, short recursively saturated models of PA were only studied indirectly when initial segments of recursively saturated models (and thus initial segments of short recursively saturated models) were investigated (see, for example, Bigorajska et al. [2], Schmerl [21], and Nurkhaidarov [18]).

In this paper, we present some results from the author's thesis, Shochat [22], regarding the automorphism groups of countable short recursively saturated models of PA. In Section 2, we introduce notation, list definitions, and survey basic results. In Section 3, we discuss automorphisms of countable short recursively saturated models of PA. In Section 4, we prove that any automorphism of a countable short recursively saturated model of PA has continuum many conjugates. In Section 5, we prove the existence of automorphisms of a countable short recursively saturated model of PA which do not extend to automorphisms of the countable recursively saturated elementary end extension of the model. This leads to the investigation of the subgroup $G|_{M(a)}$ which consists of all the automorphisms of the model which can be extended. We note that the extensions of the automorphisms in $G|_{M(a)}$ are exactly the automorphisms of the recursively saturated extension of the short recursively saturated model which fix the short recursively saturated model setwise. Subgroups which fix setwise an initial segment, a gap, or an intersticial gap have been studied in the past, for example, in Kossak et al. [12] and in Nurkhaidarov [18]. In Section 5, we show that the subgroup $G|_{M(a)}$ is dense (with the usual topology) but is neither open nor closed. Finally, we show that it has an uncountable index and that it is not normal.

2 Background and Notation

We assume that the reader is familiar with the basic model theory of PA. The book by Kaye [7] is a good reference. For additional advanced material and recent results related to this paper, see Kossak and Schmerl [13]. Unless otherwise stated, all models in this paper are models of PA and the language is the language of the theory of PA, $\mathcal{L}_{\mathsf{PA}}$.

Let $\mathcal{L}^*$ be an extension of $\mathcal{L}_{\mathsf{PA}}$ and M^* be an expansion of a model M to the language $\mathcal{L}^*$. Then $M^* \models \mathsf{PA}^*$ if and only if the induction scheme is true in M^* for all formulas of $\mathcal{L}^*$. Most of the results in this paper are true for PA^* when $\mathcal{L}^*$ is recursively presented and could be proven in exactly the same way. In particular, we will use this fact when we need to extend the language to include a finite number of constant symbols.

Let I be a subset of M. If for every $x \in I$ and for every $y \in M$, $y < x \rightarrow y \in I$, we shall write $I \subseteq_{\mathrm{end}} M$. In this case, I is said to be an *initial segment* of M, and M is an *end extension* of I. An initial segment closed under the successor function is called a *cut*.

Let J be a subset of M. If for every $x \in M$ there is a $y \in J$ with $y \geq x$, then we write $J \subseteq_{\mathrm{cof}} M$ and say that J is *cofinal* in M.

Let N be a submodel of M. If N is an elementary submodel of M, we will write $N \preceq M$ (if, in addition, N is a proper subset, we will write $N \prec M$). $N \preceq_{\mathrm{end}} M$ denotes that M is an elementary end extension of N. In this case, N is an elementary initial segment (or, equivalently, an elementary cut) of M. $N \preceq_{\mathrm{cof}} M$ denotes that M is an elementary cofinal extension of N.

Let $\bar{a} = a_1, \ldots, a_n$ where n is a standard number and $a_1, \ldots, a_n \in M$ (this will be denoted by $\bar{a} \in M$). By $\mathrm{tp}_M(\bar{a})$ we denote the type of $\bar{a}$ in M. If there is no ambiguity, we shall write $\mathrm{tp}(\bar{a})$ instead of $\mathrm{tp}_M(\bar{a})$.

Let $\bar{a} \in M$. A Skolem term $t_\varphi(\bar{a})$ is a term defined by the formula $\varphi(\bar{y}, x)$ in the following manner:

$$t_\varphi(\bar{a}) = \begin{cases} \min\{x : \varphi(\bar{a}, x)\} & \text{if } \exists x \varphi(\bar{a}, x), \\ 0 & \text{otherwise.} \end{cases}$$

By $\mathrm{Scl}^M(\bar{a})$ we denote the Skolem closure of $\bar{a}$ in M; that is, $\mathrm{Scl}^M(\bar{a}) = \{t(\bar{a}) : t$ is a Skolem term$\}$. If there is no ambiguity, we shall write $\mathrm{Scl}(\bar{a})$. Models of this form are called *simple* models.

A subset of M is said to be *M-finite* if it is definable and bounded in M. For every $a \in M$ there is a unique M-finite set A such that

$$a = \sum_{x \in A} 2^x.$$

In this case we will say that a codes A. Also, for any M-finite set A, there is an $a \in M$ such that a codes A. If a codes the set A, we will write $x \in a$ if and only if $x \in A$.

Every M-finite sequence of M is also coded by some $a \in M$. The length of the sequence coded by a is denoted by $\mathrm{lh}(a)$, and by $(a)_i$ we denote the $i + 1$ term in the sequence coded by a. An ω-sequence $x_0, x_1, x_2, \ldots$ is said to be ω-coded by c if c codes a sequence of nonstandard length, and for any $i \in \mathbb{N}$, $(c)_i = x_i$. In this paper, we also use Gödel's pairing function, $\langle x, y \rangle = \frac{1}{2}[(x + y)^2 + 3x + y]$, as a definable bijection from $M \times M$ into M.

2.1 Recursive and short recursive saturation

Many of the definitions and results (including proofs) in this subsection could be found in [23].

Definition 2.1 A type $p(v, \bar{a}) = \{\varphi_n(v, \bar{a}) : n \in \mathbb{N}\}$ is *recursive* if the set $\{\ulcorner \varphi_n(x, \bar{y}) \urcorner : n \in \mathbb{N}\}$ is recursive, where $\ulcorner \varphi(x, \bar{y}) \urcorner$ denotes the Gödel number of the formula $\varphi(x, \bar{y})$. A model M is *recursively saturated* if and only if every recursive type $p(v, \bar{a})$, with $\bar{a} \in M$, which is finitely realized in M, is realized in M.

Definition 2.2 A type $p(v, \bar{a})$ over a model M is *bounded* if it contains the formula $v < t(\bar{a})$ for some Skolem term t. A model M is *short recursively saturated* if and only if every bounded recursive type $p(v, \bar{a})$, with $\bar{a} \in M$, which is finitely realized in M, is realized in M.

Clearly, every recursively saturated model is short recursively saturated. Moreover, it is easy to prove the following.

Proposition 2.3 *Let M be a recursively saturated model of* PA*, and let $N \preceq_{\mathrm{end}} M$. Then N is short recursively saturated.*

Kossak [10] showed that the converse to the above theorem holds as well. That is, any short recursively saturated model of PA has a recursively saturated elementary end extension of the same cardinality.

2.2 Short models   Let M be a model of PA and $a \in M$. Let

$$M(a) = \{b \in M : b < t(a) \text{ for some Skolem term } t\}.$$

Proposition 2.4   *Let M be a model of PA and $a \in M$. Then*

$$\mathrm{Scl}(a) \preceq_{\mathrm{cof}} M(a) \preceq_{\mathrm{end}} M.$$

Definition 2.5   If $M = M(a)$ for some $a \in M$, we say that M is *short*. Otherwise, we say that M is *tall*.

Proposition 2.6 ([23])   *If M is a short model, then M is not recursively saturated.*

Since for any $a \in M$, $M(a)$ is a short model, it follows from Propositions 2.3 and 2.6 that there are short recursively saturated models which are not recursively saturated. On the other hand, we have the following.

Proposition 2.7 ([23])   *Let M be a tall model. If M is short recursively saturated, then M is recursively saturated.*

Notation   From this point on, since short recursively saturated models that are tall are recursively saturated, whenever we refer to a model as short recursively saturated we mean a short recursively saturated model which is short.

With this convention, Proposition 2.3 and the remark following the proposition imply the following.

Proposition 2.8   *A model N is short recursively saturated if and only if $N \cong M(a)$ for some recursively saturated model M and $a \in M$.*

It follows from the above results that any countable recursively saturated model of PA has only countably many short recursively saturated elementary initial segments.

Proposition 2.9 ([14])   *Let M be a recursively saturated model of PA. Then the family of short recursively saturated elementary initial segments of the model forms a dense linear order with a least element and no last element. In particular, if M is countable, then the family of short recursively saturated elementary initial segments of M has the order type $1 + \mathbb{Q}$.*

In contrast to this result, in the same paper, Kotlarski [14] showed that the family of recursively saturated initial cuts of M has the order type of the Cantor set.

We will say that a model is *extremely short* if $M = M(0)$. Since $\mathrm{Scl}(0)$ is cofinal in $M(0)$, extremely short models have no proper elementary initial segments. On the other hand, all short recursively saturated models that are not extremely short have elementary recursively saturated initial segments. To show this, we introduce the following notation. Let M be a model which is not extremely short. For any $a \in M \setminus M(0)$, let

$$M[a] = \{b \in M : t(b) < a \text{ for all Skolem terms } t\}.$$

Notice that $M[a]$ is closed under Skolem terms; hence it is an elementary initial segment of M (and of $M(a)$ as well). In fact, it is the largest elementary initial segment of M which does not contain a. Moreover, we have the following.

Proposition 2.10   *Let M be a recursively saturated model of PA, and $a \in M \setminus M(0)$. Then $M[a]$ is tall; hence it is recursively saturated.*

Another notion which is used throughout this paper is that of gaps.

Definition 2.11   Let $a \in M \models \mathsf{PA}$. Then the *gap* of a is defined as follows:

$$\mathrm{gap}(a) = M(a) \backslash M[a].$$

We call gap(0) the *least gap* of M. If $M = M(a)$ for some $a \in M$, we call gap(a) the *last gap* of M. Notice that M is short if and only if M has a last gap.

2.3 Standard systems and isomorphism conditions

Definition 2.12   Let $M \models \mathsf{PA}$. The *standard system* of M, SSy(M), is the family of all subsets of $\mathbb{N}$ that are coded in M. That is, $X \in \mathrm{SSy}(M)$ if and only if $X = \{x \in \mathbb{N} : x \in a\}$ for some $a \in M$. (Equivalently, $X \in \mathrm{SSy}(M)$ if and only if $X = Y \cap \mathbb{N}$ where $Y = \{x : \varphi(x, a)\}$ for some $\varphi \in \mathcal{L}_{\mathsf{PA}}$ and $a \in M$.)

For any nonstandard $M \models \mathsf{PA}$, if $X \in \mathrm{SSy}(M)$, then X has arbitrarily small nonstandard codes. Therefore, whenever $N \subseteq_{\mathrm{end}} M$ and N is nonstandard, $\mathrm{SSy}(N) = \mathrm{SSy}(M)$.

Definition 2.13   Let M be a model of PA. We say that M is (*short*) SSy(M)-*saturated* if and only if

1. for every (bounded) type $p(v, \bar{w})$ whose set of Gödel numbers of formulas in p is in SSy(M), and for every tuple $\bar{b}$ in M, if $p(v, \bar{b})$ is finitely realized in M, then $p(v, \bar{b})$ is realized in M; and
2. for every $a \in M$, $\mathrm{tp}(a) \in \mathrm{SSy}(M)$.

Wilmers [25] (and Smoryński in the short case, see proof in [23]) proved that $M \models \mathsf{PA}$ is (short) recursively saturated if and only if it is (short) SSy(M)-saturated. Using this result we can interchange these notions, and many times when proving results about (short) recursively saturated models we will use (bounded) coded types instead of (bounded) recursive types.

 Another important observation is that every countable recursively saturated model of PA is uniquely determined by its complete theory and its standard system. That is, if M and N are countable recursively saturated models of PA with $\mathrm{Th}(M) = \mathrm{Th}(N)$ and $\mathrm{SSy}(M) = \mathrm{SSy}(N)$, then M and N are isomorphic. For countable short recursively saturated models we need a third condition (see [23]).

Theorem 2.14   *Let M and N be countable short recursively saturated models. Then $M \cong N$ if and only if $\mathrm{Th}(M) = \mathrm{Th}(N)$, $\mathrm{SSy}(M) = \mathrm{SSy}(N)$, and there are isomorphic simple models $\mathrm{Scl}(a)$ and $\mathrm{Scl}(b)$ with $\mathrm{Scl}(a) \prec_{\mathrm{cof}} M$ and $\mathrm{Scl}(b) \prec_{\mathrm{cof}} N$.*

From the work of Gaifman [5] and later results on types (see, for example, [12]), it is known that any countable recursively saturated model of arithmetic has countably many gaps such that whenever gap$(a) \neq$ gap(b), for any $c \in$ gap(a) and $d \in$ gap(b), $\mathrm{Scl}(c) \not\cong \mathrm{Scl}(d)$. It follows from the above theorem that any countable recursively saturated model of PA has countably many nonisomorphic short recursively saturated elementary initial segments. In the author's thesis [22], it was shown that some of these nonisomorphic models have also nonisomorphic automorphism groups (as topological groups).

2.4 Automorphisms   Let M be a countable first-order structure. Let $G = \mathrm{Aut}(M)$; that is, G is the automorphism group of M. We start with a few definitions.

Definition 2.15   The *pointwise stabilizer* of a set $X \subseteq M$, denoted $G_{(X)}$, is the subgroup of G containing all automorphisms of M which fix X pointwise; that is,

$$G_{(X)} = \{g \in G : g(a) = a \text{ for all } a \in X\}.$$

The *setwise stabilizer* of a set $X \subseteq M$, denoted $G_{\{X\}}$, is the subgroup of G containing all automorphisms of M which fix X setwise; that is,

$$G_{\{X\}} = \{g \in G : g(X) = X\}.$$

Remark   If $X = \{\bar{a}\}$, we will denote $G_{(X)}$ and $G_{\{X\}}$ by $G_{(\bar{a})}$ and $G_{\{\bar{a}\}}$, respectively.

We define a topology on G the usual way by letting the basic open subgroups of G be the pointwise stabilizers of tuples of M. That is, H is a basic open subgroup of G if and only if $H = G_{(\bar{a})}$ for some $\bar{a} \in M$. The basic open sets of G are cosets of the basic open subgroups. Equivalently, the basic open sets are the sets of the form $S_{\bar{a},\bar{b}} = \{g \in G : g(\bar{a}) = \bar{b}\}$. This definition makes G a topological group, that is, a group in which the operations multiplication and inversion are continuous.

Some important facts about this topology are

1. a subgroup is open if and only if it contains a basic open subgroup;
2. if a subgroup is open then it is closed;
3. a subgroup H is closed if and only if whenever $g \in G$ has the property that for any $\bar{a} \in M$ there is an $h \in H$ such that $g(\bar{a}) = h(\bar{a})$, then $g \in H$.

For any countable first-order structure M, $\mathrm{Aut}(M)$ is metrizable. Moreover, $\mathrm{Aut}(M)$ is a Polish group; that is, $\mathrm{Aut}(M)$ with this topology is a complete, separable metric space.

Remark   When M is a model of PA, tuples are coded by single elements. Hence, the basic open subgroups of $\mathrm{Aut}(M)$ are exactly the stabilizers of single elements.

We conclude this section with a well-known result concerning countable recursively saturated structures. This result (and its modifications) will be used extensively in this paper.

Proposition 2.16   *Let M be a countable recursively saturated structure, and let $\bar{a}, \bar{b} \in M$. If $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, then there is an automorphism $f \in \mathrm{Aut}(M)$ which sends $\bar{a}$ to $\bar{b}$.*

This property is called ω-homogeneity, and models which have this property are said to be ω-homogeneous.

For further results on automorphisms of first-order structures see Hodges [6] and Kaye and Macpherson [9]. For results on the automorphisms of models of PA, see [13].

3   Automorphisms of Countable Short Recursively Saturated Models of PA

In this section, we discuss automorphisms of countable short recursively saturated models of PA. We show that, unlike recursively saturated models, whenever a short recursively saturated model is not extremely short, the model is not ω-homogeneous. We then discuss alternative ways of proving the existence of automorphisms of countable short recursively saturated models of PA and show that any automorphism of any short recursively saturated model of PA fixes elements cofinally high in the model. We conclude this section by stating an important lemma of Kueker and Reyes (Kueker [16]) concerning automorphic images of subsets of countable models.

For the rest of the paper, we fix a countable recursively saturated model M of PA. Let a be an element of M. Let $G = \text{Aut}(M)$ and $G(a) = \text{Aut}(M(a))$.

It is easy to see that for any model of PA and any automorphism f of the model, $f(\text{gap}(a)) = \text{gap}(f(a))$. Since a is in the last gap of $M(a)$, we get the following.

Lemma 3.1   *If $f \in G(a)$ then $f(\text{gap}(a)) = \text{gap}(a)$.*

This gives us a necessary condition for a restriction of an automorphism of M to be an automorphism of $M(a)$. It must fix $\text{gap}(a)$ setwise. Moreover, this is a sufficient condition, because any automorphism of M which fixes $\text{gap}(a)$ setwise must fix $M(a)$ setwise. Hence, it is a bijection which preserves all definable properties of $M(a)$. Thus, we have the following.

Proposition 3.2   *Let $f \in G$. The restriction of f to the domain of $M(a)$, $f|_{M(a)}$, is in $G(a)$ if and only if $f(\text{gap}(a)) = \text{gap}(a)$.*

Since M is ω-homogenous, for every $\bar{a}, \bar{b} \in M(0)$ with $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$, there is an automorphism f of M which sends $\bar{a}$ to $\bar{b}$. But since any automorphism sends gaps to gaps and preserves order, $f(\text{gap}(0)) = \text{gap}(0)$. Thus, by the above proposition, $f|_{M(0)} \in G(0)$, so it is an automorphism of $M(0)$ sending $\bar{a}$ to $\bar{b}$. Thus, countable short recursively saturated models of PA which are extremely short are ω-homogeneous. When a countable short recursively saturated model of PA is not extremely short, this is not the case.

Lemma 3.3   *Let $a > M(0)$. Then, for every $b \in \text{gap}(a)$, there exists an element $c < \text{gap}(a)$ with $\text{tp}(b) = \text{tp}(c)$.*

Proof   Let $p(v, b)$ be the following recursive type,

$$\{\varphi(v) \leftrightarrow \varphi(b) : \varphi \in \mathcal{L}_{\text{PA}}\} \cup \{t(v) < a : t \text{ is a Skolem term}\}.$$

An element c realizing this type will have the same type as b and will be below $\text{gap}(a)$.

Since $p(v, b)$ is bounded and recursive, to finish the proof we need to show that it is finitely realized. Suppose for a contradiction that it is not finitely realized. Then there must be a finite conjunction of formulas $\Phi(x)$ such that $M(a) \models \Phi(b)$, but for all elements $v < \text{gap}(a)$, $M(a) \nvDash \Phi(v)$. But then $x = \min\{v : \Phi(v)\}$ defines an element in $\text{gap}(a)$, which is impossible since $\text{gap}(a)$ contains no definable elements. ☐

Proposition 3.4   *Let $a > M(0)$. Then $M(a)$ is not ω-homogeneous.*

Proof   Let b and c be as in Lemma 3.3. Since by Lemma 3.1, $\text{gap}(a)$ is fixed setwise by $G(a)$, there is no automorphism of $M(a)$ which sends b to c. Since $\text{tp}(b) = \text{tp}(c)$, $M(a)$ is not ω-homogeneous. ☐

Since countable short recursively saturated models that are not extremely short are not ω-homogeneous, we need a stronger condition to establish the existence of automorphisms.

Proposition 3.5   *Let $b, c \in M(a)$. If there are $d, e \in \text{gap}(a)$ (not necessarily distinct) with $\text{tp}(b, d) = \text{tp}(c, e)$, then there is an automorphism of $M(a)$ sending b to c. In particular, if $b, c \in \text{gap}(a)$ and $\text{tp}(b) = \text{tp}(c)$ then there is $f \in G(a)$ with $f(b) = c$.*

Proof Suppose that there are such d and e. Since M is ω-homogeneous and $\mathrm{tp}(b, d) = \mathrm{tp}(c, e)$, there is an automorphism $f \in G$ with $f(b) = c$ and $f(d) = e$. Since d and e are in $\mathrm{gap}(a)$ and automorphisms send gaps to gaps, $f(\mathrm{gap}(a)) = \mathrm{gap}(a)$. Hence, by Proposition 3.2, $f|_{M(a)}$ is an automorphism of $M(a)$, and $f|_{M(a)}(b) = c$.   □

This proposition gives us a way of showing existence of automorphisms of $M(a)$ with a given property without explicitly constructing the automorphisms.

Another way of showing existence of automorphisms of $M(a)$ is the following: we can extend the language of PA to have a constant symbol for a (or any other element in $\mathrm{gap}(a)$). Then the expansion (M, a) is still recursively saturated. Clearly, $\mathrm{gap}(a)$ is fixed setwise by all automorphisms of (M, a). Hence, by Proposition 3.2, the restriction of any automorphism of (M, a) to $M(a)$ is an automorphism of $M(a)$ (which, in addition, fixes a).

We now turn our attention to proving a property shared by all automorphisms of countable short recursively saturated models of PA. This result is not true in the recursively saturated case. We prove this result using a well-known lemma of Blass [3] and Gaifman [5].

Lemma 3.6 (Blass-Gaifman Lemma)   *Let K be a model of* PA. *Let $a < b \in K$. If $b \in \mathrm{gap}(a)$ then there is a Skolem term $t(x)$ such that $K \models a < b \leq t(a) = t(b)$.*

The above lemma implies the following proposition, which can also be found in [13].

Proposition 3.7   *Let K be a model of* PA. *Let $f \in \mathrm{Aut}(K)$ and let $a > \mathrm{Scl}(0)$ be such that $f(a) \in \mathrm{gap}(a)$. Then there is $c \in \mathrm{gap}(a)$ such that $f(c) = c$.*

Proof   Let $b = f(a)$. Since $b \in \mathrm{gap}(a)$, by the Blass-Gaifman lemma, there is a Skolem term t such that $t(a) = t(b)$. Let $c = t(a) = t(b)$. Then, $f(c) = f(t(a)) = t(f(a)) = t(b) = c$. Hence, c is fixed.   □

Let N be a model of PA. Let $f \in \mathrm{Aut}(N)$. By $\mathrm{fix}(f)$ we denote the set of elements in N fixed by f. Since $\mathrm{fix}(f)$ is closed under Skolem functions, $\mathrm{fix}(f) \preceq N$.

Corollary 3.8   *Let N be a short recursively saturated model of* PA. *Then for every $f \in \mathrm{Aut}(N)$, $\mathrm{fix}(f) \preceq_{\mathrm{cof}} N$.*

Proof   Let a be in the last gap of N. Thus, for all $f \in \mathrm{Aut}\,N$, $f(a) \in \mathrm{gap}(a)$. Therefore, by the above result, there is $c \in \mathrm{gap}(a)$ such that $f(c) = c$. But then $\mathrm{Scl}(c) \subseteq \mathrm{fix}(f)$. Notice that the Blass-Gaifman lemma implies that for any $b \in \mathrm{gap}(a)$, $\mathrm{Scl}(b) \subset_{\mathrm{cof}} N$. Thus, $\mathrm{Scl}(c) \subset_{\mathrm{cof}} N$, so $\mathrm{fix}(f) \subseteq_{\mathrm{cof}} N$. Since $\mathrm{fix}(f) \preceq N$, $\mathrm{fix}(f) \preceq_{\mathrm{cof}} N$.   □

In the subsequent sections, we prove various results concerning automorphisms and automorphism groups of countable short recursively saturated models of PA by applying a famous lemma of Kueker and Reyes.

Let K be a model and X a subset of K. By $\mathcal{A}_{\mathrm{Aut}(K)}(X)$ we denote the set of all automorphic images of X under $\mathrm{Aut}(K)$. That is,

$$\mathcal{A}_{\mathrm{Aut}(K)}(X) = \{g(X) : g \in \mathrm{Aut}(K)\}.$$

Lemma 3.9 (Kueker-Reyes Lemma [16])   *Let K be a countable structure. Let $X \subseteq K$. Suppose that for all $\bar{a} \in K$ there are $x \in X$, $y \notin X$ such that $(K, \bar{a}, x) \cong (K, \bar{a}, y)$. Then $|\mathcal{A}_{\mathrm{Aut}(K)}(X)| = 2^{\aleph_0}$.*

Remark   If $K \models$ PA is a countable model and $X \subseteq K$, to show that X has continuum many automorphic images it is enough to show that for all $a \in K$ there are $x \in X$ and $y \notin X$ such that $(K, a, x) \cong (K, a, y)$ (since all finite sequences are coded). If, in addition, K is recursively saturated, then, by the ω-homogeneity of K, it is enough to show that for any $a \in K$ there are $x \in X$ and $y \notin X$ such that $\mathrm{tp}(a, x) = \mathrm{tp}(a, y)$. When K is a countable short recursively saturated model of PA, it is enough to show that in the expanded model (K, b), where b is an element in the last gap of K, for any $a \in K$ there are $x \in X$ and $y \notin X$ with $\mathrm{tp}(a, x) = \mathrm{tp}(a, y)$ (this follows from Proposition 3.5).

4 Conjugates

Tzouvaras [24] proved that any automorphism of a countable recursively saturated model of PA has continuum many conjugates. In this section, we prove that the same is true for automorphisms of countable short recursively saturated models. We start with the following observation.

Lemma 4.1   *Let N be a short recursively saturated model of* PA. *Then, for every $b, c \in N$, if $c \notin \mathrm{Scl}(b)$, there is $d < c \in N$ such that $\mathrm{tp}(b, c) = \mathrm{tp}(b, d)$.*

To prove the next proposition, we will use the above lemma and a result proven independently by Ehrenfeucht [4] and by Gaifman [5].

Lemma 4.2 (Ehrenfeucht-Gaifman Lemma)   *Let K be a model of* PA. *Let $a, b \in K$ and let $t(x)$ be a Skolem term such that $K \models a \neq b = t(a)$. Then, $\mathrm{tp}(a) \neq \mathrm{tp}(b)$.*

We now apply the above results to our countable short recursively saturated model $M(a)$ and its automorphism group $G(a)$.

Proposition 4.3   *Let $g \in G(a)$ be a nontrivial automorphism. Then, for any $b \in M(a)$, there exist $c \in M(a)$ and $h \in G(a)_{(b)}$ such that $hg(c) \neq gh(c) = g(c)$.*

Proof   Since $g \in G(a)$, by Corollary 3.8, there is an element $r \in \mathrm{gap}(a)$ such that $g(r) = r$. We will consider two cases.

Case 1   Suppose that $g(b) = b$. Since g is nontrivial, let $c \in M(a)$ be such that $g(c) = d \neq c$. Thus,

$$\mathrm{tp}(r, b, c) = \mathrm{tp}(g(r), g(b), g(c)) = \mathrm{tp}(r, b, d).$$

Therefore, by the Ehrenfeuch-Gaifman lemma, $d \notin \mathrm{Scl}(r, b, c)$. So, by Lemma 4.1, $\exists e < d \in M(a)$ such that $\mathrm{tp}(r, b, c, d) = \mathrm{tp}(r, b, c, e)$. Now, since $r \in \mathrm{gap}(a)$, by Corollary 3.5, there is an automorphism $h \in G(a)$ sending (r, b, c, d) to (r, b, c, e). Thus, $h(b) = b$ and $gh(c) = g(c) = d \neq e = h(d) = hg(c)$.

Case 2   Suppose that $g(b) = d \neq b$. Then $\mathrm{tp}(r, b) = \mathrm{tp}(r, d)$. Then again, by the Ehrenfeucht-Gaifman lemma, $d \notin \mathrm{Scl}(r, b)$. Hence, by Lemma 4.1, $\exists e < d \in M(a)$ such that $\mathrm{tp}(r, b, d) = \mathrm{tp}(r, b, e)$. For this case we let $c = b$. Since $\mathrm{tp}(r, b, d) = \mathrm{tp}(r, b, e)$, we get that $\mathrm{tp}(r, b, c, d) = \mathrm{tp}(r, b, c, e)$. Hence there is an $h \in G(a)$ with the same properties as in Case 1. That is, h maps (r, b, c, d) to (r, b, c, e), so $h(b) = b$ and $gh(c) = g(c) = d \neq e = h(d) = hg(c)$.   □

Since there is a definable bijection from $M(a) \times M(a)$ into $M(a)$, the graph of g, $\Gamma(g)$, can be regarded as a subset of $M(a)$. In particular, let $\Gamma(g) = \{\langle x, g(x) \rangle : x \in M(a)\}$.

Proposition 4.4 *Let $g \in G(a)$ be nontrivial. Then the graph of g, $\Gamma(g)$, has continuum many images under the action of $G(a)$.*

Proof Let r be as in the proof of the proposition above; that is, $r \in \text{gap}(a)$ and $g(r) = r$. We will use the Kueker-Reyes lemma (Lemma 3.9). By the remark after the lemma, it is enough to show that for any $b \in M(a)$, there are $x \in \Gamma(g)$ and $y \notin \Gamma(g)$ such that in the expanded structure $(M(a), r)$, $\text{tp}(b, x) = \text{tp}(b, y)$. But in the proof of Proposition 4.3, we showed that for any $b \in M(a)$, there are $c \in M(a), d = g(c) \neq c$, and $e \neq d$ such that $\text{tp}(r, b, c, d) = \text{tp}(r, b, c, e)$. Thus, in $(M(a), r)$, $\text{tp}(b, c, d) = \text{tp}(b, c, e)$. In particular, $\text{tp}(b, \langle c, d \rangle) = \text{tp}(b, \langle c, e \rangle)$. But $\langle c, d \rangle \in \Gamma(g)$ and $\langle c, e \rangle \notin \Gamma(g)$. Hence, by the Kueker-Reyes lemma, $\Gamma(g)$ has continuum many images under $G(a)$. $\square$

Now, since for any $g \in G(a)$ the image of $\Gamma(g)$ under any $f \in G(a)$ is the graph of fgf^{-1}, we get the following corollary.

Corollary 4.5 *Let $g \in G(a)$ be nontrivial. Then g has continuum many conjugates in $G(a)$.*

5 The Subgroup $G|_{M(a)}$

Let

$$G|_{M(a)} = \{g|_{M(a)} : g \in G_{\{\text{gap}(a)\}}\};$$

that is, $G|_{M(a)}$ is the set of restrictions to $M(a)$ of those automorphisms of M which fix $\text{gap}(a)$ setwise. Clearly, $G|_{M(a)}$ is a subgroup of $G(a)$. Notice also that $G|_{M(a)} \cong G_{\{M(a)\}}/G_{(M(a))}$. In this section we show that $G|_{M(a)}$ is in fact a proper subgroup of $G(a)$. This will be done by showing that there are automorphisms of $M(a)$ that do not extend to automorphisms of M. Later in the section we discuss other properties of $G|_{M(a)}$.

Lemma 5.1 *Let M, $M(a)$, G, and $G(a)$ be as above. Then there exists $X \subseteq M(a)$ such that X is ω-coded in M and $|\mathcal{A}_{G(a)}(X)| = 2^{\aleph_0}$. That is, X has continuum many automorphic images under the action of $G(a)$.*

Proof We will start by defining in $M(a)$ a cofinal sequence $\{(b)_n : n \in \mathbb{N}\}$. Let $q(w, a)$ be the type

$$q(w, a) = \{(w)_0 = a\} \cup \{(w)_{n+1} = \max(((w)_n)^2, t_n(a)) : n \in \mathbb{N}\},$$

where $\langle t_n : n \in \omega \rangle$ is some recursive enumeration of all Skolem terms. Since this type is recursive, we only need to show that the type is finitely realized. Any finite collection of formulas from $q(w, a)$ involves only finitely many terms $(w)_{n_1}, (w)_{n_2}, \ldots, (w)_{n_k}$, with $k \in \mathbb{N}$ and $n_1 < n_2 < \cdots < n_k \in \mathbb{N}$. This collection can be realized by an element c coding the finite sequence $(w)_0, (w)_1, \ldots, (w)_{n_k}$, with $(w)_0 = a$ and for $i < n_k$ $(w)_{i+1} = \max((w)_n^2, t_n(a))$ (since any finite sequence is coded in the model). Therefore, this type is realized in M. Let b realize this type. Now define the following recursive type,

$$p(v, b) = \{(b)_n < (v)_n < (b)_{n+1} : n \in \mathbb{N}\} \cup$$
$$\{\forall x < (b)_n \ (v)_n \neq t(x, (b)_n) : n \in \mathbb{N}, t \text{ is a Skolem term}\}.$$

Again, since the type is recursive, we only need to show that it is finitely realized. Take a finite set of formulas from $p(v, b)$. Let k be a natural number larger than

the largest n used in this finite set of formulas and also larger than the number of Skolem terms used. Since for every $i < k$ between $(b)_i$ and $(b)_{i+1}$ there are at least $((b)_i)^2 - (b)_i$ elements and there are no more than $(b)_i \cdot k$ many elements of the form $t(x, (b)_i)$ with $x < (b)_i$ and some Skolem term t from this finite collection, there exists a c_i between $(b)_i$ and $(b)_{i+1}$ which is not definable from this finite set of formulas. Since $c_0, \ldots, c_{k-1}$ is a finite sequence it can be coded by some $c \in M$ with $(c)_i = c_i$, for all $i < k$; hence, the type $p(v, b)$ is finitely realized so it is realized by some $c \in M$.

Let $X = \{(c)_n : n \in \mathbb{N}\}$. By our construction, $X \subseteq M(a)$ and X is ω-coded in M by c. To finish the proof, by the Kueker-Reyes lemma (working in $(M(a), a)$), we need to show that for every $d \in M(a)$ there are $x \in X$, $y \notin X$ such that $\mathrm{tp}(d, x) = \mathrm{tp}(d, y)$. We argue by contradiction. Suppose that for some $d \in M(a)$ there are no such x and y. Because b codes a cofinal ω-sequence in $M(a)$, we can find $i \in \mathbb{N}$ such that $(b)_i > d$. Since between $(b)_i$ and $(b)_{i+1}$ there is only one $x \in X$, by our assumption, for all $y \neq x$ between $(b)_i$ and $(b)_{i+1}$, $\mathrm{tp}(d, x) \neq \mathrm{tp}(d, y)$. Thus, the recursive type

$$r(u, x, b, d) = \{(b)_i < u < (b)_{i+1}\} \cup$$
$$\{u \neq x\} \cup \{\varphi(d, x) \leftrightarrow \varphi(d, u) : \varphi \in \mathcal{L}_{\mathcal{P}\mathcal{A}}\}$$

is not realized by any y so it is not finitely realized. Hence, there is formula Φ such that $M \models \Phi(d, x)$ and for any $y \neq x$ between $(b)_i$ and $(b)_{i+1}$ $M \models \neg \Phi(d, y)$. But then the term $t(d, (b)_i) = \min\{v : M \models \Phi(d, v) \wedge v > (b)_i\}$ defines x from $(b)_i$ and d which contradicts the fact that x cannot be defined by $(b)_i$ and an element less than $(b)_i$. Hence, for any $d \in M(a)$ there are $x \in X$ and $y \notin X$ such that $\mathrm{tp}(d, x) = \mathrm{tp}(d, y)$. Thus, $|\mathcal{A}_{G(a)}(X)\}| = 2^{\aleph_0}$. $\square$

Proposition 5.2   *There are continuum many automorphisms of $M(a)$ which are not extendible to M.*

Proof   We will use the set X from Lemma 5.1. Since X is ω-coded in M by an element $c \in M$ and since c has only countably many automorphic images, X has at most countably many automorphic images in M. However, by Lemma 5.1, X has continuum many automorphic images in $M(a)$. Thus, continuum many automorphisms of $M(a)$ do not extend to M. $\square$

It follows from this proposition that $G|_{M(a)}$ is a proper subgroup of $G(a)$.

We will now investigate other properties of the subgroup $G|_{M(a)}$. We will refer to the construction of X from Lemma 5.1 frequently throughout this section.

Corollary 5.3 $G|_{M(a)}$ *is not open in* $G(a)$.

Proof   To show that $G|_{M(a)}$ is not open in $G(a)$, we need to show that it does not contain any basic open subgroup of $G(a)$. Let $d \in M(a)$. We will show that there is an automorphism $g \in G(a)_{(d)}$ such that $g \notin G|_{M(a)}$. Since $d \in M(a)$, $(M(a), d)$ is a short recursively saturated initial segment of (M, d). Hence, we can repeat the construction of X from Lemma 5.1 in this expanded structure. Thus, we get an $X \subseteq M(a)$ coded in (M, d) (and hence also in M) with continuum many automorphic images in $(M(a), d)$. But since X has at most countably many automorphic images in M (since it is ω-coded), continuum many automorphisms of $(M(a), d)$ do

not extend to M. Hence, $G|_{M(a)}$ does not contain any basic open subgroup, so it is not open in $G(a)$.   □

Since any closed subgroup in this topology is also open, and $G|_{M(a)}$ is not open, it is not closed. Moreover, we have the following.

Proposition 5.4 $G|_{M(a)}$ *is dense in* $G(a)$.

Proof   Let $g \in G(a)$. We will show that g is in the closure of $G|_{M(a)}$. That is, we need to show that for any $c, d \in M(a)$ such that $g(c) = d$, there is $h \in G|_{M(a)}$ with $h(c) = d$.
     Suppose $g(c) = d$. Let $b = g(a)$. Then $\mathrm{tp}(a, c) = \mathrm{tp}(b, d)$. Now, since M is ω-homogeneous, there is an automorphism $f \in G$ such that $f(a) = b$ and $f(c) = d$. But since $g(a) = b$ and $g \in G(a)$, by Lemma 3.1, $b \in \mathrm{gap}(a)$. Thus, f fixes $\mathrm{gap}(a)$ setwise. Let $h = f|_{M(a)}$. Since $h \in G|_{M(a)}$ and $h(c) = d$, g is in the closure of $G|_{M(a)}$.   □

The next proposition shows that the index of $G|_{M(a)}$ in $G(a)$ is not small. That is, there are uncountably many cosets of $G|_{M(a)}$.

Proposition 5.5 $\left[G(a) : G|_{M(a)}\right] = 2^{\aleph_0}$.

Proof   Since $|G(a)| = 2^{\aleph_0}$, $\left[G(a) : G|_{M(a)}\right] \leq 2^{\aleph_0}$. It remains to show that $\left[G(a) : G|_{M(a)}\right] \geq 2^{\aleph_0}$. Suppose for a contradiction that $\left[G(a) : G|_{M(a)}\right] = \lambda < 2^{\aleph_0}$. Let X be as in Lemma 5.1. Recall that $|\mathcal{A}_{G|_{M(a)}}(X)| = \aleph_0$ and $|\mathcal{A}_{G(a)}(X)| = 2^{\aleph_0}$. Now, for any $g \in G(a)$ and all $h_1, h_2 \in G|_{M(a)}$,

$$h_1(X) = h_2(X) \iff gh_1(X) = gh_2(X).$$

Hence,

$$|\{gh(X) : h \in G|_{M(a)}\}| = |\{h(X) : h \in G|_{M(a)}\}| = |\mathcal{A}_{G|_{M(a)}}(X)| = \aleph_0;$$

that is, X has countably many automorphic images under the action of any coset of $G|_{M(a)}$. Now,

$$|\mathcal{A}_{G(a)}(X)| \leq |\mathcal{A}_{G|_{M(a)}}(X)| \cdot \left[G(a) : G|_{M(a)}\right] = \aleph_0 \cdot \lambda < 2^{\aleph_0}.$$

But this contradicts the fact that $|\mathcal{A}_{G(a)}(X)| = 2^{\aleph_0}$. Therefore, $\left[G(a) : G|_{M(a)}\right] \geq 2^{\aleph_0}$, and so $\left[G(a) : G|_{M(a)}\right] = 2^{\aleph_0}$.   □

We will now proceed to show that $G|_{M(a)}$ is not a normal subgroup.

Lemma 5.6   *With the set X as above, there is an automorphism $f \in G(a)$ such that* $|\mathcal{A}_{G|_{M(a)}}(f(X))| = 2^{\aleph_0}$.

Proof   We will construct an automorphism f which fixes a, using "back-and-forth" inside $M(a)$. Let b, c, X be as in Lemma 5.1; that is, X is coded by $c \in M$ with the property that $(b)_n < (c)_n < (b)_{n+1}$ for all $n \in \mathbb{N}$, and $\{(b)_n : n \in \mathbb{N}\}$ is a cofinal sequence in $M(a)$, $(b)_0 = a$, and $(b)_{n+1} = \max((b)_n^2, t_n(a))$. Notice that each $(b)_n$ is defined from a; hence, f will fix it as well. Another important fact to recall from the construction is that $(c)_n \notin \mathrm{Scl}(x, (b)_n)$ for all $x < (b)_n$.
     We are now ready to construct f. Enumerate $M(a) = \{a_0, a_1, \ldots\}$, and also enumerate $M = \{m_0, m_1, \ldots\}$. Let $\bar{d}_0 = a$ and $\bar{e}_0 = a$. Suppose that $2n$ steps have been done in the construction of f and that $\mathrm{tp}(\bar{d}_n) = \mathrm{tp}(\bar{e}_n)$.

Step $2n+1$ We will do both a "forth" and a "back." For a_n find $u \in M(a)$ such that $\text{tp}(\bar{d}_n, a_n) = \text{tp}(\bar{e}_n, u)$ and then find $v \in M(a)$ such that $\text{tp}(\bar{d}_n, a_n, v) = \text{tp}(\bar{e}_n, u, a_n)$. Such u and v exist because $\text{tp}(\bar{d}_n) = \text{tp}(\bar{e}_n)$ implies that there is an automorphism of M sending $\bar{d}_n$ to $\bar{e}_n$, fixing a (and also $\text{gap}(a)$ and $M(a)$ setwise). This automorphism will send a_n to some $u \in M(a)$ and some $v \in M(a)$ to a_n. Hence, $\text{tp}(\bar{d}_n, a_n, v) = \text{tp}(\bar{e}_n, u, a_n)$.

Step $2n + 2$ In this step we will only do a "forth." Let $d = \langle \bar{d}_n, a_n, v \rangle$ (that is, $\langle \langle \bar{d}_n, a_n \rangle, v \rangle$) and $e = \langle \bar{e}_n, u, a_n \rangle$. Since $\{(b)_n : n \in \mathbb{N}\}$ is cofinal in $M(a)$, we can find $k \in \mathbb{N}$ such that $(b)_k > d$. For this k we will use $(c)_k$, the $k + 1$ element of X, and find an x such that $\text{tp}(d, (c)_k) = \text{tp}(e, x)$ (this ensures that $\text{tp}(\bar{d}_n, a_n, v, (c)_k) = \text{tp}(\bar{e}_n, u, a_n, x)$), with the additional condition on x that there is a $y \neq x$ with $\text{tp}(m_n, a, x) = \text{tp}(m_n, a, y)$. Consider the following recursive type

$$p(w, z) = \{w \neq z\} \cup \{\varphi(d, (c)_k) \leftrightarrow \varphi(e, w) : \varphi \in \mathcal{L}_{\mathsf{PA}}\} \cup$$
$$\{\varphi(m_n, a, w) \leftrightarrow \varphi(m_n, a, z) : \varphi \in \mathcal{L}_{\mathsf{PA}}\}.$$

First, notice that $\text{tp}(d) = \text{tp}(e)$, so there exists an x such that $\text{tp}(d, (c)_k) = \text{tp}(e, x)$. Also, because $(c)_k$ is between $(b)_k$ and $(b)_{k+1}$, x is also between these elements. Moreover, by the way the set X was constructed, $(c)_k$ is not defined from d and $(b)_k$, so x is not defined from e and $(b)_k$. Therefore, since the interval $((b)_k, (b)_{k+1})$ is of nonstandard length, by recursive saturation there are infinitely many $w_i \in M(a)$ such that $\text{tp}(e, x) = \text{tp}(e, w_i)$ (since otherwise we would have been able to define x from e and $(b)_k$). Thus, there are countably many $w_i \in M(a)$ such that $\text{tp}(d, (c)_k) = \text{tp}(e, w_i)$. Now let $\Phi(w, z)$ be a finite conjunction of (say, r many) formulas of the form $\varphi(m_n, a, w) \leftrightarrow \varphi(m_n, a, z)$, where $\varphi \in \mathcal{L}_{\mathsf{PA}}$. That is,

$$\Phi(w, z) = \bigwedge_{j=1}^{r} (\varphi_j(m_n, a, w) \leftrightarrow \varphi_j(m_n, a, z)),$$

for some enumeration of the formulas of $\mathcal{L}_{\mathsf{PA}}$. Since there are only finitely (2^r) many possible truth values to r many formulas, and since there are infinitely many w_i satisfying $\text{tp}(d, (c)_k) = \text{tp}(e, w_i)$, we can pick from this set $w_{i_1} \neq w_{i_2}$ satisfying $\Phi(w_{i_1}, w_{i_2})$. Hence, $p(w, z)$ is finitely realized, so it is realized by some x and y, respectively. Therefore, we found an x such that $\text{tp}(\bar{d}_n, a_n, v, (c)_k) = \text{tp}(\bar{e}_n, u, a_n, x)$ and also a $y \neq x$ such that $\text{tp}(m_n, a, x) = \text{tp}(m_n, a, y)$. Now set $\bar{d}_{n+1} = (\bar{d}_n, a_n, v, (c)_k)$ and $\bar{e}_{n+1} = (\bar{e}_n, u, a_n, x)$ and continue to the next odd step. After ω steps we will get the automorphism $f \in G(a)$.

To see that $|\mathcal{A}_{G|_{M(a)}}(f(X))| = 2^{\aleph_0}$ we will use the Kueker-Reyes lemma. Consider the recursively saturated structure (M, a). By our construction, for any $m_n \in (M, a)$, there are $x \in f(X)$ and $y \notin f(X)$ such that $\text{tp}(m_n, x) = \text{tp}(m_n, y)$. Hence, by the Kueker-Reyes lemma, $f(X)$ has continuum many automorphic images in (M, a) (all in $\text{gap}(a)$). But since every automorphism of (M, a) is also an automorphism of M which fixes a, $\text{Aut}(M, a)|_{(M(a),a)} \leq G|_{M(a)}$. Hence, there are continuum many images of $f(X)$ under $G|_{M(a)}$. $\square$

Corollary 5.7 $G|_{M(a)}$ *is not a normal subgroup of* $G(a)$.

Proof We will use the set X and the automorphism f as above to prove the corollary. By Lemma 5.1, since X is coded in M, X has only countably many images under the action of $G|_{M(a)}$. We will show that X has continuum many images under

the action of $\{f^{-1}gf : g \in G|_{M(a)}\}$. Thus, showing that there are $g_i \in G|_{M(a)}$ with $f^{-1}g_i f \neq h$ for any $h \in G|_{M(a)}$, which implies that $G|_{M(a)}$ is not normal in $G(a)$.

Notice that since f^{-1} is an automorphism of $M(a)$, for any $Y, Z \subseteq M(a)$,

$$Y = Z \iff f^{-1}(Y) = f^{-1}(Z).$$

Hence, $|\{gf(X) : g \in G|_{M(a)}\}| = |\{f^{-1}gf(X) : g \in G|_{M(a)}\}|$. But, by Lemma 5.6, $|\{gf(X) : g \in G|_{M(a)}\}| = 2^{\aleph_0}$; thus $|\{f^{-1}gf(X) : g \in G|_{M(a)}\}| = 2^{\aleph_0}$, which completes the proof. □

We conclude this section with an open problem.

Question 5.8   Is the subgroup $G|_{M(a)}$ maximal in $G(a)$?

Remark   Recall from Section 2 that the elementary initial segments of M are either recursively saturated or short recursively saturated. By results of Kossak and Bamber [11] and Schmerl [20], there is a recursively saturated $K \prec_{end} M$ such that $G_{\{K\}}/G_{(K)}$ is isomorphic to the trivial group. This implies that the restriction of $G_{\{K\}}$ to K, $G_{\{K\}}|_K$ (which we denote for short $G|_K$) contains one element, the identity on K. It follows that $G|_K$ is not maximal in $\text{Aut}(K)$.

When K is short recursively saturated this result is not true. That is, $G_{\{K\}}/G_{(K)}$ is not isomorphic to the trivial group. Otherwise, this would imply that $G|_K$ contains only the identity. But this is impossible since we have shown in this section that when K is short recursively saturated $G|_K$ is dense (Proposition 5.4) in $\text{Aut}(K)$ and has continuum many automorphisms (as follows from Lemma 5.6). Thus, this result cannot be used for showing that $G|_{M(a)}$ is not maximal.

6 Concluding Remarks

Since automorphisms of countable short recursively saturated models of PA were never studied directly, in Section 3 we developed methods of showing existence of automorphisms of such models. We then used these methods in Sections 4 and 5 to prove various properties of automorphisms and automorphism groups of countable short recursively saturated models of PA.

When investigating the subgroup $G|_{M(a)}$, we were especially interested in the index of the subgroup and whether it is normal or not. A model has the Small Index Property if the only subgroups of its automorphism group with a countable index are the open subgroups. Since $G|_{M(a)}$ is not open, a countable index for $G|_{M(a)}$ would imply that countable short recursively saturated models of PA do not have the Small Index Property. Since we have shown that the index of $G|_{M(a)}$ is uncountable, the question of whether (some or all) countable short recursively saturated models of PA have the Small Index Property remains open.

In [22], the author had proved that Kaye's characterization of the closed normal subgroups of countable recursively saturated models of PA [8] is true for countable short recursively saturated models as well. That is, for any closed normal subgroup N of the automorphism group of a countable short recursively saturated model of PA, there is an invariant initial segment $I \subseteq_{end} M(a)$ closed under exponentiation such that

$$N = G(a)_{(I)} = \{f \in G(a) : f(x) = x \text{ for all } x \in I\}.$$

A question that remains open is the analog to Kaye's conjecture for normal subgroups in general: Does any normal subgroup N of the automorphism group of a countable

short recursively saturated model of PA have either the form $G(a)_{(I)}$ or the form

$$G(a)_{(>I)} = \{f \in G(a) : f \in G(a)_{(J)} \text{ for some } J \supsetneq_{end} I\},$$

for some invariant initial segment I closed under exponentiation? Had $G|_{M(a)}$ been normal, it would have given us a counterexample to this question. Since $G|_{M(a)}$ is not normal, this question remains open as well.

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