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Self-implications in BCI

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Abstract Humberstone asks whether every theorem of BCI provably implies $\varphi \to \varphi$ for some formula φ. Meyer conjectures that the axiom **B** does not imply any such "self-implication." We prove a slightly stronger result, thereby confirming Meyer's conjecture.

1 Introduction

The logic BCI is one of several pure implication calculi that have acquired fame. Its name comes from the connection with the *combinators* B, C, and I (see [3]). From the perspective of type theory, BCI can be viewed as the set of types of certain restricted family of λ-terms, via the Curry-Howard isomorphism. Seen in yet another way, BCI is nothing but the implication fragment of linear logic. From the point of view of formal language theory, it is natural to present BCI as the implication fragment of (associative) Lambek Calculus with exchange (see [2]).

Formally, the logic BCI can be defined as a Hilbert system by the following three axiom schemes:

(**B**)	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$	(prefixing)
(**C**)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	(commutation)
(**I**)	$\varphi \to \varphi$	(identity)

and the rule

$$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

of modus ponens. In presence of **C**, the axiom **B** is equivalent to

(**B**$'$) $\quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ (suffixing).

In [4] Humberstone takes BCI as a base logic for versions of naïve set theory in which Curry Paradox cannot be proved. An extension of BCI by the axiom

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$$(\star) \quad (\varphi \to \varphi) \to (\psi \to \psi)$$

is also considered there, and Humberstone conjectures that this logic, called BCI*, is the smallest extension of BCI in which all theorems are provably equivalent. He shows that his conjecture follows from a property that can be stated as follows. Let L be any logic with implication connective. Consider

$$L \vdash \varphi \quad \text{implies} \quad L \vdash \varphi \to (\chi \to \chi) \quad \text{for some } \chi. \tag{P}$$

If (P) holds for L, we say that *every theorem implies a self-implication in L*. Humberstone shows that if every theorem implies a self-implication in BCI*, then BCI* is the smallest extension of BCI in which all theorems are provably equivalent. In [5] it was shown that (P) indeed holds for BCI*, confirming Humberstone's conjecture. But a harder question, also posed in [4], whether every theorem implies a self-implication *in* BCI, remained open. Meyer conjectured (in private communication) that the BCI axiom **B** is a counterexample. To be precise, the conjecture states that $((x \to y) \to ((z \to x) \to (z \to y))) \to (\chi \to \chi)$, where x, y, z are variables, is not a theorem of BCI for any BCI formula χ. We will prove that a certain instance of $\mathbf{B}'$ in only two variables does not imply any self-implication. Meyer's conjecture will follow as a corollary.

2 Sequent System

We use a Gentzen-style rendering of BCI. We define sequents to be pairs $\Gamma \Rightarrow \alpha$, where Γ is a possibly empty multiset of formulas, α a formula, and $\Rightarrow$ is a separator (sequent arrow). From now on BCI will stand for the sequent calculus comprising initial sequents,

$$\alpha \Rightarrow \alpha,$$

and inference rules,

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta \Rightarrow \gamma}{\Gamma, \Delta, \alpha \to \beta \Rightarrow \gamma}\ (\to\Rightarrow) \qquad\qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to)\,.$$

Cut, in the form below,

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}\ ,$$

will also be at our disposal. The following three lemmas are folklore.

Lemma 2.1 *Cut is admissible in* BCI. *The rule* $(\Rightarrow\to)$ *is invertible in* BCI. *Moreover,* $\to$ *is monotonic in second and antitonic in first argument, with respect to the sequent arrow.*

Lemma 2.2 *For any formulas* α, β *the sequent* $\alpha \Rightarrow \beta$ *is provable in the sequent system if and only if the formula* $\alpha \to \beta$ *is provable in the axiomatic system. The sequent* $\Rightarrow \alpha$ *is provable if and only if* α *is provable.*

We use the notation $\alpha \Leftrightarrow \beta$ to denote the pair of sequents $\alpha \Rightarrow \beta$, $\alpha \Rightarrow \beta$. When the sequents $\alpha \Leftrightarrow \beta$ are provable, we call α and β provably equivalent. Recall that a logic L is called *congruential* if replacement of provably equivalent formulas in longer formulas preserves provability.

Lemma 2.3 BCI *is congruential.*

One consequence of Lemma 2.1 is that a sequent $\Gamma \Rightarrow \alpha$ is provable if and only if the sequent $\Gamma, \alpha_n, \ldots, \alpha_1 \Rightarrow \alpha_0$ is provable, where

$$\alpha = \alpha_n \to (\alpha_{n-1} \to \cdots \to (\alpha_1 \to \alpha_0) \ldots)$$

and α_0 is a variable. Clearly, each BCI formula α is of the above form. Wherever practical, we will use the shorthand notation $(\alpha_1 \cdots \alpha_n) \to \alpha_0$. This notation is also meant to admit an empty multiset of antecedents, in which case the formula simply reduces to α_0. Any permutation within the parentheses will be tacitly assumed to result in the same formula and thus we will sometimes use even shorter $\otimes\mathrm{A} \to \alpha_0$, with $\mathrm{A} = \{\alpha_1, \ldots, \alpha_n\}$. Thus $\Gamma \Rightarrow \otimes\mathrm{A} \to \alpha_0$ is provable if and only if $\Gamma, \mathrm{A} \Rightarrow \alpha_0$ is. The following rather straightforward lemma spells out some algebraic properties of our shorthand notation, which will be used frequently in proofs.

Lemma 2.4 *Suppose the sequents $\Gamma \Rightarrow \gamma$ and $\Delta \Rightarrow \delta$ are provable. Then the sequents*

$$(\gamma\,\delta) \to \varphi \Rightarrow \otimes(\Gamma \cup \Delta) \to \varphi$$
$$\gamma \to \varphi \Rightarrow \otimes\Gamma \to \varphi$$

are also provable, for any formula φ.

Proof Clearly, $(\gamma\,\delta) \to \varphi \Rightarrow \otimes(\Gamma \cup \Delta) \to \varphi$ is provable if and only if the sequent $\Gamma, \Delta, \gamma \to (\delta \to \varphi) \Rightarrow \varphi$ is provable. The following two applications of $(\to\Rightarrow)$

$$\dfrac{\Gamma \Rightarrow \gamma \qquad \dfrac{\Delta \to \delta \quad \varphi \Rightarrow \varphi}{\Delta, \delta \to \varphi \Rightarrow \varphi}}{\Gamma, \Delta, \gamma \to (\delta \to \varphi) \Rightarrow \varphi}$$

complete the proof of the first sequent. The second is similar (it would follow from the first if we allowed sequents with both sides empty as provable). $\square$

The role of Lemma 2.4 is that it enables us to use an unofficial multiplication on the right-hand side of sequents, with desirable properties of commutativity and monotonicity with respect to the sequent arrow. That is, given provable sequents $\Gamma_i \Rightarrow \gamma_i$, for $i \in \{1, \ldots, n\}$, we allow ourselves to write $\Gamma_1, \ldots, \Gamma_n \Rightarrow \otimes_{i=1}^n \gamma_i$ as long as in the next step we are going to resolve this into $\otimes_{i=1}^n \gamma_i \to \varphi \Rightarrow \otimes_{i=1}^n \Gamma_i \to \varphi$ for some formula φ (typically a variable).

This feature of BCI may be seen as a consequence of the fact that adding fusion to BCI is conservative or, speaking semantically, that BCI-algebras are subreducts of certain residuated semigroups. We will not dwell on these issues here, as they have no bearing on our result beyond what Lemma 2.4 states.

3 Split Formulas

This section exploits two facts about sequent system for BCI. One is that by the second part of Lemma 2.1 we can bring every provable sequent to the form $\Gamma \Rightarrow \upsilon$ for a variable υ. The other is that because weakening and contraction are missing, there always is a formula on the left of the sequent arrow that, so to speak, keeps track of the proof.

Lemma 3.1 *The sequent $\Gamma \Rightarrow \upsilon$, with υ a variable, is provable if and only if there exists a formula $\gamma = (\gamma_n \gamma_{n-1} \ldots \gamma_1) \to \gamma_0 \in \Gamma$ and a partition $\{\Gamma_i\}_{i=1}^n$ of $\Gamma \setminus \{\gamma\}$ such that*

1. $\gamma_0 = \upsilon$,
2. $\Gamma_i \Rightarrow \gamma_i$ *is provable for every* $i \in \{1, \dots, n\}$.

Proof Notice that only the forward direction is nontrivial. We proceed by induction on the length of a (cut-free) proof of $\Gamma \Rightarrow \upsilon$. If $\Gamma \Rightarrow \upsilon$ is an initial sequent, the lemma holds. If $\Gamma \Rightarrow \upsilon$ is not an initial sequent, the last rule in the proof must be

$$\frac{\Pi \Rightarrow \alpha \quad \Delta, \beta \Rightarrow \upsilon}{\Gamma \Rightarrow \upsilon}$$

with $\alpha \to \beta \in \Gamma$ and $\Pi \cup \Delta = \Gamma \setminus \{\alpha \to \beta\}$. Then, inductive hypothesis applies to the sequent $\Delta, \beta \Rightarrow \upsilon$ and thus there is a formula $\delta = \delta_n \to (\delta_{n-1} \to \cdots \to (\delta_1 \to \upsilon)\dots)$ and a partition of $(\Delta \cup \{\beta\}) \setminus \{\delta\}$ into $\Delta_1, \dots, \Delta_n$ such that sequents $\Delta_i \Rightarrow \delta_i$ are provable. We have two cases, according to whether $\beta = \delta$. If $\beta = \delta$, then $\alpha \to \beta$ satisfies all requirements of the lemma. If $\beta \neq \delta$, then $\beta \in \Delta_j$ for some $j \in \{1, \dots, n\}$. In this case, the sequent $\Pi, \Delta_j \setminus \{\beta\}, \alpha \to \beta \Rightarrow \delta_j$ is provable, by application of

$$\frac{\Pi \Rightarrow \alpha \quad \Delta_j \Rightarrow \delta_j}{\Pi, \Delta_j \setminus \{\beta\}, \alpha \to \beta \Rightarrow \delta_j}\ .$$

Then we have

$$\Gamma = \Delta_1, \dots, \Delta_{j-1}, \Pi, \Delta_j \setminus \{\beta\}, \alpha \to \beta, \Delta_{j+1}, \dots, \Delta_n$$

and the conditions of the lemma are satisfied again, with $\alpha \to \beta$ as split formula. This ends the proof. □

For a provable sequent $\Gamma \Rightarrow \upsilon$, we will call the formula γ from Lemma 3.1 a *split* formula. Such a γ may not be unique, so we will also say that $\Gamma \Rightarrow \upsilon$ is *provable with split formula* γ.

4 More about Split Formulas

The general setting is now as follows. We consider a formula $\rho \to (\tau \to \tau)$ and look at various ways it can be a theorem of BCI, with a particular split formula. We show that the choices for the split formula are rather limited: either τ can be shortened or the split formula has to be ρ.

Lemma 4.1 *Let* $\tau = (\tau_1 \cdots \tau_n) \to \upsilon$. *Suppose that for some* $k \leq n$ *and some formula* ρ *the sequent* $\rho, \tau, \tau_1, \dots, \tau_k \Rightarrow \upsilon$ *is provable, with split formula* $\tau_1 = (\alpha_1 \cdots \alpha_m) \to \upsilon$. *Then* $\rho, (\tau_2 \cdots \tau_n) \to \alpha_i, \tau_2, \dots, \tau_k \Rightarrow \alpha_i$ *is provable, for some* $i \in \{1, \dots, m\}$.

Proof Let $\Pi = \{\rho, \tau, \tau_1, \dots, \tau_k\}$ and $\{\Pi_i\}_{i=1}^m$ be the partition of $\Pi \setminus \{\tau_1\}$ required by Lemma 3.1. We then have $\Pi_i \Rightarrow \alpha_i$ for all $i \in \{1, \dots, m\}$. We can also assume (by renumbering) that $\tau = (((\alpha_1 \cdots \alpha_m) \to \upsilon)\tau_2 \cdots \tau_n) \to \upsilon$ belongs to partition class Π_1. Let $\Theta = \Pi_1 \setminus \{\tau\}$. By Lemma 3.1 the sequent $\tau, \Theta \Rightarrow \alpha_1$ is provable, and thus so is $\tau \Rightarrow \otimes\Theta \to \alpha_1$. By monotonicity of $\to$ in second argument

$$(\alpha_2 \cdots \alpha_m) \to \tau \Rightarrow (\alpha_2 \cdots \alpha_m) \to (\otimes\Theta \to \alpha_1)$$

is also provable. Now consider

$$(\alpha_2 \cdots \alpha_m) \to \tau = (\alpha_2 \cdots \alpha_m((\alpha_1 \cdots \alpha_m) \to \upsilon)\tau_2 \cdots \tau_n) \to \upsilon$$

and observe that, by antitonicity of $\to$ in first argument, the sequent

$$((\alpha_1 \to \upsilon)\tau_2 \cdots \tau_n) \to \upsilon \Rightarrow ((\alpha_2 \cdots \alpha_m)((\alpha_1 \cdots \alpha_m) \to \upsilon)\tau_2 \cdots \tau_n) \to \upsilon$$

is provable. Therefore, so is

$$((\alpha_1 \to \upsilon)\tau_2 \cdots \tau_n) \to \upsilon \Rightarrow (\alpha_2 \cdots \alpha_m) \to (\otimes\Theta \to \alpha_1).$$

Now the left-hand side of this sequent is $(\alpha_1 \to \upsilon) \to ((\tau_2 \cdots \tau_n) \to \upsilon)$, which is provably implied by $(\tau_2 \cdots \tau_n) \to \alpha_1$. So, by cut we conclude that

$$(\tau_2 \cdots \tau_n) \to \alpha_1 \Rightarrow (\alpha_2 \cdots \alpha_m) \to (\otimes\Theta \to \alpha_1)$$

is provable, and unwinding that we obtain

$$(\tau_2 \cdots \tau_n) \to \alpha_1, \alpha_2, \ldots, \alpha_m, \Theta \Rightarrow \alpha_1.$$

Since $\Theta \cup \bigcup_{i=2}^{m} \Pi_i = \{\rho, \tau_2, \ldots, \tau_k\}$, by $m-1$ cuts we get

$$\rho, (\tau_2 \cdots \tau_n) \to \alpha_1, \tau_2, \ldots, \tau_k \Rightarrow \alpha_1$$

as claimed. $\square$

Lemma 4.2 *Let $\tau = (\tau_1 \cdots \tau_n) \to \upsilon$. Suppose that for some formula ρ the sequent $\rho, \tau, \tau_1, \ldots, \tau_n \Rightarrow \upsilon$ is provable, with split formula τ. Then $\rho, \tau_i \Rightarrow \tau_i$ is provable, for some $i \in \{1, \ldots, n\}$.*

Proof Combinatorics on top of induction on the number of antecedents in τ. For zero antecedents the claim holds vacuously, since a variable υ cannot be a split formula (it would require $\rho \Rightarrow$ to be provable, which it is not). Assume the claim holds for any τ with number of antecedents smaller than n and let $\tau = (\tau_1 \cdots \tau_n) \to \upsilon$. By Lemma 3.1 we get n provable sequents

$$\begin{array}{ccc} \Lambda_1 & \Rightarrow & \tau_1 \\ \vdots & & \vdots \\ \Lambda_n & \Rightarrow & \tau_n \end{array}$$

where $\rho, \tau_1, \ldots, \tau_n$ are somehow distributed among the Λ_i. We can assume that $\rho \in \Lambda_1$, so in particular Λ_1 is nonempty. Observe that if Λ_i is empty, that is, $\Rightarrow \tau_i$ is provable, then using cut we can eliminate the "variable" τ_i from the "system of sequents," reducing their number by one in the process. For suppose $\tau_i \in \Lambda_j$, then by cut $\Lambda'_j \Rightarrow \tau_j$ is provable with $\Lambda'_j = \Lambda_j \setminus \{\tau_i\}$. Now applying Lemma 3.1 to the new "system" with $n-1$ sequents

$$\begin{array}{ccc} \Lambda_1 & \Rightarrow & \tau_1 \\ \vdots & & \vdots \\ \Lambda'_j & \Rightarrow & \tau_j \\ \vdots & & \vdots \\ \Lambda_n & \Rightarrow & \tau_n \end{array}$$

from which τ_i is missing altogether, we get that the sequent

$$\rho, \tau', \tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n \Rightarrow \upsilon$$

is provable, with split formula $\tau' = (\tau_1 \cdots \tau_{i-1}\tau_{i+1} \cdots \tau_n) \to \upsilon$, and so the claim follows by inductive hypothesis. Therefore, we can assume Λ_i is nonempty for every i. This leaves two cases to consider.

Case 1 $\Lambda_1 = \{\rho, \tau_{\pi(1)}\}$ and thus $\Lambda_i = \{\tau_{\pi(i)}\}$ for some permutation π of the indices $\{1, \ldots, n\}$. If $\pi(1) = 1$, the claim holds trivially. If $\pi(i) = i$ for any $i > 1$, we can dispose of the sequent $\Lambda_i \Rightarrow \tau_i$ and reduce the number of sequents, getting access to the inductive hypothesis. Thus, we can assume $\pi(i) \neq i$ for all $i \in \{1, \ldots, n\}$. Let o be the orbit of 1. Consider the sequents $\Lambda_1 \Rightarrow \tau_1$ and $\Lambda_{\pi(1)} \Rightarrow \tau_{\pi(1)}$. By assumptions they are, respectively, $\rho, \tau_{\pi(1)} \Rightarrow \tau_1$ and $\tau_{\pi(\pi(1))} \Rightarrow \tau_{\pi(1)}$. Applying cut, we then obtain $\rho, \tau_{\pi(\pi(1))} \Rightarrow \tau_1$. Repeating this successively for $k \in o$ we finally obtain $\rho, \tau_1 \Rightarrow \tau_1$, as required.

Case 2 $\Lambda_1 = \{\rho\}$ and thus for some index i we have $\Lambda_i = \{\tau_k, \tau_j\}$, with i, k not necessarily distinct. As before, if for any $\ell \in \{1, \ldots, n\}$ we have $\Lambda_\ell = \{\tau_\ell\}$ (it can only happen for $\ell \neq i$), the number of sequents gets reduced, so we can assume $\Lambda_\ell \neq \{\tau_\ell\}$. Consider the map f defined by $f(\ell) = m$ if $\tau_\ell \in \Lambda_m$. Thus, f maps $\{1, \ldots, n\}$ onto $\{2, \ldots, n\}$ and has no fixed points, except possibly $f(i)$. Therefore, there exists an m such that $|f^{-1}(f^m(1))| = 2$ and so $f^m(1) = i$. It follows that $f^{m-1}(1) \in \{k, j\}$, so assume $f^{m-1}(1) = k$. Again, we apply cut successively, this time beginning with the sequents $\rho \Rightarrow \tau_1$ and $\Lambda_{f(1)} \Rightarrow \tau_{f(1)}$. After $m - 1$ steps we obtain the sequent $\rho, \tau_j \Rightarrow \tau_i$ reducing this case to the previous one. □

Lemma 4.3 *Let $\tau = (\tau_1 \cdots \tau_n) \to \upsilon$. If the sequent $\rho, \tau, \tau_1, \ldots, \tau_n \Rightarrow \upsilon$ is provable and ρ is not a split formula for the sequent, then there is a formula τ', shorter than τ, such that the sequent $\rho, \tau' \Rightarrow \tau'$ is provable.*

Proof By Lemmas 4.1 and 4.2. □

5 Main Argument

We are now ready to prove the main result of the paper. Let D stand for the following substitution instance of $\mathbf{B}'$

$$(x \to y) \to ((y \to x) \to (x \to x))$$

where x and y are distinct variables. Consider the three element algebra

$$\langle \{-1, 0, 1\}; \to \rangle$$

with $\to$ defined by the table below.

$\to$	1	0	−1
1	1	−1	−1
0	1	0	−1
−1	1	1	1

Make it into a logical matrix by designating the values 0 and 1. Call the resulting matrix $\mathbb{M}_3$. Checking that $\mathbb{M}_3$ verifies all theorems of BCI is an easy exercise, but it also follows from the fact that the underlying algebra of $\mathbb{M}_3$ is the implicational reduct of the three element *Sugihara algebra*, which in turn is a characteristic matrix of an extension **RM3** of BCI. See, for example, [1] for this fact and more on Sugihara algebras.

Lemma 5.1 *The matrix* $\mathbb{M}_3$ *invalidates the formula*

$$((x \to y) \to ((y \to x) \to (x \to x))) \to (v \to v)$$

for any variable v*, not necessarily distinct from* x *or* y.

Proof Take $v \mapsto 0$ and $v' \mapsto 1$ for any variable $v' \neq v$. □

Theorem 5.2 *There is no formula* τ *such that* $D \to (\tau \to \tau)$ *is a theorem of* BCI.

Proof We argue by reductio. Suppose that for some formula τ of the form $(\tau_1 \cdots \tau_n) \to v$, the sequent $D, \tau, \tau_1, \ldots, \tau_n \Rightarrow v$ is provable. If τ is a variable, Lemma 5.1 provides countermodels. We can thus assume that n is positive and for all τ' shorter than τ, the formula $D \to (\tau' \to \tau')$ is not a theorem. By Lemma 4.3 we conclude that D is a split formula for the corresponding sequent. Thus, $v = x$ and the sequents

$$\mathrm{T}_1, x \Rightarrow y \qquad \mathrm{T}_2, y \Rightarrow x \qquad \mathrm{T}_3 \Rightarrow x$$

are provable, with $\mathrm{T}_1 \cup \mathrm{T}_2 \cup \mathrm{T}_3 = \{\tau, \tau_1, \ldots, \tau_n\}$. Observe that to make all the sequents above provable, none of T_1, T_2, T_3 can be empty. This observation underlies a pigeonhole argument, which divides into three cases according to where τ lives. Since the cases are similar, we deal with the first in a detailed way and ask the reader to supply the missing details for the two others.

Case 1 $\tau \in \mathrm{T}_1$. We can assume $\mathrm{T}_1 = \{\tau, \tau_1, \ldots, \tau_k\}$, $\mathrm{T}_2 = \{\tau_{k+1}, \ldots, \tau_p\}$, and $\mathrm{T}_3 = \{\tau_{p+1}, \ldots, \tau_n\}$. We then obtain

$$\tau \Rightarrow (x\tau_1 \cdots \tau_k) \to y \quad \tau_{k+1}, \ldots, \tau_p \Rightarrow y \to x \quad \tau_{p+1}, \ldots, \tau_n \Rightarrow x. \tag{1}$$

Multiplying the third of these sequents by $\tau_1 \cdots \tau_k$ on both sides (and changing $\tau_1 \cdots \tau_k$ on the left to $\tau_1, \ldots, \tau_k$, cf. the remarks following Lemma 2.4), we get

$$\tau_1, \ldots, \tau_k, \tau_{p+1}, \ldots, \tau_n \Rightarrow x\tau_1 \cdots \tau_k\,.$$

Multiplying again, this time by the second sequent of (1), yields

$$\tau_1, \ldots, \tau_n \Rightarrow x\tau_1 \cdots \tau_k(y \to x)$$

from which, by Lemma 2.4, we obtain

$$(x\tau_1 \cdots \tau_k(y \to x)) \to x \Rightarrow (\tau_1 \cdots \tau_n) \to x$$

with the right-hand side being simply τ. Unwinding the left-hand side we get

$$(y \to x) \to ((x\tau_1 \cdots \tau_k) \to x) \Rightarrow (\tau_1 \cdots \tau_n) \to x.$$

Now consider the following provable sequent

$$(x\tau_1 \cdots \tau_k) \to y \Rightarrow (y \to x) \to ((x\tau_1 \cdots \tau_k) \to x)$$

which is a sequent version of suffixing. By an application of cut, we then obtain

$$(x\tau_1 \cdots \tau_k) \to y \Rightarrow (\tau_1 \cdots \tau_n) \to x$$

and this is just

$$(x\tau_1 \cdots \tau_k) \to y \Rightarrow \tau.$$

We will refer to this type of argument as *unwinding and desuffixing* and use it as one step in the proof later on. It now follows that $(x\tau_1 \cdots \tau_k) \to y$ and τ are provably equivalent, thus, by congruentiality of BCI, they are interchangeable. Therefore,

$D \to (\tau' \to \tau')$ is provable, with $\tau' = (x\tau_1 \cdots \tau_k) \to y$. We will now show that τ' is strictly shorter than τ. Compare

$$\tau = (\tau_1 \cdots \tau_k \cdot \tau_{k+1} \cdots \tau_n) \to x$$

with

$$\tau' = (\tau_1 \cdots \tau_k \cdot x) \to y.$$

So, τ' is not shorter than τ only if $k + 1 = n$ (and τ_n is a variable). However, since both T_2 and T_3 must be nonempty, in the worst case we get $\mathrm{T}_2 = \{\tau_{n-1}\}$ and $\mathrm{T}_3 = \{\tau_n\}$. Then $k + 1 = n - 1 < n$ and so τ' is shorter than τ, contradicting the inductive hypothesis.

Case 2 $\tau \in \mathrm{T}_2$. We can set $\mathrm{T}_1 = \{\tau_1, \ldots, \tau_k\}$, $\mathrm{T}_2 = \{\tau, \tau_{k+1}, \ldots, \tau_p\}$, and $\mathrm{T}_3 = \{\tau_{p+1}, \ldots, \tau_n\}$. We get

$$\tau_1, \ldots, \tau_k, x \Rightarrow y \quad \tau \Rightarrow (\tau_{k+1} \cdots \tau_p y) \to x \quad \tau_{p+1}, \ldots, \tau_n \Rightarrow x \tag{2}$$

and thus $\tau_1, \ldots, \tau_k, \tau_{p+1}, \ldots, \tau_n \Rightarrow y$, by cut. Multiplying both sides by $\tau_{k+1} \cdots \tau_p$, we get

$$\tau_1, \ldots, \tau_n \Rightarrow \tau_{k+1} \cdots \tau_p y$$

and from that

$$(\tau_{k+1} \cdots \tau_p y) \to x \Rightarrow \tau$$

follows by Lemma 2.4, so we get

$$\tau \Leftrightarrow (\tau_{k+1} \cdots \tau_p y) \to x.$$

Now $k + 1$ must be at least 2 and p at most $n - 1$. Thus, in the worst case we get $\tau' = (\tau_2 \cdots \tau_{n-1} y) \to x$, again shorter than τ.

Case 3 $\tau \in \mathrm{T}_3$. We can set $\mathrm{T}_1 = \{\tau_1, \ldots, \tau_k\}$, $\mathrm{T}_2 = \{\tau_{k+1}, \ldots, \tau_p\}$, and $\mathrm{T}_3 = \{\tau, \tau_{p+1}, \ldots, \tau_n\}$. We get

$$\tau_1, \ldots, \tau_k, x \Rightarrow y \quad \tau_{k+1}, \ldots, \tau_p, y \Rightarrow x \quad \tau \Rightarrow (\tau_{p+1} \cdots \tau_n) \to x. \tag{3}$$

By cut we get $x, \tau_1, \ldots, \tau_p \Rightarrow x$ and thus $\tau_1, \ldots, \tau_p \Rightarrow x \to x$. Multiplying both sides by $\tau_{p+1} \cdots \tau_n$ yields

$$\tau_1, \ldots, \tau_n \Rightarrow \tau_{p+1} \cdots \tau_n (x \to x);$$

therefore, the sequent

$$(\tau_{p+1} \cdots \tau_n (x \to x)) \to x \Rightarrow (\tau_1 \cdots \tau_n) \to x$$

is also provable. By unwinding and desuffixing we obtain

$$(\tau_{p+1} \cdots \tau_n) \to x \Rightarrow (\tau_1 \cdots \tau_n) \to x = \tau.$$

As in the previous case, this proves

$$\tau \Leftrightarrow (\tau_{p+1} \cdots \tau_n) \to x.$$

Now to make T_1 and T_2 nonempty $p + 1$ must be at least 3, so in the worst case we get $\tau' = (\tau_3 \cdots \tau_n) \to x$, which is shorter than τ. This ends the proof. □

Corollary 5.3 *Let B' and B stand for the respective instances of* $\mathbf{B}'$ *and* $\mathbf{B}$ *with φ, ψ and χ replaced by distinct variables. Neither B' nor B implies any self-implication in* BCI.

6 Some Comments and a Question

It seems reasonable to expect that the notion of split formula might have some uses beyond the proof of Theorem 5.2. For example, we can obtain simple proofs of the following well-known facts.

Theorem 6.1 *If φ is a theorem of* BCI, *then each variable occurs in φ an even number of times.*

Proof Induction on the length of φ. Base case is vacuously true. Let $\varphi = (\varphi_1 \cdots \varphi_n) \to v$, for some variable v. Then, the sequent $\varphi_1, \ldots, \varphi_n \Rightarrow v$ is provable and thus, by Lemma 3.1, we reduce the number of occurrences of v by two (one on the right of the sequent arrow, the other in the split formula) getting access to inductive hypothesis. $\square$

The next theorem is a stronger version of a result proved for BCK in [6]. Its import is mainly algebraic: certain facts about (congruence lattices of) appropriate quasi-varieties are easy to prove with its help. Before we prove the theorem, we need to reprove in our setting a technical lemma from [6].

Lemma 6.2 *Let x and y be variables. Suppose the sequent $\otimes\Delta \to x, \Gamma \Rightarrow y$ is provable and $\otimes\Delta \to x$ is not a split formula for it. Then there exists a multiset Δ' of formulas such that the sequents $\otimes\Delta \to x \Rightarrow \otimes\Delta' \to x$ and $\otimes\Delta' \to x \Rightarrow \otimes\Gamma \to y$ are provable and $\otimes\Delta' \to x$ is shorter than $\otimes\Gamma \to y$.*

Proof Induction on the length of the sequent. The base case holds vacuously, because if x and y are distinct, then $x \Rightarrow y$ is unprovable, and if $x = y$, then $\otimes\Delta \to x = x$ is a split formula for the sequent. For the inductive step, Lemma 3.1 provides (after renumbering) a system of provable sequents

$$\begin{aligned} \otimes\Delta \to x, \Gamma_1' &\Rightarrow \alpha_1 \\ \Gamma_2 &\Rightarrow \alpha_2 \\ \vdots\ &\quad \vdots \\ \Gamma_k &\Rightarrow \alpha_k \end{aligned}$$

where $\Gamma_1' \cup \bigcup_{i=2}^{k} = \Gamma \setminus \{\gamma\}$ and $\gamma = (\alpha_1 \cdots \alpha_k) \to y$. Since α_1 is of the form $\otimes\Pi \to z$, the sequent $\otimes\Delta \to x, \Gamma_1', \Pi \Rightarrow z$ is provable. If $\otimes\Delta \to x$ is a split formula for this sequent, then $x = z$ and we can take Γ_1', Π for Δ'. Suppose $\otimes\Delta \to x$ is not a split formula for $\otimes\Delta \to x, \Gamma_1', \Pi \Rightarrow z$. Then, by inductive hypothesis, we get a multiset Σ of formulas such that $\otimes\Delta \to x \Rightarrow \otimes\Sigma \to x$ and $\otimes\Sigma \to x \Rightarrow \otimes(\Gamma_1' \cup \Pi) \to x$ are provable, and $\otimes\Sigma \to x$ is shorter than $\otimes(\Gamma_1' \cup \Pi) \to x$. Notice that $\otimes(\Gamma_1' \cup \Pi) \to x$ can also be written as $\otimes\Gamma_1' \to \alpha_1$. Therefore, the following is also a system of provable sequents

$$\begin{aligned} \otimes\Sigma \to x, \Gamma_1' &\Rightarrow \alpha_1 \\ \Gamma_2 &\Rightarrow \alpha_2 \\ \vdots\ &\quad \vdots \\ \Gamma_k &\Rightarrow \alpha_k \end{aligned}$$

from which, by another application of Lemma 3.1, we obtain a provable sequent $\otimes\Sigma \to x, \Gamma \Rightarrow y$. Thus $\otimes\Sigma \to x \Rightarrow \otimes\Gamma \to y$ is provable with $\otimes\Sigma \to x$ shorter than $\Gamma \to y$, as required. □

Theorem 6.3 *If the sequents $\varphi \Leftrightarrow \psi$ are provable in* BCI, *then the rightmost variable in φ and ψ is the same.*

Proof Let $\varphi = \otimes\Phi \to x$ and $\psi = \otimes\Psi \to y$. We have provable sequents

$$\otimes\Phi \to x, \Psi \Rightarrow y$$
$$\otimes\Psi \to y, \Phi \Rightarrow x.$$

If x and y are different, neither $\otimes\Phi \to x$ nor $\otimes\Psi \to y$ can be split. By Lemma 6.2, there exists a formula $\otimes\Phi_1 \to x$ shorter than $\Psi \to y$ and such that the sequents $\otimes\Phi \to x \Rightarrow \otimes\Phi_1 \to x$ and $\otimes\Phi_1 \to x \Rightarrow \otimes\Psi \to y$ are provable. It follows that the sequents $\otimes\Phi_1 \to x \Leftrightarrow \otimes\Psi \to y$ are provable. Now $\Psi \to y$ cannot be split either, so repeating the reasoning we get a $\otimes\Psi_1 \to y$ shorter than $\otimes\Phi_1 \to x$ with the sequents $\otimes\Phi_1 \to x \Leftrightarrow \otimes\Psi_1 \to y$ provable. Repeating the reasoning further produces an infinite sequence of shorter and shorter formulas, an impossibility. Thus, x and y must be the same, as claimed. □

We finish with a remaining open question. The proof of Theorem 5.2 suggests that an appropriately chosen substitution instance of **B** (or **B**$'$) might provide a positive answer to the following question.

Question 6.4 *Is there any theorem of the one variable fragment of* BCI *that does not imply a self-implication?*

However, a number of attempts at finding such an instance ended in failure, so the question, although admittedly technical, appears interesting and we do not propose any conjecture here.

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