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Tennenbaum's Theorem and Unary Functions

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Abstract It is well known that in any nonstandard model of PA (Peano arithmetic) neither addition nor multiplication is recursive. In this paper we focus on the recursiveness of unary functions and find several pairs of unary functions which cannot be both recursive in the same nonstandard model of PA (e.g., $\{2x, 2x+1\}$, $\{x^2, 2x^2\}$, and $\{2^x, 3^x\}$). Furthermore, we prove that for any computable injection $f(x)$, there is a nonstandard model of PA in which $f(x)$ is recursive.

1 Introduction

Let $\{0, 1, +, \times, <\}$ be the usual language of arithmetic. PA is Peano arithmetic. We write $\overline{n}$ for the term corresponding to the natural number n, that is, $\overline{0} := 0$, and $\overline{n+1} := \overbrace{1 + \cdots + 1}^{n+1 \text{ 1s}}$. Let $\pi(x)$ be the function which denotes the xth prime.

In this paper, we only consider computable functions (i.e., recursive functions). When $f(x_1, \ldots, x_j)$ is computable, there is a formula $F(x_1, \ldots, x_j, y)$ such that

$$\mathsf{PA} \vdash \forall y(y = \overline{f(n_1, \ldots, n_j)} \leftrightarrow F(\overline{n_1}, \ldots, \overline{n_j}, y)),$$

for all $n_i \in \mathbb{N}$. In particular, if f is primitive recursive, F is provably total in PA; that is,

$$\mathsf{PA} \vdash \forall x_1 \ldots \forall x_j \exists! y F(x_1, \ldots, x_j, y).$$

Definition 1.1 Suppose $F_0, \ldots, F_j$ are provably total in PA. We say that $F_0, \ldots, F_j$ are recursive in some model $\mathcal{M} = (M, \ldots)$ of PA if there are recursive functions $f_0, \ldots, f_j$ such that

$$(M, F_0^{\mathcal{M}}, \ldots, F_j^{\mathcal{M}}) \cong (\mathbb{N}, f_0, \ldots, f_j).$$

In the rest of the paper, we shall use M instead of $\mathcal{M}$ to simplify notation.

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Tennenbaum's theorem says that there is no nonstandard model of PA in which addition or multiplication is recursive (see Boolos et al. [1]). Afterward, many studies about other binary functions were carried out. D'Aquino [2] showed the nonrecursiveness of x^y and $x^{\lfloor \log_2 y \rfloor}$. On the other hand, Schmerl [4] constructed a nonstandard model in which both $\gcd(x, y)$ and $\operatorname{lcm}(x, y)$ are recursive.

In this paper, we study about unary functions and prove two results. First, we find several pairs of functions which cannot be both recursive in the same nonstandard model. For example, $\{2x, 2x+1\}$, $\{x^2, 2x^2\}$, and $\{2^x, 3^x\}$, which strengthen the previous results about addition, multiplication, and exponentiation, respectively. Second, we show that for all computable injections $f(x)$, there is a nonstandard model in which f is recursive. This is an extension of the fact proved in [2], which says that there is a nonstandard model where $f(x)$ is recursive if f has the following properties: (1) f is a computable injection, (2) $\mathbb{N} - \operatorname{range}(f)$ is infinite, (3) f has no cycles.

The following is the well-known condition which is sufficient to deduce that functions cannot be recursive in any nonstandard model (see Kaye [3]).

Theorem 1.2 *Let $f_1, \ldots, f_j$ be functions which are provably total in PA. Let $\alpha(x, y)$ be a formula, and let $\{E_n(x)\}_{n\in\mathbb{N}}$ be a recursive sequence of existential formulas which contain only nonlogical symbols among f_i. Furthermore, suppose the next two conditions hold.*

1. $\mathsf{PA} \vdash \exists y\big(\bigwedge_{i\in X} \alpha(\bar{\imath}, y) \wedge \bigwedge_{i\le n, i\notin X} \neg\alpha(\bar{\imath}, y)\big)$ *for all* $X \subseteq \{0, 1, \ldots, n\}$.
2. $\mathsf{PA} \vdash \forall y(\alpha(\bar{n}, y) \leftrightarrow E_n(y))$ *for each* $n \in \mathbb{N}$.

Then $f_1, \ldots, f_j$ cannot be recursive in any nonstandard model of PA.

Example 1.3 Define $\alpha(x, y) := \pi(x) \mid y$ and $E_n(x) := \exists w\; x = \overbrace{w + \cdots + w}^{\pi(n)\ ws}$. For $X \subseteq \{0, 1, \ldots, n\}$, let $m = \prod_{i\in X} \pi(i)$. Obviously,

$$\mathsf{PA} \vdash \bigwedge_{i\in X} \alpha(\bar{\imath}, \bar{m}) \wedge \bigwedge_{i\le n, i\notin X} \neg\alpha(\bar{\imath}, \bar{m}).$$

Also, $\mathsf{PA} \vdash \pi(\bar{n}) \mid y \leftrightarrow E_n(y)$. Hence, by Theorem 1.2, the addition cannot be recursive in any nonstandard model of PA.

The rest of this section describes the proof of Theorem 1.2. Let M be a nonstandard model of PA.

Lemma 1.4 *Suppose $F(x, x_1, \ldots, x_k)$ is a formula and $t_1, \ldots, t_k \in M$. Then there exists $s \in M$ such that $M \vDash F(\bar{n}, t_1, \ldots, t_k) \leftrightarrow \alpha(\bar{n}, s)$ for each $n \in \mathbb{N}$.*

Proof We write $F_t(x)$ for $F(x, t_1, \ldots, t_k)$. For all n, let $X = \{i \le n : M \vDash F_t(\bar{\imath})\}$. By assumption of Theorem 1.2, $M \vDash \exists y\big(\bigwedge_{i\in X} \alpha(\bar{\imath}, y) \wedge \bigwedge_{i\le n, i\notin X} \neg\alpha(\bar{\imath}, y)\big)$. Therefore,

$$M \vDash \exists y \forall z \le \bar{n}(F_t(z) \leftrightarrow \alpha(z, y)).$$

By the overspill principle, there exists a nonstandard $c \in M$ such that

$$M \vDash \exists y \forall z \le c(F_t(z) \leftrightarrow \alpha(z, y)).$$

Hence, $M \vDash \forall z \le c(F_t(z) \leftrightarrow \alpha(z, s))$ for some $s \in M$. For all n, $M \vDash \bar{n} < c$, and thus $M \vDash F_t(\bar{n}) \leftrightarrow \alpha(\bar{n}, s)$. □

Lemma 1.5 *There exists $s \in M$ such that $\{n : M \vDash \alpha(\overline{n}, s)\}$ is not semirecursive (i.e., not recursively enumerable).*

Proof Let X, Y be two recursively inseparable semirecursive sets. PA can separate X and Y; that is, there is a formula $U(x)$ such that if $n \in X$ then PA $\vdash U(\overline{n})$, and if $n \in Y$ then PA $\vdash \neg U(\overline{n})$. Thus, $M \vDash U(\overline{n})$ if $n \in X$, and $M \vDash \neg U(\overline{n})$ if $n \in Y$. By Lemma 1.4, $M \vDash U(\overline{n}) \leftrightarrow \alpha(\overline{n}, s)$ for some $s \in M$. $\{n : M \vDash \alpha(\overline{n}, s)\} = \{n : M \vDash U(\overline{n})\}$ separates X and Y; thus is not recursive.

Using Lemma 1.4 again, there exists $t \in M$ such that $M \vDash \neg\alpha(\overline{n}, s) \leftrightarrow \alpha(\overline{n}, t)$. Since $\{n : M \vDash \alpha(\overline{n}, s)\}$ is not recursive, either $\{n : M \vDash \neg\alpha(\overline{n}, s)\} = \{n : M \vDash \alpha(\overline{n}, t)\}$ or $\{n : M \vDash \alpha(\overline{n}, s)\}$ is not semirecursive. □

We assume that $f_1, f_2, \ldots, f_j$ are all recursive in M. Let $s \in M$ arbitrary. By assumption, $M \vDash \alpha(\overline{n}, s) \leftrightarrow E_n(s)$ for all n. Since $E_n(s)$ is an existential formula whose only nonlogical symbols are functions which are recursive in M, $\{n : M \vDash E_n(s)\}$ is semirecursive. Hence $\{n : M \vDash \alpha(\overline{n}, s)\}$ is also semirecursive. This is a contradiction to Lemma 1.5.

2 On Pairs of Unary Functions

First, we give a short proof of nonrecursiveness of $\{2x, 2x + 1\}$.

Theorem 2.1 *There is no nonstandard model of PA in which both $2x$ and $2x + 1$ are recursive.*

Proof We define $\alpha(z, x)$ as the formula $\exists w, u(u < \overline{2}^z \wedge x = w\overline{2}^{z+1} + \overline{2}^z + u)$. $\alpha(z, x)$ means that the zth digit of the binary expansion of x is 1. For $X \subseteq \{0, 1, \ldots, n\}$, let $m_i = 1$ if $i \in X$, $m_i = 0$ if $i \notin X$, and $m = \sum_{i=0}^{n} m_i 2^i$. Then clearly,

$$\mathrm{PA} \vdash \bigwedge_{i \in X} \alpha(\overline{\imath}, \overline{m}) \wedge \bigwedge_{i \leq n, i \notin X} \neg\alpha(\overline{\imath}, \overline{m}).$$

Moreover,

$$\begin{aligned} \alpha(\overline{n}, x) &\leftrightarrow \exists w, u(u < \overline{2^n} \wedge x = w\overline{2^{n+1}} + \overline{2^n} + u) \\ &\leftrightarrow \exists w \bigvee_{0 \leq k < 2^n} x = w\overline{2^{n+1}} + \overline{2^n} + \overline{k} \\ &\leftrightarrow \exists w \bigvee_{k_i = 0,1} x = w\overline{2^{n+1}} + \overline{2^n} + \overline{k_{n-1}2^{n-1}} + \overline{k_{n-2}2^{n-2}} + \cdots + \overline{k_0} \\ &\leftrightarrow \exists w \bigvee_{k_i = 0,1} x = \overline{2}(\cdots \overline{2}(\overline{2}(\overline{2}w + \overline{1}) + \overline{k_{n-1}}) + \overline{k_{n-2}} \cdots) + \overline{k_0} \end{aligned}$$

in PA. Hence $\alpha(\overline{n}, x)$ is equivalent to an existential formula of $2x$ and $2x + 1$. By Theorem 1.2, $2x$ and $2x + 1$ cannot be recursive. □

Remark 2.2 If $x + y$ is recursive in a nonstandard model M, then $2x$ $(= x + x)$ and $2x + 1$ $(= x + x + 1)$ are also recursive in M. Hence, Theorem 2.1 implies nonrecursiveness of addition.

Remark 2.3 From the proof of Theorem 2.1, we can find some more function pairs which cannot be recursive. Let $f(x) = \lfloor x/2 \rfloor$, and $g(x) = x \bmod 2$. Then obviously PA $\vdash \alpha(\overline{n}, x) \leftrightarrow g(f^n(x)) = 1$. Thus $\{f(x), g(x)\}$ cannot be recursive. Moreover, as $g(x) = 1$ is equivalent to $f(x + 1) \neq f(x)$,

$\mathsf{PA} \vdash \alpha(\overline{n}, x) \leftrightarrow f(f^n(x)+1) \neq f^{n+1}(x)$. Hence $\{f(x), x+1\}$ cannot be recursive.

Indeed, Theorem 2.1 can be extended to a more general form.

Theorem 2.4 *Let $f(x)$ and $g(x)$ be distinct functions which are provably total in* PA. *Suppose for all n and $h_0, \ldots, h_n \in \{f, g\}$,*

$$\mathsf{PA} \vdash \exists z \forall w \bigwedge_{i \leq n} \bigwedge_{\substack{h'_0,\ldots,h'_i \in \{f,g\} \\ (h'_0,\ldots,h'_i) \neq (h_0,\ldots,h_i)}} h_0 \circ \cdots \circ h_n(z) \neq h'_0 \circ \cdots \circ h'_i(w).$$

Then there is no nonstandard model of PA *in which both $f(x)$ and $g(x)$ are recursive.*

Proof Let $\alpha(z, x)$ be a formula such that

$$\mathsf{PA} \vdash \alpha(\overline{n}, x) \leftrightarrow \exists w \bigvee_{h_i = f,g} x = h_0 \circ \cdots \circ h_{n-1} \circ f(w).$$

(Using primitive recursion, the right side can be expressed by a finite formula of x and n.) For $X \subseteq \{0, \ldots, n\}$, let $h_i = f$ if $i \in X$ and $h_i = g$ if $i \notin X$. In PA, choose z as guaranteed for $h_0, \ldots, h_n$, and let $x = h_0 \circ \cdots \circ h_n(z)$. If $i \in X$, then $x = h_0 \circ \cdots \circ h_{i-1} \circ f(h_{i+1} \circ \cdots \circ h_n(z))$; thus $\mathsf{PA} \vdash \alpha(\overline{\imath}, x)$. Assume $i \leq n$ and $i \notin X$. Since $h_i = g$, $(h'_0, \ldots, h'_{i-1}, f) \neq (h_0, \ldots, h_i)$ for all $h'_0, \ldots, h'_{i-1} \in \{f, g\}$. Thus by assumption of z, $\mathsf{PA} \vdash \forall w \bigwedge_{h'_j = f,g} x \neq h'_0 \circ \cdots \circ h'_{i-1} \circ f(w)$. Hence $\mathsf{PA} \vdash \neg\alpha(\overline{\imath}, x)$. From Theorem 1.2, $\{f, g\}$ cannot be recursive. □

Example 2.5 Suppose in PA, f, g are both provably total and injective, and $f(x) \neq g(y)$ for all x, y. In this case, we can prove

$$\mathsf{PA} \vdash \forall w \bigwedge_{i \leq n} \bigwedge_{\substack{h'_0,\ldots,h'_i \in \{f,g\} \\ (h'_0,\ldots,h'_i) \neq (h_0,\ldots,h_i)}} h_0 \circ \cdots \circ h_n(0) \neq h'_0 \circ \cdots \circ h'_i(w).$$

Thus $\{f, g\}$ cannot be recursive from Theorem 2.4. For example, $f(x) = ax + b$ and $g(x) = ax + c$ such that $a \geq 2$ and $b \not\equiv c \pmod{a}$ have the properties above.

Example 2.6 $f(x) = 2x^2$ and $g(x) = x^2$ (respectively, $f(x) = 2^x$ and $g(x) = 3^x$) cannot be recursive. This extends nonrecursiveness of $x \times y$ (respectively, x^y). Since $f(0) = g(0)$, the previous example is not applicable. But it is easy to see

$$\mathsf{PA} \vdash \forall w \bigwedge_{i \leq n} \bigwedge_{\substack{h'_0,\ldots,h'_i \in \{f,g\} \\ (h'_0,\ldots,h'_i) \neq (h_0,\ldots,h_i)}} h_0 \circ \cdots \circ h_n(1) \neq h'_0 \circ \cdots \circ h'_i(w).$$

3 On Single Unary Function

Our goal in this section is to prove the following theorem.

Theorem 3.1 *For all computable injections $f(x)$, there exists a nonstandard model M of* PA *such that f is recursive in M.*

Remark 3.2 The "injections" in Theorem 3.1 cannot be replaced by "functions." There exists a primitive recursive function $f(x)$ which cannot be recursive in any nonstandard model. $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x/4 & \text{if } x = 2^{2m+2}w \text{ and } 2 \nmid w \text{ for some } m, w, \\ 2x & \text{if } x = 2^{2m+1}w \text{ and } 2 \nmid w \text{ and } \pi(m+1) \mid w \text{ for some } m, w, \\ x & \text{otherwise.} \end{cases}$$

Let $\alpha(x, y) := \overline{2} \nmid y \wedge \pi(x+1) \mid y$. For $X \subseteq \{0, 1, \ldots, n\}$, let $m = \prod_{i \in X} \pi(i+1)$. It is clear that $\mathsf{PA} \vdash \alpha(\overline{\imath}, \overline{m})$ if $i \in X$, and $\mathsf{PA} \vdash \neg\alpha(\overline{\imath}, \overline{m})$ if $i \notin X$. Moreover, by Lemma 3.3, $\alpha(\overline{n}, y)$ is equivalent to an existential formula of f. Hence, by Theorem 1.2, $f(x)$ cannot be recursive.

Lemma 3.3 *The following are equivalent in* PA.

1. $\alpha(\overline{n}, y)$.
2. $\exists u, v(f(y) = y \wedge f^{n+1}(u) \neq y \wedge f^{n+2}(u) = y \wedge u \neq v \wedge f(u) = f(v))$.

Proof We give an informal proof. We first show the implication (1) $\rightarrow$ (2). By $2 \nmid y$, $f(y) = y$ and $y \neq 0$. Let $u = 2^{2n+4}y$, $v = 2^{2n+1}y$. Since $y \neq 0$, $u \neq v$. By $\pi(n+1) \mid y$, $f(u) = f(v) = 2^{2n+2}y$. Also, $f^{n+1}(u) = 2^{2n+4}y/4^{n+1} = 4y \neq y$, $f^{n+2}(u) = f(4y) = y$.

We next prove the implication (2) $\rightarrow$ (1). By $f^{n+1}(u) \neq f^{n+2}(u) = y$, $y = f^{n+1}(u)/4$ or $y = 2f^{n+1}(u)$. If $y = 2f^{n+1}(u)$, then $y = 2^{2m+2}w$ and $2 \nmid w$ for some w, m. Thus, $f(y) = y/4 \neq y$, which contradicts $f(y) = y$. Hence, $y = f^{n+1}(u)/4 = 2^{2m}w$, $2 \nmid w$ for some w, m. If $m \geq 1$, then $f(y) = y/4 \neq y$. Thus $m = 0$ and $2 \nmid y$.

Since $f^{n+1}(u) = f^{n+1}(v) \neq f^{n+2}(u) = f^{n+2}(v)$, $f(u) \neq u$ and $f(v) \neq v$. Moreover, $u \neq v$ and $f(u) = f(v)$; thus we may assume $f(u) = u/4 = 2v = f(v)$ (the case when $f(u) = 2u = v/4 = f(v)$ is symmetric). Then for some m and w, $v = 2^{2m+1}w$, $2 \nmid w$, $\pi(m+1) \mid w$, and $u = 8v = 2^{2m+4}w$. If $m > n$, then $y = f^{n+2}(u) = 4^{m-n}w$, which gives a contradiction to $2 \nmid y$. Thus, $m \leq n$ (m is standard) and $f^{m+2}(u) = w$. Assume $m < n$. By $2 \nmid w$, $f(w) = w$. Moreover, by $f^{m+2}(u) = w$ and $m+2 \leq n+1$, $f^{n+1}(u) = f^{n+2}(u) = w$. This is a contradiction to $f^{n+1}(u) \neq y = f^{n+2}(u)$. Thus $m = n$. Furthermore, $y = f^{n+2}(u) = f^{m+2}(u) = w$. By $\pi(m+1) \mid w$, $\pi(n+1) \mid y$. $\square$

Before proving Theorem 3.1, we observe some properties of injections. Let X be a countable set and let f be an injection on X. Consider the relation on X which holds if $f^n(x) = y$ or $f^n(y) = x$ for some $n \geq 0$. Clearly, this is an equivalence relation on X. Every equivalence class is classified into the following types:

1. $\{x, f(x), f^2(x), \ldots\}$ ($\mathbb{N}$);
2. $\{\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots\}$ ($\mathbb{Z}$);
3. $\{x = f^m(x), f(x), f^2(x), \ldots, f^{m-1}(x)\}$ ($\mathbb{Z}_m$, $m = 1, 2, \ldots$).

For $D \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Z}_1, \mathbb{Z}_2, \ldots\}$, let $0 \leq K_D^{X,f} \leq \infty$ be the number of the equivalence classes having the type D. It is easy to see that the structure of (X, f) is completely determined by $K_D^{X,f}$; that is, $(X, f) \cong (Y, g)$ if and only if $K_D^{X,f} = K_D^{Y,g}$ for all D.

Example 3.4 For $1 \le n \le \infty$, there is a computable function $h_n(x)$ such that $K_{\mathbb{Z}}^{\mathbb{N},h_n} = n$ and $K_D^{\mathbb{N},h_n} = 0$ for all $D \neq \mathbb{Z}$. First, h_1 is defined by the following:

$$h_1(x) = \begin{cases} x+2 & \text{if } 2 \mid x\,, \\ 0 & \text{if } x = 1\,, \\ x-2 & \text{otherwise.} \end{cases}$$

$(\mathbb{N}, h_1)$ has only one equivalence class $\{\ldots, 5, 3, 1, 0, 2, 4, \ldots\}$. For $2 \le n < \infty$, define $h_n(x) = h_1(p)n + q$ if $x = pn + q$ and $q < n$. Let $J(x, y)$ be a pairing function (computable bijection from $\mathbb{N}^2$ to $\mathbb{N}$). We define $h_\infty(x) = J(h_1(p), q)$ if $x = J(p, q)$.

Claim 3.5 *Let $f(x)$ be an injection on X, and (Y, f) is a substructure of (X, f). Then $K_{\mathbb{Z}}^{X,f} \ge K_{\mathbb{Z}}^{Y,f}$.*

Proof If $\{\ldots, f^{-1}(x), x, f(x), \ldots\} \subseteq Y$, $\{\ldots, f^{-1}(x), x, f(x), \ldots\} \subseteq X$ by $Y \subseteq X$. Thus $K_{\mathbb{Z}}^{X,f} \ge K_{\mathbb{Z}}^{Y,f}$. (Analogously, we can prove $K_{\mathbb{Z}_m}^{X,f} \ge K_{\mathbb{Z}_m}^{Y,f}$. But $K_{\mathbb{N}}^{X,f}$ may be less than $K_{\mathbb{N}}^{Y,f}$ when $x \in Y$ and $\{\ldots, f^{-2}(x), f^{-1}(x)\} \subseteq X - Y$ for some x.) □

Claim 3.6 *Assume $f(x)$ is an injection on X, $g(x)$ is an injection on Y, and* $\mathrm{Th}(X, f) = \mathrm{Th}(Y, g)$. *Then $K_D^{X,f} = K_D^{Y,g}$ for all $D \neq \mathbb{Z}$.*

Proof Let A_n be the sentence that asserts that there are precisely n objects that are not in the range of f. Then clearly,

$$K_{\mathbb{N}}^{X,f} = n \Leftrightarrow A_n \in \mathrm{Th}(X, f),$$
$$K_{\mathbb{N}}^{X,f} = \infty \Leftrightarrow A_n \notin \mathrm{Th}(X, f) \text{ for all } n.$$

Moreover, for $m \ge 1$, let B_n^m be the sentence which asserts that there exist exactly mn xs such that $f^l(x) \neq x$ for all $1 \le l < m$ and $f^m(x) = x$. Then

$$K_{\mathbb{Z}_m}^{X,f} = n \Leftrightarrow B_n^m \in \mathrm{Th}(X, f),$$
$$K_{\mathbb{Z}_m}^{X,f} = \infty \Leftrightarrow B_n^m \notin \mathrm{Th}(X, f) \text{ for all } n.$$

Thus, every $K_D^{X,f}$ but $K_{\mathbb{Z}}^{X,f}$ is determined by $\mathrm{Th}(X, f)$. □

Now we prove Theorem 3.1. Let $f(x)$ be a computable injection on $\mathbb{N}$. $(\mathbb{N}, f)$ has a countable recursively saturated elementary extension (M, f). (f is also injective on M since $\mathrm{Th}(\mathbb{N}, f) = \mathrm{Th}(M, f)$.)

We show that (M, f) is recursive. By definition of (M, f), $(\mathbb{N}, f)$ is a substructure of (M, f), and $\mathrm{Th}(\mathbb{N}, f) = \mathrm{Th}(M, f)$. Thus by the above claims, $K_{\mathbb{Z}}^{M,f} \ge K_{\mathbb{Z}}^{\mathbb{N},f}$, and $K_D^{M,f} = K_D^{\mathbb{N},f}$ for all $D \neq \mathbb{Z}$. We can define a computable injection $g(x)$ such that $K_D^{\mathbb{N},g} = K_D^{M,f}$ for every D. If $K_{\mathbb{Z}}^{M,f} = K_{\mathbb{Z}}^{\mathbb{N},f}$ then $g = f$. If $K_{\mathbb{Z}}^{M,f} > K_{\mathbb{Z}}^{\mathbb{N},f}$, let

$$p = \begin{cases} K_{\mathbb{Z}}^{M,f} - K_{\mathbb{Z}}^{\mathbb{N},f} & \text{if } K_{\mathbb{Z}}^{M,f} < \infty\,, \\ \infty & \text{if } K_{\mathbb{Z}}^{M,f} = \infty\,. \end{cases}$$

and

$$g(x) = \begin{cases} 2f(x/2) & \text{if } 2 \mid x\,, \\ 2h_p(\lfloor x/2 \rfloor) + 1 & \text{if } 2 \nmid x\,. \end{cases}$$

$h_p(x)$ is defined in the above example. Clearly, for each D, $K_D^{\mathbb{N},g} = K_D^{\mathbb{N},f} + K_D^{\mathbb{N},h_p}$, and thus, $K_D^{\mathbb{N},g} = K_D^{M,f}$. Therefore, $(M, f) \cong (\mathbb{N}, g)$, which means (M, f) is recursive.

Finally, we expand (M, f) to a model of PA. Add $\{0, 1, +, \times, <, c\}$ to the language. c is a constant. Let $F(x, y)$ be the formula defining the graph of $f(x)$ on $\mathbb{N}$. Consider the following theory T:

$$T = \mathrm{Th}(M, f) \cup \mathsf{PA} \cup \{\forall x, y(F(x, y) \leftrightarrow y = f(x))\} \cup \{\overline{n} \neq c \ : n \in \mathbb{N}\}.$$

Since $\mathrm{Th}(\mathbb{N}, f) = \mathrm{Th}(M, f)$, $\mathbb{N}$ is a model of every finite fragment of T. Hence, by resplendency, we can expand (M, f) to a model of T. The model we get is a nonstandard model of PA in which $f(x)$ is recursive.

References

[1] Boolos, G. S., J. P. Burgess, and R. C. Jeffrey, *Computability and Logic*, 4th edition, Cambridge University Press, Cambridge, 2002. Zbl 1014.03001. MR 1898463. 178

[2] D'Aquino, P., "Toward the limits of the Tennenbaum phenomenon," *Notre Dame Journal of Formal Logic*, vol. 38 (1997), pp. 81–92. Zbl 0889.03052. MR 1479370. 178

[3] Kaye, R., "Tennenbaum's theorem for models of arithmetic," http://web.mat.bham.ac.uk/R.W.Kaye/papers/tennenbaum/tennenbaum.pdf, 2006. 178

[4] Schmerl, J. H., "Recursive models and the divisibility poset," *Notre Dame Journal of Formal Logic*, vol. 39 (1998), pp. 140–48. Zbl 0968.03078. MR 1671738. 178

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