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A Note on the Logic of Eventual Permanence for Linear Time

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Abstract In a paper from the 1980s, Byrd claims that the logic of "eventual permanence" for linear time is **KD5**. In this note we take up Byrd's novel argument for this and, treating the problem as one concerning translational embeddings, show that rather than **KD5** the correct logic of "eventual permanence" is **KD45**.

1 Introduction

In Byrd [1], an attempt is made to determine what the logic of the notion of "eventual permanence" is. This notion was first mentioned in Rescher and Urquhart [3], p. 135, under the name of "Marxist Necessity," where it was conjectured that the logic of this notion when time is considered to be linear and unending was the logic **KD5**. The logic of this notion, whatever it may be, would be one whose primitive modal operator exhibited the same behavior as the tense operator $\blacksquare$ below.[1]

$$\mathcal{M} \models_x \blacksquare A \iff \exists y(x < y \text{ and } \forall z(\text{ if } y < z \text{ then } \mathcal{M} \models_z A)). \tag{1}$$

Here we are treating models $\mathcal{M}$ as ordered triples $\langle W, <, V\rangle$ with W a nonempty set, $<$ the "earlier than" relation (our model's *accessibility* relation), and V a function assigning to each p_i a subset of W. In particular, we will be concerned here with the case where, as in [3], time is taken to be linear and unending. The logic of (future directed) unending linear time is the modal logic **KD4.3** (called $K_l^{\infty+}$ in [3]), which has the following axioms and rules in addition to those of classical logic ($\Diamond$ being taken as defined as $\neg\Box\neg$ in the usual way.[2])

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Axioms

$$
\begin{array}{rcl}
\mathbf{K} & : & \Box(p \to q) \to (\Box p \to \Box q) \\
\mathbf{D} & : & \Box p \to \Diamond p \\
\mathbf{4} & : & \Box p \to \Box\Box p \\
\mathbf{.3} & : & (\Box(p \vee q) \wedge \Box(p \vee \Box q) \wedge \Box(\Box p \vee q)) \to (\Box p \vee \Box q).
\end{array}
$$

Rules

$$
\begin{array}{rcl}
\text{Necessitation} & : & \text{From } \vdash A \text{ infer } \vdash \Box A. \\
\text{Modus Ponens} & : & \text{From } \vdash A \to B \text{ and } \vdash A \text{ infer } \vdash B. \\
\text{Uniform Substitution} & : & \text{From } \vdash A \text{ infer } \vdash A[B/p].
\end{array}
$$

Here note that we are taking the formula $A[B/p]$ to be the result of uniformly replacing all occurrences of the propositional variable p in the formula A, with the formula B. Throughout we will be concerned with models for **KD4.3**, where the relation $<$ is taken to be serial, irreflexive, transitive, and connected.

For the sake of definiteness we will take Byrd's task—that is, determining the logic of the operator $\blacksquare$—as being that of finding a suitable source logic for a particular translational embedding. What is meant by a *translation* in this context is a mapping τ, which is the identity map on all propositional variables, and translates the classical connectives—which we will fix for the remainder as the set $\{\to, \neg\}$—homonymously. Additionally, our translation must translate $\Box p$ in terms of some formula $C(p)$ constructed solely out of the variable p. That is to say, for a mapping τ to be a translation in our (restrictive) sense it must fulfill the following conditions.

$$
\begin{aligned}
\tau(p_i) &= p_i. \\
\tau(\neg A) &= \neg\tau(A). \\
\tau(A \to B) &= \tau(A) \to \tau(B). \\
\tau(\Box B) &= C(\tau(B)).
\end{aligned}
$$

Given two logics S_0 and S_1 considered as sets of formulas, we will think of a translation τ as *embedding* S_0 *faithfully into* S_1 whenever, for all formulas A of the language of S_0, we have that

$$\vdash_{S_0} A \text{ if and only if } \vdash_{S_1} \tau(A). \tag{2}$$

Additionally, whenever the "only if" direction of the above is fulfilled we will say that τ is an *embedding of* S_0 *into* S_1. Here we will think—for convenience's sake—of our logics S_0 and S_1 as sharing the same (propositional) language, whose formulas are constructed out of a countable supply of propositional variables $p_1, p_2, p_3, \ldots$ and the connectives $\{\neg, \to, \Box\}$. To see how we get from trying to determine what the logic of the operator $\blacksquare$ over linear time is to discussing translational embeddings it is worth noting that the above operator can be defined in **KD4.3** as $\blacksquare A =_{\mathrm{Df}} \Diamond\Box A$. If we consider now the translation $\tau_{\Diamond\Box}$, a translation like τ above for which the formula $C(p)$ is $\Diamond\Box(p)$, we are able to clearly state Byrd's task. Calling S_0 in (2) the *source* logic of our embedding, and S_1 the *target* logic, we can state Byrd's task as being that of finding a source logic L which can be embedded faithfully into **KD4.3** by the translation $\tau_{\Diamond\Box}$. That is to say, we wish to find the logic L such that, for all formulas A, we have that

$$\vdash_L A \text{ if and only if } \vdash_{\mathbf{KD4.3}} \tau_{\Diamond\Box}(A). \tag{3}$$

It is clear to see that there will be a unique such L.[3]

In [1] we are presented with a novel argument for why L is **KD5**—as had been conjectured in [3], p. 137. As it turns out, Byrd's argument is unsound, the essential theorem involved in his proof being faulty, as pointed out in Humberstone [2] (p. 286, n. 11). Byrd's argument strategy, cast into the present terminology, involved showing that all of the proper extensions of **KD5** (there falsely claimed to be exactly the logics **S5**, $\mathbf{S5Alt}_n$, and $\mathbf{KD5Alt}_n$ for some $n \in Nat$) prove some formula A such that $\tau_{\Diamond\Box}(A)$ is not provable in **KD4.3**. Byrd then reasoned that **KD5** was exactly the logic of eventual permanence. In [2], the following counterexample to the "if" direction of (3) with $L = \mathbf{KD5}$ is given. Consider the following formula.

$$\Diamond\Box p \rightarrow \Diamond\Box\Diamond\Box p. \tag{4}$$

This formula can quite easily be shown to be provable in **KD4.3**. As this is a formula of the form $\tau_{\Diamond\Box}(A)$ the "if" direction of (3) would indicate that A—the corresponding "untranslated" formula—should be a theorem of **KD5**. In this case A happens to be the formula $\blacksquare p \rightarrow \blacksquare\blacksquare p$, which is quite clearly not a theorem of **KD5**. Additionally, it seems intuitive that the temporal notion of eventual permanence is one for which "transitivity" (as encoded in the **4** axiom) should hold. Hence we can see that the logic of eventual permanence must be at least as strong as **KD45**.

2 The Correct Logic of Eventual Permanence

Our aim in this section is to show that **KD45** can be faithfully embedded into **KD4.3** by the translation $\tau_{\Diamond\Box}$. To do this we will first show that this translation fulfills the "only if" condition of (3) (Theorem 2.1). Then we will show that **KD45** is the strongest logic which $\tau_{\Diamond\Box}$ embeds into **KD4.3** (Theorem 2.3) and consequently that **KD45** can be faithfully embedded into **KD4.3** and thus is the correct logic of "eventual permanence" (Theorem 2.4).

Theorem 2.1 *For all A, $\vdash_{\mathbf{KD45}} A$ only if $\vdash_{\mathbf{KD4.3}} \tau_{\Diamond\Box}(A)$.*

Proof By induction on the length of derivations of A. From [3], pp. 136–37, we know that the translations of axioms **K**, **D**, and **5** are all provable in **KD4.3** and that the rule of necessitation for $\Diamond\Box$ is admissible in **KD4.3**. The only case to check then is in the base case, namely, that of the **4** axiom. It is quite easy to verify that $\Diamond\Box p \rightarrow \Diamond\Box\Diamond\Box p$ is provable in **KD4.3** and the result follows. □

Recall now for the following theorem that, following [4], the formula $\mathbf{Alt}_n$ for $n \in Nat$ is as follows and is canonical for the condition on frames $\langle W, R\rangle$ that each point in W have no more than n R-successors (i.e., that $\forall x \in W : |R(x)| \leq n$).

$$\mathbf{Alt}_n : \quad \Box p_1 \vee \Box(p_1 \rightarrow p_2) \vee \cdots \vee \Box(p_1 \wedge \cdots \wedge p_n \rightarrow p_{n+1}).$$

Theorem 2.2 *Every modal logic $L \supseteq$ **KD45** is either one of **KD45**, **S5**, **Triv**, or one of $\mathbf{KD45Alt}_n$, $\mathbf{S5Alt}_n$ for some $n \in Nat$.*

Proof From [4], p. 127, we know that the above logics are all of the normal extensions of **KD45**. By the result listed at [4], p. 190, we also know that every extension of **KD45** is normal and, hence, will be one of the logics listed above. □

To show the faithfulness of the translation given in Theorem 2.1 we need to show that $\tau_{\Diamond\Box}$ does not embed (in the sense not requiring faithfulness) any of the proper extensions of **KD45** (as listed in Theorem 2.2) into **KD4.3**. The case of **S5** is the easiest to deal with, by noting that the translation of the **T** axiom, namely, $\Diamond\Box p \rightarrow p$, is not provable in **KD4.3**. This removes **S5** and **S5Alt**$_n$ from contention, leaving only **KD45Alt**$_n$. For this last case we recall the model used in [1], p. 592, and use the method hinted at there to show that $\tau_{\Diamond\Box}(\mathbf{Alt}_n)$ is not **KD4.3**-provable. Let $\mathcal{M}_n = \langle Nat, <, V_n \rangle$ be a model for **KD4.3**, where $w \notin V_n(p_i) \iff i \equiv w \bmod (n+1)$. Consider now the translation of **Alt**$_n$.

$$\Diamond\Box p_1 \vee \Diamond\Box(p_1 \rightarrow p_2) \vee \Diamond\Box((p_1 \wedge p_2) \rightarrow p_3) \vee \cdots \vee \Diamond\Box((p_1 \wedge \cdots \wedge p_n) \rightarrow p_{n+1}). \quad (5)$$

No disjunct of $\tau_{\Diamond\Box}(\mathbf{Alt}_n)$ (inset formula (5)) will be true at 0 in $\mathcal{M}_n$. To see this, consider the kth disjunct: $\Diamond\Box((p_1 \wedge \cdots \wedge p_{k-1}) \rightarrow p_k)$. For this to be true at 0 there must be some point y such that for all $z > y$, $\mathcal{M}_n \models_z (p_1 \wedge \cdots \wedge p_{k-1}) \rightarrow p_k$. Now consider those points $j_m = [k + (n+1) \cdot m]$, for $m \in Nat$ such that $j_m > y$. It is quite easy to see that p_k will be false at j_m—consider that this would mean that $k \equiv k + [(n+1) \cdot m](\bmod (n+1)))$, and as both 0 and $(n+1) \cdot m$ are integer multiples of the modulus it is trivial to note that this is the case. We can also see that $p_1, \ldots, p_{k-1}$ will be true at j_m. To see this, suppose that they weren't. That is, suppose that there exists some l such that $1 \leq l \leq k-1$ and $l \equiv j_m \bmod (n+1)$—making p_l false at j_m. By the transitivity and symmetry of $\equiv$ this would mean that $k \equiv l \bmod (n+1)$, which representing l as $k - c$ for some c ($1 \leq c \leq n$) means that $k \equiv k - c \bmod (n+1)$. As this clearly cannot be, we can conclude that $p_1, \ldots, p_{k-1}$ will be true at j_m and, thus, that $\mathcal{M}_n \not\models_{j_m} (p_1 \wedge \cdots \wedge p_{k-1})) \rightarrow p_k$. Consequently, there is no such point y, and the result follows.

Thus we can see that the translation $\tau_{\Diamond\Box}$ doesn't satisfy the "only if" direction of (2) for any of the extensions of **KD45**, giving us the following result.

Theorem 2.3 **KD45** *is the maximal logic L which satisfies the "only if" direction of* (3).

We are now in a position to show that the translation $\tau_{\Diamond\Box}$ faithfully embeds **KD45** into **KD4.3** and that consequently **KD45** is the correct logic of eventual permanence (for linear unending time).

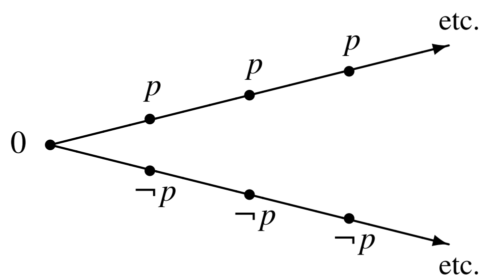
Theorem 2.4 *For all formulas A,* $\vdash_{\mathbf{KD45}} A$ *if and only if* $\vdash_{\mathbf{KD4.3}} \tau_{\Diamond\Box}(A)$.

Proof The "only if" direction of the proof has already been given as Theorem 2.1. All that remains then is to show that the "if" direction holds. Suppose, for a reductio, that it doesn't. This would mean that there is some formula A such that $\vdash_{\mathbf{KD4.3}} \tau_{\Diamond\Box}(A)$, while $\nvdash_{\mathbf{KD45}} A$. From this we can see that $\tau_{\Diamond\Box}$ will (not necessarily faithfully) embed the extension of **KD45** by A into **KD4.3**. But by Theorem 2.3 this will only occur when the result of extending **KD45** by A is simply **KD45**. Consequently, we know that $\vdash_{\mathbf{KD45}} A$, giving us a contradiction, and the result follows. □

3 Concluding Remarks

As we have shown above, the problem with Byrd's result was a minor one, the simple misquotation of the essential theorem upon which it was built, but this simple mistake caused Byrd's result to be incorrect. Thus, contrary to what Byrd claims, the logic of eventual permanence for unending linear time can be seen to be **KD45**, thus refuting the conjecture in [3] and followed up in [1] that it was **KD5**.

Byrd does not just consider the case where time is linear and unending, the majority of his paper being concerned with determining what the logic of eventual permanence is for unending branching time. Byrd proves that the logic of eventual permanence for branching time is the logic **KD4U**. In the light of the error in Byrd's result concerning linear unending time, one might then wonder whether there is a similar error of reasoning in his result concerning the logic of eventual permanence for unending branching time. In particular, one might wonder whether the logic of eventual permanence for unending branching time is again **KD45**. Indeed, the fact that **KD4U** is a proper subsystem of **KD45** might start alarm bells ringing that perhaps a mistake of the same nature has been made. The author was unable to find any error in Byrd's proof concerning unending branching time but can at the very least present the following evidence as to why it most certainly isn't going to be **KD45**.[4] It is quite clear that the **5** axiom (=$\blacklozenge p \rightarrow \blacksquare\blacklozenge p$) is false at point 0 in the following structure, the top branch being sufficient to secure the truth of $\blacklozenge p$, and the fact that there is no point on the lower branch such that p (and hence such that $\blacklozenge p$) is true securing the falsity of $\blacksquare\blacklozenge p$.



Byrd also considers the question as to what the logic of eventual permanence is when time is considered to be convergent toward the future, stating that this logic too is **KD5**. As it happens, Byrd's mistake runs deep, and the resulting logic here can also be shown to be **KD45**—not **KD5** as stated by Byrd. It is worth noting first that the logic of (future directed) convergent time is none other than **KD4.2** (**KD4G**), the logic we get by replacing the **.3** axiom in **KD4.3** with the following axiom, often called **G** for Geach.

$$\mathbf{G}: \quad \Diamond\Box p \rightarrow \Box\Diamond p.$$

To see that the logic of eventual permanence for future convergent time is **KD45** note that we can prove the translations of all of the axioms of **KD45** in **KD4.2**. The translation of **D** is the **G** axiom, the translation of **K** following from the normality of $\Diamond\Box$ in **KD4.2**, and the translations of **4** and **5** provable from **4** using **D** to weaken $\Box$ to $\Diamond$ as necessary. This allows us to prove the relevant analogue of Theorem 2.1 with **KD4.2** replacing **KD4.3**, and from this we are able to use the argument in Theorem 2.4 to conclude that the logic of eventual permanence for future convergent time is **KD45**.

Notes

1. Here we will notate the modal operators for the logic of eventual permanence as ■ and ♦ and those of our future-directed tense logic as □ and ◊ (as opposed to F and G).

2. As noted above we are considering the embedding of a monomodal logic into a future directed tense logic, allowing us to think of our logics as being on the same language. The following argument can be worked equally well if we consider the "target" logic of our translation as being on the standard (propositional) tense logical language with the sole primitive modal connectives being G and H.

3. To see that (3) secures that there will be a unique logic L simply consider the set of formulas A such that $\tau_{\Diamond\Box}(A)$ is provable in **KD4.3**.

4. It is worth reminding the reader here that Byrd uses different truth conditions for eventual permanence over branching time—the truth of $\blacksquare A$ amounting to A becoming permanently true on all temporal branches (roughly conceived as maximal sets of point B such that for $x, y \in B$ such that $x \neq y$ either $x > y$ or $y > x$). For more information see [1], p. 594.

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