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General Models and Entailment Semantics for Independence Logic

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Abstract We develop a semantics for independence logic with respect to what we will call *general models*. We then introduce a simpler *entailment semantics* for the same logic, and we reduce the validity problem in the former to the validity problem in the latter. Then we build a proof system for independence logic and prove its soundness and completeness with respect to entailment semantics.

1 Introduction

Logics of imperfect information are extensions of first-order logic (or, sometimes, of other logics; see, e.g., Tulenheimo [15] and Väänänen [18]) which allow us to reason about patterns of dependence and independence between variables.

Historically, the earliest such logic was *branching quantifier logic* (see Henkin [7]), which adds to the language of first-order logic branching quantifiers such as

$$\begin{pmatrix} \forall x & \exists y \\ \forall z & \exists w \end{pmatrix} \varphi(x, y, z, w),$$

whose interpretation, informally speaking, states that the choice of y is not dependent on the choice of z and the choice of w is not dependent on the choice of x. A significant breakthrough in the study of this class of logics occurred with the development of *independence-friendly logic* (see Hintikka and Sandu [8]).

1. The syntax of branching quantifier logic was significantly simplified, doing away with complex structures of quantifiers such as the above one and introducing instead *slashed quantifiers* $(\exists x / W)\varphi$, whose informal interpretation is "there exists an x, not dependent on any variables in W, such that φ".
2. The *game-theoretic semantics* of logics of imperfect information was defined formally, and its properties were examined in detail.

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These developments made it possible to define, in Hodges [9], a *compositional semantics* for independence-friendly logic which is equivalent to its game-theoretic semantics modulo the axiom of choice. This semantics, called *team semantics* or *trump semantics*, differs from Tarski's semantics for first-order logic in that the satisfaction relation is predicated not over single assignments but over *sets* of assignments[1] (which we will henceforth call *teams*, after the terminology of Väänänen [16]).

This alternative semantics provided one of the main impulses toward the development of *dependence logic* (see [16]), which separates the notion of dependence and independence from the notion of quantification by doing away with slashed quantifiers and introducing instead *dependence atoms* of the form $=(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms, which are satisfied by a team X if and only if the value of t_n is a function of the values of $t_1 \ldots t_{n-1}$ in it. This—only at first sight minor—innovation led to a number of significant advances in the study of the properties of logics of imperfect information and, in particular, of their model theory; apart from the aforementioned [16], we can refer here for example to the results of (Juha) Kontinen and Väänänen [11] and (Jarmo) Kontinen [10].

Furthermore, a recent direction of research in the field of logics of imperfect information consists in the study of the model-theoretical properties of variants of dependence logic obtained by substituting the dependence atoms with other kinds of non-first-order atomic formulas. The earliest work along these lines was done by Grädel and Väänänen [5], whose *independence logic* is equivalent to dependence logic on the level of sentences (but expressively stronger on the level of formulas and definability of classes of teams) and will be the main logical formalism examined in the rest of this work; furthermore, we have *multivalued dependence logic* from Engström [3] and *inclusion logic* and *exclusion logic* from Galliani [4].[2]

One property common to all these papers is that they are essentially concerned only with the *semantics* of logics of imperfect information and its model-theoretic properties. The corresponding proof theories, instead, are still relatively undeveloped. Nurmi's Ph.D. thesis [14] discusses a proof system for a fragment of dependence logic, but without proving its completeness, and the recent article by Kontinen and Väänänen [12] presents a sound and complete deduction system for extracting the first-order consequences of a dependence-logic theory; however, due to the equivalence between dependence logic and existential second-order logic there exists no hope of finding a sound and complete proof system for dependence logic or independence logic with respect to their standard semantics. The present paper, drawing inspiration from Henkin's treatment of second-order logic (see [6]) and from the analysis of branching quantifiers of López-Escobar [13], may be seen as a different approach to the study of the proof theories of logics of imperfect information: instead of restricting our language, we will weaken the semantics and consider a more general class of models, and then we will develop a proof system capable of extracting all valid formulas for this new semantics.

2 Independence Logic

In this section, we will briefly recall the syntax and the semantics of independence logic, plus a few of its basic properties. It can be safely skipped by anyone who is already familiar with the results of [5].

As is often done in the field of logics of imperfect information, we will assume that our expressions are always in negation normal form.

Definition 2.1 (Syntax) Let Σ be a first-order signature. Then the set NNF_Σ of the *negation normal form formulas* of our logic is the smallest set such that the following hold:

NNF-lit: If φ is a first-order literal over the signature Σ, then $\varphi \in \mathrm{NNF}_\Sigma$.
NNF-ind: If $\vec{t}_1$, $\vec{t}_2$, and $\vec{t}_3$ are tuples of terms with signature Σ, then $\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ is in NNF_Σ.
NNF-∨: If φ and ψ are in NNF_Σ, then $(\varphi \vee \psi)$ is also in NNF_Σ.[3]
NNF-∧: If φ and ψ are in NNF_Σ, then $(\varphi \wedge \psi)$ is also in NNF_Σ.
NNF-∃: If φ is in NNF_Σ and x is a variable, then $\exists x \varphi$ is in NNF_Σ.
NNF-∀: If φ is in NNF_Σ and x is a variable, then $\forall x \varphi$ is in NNF_Σ.

The set $\mathrm{Free}(\varphi)$ of the *free variables* of a formula φ is defined similarly to the case of first-order logic.

Definition 2.2 (Free variables) Let Σ be a first-order signature, and let $\varphi \in \mathrm{NNF}_\Sigma$. Then the set $\mathrm{Free}(\varphi)$ of the *free variables* of φ is defined by structural induction on φ as follows:

Free-lit: If φ is a first-order literal, then $\mathrm{Free}(\varphi)$ is the set of all variables occurring in φ.
Free-ind: If φ is $\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$, then $\mathrm{Free}(\varphi)$ is the set of all variables occurring in $\vec{t}_1$, $\vec{t}_2$, or $\vec{t}_3$.
Free-∨: If φ is $\psi \vee \theta$ for some formulas $\psi, \theta \in \mathrm{NNF}_\Sigma$, then $\mathrm{Free}(\varphi)$ is $\mathrm{Free}(\psi) \cup \mathrm{Free}(\theta)$.
Free-∧: If φ is $\psi \wedge \theta$ for some formulas $\psi, \theta \in \mathrm{NNF}_\Sigma$, then $\mathrm{Free}(\varphi)$ is $\mathrm{Free}(\psi) \cup \mathrm{Free}(\theta)$.
Free-∃: If φ is $\exists x \psi$ for some variable x and some $\psi \in \mathrm{NNF}_\Sigma$, then $\mathrm{Free}(\varphi) = \mathrm{Free}(\psi) \backslash \{x\}$.
Free-∀: If φ is $\forall x \psi$ for some variable x and some $\psi \in \mathrm{NNF}_\Sigma$, then $\mathrm{Free}(\varphi) = \mathrm{Free}(\psi) \backslash \{x\}$.

The following definition is standard.

Definition 2.3 (Team) Let V be a finite set of variables, and let M be a first-order model. A *team* over M with domain V is a set of first-order assignments over M with domain V.

The next definition will be useful to give the semantics for the "lax" (in the sense of [4]) version of the existential quantifier that we will use.

Definition 2.4 (x-variation) Let M be a first-order model, let X be a team over M, and let x be a variable symbol (not necessarily in $\mathrm{Dom}(X)$). Then a team X' of M with domain $\mathrm{Dom}(X') = \mathrm{Dom}(X) \cup \{x\}$ is said to be an x-*variation* of X, and we write $X[x]X'$ if and only if the restrictions of X and X' to $\mathrm{Dom}(X)\backslash\{x\}$ are the same.

At this point, we have all that we need in order to define the team semantics for independence logic.

Definition 2.5 (Team semantics for independence logic) Let Σ be a first-order signature, let M be a first-order model of signature Σ, let $\varphi \in \mathrm{NNF}_\Sigma$, and let X be a team with domain containing $\mathrm{Free}(\varphi)$. Then we say that X *satisfies* φ in M, and we write $M \models_X \varphi$, if and only if

TS-lit: φ is a first-order literal and, for all $s \in X$, $M \models_s \varphi$ in the usual first-order sense;

TS-ind: φ is $\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ for some tuples of terms $\vec{t}_1, \vec{t}_2$, and $\vec{t}_3$, and for all $s, s' \in X$ with $\vec{t}_1\langle s\rangle = \vec{t}_1\langle s'\rangle$ there exists an $s'' \in X$ with $\vec{t}_1\vec{t}_2\langle s''\rangle = \vec{t}_1\vec{t}_2\langle s\rangle$ and $\vec{t}_1\vec{t}_3\langle s''\rangle = \vec{t}_1\vec{t}_3\langle s'\rangle$;

TS-∨: φ is $\psi_1 \vee \psi_2$ for $\psi_1, \psi_2 \in \mathrm{NNF}_\Sigma$ and $X = Y \cup Z$ for some teams Y and Z such that $M \models_Y \psi_1$ and $M \models_Z \psi_2$;

TS-∧: φ is $\psi_1 \wedge \psi_2$ for $\psi_1, \psi_2 \in \mathrm{NNF}_\Sigma$, $M \models_X \psi_1$, and $M \models_X \psi_2$;

TS-∃: φ is $\exists x \psi$ for some variable x and some $\psi \in \mathrm{NNF}_\Sigma$ and there exists a team X' such that $X[x]X'$ (i.e., X' is an x-variation of X) and such that $M \models_{X'} \psi$;

TS-∀: φ is $\forall x \psi$ for some suitable x and $M \models_{X[M/x]} \psi$, where

$$X[M/x] = \{s[m/x] : s \in X, m \in \mathrm{Dom}(M)\}.$$

As [5] shows, the dependence atom $=(t_1 \dots t_n)$ is equivalent to the independence atom $t_n \perp_{t_1 \dots t_{n-1}} t_n$. Therefore, dependence logic is contained in independence logic. The following result is also in [5].

Theorem 2.6 (see [5]) *Let Σ be a first-order signature, let $V = \{v_1, \dots, v_n\}$ be a finite set of variables, let $\vec{v} = v_1 \dots v_n$, and let $\varphi(\vec{v}) \in \mathrm{NNF}_\Sigma$ be an independence logic formula with signature Σ and free variables in V. Then there exists an existential second-order logic formula $\Phi(R)$ such that, for all models M with signature Σ and all teams X over M with domain V,*

$$M \models_X \varphi \Leftrightarrow M \models \Phi(\mathrm{Rel}(X)),$$

where $\mathrm{Rel}(X) = \{s(\vec{v}) : s \in X\}$.

In [4], the converse of this result is proved.

Theorem 2.7 (see [4]) *Let Σ be a first-order signature, let $V = \{v_1, \dots, v_n\}$ be a finite set of variables, and let $\Phi(R)$ be an existential second-order formula with signature Σ and with R as its only free variable, where R is a relational variable of arity n. Then there exists an independence logic formula $\varphi(\vec{v})$, over the signature Σ and with free variables in V, such that*

$$M \models_X \varphi \Leftrightarrow M \models \Phi(\mathrm{Rel}(X))$$

for all models M with signature Σ and all nonempty teams X over M with domain V.

3 General Models for Independence Logic

In this section, we will develop a generalization of team semantics along the lines of Henkin's treatment of second-order logic. As we will see, the fact that independence logic corresponds to existential second-order logic (and not to full second-order logic) means that we will be able to restrict ourselves to considering only a very specific kind of general model.

Definition 3.1 (General model) Let Σ be a first-order signature. A *general model* with signature Σ is a pair $(M, \mathcal{G})$, where M is a first-order model with signature Σ and $\mathcal{G}$ is a set of teams over finite—but not necessarily identical, nor of the same size—domains, respecting the following condition.

- If $n \in \mathbb{N}$ and $\varphi(x_1 \dots x_n, \vec{m}, \vec{R})$ is a first-order formula, where $\vec{m}$ is a tuple of constant parameters in $\mathrm{Dom}(M)$ and where $\vec{R}$ is a tuple of "relation parameters" corresponding to teams in $\mathcal{G}$, in the sense that each R_i is of the form
$$R_i = \mathrm{Rel}(X_i) = \{s(\vec{z}) : s \in X_i\}$$
for some $X_i \in \mathcal{G}$, then for
$$\|\varphi(x_1 \dots x_n, \vec{m}, \vec{R})\|_M = \{s : \mathrm{Dom}(s) = \{x_1, \dots, x_n\}, M \models_s \varphi(x_1 \dots x_n, \vec{m}, \vec{R})\}$$
it holds that $\|\varphi(x_1 \dots x_n, \vec{m}, \vec{R})\|_M \in \mathcal{G}$.

Lemma 3.2 *Let Σ be a first-order signature, and let $(M, \mathcal{G})$ be a general model with signature Σ. Then for all $X \in \mathcal{G}$ and all variables y, $X[M/y] \in \mathcal{G}$.*

Proof Let $\mathrm{Dom}(X) = \{x_1, \dots, x_n\}$, let $R = \mathrm{Rel}(X)$, and consider the formula $\varphi(x_1 \dots x_n, y) = \exists y R(x_1 \dots x_n)$. Then take any assignment s with domain $\{x_1, \dots, x_n, y\}$: by construction, $M \models_s \varphi(x_1 \dots x_n, y)$ if and only if there exists a $m \in \mathrm{Dom}(M)$ such that $s[m/y]_{|\vec{x}} \in X$, or, in other words, if and only if $s \in X[M/y]$.[4] □

We can easily adapt the team semantics of the previous section to general models. We report all the rules here, for ease of reference; but the only differences between this semantics and the previous one are in the cases **GMS-**$\vee$ and **GMS-**$\exists$.

Definition 3.3 (General model semantics for independence logic) Let Σ be a first-order signature, let $(M, \mathcal{G})$ be a general model of signature Σ, let $\varphi \in \mathrm{NNF}_\Sigma$ be a formula of independence logic, and let $X \in \mathcal{G}$ be a team with domain containing $\mathrm{Free}(\varphi)$. Then we say that X *satisfies* φ in $(M, \mathcal{G})$, and we write $(M, \mathcal{G}) \models_X \varphi$ if and only if

GMS-lit: φ is a first-order literal and, for all $s \in X$, $M \models_s \varphi$ in the usual first-order sense;

GMS-ind: φ is $\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ for some tuples of terms $\vec{t}_1$, $\vec{t}_2$, and $\vec{t}_3$, and for all $s, s' \in X$ with $\vec{t}_1\langle s\rangle = \vec{t}_1\langle s'\rangle$ there exists an $s'' \in X$ with $\vec{t}_1\vec{t}_2\langle s''\rangle = \vec{t}_1\vec{t}_2\langle s\rangle$ and $\vec{t}_1\vec{t}_3\langle s''\rangle = \vec{t}_1\vec{t}_3\langle s'\rangle$;

GMS-$\vee$: φ is $\psi_1 \vee \psi_2$ for $\psi_1, \psi_2 \in \mathrm{NNF}_\Sigma$ and $X = Y \cup Z$ for some teams $Y, Z \in \mathcal{G}$ such that $(M, \mathcal{G}) \models_Y \psi_1$ and $(M, \mathcal{G}) \models_Z \psi_2$;

GMS-$\wedge$: φ is $\psi_1 \wedge \psi_2$ for $\psi_1, \psi_2 \in \mathrm{NNF}_\Sigma$, $(M, \mathcal{G}) \models_X \psi_1$ and $(M, \mathcal{G}) \models_X \psi_2$;

GMS-$\exists$: φ is $\exists x \psi$ for some variable x and some $\psi \in \mathrm{NNF}_\Sigma$ and there exists a team $X' \in \mathcal{G}$ such that $X[x]X'$ and such that $(M, \mathcal{G}) \models_{X'} \psi$;

GMS-$\forall$: φ is $\forall x \psi$ for some suitable x and $(M, \mathcal{G}) \models_{X[M/x]} \psi$.

The usual semantics for independence logic satisfies a *locality principle*: in brief, the satisfiability of a formula φ in a team depends only on the restriction of the team to $\mathrm{Free}(\varphi)$. Let us verify that the same holds for general model semantics.

Lemma 3.4 *Let $(M, \mathcal{G})$ be a general model, and let $X \in \mathcal{G}$ be such that* $\mathrm{Dom}(X) = \vec{x}\vec{y}$. *Then $X_{|\vec{x}} = \{s : \mathrm{Dom}(s) = \vec{x}, \exists \vec{m}$ such that $s[\vec{m}/\vec{y}] \in X\}$ is in $\mathcal{G}$.*

Furthermore, let $Y \subseteq X_{|\mathscr{G}}$ be such that $Y \in \mathscr{G}$. Then the team

$$X(\vec{x} \in Y) = \{s \in X : s_{|\vec{x}} \in Y\}$$

is in $\mathscr{G}$.

Proof By definition, $X_{|\vec{x}}$ is $\|\varphi(\vec{x}, R)\|_M$, where φ is $\exists\vec{y}(R\vec{x}\vec{y})$ and $R = \mathrm{Rel}(X)$. Therefore, $X_{|\vec{x}} \in \mathscr{G}$.

Similarly, $X(\vec{x} \in Y)$ is $\|\varphi(\vec{x}\vec{y}, R_1, R_2)\|_M$, where φ is $R_1\vec{x}\vec{y} \wedge R_2\vec{x}$, R_1 is $\mathrm{Rel}(X)$, and R_2 is $\mathrm{Rel}(Y)$. □

Theorem 3.5 (Locality) *Let $(M, \mathscr{G})$ be a general model, let $X \in \mathscr{G}$, and let φ be an independence logic formula over the signature of M with* $\mathrm{Free}(\varphi) \subseteq \vec{z} \subseteq \mathrm{Dom}(X)$. *Then $(M, \mathscr{G}) \models_X \varphi$ if and only if $(M, \mathscr{G}) \models_{X_{|\vec{z}}} \varphi$.*

Proof The proof is by structural induction on φ. We present only the passages corresponding to disjunction and existential quantification, as the others are trivial.

- Suppose that $(M, \mathscr{G}) \models_X \psi_1 \vee \psi_2$. Then, by definition, there exist teams Y and Z in $\mathscr{G}$ such that $X = Y \cup Z$, $(M, \mathscr{G}) \models_Y \psi_1$, and $M \models_Z \psi_2$. By induction hypothesis, this means that $(M, \mathscr{G}) \models_{Y_{|\vec{z}}} \psi_1$ and $(M, \mathscr{G}) \models_{Z_{|\vec{z}}} \psi_2$. But $Y_{|\vec{z}} \cup Z_{|\vec{z}} = X_{|\vec{z}}$, and hence $(M, \mathscr{G}) \models_{X_{|\vec{z}}} \psi_1 \vee \psi_2$.

  Conversely, suppose that $(M, \mathscr{G}) \models_{X_{|\vec{z}}} \psi_1 \vee \psi_2$. Then there exist teams Y', Z' in $\mathscr{G}$ such that $(M, \mathscr{G}) \models_{Y'} \psi_1$, $(M, \mathscr{G}) \models_{Z'} \psi_2$, and $X_{|\vec{z}} = X' \cup Y'$. Now let Y be $X(\vec{z} \in Y')$, and let Z be $X(\vec{z} \in Z')$; by construction, $Y \cup Z = X$, and furthermore $Y' = Y_{|\vec{z}}$ and $Z' = Z_{|\vec{z}}$, and, by Lemma 3.4, Y and Z are in $\mathscr{G}$. Thus, by induction hypothesis, $(M, \mathscr{G}) \models_Y \psi_1$ and $(M, \mathscr{G}) \models_Z \psi_2$, and finally $(M, \mathscr{G}) \models_X \psi_1 \vee \psi_2$, as required.
- Suppose that $(M, \mathscr{G}) \models_X \exists x\psi$. Then there exists a team $Y \in \mathscr{G}$ such that $X[x]Y$, and $(M, \mathscr{G}) \models_Y \psi$. By induction hypothesis, this means that $(M, \mathscr{G}) \models_{Y_{|\vec{z}x}} \psi$ too; and since $X_{|\vec{z}}[x]Y_{|\vec{z}x}$, this implies that $M \models_{X_{|\vec{z}}} \exists x\psi$, as required.

  Conversely, suppose that $(M, \mathscr{G}) \models_{X_{|\vec{z}}} \exists x\psi$. Then there exists a team Y', with domain $\vec{z}x$, such that $M \models_{Y'} \psi$, and $X_{|\vec{z}}[x]Y'$. Now let Y be $(X[M/x])(\vec{z}x \in Y')$. By Lemma 3.4, $Y \in \mathscr{G}$; furthermore, $Y_{|\vec{z}x} = Y'$, and hence by induction hypothesis, $(M, \mathscr{G}) \models_Y \psi$. Finally, $X[x]Y$; indeed, if $s \in X$, then $s_{|\vec{z}}[m/x] \in Y'$ for some $m \in \mathrm{Dom}(M)$, and hence $s[m/x] \in Y$ for the same m, and on the other hand, Y is contained in $X[M/x]$, and hence if $s[m/x] \in Y$, it follows that $s \in X$.

  Therefore $(M, \mathscr{G}) \models_X \exists x\psi$, as required. □

As in the case of second-order logic, first-order models can be represented as a special kind of general model.

Definition 3.6 (Full models) Let $(M, \mathscr{G})$ be a general model. Then it is said to be *full* if and only if $\mathscr{G}$ contains all teams over M.

The following result is then trivial.

Proposition 3.7 *Let $(M, \mathscr{G})$ be a full model. Then for all suitable teams X and formulas φ, $(M, \mathscr{G}) \models_X \varphi$ in general model semantics if and only if $M \models_X \varphi$ in the usual team semantics.*

Proof This follows at once by comparing the rules of team semantics and general model semantics for the case when $\mathscr{G}$ contains all teams. □

How does the satisfaction relation in general model semantics change if we vary the set $\mathscr{G}$? The following definition and result give us some information about this.

Definition 3.8 (Refinement) Let $(M, \mathscr{G})$ and $(M, \mathscr{G}')$ be two general models. Then we say that $(M, \mathscr{G}')$ is a *refinement* of $(M, \mathscr{G})$, and we write $(M, \mathscr{G}) \subseteq (M, \mathscr{G}')$, if and only if $\mathscr{G} \subseteq \mathscr{G}'$.

Intuitively speaking, a refinement of a general model is another general model, over the same first-order structure, with more teams. The following result shows that refinements preserve satisfaction relations.

Theorem 3.9 *Let $(M, \mathscr{G})$ and $(M, \mathscr{G}')$ be two general models with $(M, \mathscr{G}) \subseteq (M, \mathscr{G}')$, let $X \in \mathscr{G}$, and let φ be a formula over the signature of M with* $\mathrm{Free}(\varphi) \subseteq \mathrm{Dom}(X)$. *Then*

$$(M, \mathscr{G}) \models_X \varphi \Rightarrow (M, \mathscr{G}') \models_X \varphi.$$

Proof The proof is an easy induction on φ.

1. If φ is a first-order literal, the result is obvious, as the choice of the set of teams $\mathscr{G}$ (or $\mathscr{G}'$) does not enter into the definition of satisfaction condition **GMS-lit**.
2. If φ is an independence atom, the result is also obvious, for the same reason.
3. If $(M, \mathscr{G}) \models_X \psi_1 \vee \psi_2$, then there exist two teams $Y, Z \in \mathscr{G}$ such that $X = Y \cup Z$, $(M, \mathscr{G}) \models_Y \psi_1$, and $(M, \mathscr{G}) \models_Z \psi_2$. But Y and Z are also in $\mathscr{G}'$, and by induction hypothesis we have that $(M, \mathscr{G}') \models_Y \psi_1$ and $(M, \mathscr{G}') \models_Z \psi_2$, and therefore $(M, \mathscr{G}') \models_X \psi_1 \vee \psi_2$.
4. If $(M, \mathscr{G}) \models_X \psi_1 \wedge \psi_2$, then $(M, \mathscr{G}) \models_X \psi_1$ and $(M, \mathscr{G}) \models_X \psi_2$. Then, by induction hypothesis, $(M, \mathscr{G}') \models_X \psi_1$ and $(M, \mathscr{G}') \models_X \psi_2$, and finally $(M, \mathscr{G}') \models_X \psi_1 \wedge \psi_2$.
5. If $(M, \mathscr{G}) \models_X \exists x \psi$, then there exists an $X' \in \mathscr{G}$ such that $X[x]X'$ and $(M, \mathscr{G}) \models_{X'} \psi$. But then X' is also in $\mathscr{G}'$, and by induction hypothesis $(M, \mathscr{G}') \models_{X'} \psi$, and finally $(M, \mathscr{G}') \models_X \exists x \psi$.
6. If $(M, \mathscr{G}) \models_X \forall x \psi$, then $(M, \mathscr{G}) \models_{X[M/x]} \psi$. Then, by induction hypothesis, $(M, \mathscr{G}') \models_{X[M/x]} \psi$, and finally $(M, \mathscr{G}') \models_X \forall x \psi$. □

This result shows us that, as was to be expected from the equivalence between independence logic and existential second-order logic, if we are interested in formulas which hold in *all* general models over a certain first-order model, we only need to pay attention to the *smallest* (in the sense of the refinement relation) ones. But do such "least general models" exist? As the following result shows, this is indeed the case.

Proposition 3.10 *Let $\{(M, \mathscr{G}_i) : i \in I\}$ be a family of general models with signature Σ and over the same first-order model M. Then $\left(M, \bigcap_{i \in I} \mathscr{G}_i\right)$ is also a general model.*

Proof Let $\varphi(x_1 \ldots x_n, \vec{m}, \vec{R})$ be a first-order formula with parameters, where each R_i is of the form $\mathrm{Rel}(X)$ for some $X \in \bigcap_i \mathscr{G}_i$. Then the team $\|\varphi(x_1 \ldots x_n, \vec{m}, \vec{R})\|_M$ is in $\mathscr{G}_i$ for all $i \in I$, and therefore it is in $\bigcap_{i \in I} \mathscr{G}$, as required. □

Therefore, it is indeed possible to talk about the *least general model* over a first-order model.

Definition 3.11 (Least general model) Let M be a first-order model. Then the *least general model* over M is the $(M, \mathcal{L})$, where

$$\mathcal{L} = \bigcap \left\{ \mathcal{G} : (M, \mathcal{G}) \text{ is a general model} \right\}.$$

As an example of a least general model, let $n \in \mathbb{N}$, and let M_n be a model with empty signature and domain $\{1, \ldots, n\}$. Then the least general model over M_n is actually the full general model $(M_n, \mathcal{G}_n)$, where $\mathcal{G}_n$ contains all teams over M_n. Indeed, let $\{v_1, \ldots, v_k\}$ be a finite set of variables, and let

$$X = \{s_1, \ldots, s_q\} = \begin{array}{c|ccc} & v_1 & \ldots & v_k \\ \hline s_1 & a_{11} & \ldots & a_{1k} \\ \ldots & \ldots & & \ldots \ldots \\ s_q & a_{q1} & \ldots & a_{qk} \end{array}$$

be any team over M_n with domain $\{v_1, \ldots, v_k\}$, where $s_i(v_j) = a_{ij}$ for all $i \in 1, \ldots, q$ and all $j \in 1, \ldots, k$. Then clearly $q \leq n^k$, and furthermore, for $\varphi(v_1 \ldots v_k) = \bigvee_{i=1}^{q} \bigwedge_{j=1}^{k} v_i = a_{qi}$, we have

$$\|\varphi(v_1 \ldots v_k, a_{11} \ldots a_{qk})\|_M = \left\{ s : \mathrm{Dom}(s) = \{v_1 \ldots v_k\}, M \models_s \varphi \right\} = X,$$

as required.

As this example shows, if M is finite, then the least (and only) general model over it is the full one. Hence, if we are only interested in finite models, general model semantics is equivalent to the standard team semantics, and the same can be said about the entailment semantics which we will develop later in this paper.

What is the purpose of least general models? The answer comes as a consequence of Theorem 3.9 and can be summarized by the following corollary.

Corollary 3.12 *Let Σ be a first-order signature, let M be a first-order model over it, and let $(M, \mathcal{L})$ be the least general model over it. Then, for all teams $X \in \mathcal{L}$ and all formulas φ with signature Σ and with free variables in* $\mathrm{Dom}(X)$*,*

$$(M, \mathcal{L}) \models_X \varphi \Leftrightarrow (M, \mathcal{G}) \models_X \varphi \quad \textit{for all general models } (M, \mathcal{G}) \textit{ over } M.$$

Proof Suppose that $(M, \mathcal{L}) \models_X \varphi$. Then take any general model $(M, \mathcal{G})$: by definition we have $(M, \mathcal{L}) \subseteq (M, \mathcal{G})$, and hence by Theorem 3.9 we have $(M, \mathcal{G}) \models_X \varphi$.

Conversely, suppose that $(M, \mathcal{G}) \models_X \varphi$ for all general models $(M, \mathcal{G})$; then in particular, $(M, \mathcal{L}) \models_X \varphi$, as required. □

We can also find a more practical characterization of this “least general model.”

Proposition 3.13 *Let M be a first-order model. Then the least general model over it is $(M, \mathcal{L})$, where $\mathcal{L}$ is the set of all $\|\varphi(\vec{x}, \vec{m})\|_M$, where φ ranges over all first-order formulas and $\vec{m}$ ranges over all tuples of variables of suitable length.*

Proof Let $(M, \mathcal{G})$ be a general model, and let $\varphi(\vec{x}, \vec{m})$ be a first-order formula with parameters. Then, by the definition of a general model, we have that $\|\varphi(\vec{x}, \vec{m})\|_M$ is in $\mathcal{G}$; and since this is the case for all φ, all $\vec{x}$, and all $\vec{m}$, it follows at once that $\mathcal{L} \subseteq \mathcal{G}$. Therefore, we only need to prove that $(M, \mathcal{L})$ is a general model.

Now, let $\varphi(\vec{x}, \vec{m}, \vec{R})$ be a first-order formula, and let each R_i be $\mathrm{Rel}(X_i)$ for some $X_i \in \mathcal{L}$. So for each R_i, any assignment s, and any suitable tuple of terms

t, $M \models_s R_i\vec{t}$ if and only if $M \models_s \psi_i(\vec{t}, \vec{n}_i)$ for some first-order formula ψ_i with parameters $\vec{n}_i$. Now let $\varphi'(\vec{x}, \vec{m}, \vec{n}_1, \vec{n}_2, \ldots)$ be the expression obtained by substituting, in φ, each instance of $R_i\vec{t}$ with $\psi_i(\vec{t}, \vec{n}_i)$; by construction, we have $M \models_s \varphi(\vec{x}, \vec{m}, \vec{R})$ if and only if $M \models_s \varphi'(\vec{x}, \vec{m}, \vec{n}_1, \ldots)$, and therefore

$$\|\varphi(\vec{x}, \vec{m}, \vec{R})\|_M = \|\varphi'(\vec{x}, \vec{m}, \vec{n}_1, \vec{n}_2, \ldots)\|_M \in \mathcal{L},$$

as required. □

Definition 3.14 (Validity with respect to general models) Let Σ be a first-order signature, let V be a finite set of variables, and let $\varphi \in \mathrm{NNF}_\Sigma$ be a formula of our language with free variables in V. Then φ is *valid* with respect to general models if and only if $(M, \mathcal{G}) \models_X \varphi$ for all general models $(M, \mathcal{G})$ with signature Σ and for all teams $X \in \mathcal{G}$ with $\mathrm{Dom}(X) \supseteq \mathrm{Free}(\varphi)$. If this is the case, we write GMS $\models \varphi$.

Definition 3.15 (Validity with respect to least general models) Let Σ be a first-order signature, let V be a finite set of variables, and let $\varphi \in \mathrm{NNF}_\Sigma$ be a formula of our language with free variables in V. Then φ is *valid* with respect to least general models if and only if $(M, \mathcal{L}) \models_X \varphi$ for all least general models $(M, \mathcal{L})$ with signature Σ and for all teams $X \in \mathcal{L}$ with $\mathrm{Dom}(X) \supseteq \mathrm{Free}(\varphi)$. If this is the case, we write LMS $\models \varphi$.

Lemma 3.16 *Let M be a first-order model with signature Σ, and let M' be another first-order model with signature $\Sigma' \supseteq \Sigma$ such that the restriction of M' to Σ is precisely M. Then for all general models $\mathcal{G}$ for M', for all formulas φ with signature Σ, and for all $X \in \mathcal{G}$,*

$$(M, \mathcal{G}) \models_X \varphi \Leftrightarrow (M', \mathcal{G}) \models_X \varphi.$$

Proof First of all, if $(M', \mathcal{G})$ is a general model, then $(M, \mathcal{G})$ is also a general model. Then, the result is proved by observing that the truth conditions of our semantics depend only on the interpretations of the symbols in the signature of the formula (and on the choice of $\mathcal{G}$, of course). □

Lemma 3.17 *Let $(M, \mathcal{G})$ be a general model with signature Σ, let $S \notin \Sigma$ be a new relation symbol, and let $X \in \mathcal{G}$. Furthermore, let $M' = M[\mathrm{Rel}(X)/S]$ be the extension of M to the signature $\Sigma \cup \{S\}$ such that $S^{M'} = \mathrm{Rel}(X)$. Then $(M', \mathcal{G})$ is a general model.*

Proof Let $\varphi(\vec{x}, \vec{m}, \vec{R})$ be a first-order formula with signature $\Sigma \cup \{S\}$ and parameters $\vec{m}$ and $\vec{R}$, where each R_i is $\mathrm{Rel}(X_i)$ for some $X_i \in \mathcal{G}$. Then let $\varphi'(\vec{x}, \vec{m}, \vec{R}, S)$ be the first-order formula with signature Σ, where S now stands for the relation $\mathrm{Rel}(X)$. Now clearly

$$\|\varphi(\vec{x}, \vec{m}, \vec{R})\|_{M'} = \|\varphi'(\vec{x}, \vec{m}, \vec{R}, S)\|_M \in \mathcal{G},$$

as required. □

Theorem 3.18 *A formula φ is valid with respect to general models if and only if it is valid with respect to least general models.*

Proof The left-to-right direction is obvious. For the right-to-left direction, suppose that LMS $\models \varphi$, let $(M, \mathcal{G})$ be a general model whose signature contains the signature of φ, and let $X \in \mathcal{G}$ be a team whose domain $\{x_1, \ldots, x_n\}$ contains all free variables of φ. Then consider the first-order model $M' = M[\mathrm{Rel}(X)/S]$, where S is a new relation symbol, and take the least general model $(M', \mathcal{L})$ over it. We clearly have that $X \in \mathcal{L}$, since

$$X = \{s : \mathrm{Dom}(s) = \{x_1, \dots, x_n\}, M' \models_s Sx_1 \dots x_n\}$$

and, therefore, $(M', \mathcal{L}) \models_X \varphi$ by hypothesis. Now, by Lemma 3.17, $(M', \mathcal{G})$ is a general model, and therefore by definition $\mathcal{L} \subseteq \mathcal{G}$, and hence by Theorem 3.9 $(M', \mathcal{G}) \models_X \varphi$ too. Finally, the relation symbol S does not occur in φ, and therefore by Lemma 3.16, $(M, \mathcal{G}) \models_X \varphi$, as required. □

What this result tells us is that in order to check whether a formula is valid in all general models, it suffices to check *least* general models. This is a direct consequence of the correspondence between independence logic and Σ_1^1: since in order to verify whether a formula holds in a general model we have to verify the existence of teams satisfying certain first-order properties, it follows at once that in order to verify whether a formula holds in *all* general models, we can limit ourselves to examining the *smallest* such models.

The same argument can be used for other logics of imperfect information, such as, for example, IF logic or dependence logic; and our proof system will be easily adaptable to such logics, either directly or by first translating formulas from these logics into independence logic. However, the same cannot be said about logics of imperfect information which involve *universal* quantifications over teams, such as, for example, team logic (see Väänänen [17]) or intuitionistic dependence logic (see Abramsky and Väänänen [1], Yang [19]); while an approach based on general models seems to be worth pursuing also for such formalisms, the equivalence between validity in general models and validity in least general models—or validity with respect to the *entailment semantics* which we will develop in the next section—will no longer be available.

4 Entailment Semantics

Let M be a first-order structure, and let $(M, \mathcal{L})$ be the least general model over it. Then, as we saw, $\mathcal{L}$ is the set of all teams corresponding to first-order formulas with parameters. Therefore, in order to reason about satisfaction in a least general model, there is no need to carry around the teams themselves; rather, we can use the corresponding first-order formulas. In this section, we will develop this idea, building up a new "entailment semantics" and proving its correspondence with general model semantics over least general models.

We will then construct a proof system and prove its soundness and completeness with respect to this semantics. Then, since—as we saw already—validity with respect to least general models is equivalent to validity with respect to general models, the proof system will also be shown to be sound and complete with respect to general model semantics.

For the purposes of this work, entailment semantics acts as a bridge between general model semantics and our proof system: by allowing us to abstract away from higher-order objects such as teams, it will make it significantly easier for us to establish a connection between semantics and proof theory.

Furthermore, the semantics which we will build, with its more syntactic flavor, is of independent interest. The phenomena of dependence and independence whose study is among the principal reasons for being of dependence logic and independence logic are present in it, but the intrinsically higher-order nature of the usual team semantics is not. Entailment semantics, in other words, can be seen as an attempt to examine the content of the notions of dependence and independence from a first-order perspective, rather than from the higher-order perspective implicit in the formulation of team semantics.

Definition 4.1 (Parameter and team variables) Let $\mathbf{V}_P = \{p_1, \ldots, p_n, \ldots\}$ and $\mathbf{V}_T = \{x, y, z, \ldots\}$ be fixed, disjoint, countably infinite sets of variables. We will call any $p \in V_P$ a *parameter variable*, and we will call any $x \in V_T$ a *team variable*. Furthermore, we will assume that any variable which occurs in any of our formulas is a team variable or a parameter variable.

Definition 4.2 (Free parameter and team variables) Let φ be any formula. Then $\mathrm{Free}_P(\varphi) = \mathrm{Free}(\varphi) \cap \mathbf{V}_P$ and $\mathrm{Free}_T(\varphi) = \mathrm{Free}(\varphi) \cap \mathbf{V}_T$.

Parameter variables clarify the interpretation of expressions such as $M \models_s \gamma(\vec{x}, \vec{m})$: this is simply a shorthand $M \models_{h \cup s} \gamma(\vec{x}, \vec{p})$, where h is a *parameter assignment* with domain $\vec{p}$ and with $h(\vec{p}) = \vec{m}$. Team variables, instead, are going to be used in order to describe the variables in the domain of the team corresponding to a given first-order expression: for any first-order $\gamma(\vec{x}, \vec{p})$, where $\vec{x}$ are team variables and $\vec{p}$ are parameter variables, and for any h with domain $\vec{p}$, we will therefore have $\|\gamma(\vec{x}, \vec{p})\|_{M,h} = \|\gamma(\vec{x}, h(\vec{p}))\|_M = \{s : \mathrm{Dom}(s) = \vec{x}, M \models_{h \cup s} \gamma\}$. For this reason, parameter variables will never occur in the domain of a team, and, hence, from this point on we will always assume that parameter variables never occur in independence logic formulas but only in the first-order team definitions.

After these preliminaries, we can now give our main definition for this section.

Definition 4.3 (Entailment semantics for independence logic) Let M be a first-order model with signature Σ, let $\gamma(\vec{x}, \vec{p})$ be a first-order formula for the same signature with $\mathrm{Free}_T = \vec{x}$ and $\mathrm{Free}_P = \vec{p}$, let h be a parameter assignment with domain $\vec{p}$, and let $\varphi \in \mathrm{NNF}_\Sigma$ be an independence logic formula.

Then we say that γ *satisfies* φ in M *under* h, and we write $M \models_{\gamma(h)} \varphi$, if and only if

- **ES-lit:** φ is a first-order literal, and for all assignments s with domain $\mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\varphi)$ such that $M \models_{h \cup s} \gamma$, it holds that $M \models_s \varphi$;
- **ES-ind:** φ is $\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ for some tuples of terms $\vec{t}_1$, $\vec{t}_2$, and $\vec{t}_3$ and for all assignments s and s' with domain $\mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\vec{t}_1\vec{t}_2\vec{t}_3)$ such that $M \models_{h \cup s} \gamma$, $M \models_{h \cup s'} \gamma$, and $\vec{t}_1\langle s \rangle = \vec{t}_1\langle s' \rangle$ there exists an s'' such that $M \models_{h \cup s''} \gamma$, $\vec{t}_1\vec{t}_2\langle s'' \rangle = \vec{t}_1\vec{t}_2\langle s \rangle$, and $\vec{t}_1\vec{t}_3\langle s'' \rangle = \vec{t}_1\vec{t}_3\langle s' \rangle$;
- **ES-∨:** φ is $\psi_1 \vee \psi_2$ and there exists a parameter assignment h' extending[5] h and two first-order formulas γ_1 and γ_2 such that
  - $\mathrm{Free}_P(\gamma_1), \mathrm{Free}_P(\gamma_2) \subseteq \mathrm{Dom}(h')$;
  - $M \models_{\gamma_1(h')} \psi_1$;
  - $M \models_{\gamma_2(h')} \psi_2$;
  - $M \models_{h'} \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \vee \gamma_2)$, where $\vec{v}$ is $\mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\gamma_1) \cup \mathrm{Free}_T(\gamma_2)$;
- **ES-∧:** φ is $\psi_1 \wedge \psi_2$, $M \models_{\gamma(h)} \psi_1$ and $M \models_{\gamma(h)} \psi_2$;
- **ES-∃:** φ is $\exists x_n \psi$ and there exist a parameter assignment h' extending h and a first-order formula γ' with $\mathrm{Free}_P(\gamma') \subseteq \mathrm{Dom}(h')$ such that

- $M \models_{\gamma'(h')} \psi$;
- $M \models_{h'} \forall \vec{v}(\exists x_n \gamma' \leftrightarrow \exists x_n \gamma)$, where $\vec{v}$ is $\mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\gamma')$;

ES-∀: φ is $\forall x_n \psi$ and there exists a parameter assignment h' extending h and a first-order formula γ' with $\mathrm{Free}_P(\gamma') \subseteq \mathrm{Dom}(h')$ such that

- $M \models_{\gamma'(h')} \psi$;
- $M \models_{h'} \forall \vec{v}(\gamma' \leftrightarrow \exists x_n \gamma)$, where $\vec{v}$ is $\mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\gamma')$.

The reason why the above semantics is called "entailment semantics" is because its satisfaction relation describes a sort of *entailment relation* between a first-order formula with parameters, which takes the role that teams have in the usual team semantics, and an independence logic formula. In particular, it is easy to see that according to our rule **ES-lit**, for all first-order literals $\varphi(\vec{x}, \vec{p})$, first-order formulas with parameters $\gamma(\vec{x}, \vec{y})$, and parameter assignments h, $M \models_{\gamma(h)} \varphi$ if and only if $M \models_h \forall \vec{x}\vec{y}(\gamma(\vec{x}) \rightarrow \varphi(\vec{x}, \vec{y}))$.

Proposition 4.4 *Let M be a first-order model with signature Σ, let $\gamma(\vec{x}, \vec{p})$ be a first-order formula with $\mathrm{Free}_P(\gamma) = \vec{p}$, and let h, h' be two parameter assignments with domains containing $\vec{p}$ such that $h(\vec{p}) = h'(\vec{p})$. Then, for all independence logic formulas φ,*

$$M \models_{\gamma(h)} \varphi \Leftrightarrow M \models_{\gamma(h')} \varphi.$$

Proof The proof is a straightforward induction over φ. □

As the next result shows, entailment semantics is entirely equivalent to least general model semantics.

Theorem 4.5 *Let Σ be a first-order model, let $\gamma(\vec{x}, \vec{p})$ be a first-order formula with $\mathrm{Free}_P(\gamma) = \vec{p}$, let h be a parameter assignment with domain $\vec{p}$, and let $\varphi \in \mathrm{NNF}_\Sigma$ be an independence logic formula with free variables in $\vec{x}$.*

Furthermore, let $(M, \mathcal{L})$ be the least general model over M, and let $X = \|\gamma(\vec{x}, \vec{p})\|_{M,h} = \{s : \mathrm{Dom}(s) = \{\vec{x}\}, M \models_{h \cup s} \gamma(\vec{x}, \vec{m})\}$. Then

$$(M, \mathcal{L}) \models_X \varphi \Leftrightarrow M \models_{\gamma(h)} \varphi.$$

Proof The proof is by structural induction on φ and presents no difficulties.

1. If φ is a first-order literal, $(M, \mathcal{L}) \models_X \varphi$ if and only if, for all $s \in X$, it holds that $M \models_s \varphi$. But $s \in X$ if and only if $M \models_s \gamma(\vec{x}, h(\vec{p}))$, and hence $(M, \mathcal{L}) \models_X \varphi$ if and only if $M \models_\gamma \varphi$, as required.
2. If φ is an independence atom, the result is also obvious and follows at once from a comparison of the rules **GMS-ind** and **ES-ind**.
3. If φ is $\psi_1 \vee \psi_2$,

$(M, \mathcal{L}) \models_X \psi_1 \vee \psi_2$

$\Leftrightarrow \exists Y, Z \in \mathcal{L}$ such that $X = Y \cup Z$, $(M, \mathcal{L}) \models_Y \psi_1$ and $(M, \mathcal{L}) \models_Z \psi_2$

$\Leftrightarrow \exists h' = h[\vec{m}/\vec{q}]$ extending h and $\exists \gamma_1 \gamma_2$ such that, for $Y = \|\gamma_1(\vec{x}, \vec{p}\vec{q})\|_{M,h'}$, $Z = \|\gamma_2(\vec{x}, \vec{p}\vec{q})\|_{M,h'}$, $X = \|\gamma(\vec{x}, \vec{p})\|_{M,h} = \|\gamma(\vec{x}, \vec{p})\|_{M,h'} = Y \cup Z$, $(M, \mathcal{L}) \models_Y \psi_1$, and $(M, \mathcal{L}) \models_Z \psi_2$

$\Leftrightarrow \exists h' = h[\vec{m}/\vec{q}]$ extending h and $\exists \gamma_1 \gamma_2$ such that $M \models_{h'} \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \vee \gamma_2)$, $M \models_{\gamma_1(h')} \psi$, and $M \models_{\gamma_2(h')} \theta$

$\Leftrightarrow M \models_{\gamma(h)} \psi \vee \theta$.

4. If φ is $\psi \wedge \theta$,

$$(M, \mathcal{L}) \models_X \psi \wedge \theta \Leftrightarrow (M, \mathcal{L}) \models_X \psi \text{ and } (M, \mathcal{L}) \models_X \theta$$
$$\Leftrightarrow M \models_{\gamma(h)} \psi \text{ and } M \models_{\gamma(h)} \theta \Leftrightarrow M \models_{\gamma(h)} \psi \wedge \theta.$$

5. If φ is $\exists x_n \psi$,

$(M, \mathcal{L}) \models_X \exists x_n \psi \Leftrightarrow \exists X' \in \mathcal{L}$ such that $X[x_n]X'$ and $(M, \mathcal{L}) \models_{X'} \psi$
$\Leftrightarrow \exists h' = h[\vec{m}/\vec{q}]$ extending h and $\exists \gamma'$ such that, for $X' = \|\gamma'(\vec{x}, \vec{p}\vec{q})\|_{M,h'}$, $X[x_n]X'$, and $(M, \mathcal{L}) \models_{X'} \psi$
$\Leftrightarrow \exists h' = h[\vec{m}/\vec{q}]$ extending h and $\exists \gamma'$ such that $M \models_{h'} \forall \vec{v}(\exists x_n \gamma \leftrightarrow \exists x_n \gamma')$ and $M \models_{\gamma'(h')} \psi$
$\Leftrightarrow M \models_{\gamma(h)} \exists x_n \psi$.

6. If φ is $\forall x_n \psi$,

$(M, \mathcal{L}) \models_X \forall x_n \psi \Leftrightarrow \exists X' \in \mathcal{L}$ such that $X' = X[M/x_n]$ and $(M, \mathcal{L}) \models_{X'} \psi$
$\Leftrightarrow \exists h' = h[\vec{m}/\vec{q}]$ extending h and $\exists \gamma'$ such that, for $X' = \|\gamma'(\vec{x}, \vec{p}q)\|_{M,h'}$, $X' = X[M/x_n]$, and $(M, \mathcal{L}) \models_{X'} \psi$
$\Leftrightarrow \exists h' = h[\vec{m}/\vec{q}]$ extending h and $\exists \gamma'$ such that $M \models_{h'} \forall \vec{v}(\gamma' \leftrightarrow \exists x_n \gamma)$ and $M \models_{\gamma'(h')} \psi$
$\Leftrightarrow M \models_{\gamma(h)} \forall x_n \psi$. □

Definition 4.6 (Validity in entailment semantics) Let φ be an independence logic formula. Then φ is *valid* in entailment semantics if and only if $M \models_{\gamma(h)} \varphi$ for all first-order models M with signature containing that of φ, for all first-order formulas $\gamma(\vec{x}, \vec{p})$ over the signature of M, and for all parameter assignments h with domain $\vec{p}$. If this is the case, we write ENS $\models \varphi$.

Corollary 4.7 *For all formulas φ,* ENS $\models \varphi$ *if and only if* LMS $\models \varphi$ *if and only if* GMS $\models \varphi$.

It will also be useful to have a slightly more general notion of validity in entailment semantics.

Definition 4.8 (Validity with respect to a team definition) Let $\gamma(\vec{x}, \vec{p})$ be a first-order formula, and let φ be an independence logic formula. Then φ is *valid* with respect to γ if and only if $M \models_{\gamma(h)} \varphi$ for all first-order models M with signature containing those of γ and φ and for all parameter assignments h with domain $\vec{p}$. If this is the case, we write $\models_\gamma \varphi$.

Proposition 4.9 *Let φ be an independence logic formula with* $\mathrm{Free}_T(\varphi) = \{x_1, \ldots, x_k\}$*, let $\vec{x} = x_1, \ldots, x_k$, and let R be a k-ary relation symbol not occurring in γ. Then* ENS $\models \varphi$ *if and only if* $\models_{R\vec{x}} \varphi$.

Proof Suppose that ENS $\models \varphi$. Then in particular, for any model M whose signature contains that of φ and R we have $M \models_{R\vec{x}} \varphi$, and hence $\models_{R\vec{x}} \varphi$.

Conversely, suppose that $\models_{R\vec{x}} \varphi$, let M be a first-order model,[6] and let $X \in \mathcal{L}$ be any team with domain $\{x_1, \ldots, x_k\}$. Let us then consider the model M' obtained by adding to M the k-ary symbol R with $R^{M'} = \mathrm{Rel}(X)$. By hypothesis, $M' \models_{R\vec{x}} \varphi$, and furthermore, since $R^{M'}$ is in $\mathcal{L}$ already, the least general model over M' is $(M', \mathcal{L})$ for the same $\mathcal{L}$.

Now $(M', \mathcal{L}) \models_X \varphi$, and therefore, as R occurs nowhere in φ, $(M, \mathcal{L}) \models_X \varphi$ too. This holds for all X with domain $\{x_1, \ldots, x_k\}$; therefore by the locality theorem (Theorem 3.5), the same holds for all domains containing $\mathrm{Free}_T(\varphi)$, and hence LMS $\models \varphi$. This implies that ENS $\models \varphi$, as required. □

In the next section, we will develop a sound and complete proof system for this notion of validity with respect to a team definition.

5 The Proof System

In this section, we will develop a proof system for independence logic (with entailment semantics) and prove its soundness and completeness.

Definition 5.1 (Sequent) Let Γ be a finite first-order theory with only parameter variables among its free ones, let $\gamma(\vec{x}, \vec{p})$ be a first-order formula, and let φ be an independence logic formula with free variables in $\mathbf{V}_T$. Then the expression

$$\Gamma \mid \gamma \vdash \varphi$$

is a sequent.

The intended semantics of a sequent is the following.

Definition 5.2 (Valid sequents) Let $\Gamma \mid \gamma \vdash \varphi$ be a sequent. Then $\Gamma \mid \gamma \vdash \varphi$ is *valid* if and only if for all models M and all parameter assignments h with domain $\mathrm{Free}_P(\Gamma) \cup \mathrm{Free}_P(\gamma)$ such that $M \models_h \Gamma$, it holds that

$$M \models_{\gamma(h)} \varphi.$$

The following result is then clear.

Proposition 5.3 *For all γ and φ, $\models_\gamma \varphi$ if and only if $\emptyset \mid \gamma \vdash \varphi$ is valid.*

Now, all we need to do is develop some syntactic rules for finding valid sequents.

We can do this as follows.

Definition 5.4 (Axioms and rules) The axioms of our proof system are the following:

PS-lit: If φ is a first-order literal with no free parameter variables (i.e., $\mathrm{Free}_P(\varphi) = \emptyset$), then

$$\forall \vec{v}(\gamma \to \varphi) \mid \gamma \vdash \varphi$$

for all first-order formulas γ, where $\vec{v} = \mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\varphi)$.

PS-ind: If $\vec{t}_1$, $\vec{t}_2$, and $\vec{t}_3$ are first-order terms with no free parameter variables, then

$$\begin{aligned}&\forall \vec{v}_1 \vec{v}_2 \big((\gamma(v_1) \wedge \gamma(v_2) \wedge \vec{t}_1(\vec{v}_1) = \vec{t}_1(\vec{v}_2)) \\ &\quad \to \exists \vec{v}_3(\gamma(v_3) \wedge \vec{t}_1\vec{t}_2(\vec{v}_3) = \vec{t}_1\vec{t}_2(\vec{v}_1) \wedge \vec{t}_1\vec{t}_3(\vec{v}_3) = \vec{t}_1\vec{t}_3(\vec{v}_2))\big) \mid \gamma \vdash \vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3\end{aligned}$$

for all γ, where $\vec{v}_1$ and $\vec{v}_2$ are tuples of variables of the same lengths of $\vec{v} = \mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\vec{t}_1\vec{t}_2\vec{t}_3)$, $\vec{t}_i(\vec{v}_j)$ is the tuple obtained by substituting $\vec{v}$ with $\vec{v}_j$ in $\vec{t}_i$, and the same holds for $\gamma(\vec{v}_j)$.

The rules of our proof system are the following:

PS-∨: If $\Gamma_1 \mid \gamma_1 \vdash \varphi_1$ and $\Gamma_2 \mid \gamma_2 \vdash \varphi_2$, then, for all γ, we have

$$\Gamma_1, \Gamma_2, \forall \vec{v}\big(\gamma \leftrightarrow (\gamma_1 \vee \gamma_2)\big) \mid \gamma \vdash \varphi_1 \vee \varphi_2,$$

where $\vec{v}$ is $\mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\gamma_1) \cup \mathrm{Free}_T(\gamma_2)$.

PS-∧: If $\Gamma_1 \mid \gamma \vdash \varphi_1$ and $\Gamma_2 \mid \gamma \vdash \varphi_2$, then $\Gamma_1, \Gamma_2 \mid \gamma \vdash \varphi_1 \wedge \varphi_2$.

PS-∃: If $\Gamma \mid \gamma' \vdash \varphi$ and x is a team variable, then, for all γ,

$$\Gamma, \quad \forall \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma) \mid \gamma \vdash \exists x \varphi,$$

where $\vec{v} = \mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\gamma')$.

PS-∀: If $\Gamma \mid \gamma' \vdash \varphi$ and x is a team variable, then, for all γ,

$$\Gamma, \quad \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma) \mid \gamma \vdash \forall x \varphi,$$

where, as in the previous case, $\vec{v} = \mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\gamma')$.

PS-ent: If $\Gamma \mid \gamma \vdash \varphi$ and $\bigwedge \Gamma' \models \bigwedge \Gamma$ holds in first-order logic, then $\Gamma' \mid \gamma \vdash \varphi$.

PS-depar: If $\Gamma \mid \gamma \vdash \varphi$ and p is a parameter variable which does not occur free in γ, then $\exists p \bigwedge \Gamma \mid \gamma \vdash \varphi$.

PS-split: If $\Gamma_1 \mid \gamma \vdash \varphi$ and $\Gamma_2 \mid \gamma \vdash \varphi$, then $\big(\bigwedge \Gamma_1\big) \vee \big(\bigwedge \Gamma_2\big) \mid \gamma \vdash \varphi$.

Definition 5.5 (Proofs and proof lengths) Let $\Gamma \mid \gamma \vdash \varphi$ be a sequent. A *proof* of this sequent is a finite list of sequents

$$(\Gamma_1 \mid \gamma_1 \vdash \varphi_1), \ldots, (\Gamma_n \mid \gamma_n \vdash \varphi_n) = (\Gamma \mid \gamma \vdash \varphi)$$

such that, for all $i = 1, \ldots, n$, $\Gamma_i \mid \gamma_i \vdash \varphi_i$ is either an instance of **PS-lit** or **PS-ind** or it follows from $\{\Gamma_j \mid \gamma_j \vdash \varphi_j : j < i\}$ through one application of the rules of our proof system.

Given a proof $P = S_1 \ldots S_n$, where each S_i is a sequent, we define its *length* $|P|$ as $n - 1$, that is, as the number of sequents in the proof minus one.

Before examining soundness and completeness for this proof system, it will be useful to obtain a couple of derived rules.

Proposition 5.6 *The following rules hold:*

PS-FO: *If φ is a first-order formula with no free parameter variables, $\forall \vec{v}(\gamma \rightarrow \varphi) \mid \gamma \vdash \varphi$ is provable for all γ, where $\vec{v} = \mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\varphi)$.*

PS-dep: *If $\vec{t}$ is a tuple of terms, t' is another term, and $=(\vec{t}, t')$ stands for $t' \perp_{\vec{t}} t'$, then*

$$\forall \vec{v}_1 \vec{v}_2 \big(\gamma(\vec{v}_1) \wedge \gamma(\vec{v}_2) \wedge \vec{t}(\vec{v}_1) = \vec{t}(\vec{v}_2)\big) \rightarrow t'(\vec{v}_1) = t'(\vec{v}_2) \mid \gamma \vdash =(\vec{t}, t')$$

is provable for all γ, where $\vec{v}_1$, $\vec{v}_2$ are tuples of the same length of $\vec{v} = \mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\vec{t} t')$.

Proof

PS-FO: The proof is by structural induction on φ.

1. If φ is a first-order literal, this follows at once from rule **PS-lit**.
2. If φ is $\psi_1 \vee \psi_2$, by induction hypothesis we have that $\forall \vec{v}((\gamma \wedge \psi_1) \rightarrow \psi_1) \mid \gamma \wedge \psi_1 \vdash \psi_1$ and $\forall \vec{v}((\gamma \wedge \psi_2) \rightarrow \psi_2) \mid \gamma \wedge \psi_2 \vdash \psi_2$ are provable. But then we can prove $\forall \vec{v}(\gamma \rightarrow \varphi_1 \vee \varphi_2) \mid \gamma \vdash \varphi$ as follows:
   (a) $\forall \vec{v}((\gamma \wedge \psi_1) \rightarrow \psi_1) \mid \gamma \wedge \psi_1 \vdash \psi_1$ (derived before);
   (b) $\forall \vec{v}((\gamma \wedge \psi_2) \rightarrow \psi_2) \mid \gamma \wedge \psi_2 \vdash \psi_2$ (derived before);
   (c) $\mid \gamma \wedge \psi_1 \vdash \psi_1$ (**PS-ent**, from (a), because $\models \forall \vec{v}((\gamma \wedge \psi_1) \rightarrow \psi_1)$ in first-order logic);

   (d) $\mid \gamma \wedge \psi_2 \vdash \psi_2$ (**PS-ent**, from (b), because $\models \forall \vec{v}((\gamma \wedge \psi_2) \to \psi_2)$ in first-order logic);
   (e) $\forall \vec{v}(\gamma \leftrightarrow (\gamma \wedge \psi_1) \vee (\gamma \wedge \psi_2)) \mid \gamma \vdash \psi_1 \vee \psi_2$ (**PS-**$\vee$, from (c) and (d));
   (f) $\forall \vec{v}(\gamma \to (\psi_1 \vee \psi_2)) \mid \gamma \vdash \psi_1 \vee \psi_2$ (**PS-ent**: from (e), because $\forall \vec{v}(\gamma \to (\psi_1 \vee \psi_2))$ entails $\forall \vec{v}(\gamma \leftrightarrow (\gamma \wedge \psi_1) \vee (\gamma \wedge \psi_2))$ in first-order logic).
3. If φ is $\psi_1 \wedge \psi_2$, by induction hypothesis we have that $\forall \vec{v}(\gamma \to \psi_1) \mid \gamma \vdash \psi_1$ and $\forall \vec{v}(\gamma \to \psi_2) \mid \gamma \vdash \psi_2$ are provable. But then
   (a) $\forall \vec{v}(\gamma \to \psi_1) \mid \gamma \vdash \psi_1$ (derived before);
   (b) $\forall \vec{v}(\gamma \to \psi_2) \mid \gamma \vdash \psi_2$ (derived before);
   (c) $\forall \vec{v}(\gamma \to \psi_1), \forall \vec{v}(\gamma \to \psi_2) \mid \gamma \vdash \psi_1 \wedge \psi_2$ (**PS-**$\wedge$, (a), (b));
   (d) $\forall \vec{v}(\gamma \to \psi_1 \wedge \psi_2) \mid \gamma \vdash \psi_1 \wedge \psi_2$ (**PS-ent**, (c)),

   as required.
4. If φ is $\exists x \psi$, by induction hypothesis we have that $\forall \vec{v} \forall x\big(((\exists x \gamma) \wedge \psi) \to \psi\big) \mid (\exists x \gamma) \wedge \psi \vdash \psi$ is provable. But then
   (a) $\forall \vec{v} \forall x\big(((\exists x \gamma) \wedge \psi) \to \psi\big) \mid (\exists x \gamma) \wedge \psi \vdash \psi$ (derived before);
   (b) $\mid (\exists x \gamma) \wedge \psi \vdash \psi$ (**PS-ent**, from (a));
   (c) $\forall \vec{v}(\exists x((\exists x \gamma) \wedge \psi) \leftrightarrow \exists x \gamma) \mid \gamma \vdash \exists x \psi$ (**PS-**$\exists$, from (b));
   (d) $\forall \vec{v}\big(((\exists x \gamma) \wedge (\exists x \psi)) \leftrightarrow \exists x \gamma\big) \mid \gamma \vdash \exists x \psi$ (**PS-ent**, from (c));
   (e) $\forall \vec{v}(\gamma \to \exists x \psi) \mid \gamma \vdash \psi$ (**PS-ent**, from (d)),

   as required, where the last passage uses the fact that $\forall \vec{v}(\gamma \to \exists x \psi) \models \forall \vec{v}\big(((\exists x \gamma) \wedge (\exists x \psi)) \leftrightarrow \exists x \gamma\big)$ in first-order logic.
5. If φ is $\forall x \psi$, by induction hypothesis we have that $\forall \vec{v} \forall x((\exists x \gamma) \to \psi) \mid \exists x \gamma \vdash \psi$ is provable. But then
   (a) $\forall \vec{v} \forall x((\exists x \gamma) \to \psi) \mid \exists x \gamma \vdash \psi$ (derived before);
   (b) $\forall \vec{v} \forall x((\exists x \gamma) \to \psi), \forall \vec{v}(\exists x \gamma \leftrightarrow \exists x \gamma) \mid \gamma \vdash \forall x \psi$ (**PS-**$\forall$, from (a));
   (c) $\forall \vec{v} \forall x((\exists x \gamma) \to \psi) \mid \gamma \vdash \forall x \psi$ (**PS-ent**, from (c));
   (d) $\forall \vec{v}(\gamma \to \forall x \psi) \mid \gamma \vdash \forall x \psi$ (**PS-ent**, from (d)),

   where the last two passages hold because $\forall \vec{v}(\exists x \gamma \leftrightarrow \exists x \gamma)$ is valid and because $\forall \vec{v}(\gamma \to \forall x \psi)$ entails $\forall \vec{v} \forall x((\exists x \gamma) \to \psi)$ in first-order logic, where $\vec{v} = \mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\psi)$ (and, therefore, if x is free in γ, then x is in $\vec{v}$).

PS-dep: By definition, $=(\vec{t}, t')$ stands for $t' \perp_{\vec{t}} t'$; therefore, by rule **PS-ind** we have

$$\forall \vec{v}_1 \vec{v}_2\big((\gamma(\vec{v}_1) \wedge \gamma(\vec{v}_2) \wedge \vec{t}(\vec{v}_1) = \vec{t}(\vec{v}_2)) \\ \to \exists \vec{v}_3(\gamma(\vec{v}_3) \wedge \vec{t}t'(\vec{v}_3) = \vec{t}t'(\vec{v}_1) \wedge \vec{t}t'(\vec{v}_3) = \vec{t}t'(\vec{v}_2))\big) \mid \gamma \vdash =(\vec{t}, t').$$

But the formula

$$\forall \vec{v}_1 \vec{v}_2\big((\gamma(\vec{v}_1) \wedge \gamma(\vec{v}_2) \wedge \vec{t}(\vec{v}_1) = \vec{t}(\vec{v}_2)) \to t'(\vec{v}_1) = t'(\vec{v}_2)\big)$$

entails the premise, and therefore by rule **PS-ent** we have our conclusion. □

Theorem 5.7 (Soundness) *Suppose that* $\Gamma \mid \gamma \vdash \varphi$ *is provable. Then it is valid.*

Proof If S is a provable sequent, then there exists a proof $S_1 \dots S_n S$ for it. Then we go by induction of the length n of this proof.

Base case: Suppose that the proof has length 0. Then S is an instance of **PS-lit** or of **PS-ind**. Suppose first that it is the former, that is, that

$$S = \forall \vec{v}(\gamma \to \varphi) \mid \gamma \vdash \varphi$$

for some first-order γ and some first-order literal φ, where $\vec{v} = \mathrm{Free}_T(\gamma) \cup \mathrm{Free}_T(\varphi)$ and φ has no parameter variables. Now suppose that $M \models_h \forall \vec{x}(\gamma \to \varphi)$; then, by definition, if s is an assignment over team variables such that $M \models_{h \cup s} \gamma$, then $M \models_s \varphi$. Therefore, by **ES-lit**, $M \models_{\gamma(s)} \varphi$ in entailment semantics, as required.

The case corresponding to **PS-ind** and **ES-ind** is entirely similar.

Induction case: Let $S_1 S_2 \ldots S_n S$ be our proof. For each $i \leq n$ we have that $S_1 \ldots S_i$ is a valid proof for S_i, and hence by induction hypothesis that S_i is valid. Now let us consider which rule r was been used to derive S from $S_1 \ldots S_n$.

1. If r was **PS-lit** or **PS-ind**, then (S) is a proof for S already, and hence by our base case S is valid.
2. If r was **PS-**$\vee$, then S is $\Gamma_1, \Gamma_2, \forall \vec{v}(\gamma \leftrightarrow (\gamma_1 \vee \gamma_2)) \mid \gamma \vdash \varphi_1 \vee \varphi_2$, and there exist two $i, j \leq n$ such that $S_i = (\Gamma_1 \mid \gamma_1 \vdash \varphi_1)$ and $S_j = (\Gamma_2 \mid \gamma_2 \vdash \varphi_2)$. By induction hypothesis, these sequents are valid.
   Now suppose that $M \models_h \Gamma_1, \Gamma_2, \forall \vec{v}(\gamma \leftrightarrow (\gamma_1 \vee \gamma_2))$. Then, since $M \models_h \Gamma_1$, we have $M \models_{\gamma_1(h)} \varphi_1$, and, analogously, since $M \models_h \Gamma_2$ we have $M \models_{\gamma_2(h)} \varphi_2$. Furthermore, $M \models_h \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \vee \gamma_2)$, and therefore by rule **ES-**$\vee$ we have $M \models_\gamma \varphi_1 \vee \varphi_2$, as required.
3. If r was **PS-**$\wedge$, then S_n is of the form $\Gamma_1, \Gamma_2 \mid \gamma \vdash \varphi_1 \wedge \varphi_2$ and, by induction hypothesis, $\Gamma_1 \mid \gamma \vdash \varphi_1$ and $\Gamma_2 \mid \gamma \vdash \varphi_2$ are valid. Now suppose that $M \models_h \Gamma_1, \Gamma_2$; then $M \models_{\gamma(h)} \varphi_1$ and $M \models_{\gamma(h)} \varphi_2$, and therefore $M \models_{\gamma(h)} \varphi_1 \wedge \varphi_2$ by **ES-**$\wedge$.
4. If r was **PS-**$\exists$, then S_n is of the form $\Gamma, \forall \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma) \mid \gamma \vdash \exists x \varphi$, where $\Gamma \mid \gamma' \vdash \varphi$ is valid by induction hypothesis. Now suppose that $M \models_h \Gamma, \forall \vec{v}(\exists x \gamma \leftrightarrow \exists x \gamma')$; then $M \models_{\gamma'(h)} \varphi$ and $M \models_h \forall \vec{v}(\exists x \gamma \leftrightarrow \exists x \gamma')$, and therefore $M \models_{\gamma(h)} \exists x \varphi$ by rule **ES-**$\exists$.
5. If r was **PS-**$\forall$, then S_n is of the form $\Gamma, \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma) \mid \gamma \vdash \forall x \varphi$, where $\Gamma \mid \gamma' \vdash \varphi$ is valid by induction hypothesis. Now, suppose that $M \models_h \Gamma, \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma)$. Then $M \models_{\gamma'(h)} \varphi$, and furthermore $M \models_h \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma)$. Therefore, by rule **ES-**$\forall$, $M \models_{\gamma(h)} \forall x \varphi$, as required.
6. If r was **PS-ent**, then S_n is of the form $\Gamma' \mid \gamma \vdash \varphi$, where $\Gamma \mid \gamma \vdash \varphi$ is valid by induction hypothesis and where $\bigwedge \Gamma \models \bigwedge \Gamma'$ holds in first-order logic. Now suppose that $M \models_h \Gamma'$; then $M \models_h \Gamma$, and hence $M \models_{\gamma(h)} \varphi$, as required.
7. If r was **PS-depar**, then S_n is of the form $\exists p \bigwedge \Gamma \mid \gamma \vdash \varphi$, where $\Gamma \mid \gamma \vdash \varphi$ holds by induction hypothesis and where the parameter variable p does not occur free in γ. Now suppose that $M \models_h \exists p \bigwedge \Gamma$; then there exists an element $m \in \mathrm{Dom}(M)$ such that, for $h' = h[m/p]$, $M \models_{h'} \Gamma$. Then $M \models_{\gamma(h')} \varphi$; but as p does not occur free in γ we then have, by Proposition 4.4, that $M \models_{\gamma(h)} \varphi$, as required.

8. If r was **PS-split**, then S_n is of the form $(\bigwedge \Gamma_1) \vee (\bigwedge \Gamma_2) \mid \gamma \vdash \varphi$, where $\Gamma_1 \mid \gamma \vdash \varphi$ and $\Gamma_2 \mid \gamma \vdash \varphi$ by induction hypothesis. Now suppose that $M \models_h (\bigwedge \Gamma_1) \vee (\bigwedge \Gamma_2)$. Then $M \models_h \Gamma_1$ or $M \models_h \Gamma_2$; and in either case, $M \models_{\gamma(h)} \varphi$, as required. □

In order to prove completeness, we first need a lemma.

Lemma 5.8 *Suppose that $M \models_{\gamma(h)} \varphi$. Then there exists a finite Γ such that $\Gamma \mid \gamma \vdash \varphi$ is provable and such that $M \models_h \Gamma$.*

Proof The proof is by structural induction on φ.

1. If φ is a first-order literal or an independence atom, this follows immediately from a comparison of **ES-lit** and **PS-lit** and of **ES-ind** and **PS-ind**.
2. If φ is $\psi_1 \vee \psi_2$ and $M \models_{\gamma(h)} \varphi$, then, by definition, there exists an assignment h' extending h and two first-order formulas γ_1, γ_2 such that $M \models_{\gamma_1(h')} \psi_1$, $M \models_{\gamma_2(h')} \psi_2$, and $M \models_{h'} \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \vee \gamma_2)$. Let $\vec{p}$ be the tuple of parameters in $\mathrm{Dom}(h') \setminus \mathrm{Dom}(h)$; now, by induction hypothesis we have that there exist Γ_1 and Γ_2 such that $\Gamma_1 \mid \gamma_1 \vdash \psi_1$ and $\Gamma_2 \mid \gamma_2 \vdash \psi_2$ are provable, and such that furthermore $M \models_{h'} \Gamma_1$ and $M \models_{h'} \Gamma_2$.

   But then the following is a correct proof:
   (a) $\Gamma_1 \mid \gamma_1 \vdash \psi_1$ (derived before);
   (b) $\Gamma_2 \mid \gamma_2 \vdash \psi_2$ (derived before);
   (c) $\Gamma_1, \Gamma_2, \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \vee \gamma_2) \mid \gamma \vdash \varphi$ (**PS-**$\vee$, (a), (b));
   (d) $\exists \vec{p}(\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2 \wedge \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \vee \gamma_2)) \mid \gamma \vdash \varphi$ (**PS-depar**, (c)).[7]

   Finally, $M \models_h \exists \vec{p}(\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2 \wedge \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \vee \gamma_2))$, as required, because there exists a tuple of elements $\vec{m}$ such that $h[\vec{m}/\vec{p}] = h'$.
3. If φ is $\psi_1 \wedge \psi_2$ and $M \models_{\gamma(h)} \varphi$, then $M \models_{\gamma(h)} \psi_1$ and $M \models_{\gamma(h)} \psi_2$. Then, by induction hypothesis, there exist Γ_1 and Γ_2 such that $\Gamma_1 \mid \gamma \vdash \psi_1$ and $\Gamma_2 \mid \gamma \vdash \psi_2$ are provable and such that $M \models_h \Gamma_1 \Gamma_2$. Then by rule **PS-**$\wedge$, $\Gamma_1 \Gamma_2 \mid \gamma \vdash \psi_1 \wedge \psi_2$, as required.
4. If φ is $\exists x \psi$ and $M \models_{\gamma(h)} \varphi$, then there exists a tuple $\vec{p}$ of parameter variables not in the domain of h, a tuple $\vec{m}$ of elements of the model, and a formula γ' such that, for $h' = h[\vec{m}/\vec{p}]$, $M \models_{\gamma'(h')} \psi$ and $M \models_{h'} \forall \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma)$. By induction hypothesis, we then have a Γ' such that $\Gamma' \mid \gamma' \vdash \psi$ and $M \models_{h'} \Gamma'$.

   Then the following is a valid proof:
   (a) $\Gamma' \mid \gamma' \vdash \psi$ (derived before);
   (b) $\Gamma', \forall \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma) \mid \gamma \vdash \exists x \psi$ (**PS-**$\exists$);
   (c) $\exists \vec{p}(\bigwedge \Gamma' \wedge \forall \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma)) \mid \gamma \vdash \exists x \psi$ (**PS-depar**).

   Furthermore, $M \models_h \exists \vec{p}(\bigwedge \Gamma' \wedge \forall \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma))$, as required.
5. If φ is $\forall x \psi$ and $M \models_{\gamma(h)} \varphi$, then there exists a tuple $\vec{p}$ of parameter variables not in the domain of h, a tuple $\vec{m}$ of elements of the model, and a formula γ' such that $M \models_{\gamma'(h')} \psi$ and $M \models_{h'} \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma)$, where $h' = h[\vec{m}/\vec{p}]$. By induction hypothesis, we can then find a Γ' such that $\Gamma' \mid \gamma' \vdash \psi$ is provable and $M \models_{h'} \Gamma'$.

   Then the following is a valid proof:
   (a) $\Gamma' \mid \gamma' \vdash \psi$ (derived before);
   (b) $\Gamma', \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma) \mid \gamma \vdash \forall x \psi$ (**PS-**$\forall$);
   (c) $\exists \vec{p}(\bigwedge \Gamma' \wedge \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma)) \mid \gamma \vdash \forall x \psi$ (**PS-depar**).

And, once again, the assignment h satisfies the antecedent of the last sequent, as required. □

The completeness of our proof system follows from the above lemma and from the compactness and the Löwenheim–Skolem theorem for first-order logic.

Theorem 5.9 (Completeness) *Suppose that* $\Gamma \mid \gamma \vdash \varphi$ *is valid, where* Γ *is finite. Then it is provable.*

Proof Since $\Gamma \mid \gamma \vdash \varphi$ is valid, for any first-order model M over the signature of Γ, γ, and φ, and for all h such that $M \models_h \Gamma$, we have $M \models_{\gamma(h)} \varphi$, and hence by the lemma, $M \models_h \Gamma_{M,h}$ for some finite $\Gamma_{M,h}$ such that $\Gamma_{M,h} \mid \gamma \vdash \varphi$ is provable.

Then consider the first-order, countable[8] theory

$$T = \left\{\bigwedge \Gamma\right\} \cup \left\{\neg \bigwedge \Gamma_{M,h} : M \text{ is a countable model,}\right.$$
$$\left. h \text{ is an assignment such that } M \models_h \Gamma\right\}.$$

This theory is unsatisfiable. Indeed, suppose that M_0 is a model that satisfies $\bigwedge \Gamma$ under the assignment h_0; then, by the Löwenheim–Skolem theorem, there exists a countable elementary submodel (M'_0, h'_0) of (M_0, h_0).

Now, $M'_0 \models_{h'_0} \Gamma$ and M'_0 is countable, and hence by definition $M'_0 \models_{h'_0} \Gamma_{M'_0,h'_0}$. But then $M_0 \models_{h_0} \Gamma_{M'_0,h'_0}$ too, and therefore M_0 is not a model of T.

By the compactness theorem, this implies that there exists a finite subset $T_0 = \{\neg \bigwedge \Gamma_{M_1,h_1}, \ldots, \neg \bigwedge \Gamma_{M_n,h_n}\}$ of T such that $\{\bigwedge \Gamma\} \cup T_0$ is unsatisfiable, that is, such that

$$\Gamma \models \left(\bigwedge \Gamma_{M_1,h_1}\right) \vee \cdots \vee \left(\bigwedge \Gamma_{M_n,h_n}\right).$$

Now, for each i, $\Gamma_{M_i,h_i} \mid \gamma \vdash \varphi$ can be proved. Therefore, by rule **PS-split**, we have that $\left(\bigwedge \Gamma_{M_1,s_1}\right) \vee \cdots \vee \left(\bigwedge \Gamma_{M_n,s_n}\right) \mid \gamma \vdash \varphi$ is also provable; and finally, by rule **PS-ent** we can prove that $\Gamma \mid \gamma \vdash \varphi$, as required. □

Using essentially the same method, it is also possible to prove a "compactness" result for our semantics.

Theorem 5.10 *Suppose that* $\Gamma \mid \gamma \vdash \varphi$ *is valid. Then there exists a finite* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \mid \gamma \vdash \varphi$ *is provable (and valid).*

Proof Let $\kappa = \max(|\Gamma|, \aleph_0)$, and consider the theory

$$T = \Gamma \cup \left\{\neg \bigwedge \Gamma_{M,h} : |M| \leq \kappa, M \models_h \Gamma\right\},$$

where, as in the previous proof, $\Gamma_{M,h}$ is a finite theory such that $M \models_h \Gamma_{M,h}$ and such that $\Gamma_{M,h} \mid \gamma \vdash \varphi$ is provable in our system.

Then T is unsatisfiable; indeed, if T had a model, then it would have a model (M, h) of cardinality at most κ, and since that model would satisfy Γ it would satisfy $\Gamma_{M,h}$ too, which contradicts our hypothesis.

Hence, by the compactness theorem, there exists a finite set $\left\{\bigwedge \Gamma_{M_1,h_1}, \ldots, \bigwedge \Gamma_{M_n,h_n}\right\}$ and a finite $\Gamma_0 \subseteq \Gamma$ such that

$$\Gamma_0 \models \bigwedge \Gamma_{M_1,h_1} \vee \cdots \vee \bigwedge \Gamma_{M_n,h_n}.$$

But by rule **PS-split**, we have that $\bigwedge \Gamma_{M_1,h_1} \vee \cdots \vee \bigwedge \Gamma_{M_n,h_n} \mid \gamma \vdash \varphi$ is provable, and hence by rule **PS-ent**, $\Gamma_0 \mid \gamma \vdash \varphi$ is also provable, as required. □

6 Adding More Teams

The proof system that we developed in the previous section is, as we saw, sound and complete with respect to its intended semantics. However, this semantics is perhaps quite weak. All that we know is that the teams which correspond to parameterized first-order formulas belong to our general models.

Rather than adding more and more axioms to our proof system in order to guarantee the existence of more teams, in this section we will attempt to separate our assumptions about team existence from our main proof system. This will allow us to *modulate* our formalism: depending on our needs, we may want to assume the existence of more or of less teams in our general model.

The natural language for describing assertions about the existence of relations is, of course, existential second-order logic. The following definitions show how it can be used for our purposes.

Definition 6.1 (Relation existence theory) A *relation existence theory* Θ is a set of existential second-order sentences of the form $\exists \vec{R}\varphi(\vec{R})$, where φ is first order.

Definition 6.2 (Θ-closed general models) Let $(M, \mathscr{G})$ be any general model, and let Θ be a relation existence theory. Then $(M, \mathscr{G})$ is Θ-*closed* if and only if for all $\exists \vec{R}\varphi(\vec{R})$ in Θ there exists a tuple of teams $\vec{X} \in \mathscr{G}$ such that $M \models \varphi[\vec{\mathrm{Rel}}(\vec{X})/\vec{R}]$.

Definition 6.3 (Θ-valid sequents) Let $\Gamma \mid \gamma \vdash \varphi$ be a sequent, and let Θ be a relation existence theory. Then $\Gamma \mid \gamma \vdash \varphi$ is *valid* if and only if for all Θ-closed models $(M, \mathscr{G})$ and all parameter assignments h with domain $\mathrm{Free}_P(\Gamma) \cup \mathrm{Free}_P(\gamma)$ such that $M \models_h \Gamma$ it holds that

$$(M, \mathscr{G}) \models_{\|\gamma\|_h} \varphi.$$

Our proof system for Θ-closed general models can then be obtained by adding the following rule to our system.

PS-Θ: If $\Gamma_1(\vec{S}), \Gamma_2 \mid \gamma \vdash \varphi$ is provable, where the relation symbols $\vec{S}$ do not occur in Γ_2, in γ, or in φ, and $\exists \vec{R} \bigwedge \Gamma_1(\vec{R})$ is in Θ for some $\vec{R}$, then $\Gamma_2 \mid \gamma \vdash \varphi$ is provable.

Theorem 6.4 (Soundness) *Let $\Gamma \mid \gamma \vdash \varphi$ be a sequent which is provable in our proof system plus* ***PS-***Θ*. Then it is Θ-valid.*

Proof The proof is by induction on the length of the proof and follows very closely the one given already. Hence, we only examine the case in which the last rule used in the proof is **PS-**Θ. Then, by induction hypothesis, we have that $\Gamma_1(\vec{S}), \Gamma \mid \gamma \vdash \varphi$, is Θ-valid for some Γ_1 and some $\vec{S}$ which does not occur in Γ, in γ, or in φ, and moreover, $\exists \vec{R} \bigwedge \Gamma_1(\vec{R})$ is in Θ.

Now, let $(M, \mathscr{G})$ be any Θ-closed general model, and let us assume without loss of generality that the relation symbols in $\vec{S}$ are not part of its signature. Furthermore, let h be a parameter assignment (with domain $\mathrm{Free}(\Gamma) \cup \mathrm{Free}(\gamma)$) such that $M \models_h \Gamma$. By definition, there exists a tuple of teams $\vec{X} \in \mathscr{G}$ such that $M \models \bigwedge \Gamma_1[\vec{\mathrm{Rel}}(\vec{X})/\vec{S}]$. Now let M' be $M[\vec{\mathrm{Rel}}(\vec{X})/\vec{S}]$; since $\vec{X}$ is in $\mathscr{G}$, it is not difficult to see that $(M', \mathscr{G})$

is a general model. Furthermore, it is Θ-closed, $M' \models \Gamma_1$, and $M' \models_h \Gamma$. Hence, $(M', \mathcal{G}) \models_{\|\gamma\|_h} \varphi$, but since the relation symbols $\vec{S}$ do not occur in γ or in φ, this implies that $(M, \mathcal{G}) \models_{\|\gamma\|_h} \varphi$. □

In order to prove completeness, we first need a definition and a simple lemma.

Definition 6.5 (Θ^{FO}) Let Θ be a relation existence theory. Then Θ^{FO} is the theory $\{\theta_i[\vec{S}_i/\vec{R}] : \exists \vec{R}\theta_i(\vec{R}) \in \Theta\}$, where the tuples of symbols $\vec{S}_i$ are all disjoint and otherwise unused.

Lemma 6.6 *Let Θ be a relation existence theory, and let M be a model such that $M \models \Theta^{FO}$. Then the least general model over it $(M, \mathcal{L})$ is Θ-closed.*

Proof Consider any $\exists \vec{R}\theta(\vec{R}) \in \Theta$. Then $M \models \theta(\vec{S}_i)$ for some tuple of relation symbols $\vec{S}_i$ in the signature of M. Then, the teams $\vec{X}$ associated to the corresponding relations are in $\mathcal{L}$, and for these teams we have $M \models \theta[\vec{\mathrm{Rel}}(\vec{X})/\vec{R}]$, as required. □

Theorem 6.7 (Completeness) *Suppose that $\Gamma \mid \gamma \vdash \varphi$ is Θ-valid. Then it is provable in our proof system plus **PS**-Θ.*

Proof Let M be any first-order model satisfying Θ^{FO}, where we assume that the relation symbols used in the construction of Θ^{FO} do not occur in Γ, in γ, or in φ. Then, by Lemma 6.6, $(M, \mathcal{L})$ is Θ-closed, and this implies that, for all assignments h such that $M \models_h \Gamma$, $M \models_{\|\gamma\|_h} \varphi$.

Therefore, $\Theta^{FO}, \Gamma \mid \gamma \vdash \varphi$ is valid, and hence, for some finite $\Delta \subseteq \Theta^{FO}$ it holds that $\Delta, \Gamma \mid \gamma \vdash \varphi$ is provable. Now we can get rid of Δ through repeated applications of rule **PS**-Θ and, therefore, prove that $\Gamma \mid \gamma \vdash \varphi$, as required. □

7 Conclusions

We began this work by defining a general semantics for independence logic. Then we proved that—owing to the relationship between independence logic and existential second-order logic—in order to study validity with respect to this semantics it suffices to examine *least* general models. We then showed that, because of the correspondence between teams in least general models and first-order formulas with parameters, we could limit ourselves to study *entailments* between first-order team-defining formulas and independence logic formulas. Finally, we developed a sound and complete proof system for this semantics, and we showed that this system can be easily strengthened by assuming the existence of more teams.

As we said, the correspondence between independence logic and *existential* second-order logic is of essential importance for the construction we described: extending our approach to such logics as team logic or intuitionistic dependence logic promises to be nontrivial, although certainly not impossible. The relationship between our approach and the one developed by Kontinen and Väänänen in [12] is also certainly worth investigating.

Furthermore, entailment semantics—the key ingredient of our construction, and our "bridge" between general model semantics and the proof system—is, as we wrote, of independent interest for a more syntactic approach to the study of dependence and independence, and more in general for the study of this interesting family of logics.

Notes

1. Later, Cameron and Hodges [2] proved, through combinatorial methods, that no compositional semantics for such a logic exists in which the satisfaction relation is predicated over single assignments.

2. That paper also characterized precisely the expressive power of independence logic with respect to open formulas, thus answering an open problem of [5], and proved that inclusion and exclusion logic are strictly weaker than independence logic.

3. Disjunction and conjunction are associative in this logic; hence, we will write $\varphi \vee \psi$ and $\varphi \wedge \psi$ for $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$ wherever the intended meaning is clear.

4. Here by $s[m/y]_{|\vec{x}}$ we intend the restriction of $s[m/y]$ to the domain $\{x_1, \ldots, x_n\}$. If y is among $x_1, \ldots, x_n$, then this is the same of $s[m/y]$ itself; otherwise, it is simply s.

5. That is, $\mathrm{Dom}(h') \supseteq \mathrm{Dom}(h)$, and $h'(\vec{p}) = h(\vec{p})$.

6. Without loss of generality, we can assume that the signature of M does not contain the symbol R.

7. To be entirely formal, this passage consists of $|\vec{p}|$ distinct applications of **PS-depar**, all of which are correct because none of the parameters in $\vec{p}$ appear in γ.

8. The fact that it is countable follows at once from the fact that it is a first-order theory over a countable vocabulary.

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