

Rank Functions and Partial Stability Spectra for Tame Abstract Elementary Classes

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Abstract We introduce a family of rank functions and related notions of total transcendence for Galois types in abstract elementary classes. We focus, in particular, on abstract elementary classes satisfying the condition known as *tameness*, where the connections between stability and total transcendence are most evident. As a byproduct, we obtain a partial upward stability transfer result for tame abstract elementary classes stable in a cardinal λ satisfying $\lambda^{\aleph_0} > \lambda$, a substantial generalization of a result of Baldwin, Kueker, and VanDieren.

1 Introduction

Abstract elementary classes (AECs) were introduced by Shelah [12] as a framework in which to address fundamental questions concerning the classification theory of nonelementary classes of structures. Early work on nonelementary classes was largely rooted in syntactic considerations, unfailingly identifying types with satisfiable sets of formulas, and relying on methods closely tailored to the peculiarities of the ambient logic. Abstract elementary classes and their associated, fundamentally nonsyntactic Galois types represent a broad, unifying framework that supports a more uniform treatment of questions concerning categoricity and stability, transcending the focus on individual logics that limited the scope of earlier assays. Essentially the category-theoretic hulls of elementary classes—we discard syntax entirely but retain the essential diagrammatic properties of elementary embeddings—AECs are sufficiently general to encompass classes of models of sentences in $L_{\infty, \omega}$, $L(Q)$, and $L_{\omega_1, \omega}(Q)$, as well as homogeneous classes, but also seemingly rich enough in structure to permit the development of an interesting classification theory.

Naturally, with the added generality comes added difficulty: for AECs satisfying a variety of conditions intended to guarantee reasonable behavior, two decades of work have typically yielded only partial categoricity and stability transfer results. We here

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focus our attention on classes in which the Galois types satisfy the important locality condition known as tameness. In [6], Grossberg and VanDieren obtain a large-scale categoricity transfer result for such classes using a toolkit that includes such classical standbys as indiscernibles and Morley sequences. With these same methods, augmented with the notion of splitting, Grossberg and VanDieren also establish, in [5], a partial stability spectrum result for tame AECs. More recently, Baldwin, Kueker, and VanDieren [2] use a splitting argument to prove a striking upward stability transfer result for \aleph_0 -Galois-stable AECs (see [2, Theorem 2.1]): if κ is a cardinal of uncountable cofinality and the AEC is Galois stable in every cardinal less than κ , it must be Galois stable in κ as well.

We here introduce an entirely new set of vaguely classical methods, revolving around a family of rank functions for Galois types, with which we are able prove a generalization of the result of [2]. In particular, we show that this kind of transfer is triggered not merely by \aleph_0 -Galois stability but by stability in any cardinal λ satisfying the inequality $\lambda^{\aleph_0} > \lambda$, and significantly weaken the assumption on stability below κ .

2 Preliminaries

Essential background material on AECs can be found in Baldwin [1] or Grossberg [3], whereas the tame context, in particular, is addressed in [6], [5], and [2]. We work with an AEC $(\mathcal{K}, \prec_{\mathcal{K}})$, where \mathcal{K} is a class of structures in a fixed finitary signature $L(\mathcal{K})$, and $\prec_{\mathcal{K}}$ is the strong submodel relation. For any infinite cardinal λ , we denote by \mathcal{K}_{λ} the subclass of \mathcal{K} consisting of all models of cardinality λ (with the obvious interpretations for such notations as $\mathcal{K}_{\leq \lambda}$ and $\mathcal{K}_{> \lambda}$). We say that \mathcal{K} is λ -categorical if \mathcal{K}_{λ} contains only a single model up to isomorphism. For $M, N \in \mathcal{K}$, we say that a map $f : M \rightarrow N$ is a \mathcal{K} -embedding (or, more often, a strong embedding) if f is an injective homomorphism of $L(\mathcal{K})$ -structures, and $f[M] \prec_{\mathcal{K}} N$; that is, f induces an isomorphism of M onto a strong submodel of N . In that case, we write $f : M \hookrightarrow_{\mathcal{K}} N$. We also adopt the following notational shorthand: for any $M \in \mathcal{K}$ and any cardinal λ , we denote by $\text{Sub}_{\leq \lambda}(M)$ the set of all strong submodels $N \prec_{\mathcal{K}} M$ with $|N| \leq \lambda$. We assume throughout that \mathcal{K} satisfies the amalgamation and joint embedding properties.

Galois types, which have their origins in the work of Shelah (first appearing in [12] and [13]), are closely adapted to the syntax-neutral context of AECs. Although they can be defined in very general classes, they have a particularly simple description in AECs with amalgamation: in any \mathcal{K} of this form, we may fix a large, strongly model-homogeneous monster model $\mathbb{C} \in \mathcal{K}$. In this case, we are able to characterize Galois types as follows.

Definition 2.1 Let $M \in \mathcal{K}$, and let $a \in \mathbb{C}$. The *Galois type of a over M* , denoted $\text{ga-tp}(a/M)$, is the orbit of a in \mathbb{C} under $\text{Aut}_M(\mathbb{C})$, the group of automorphisms of \mathbb{C} that fix M . We denote by $\text{ga-S}(M)$ the set of all Galois types over M .

In case \mathcal{K} is an elementary class with $\prec_{\mathcal{K}}$ as elementary submodel, the Galois types over M correspond to the complete first-order types over M . In general, however, Galois types and syntactic types do not match up, even in cases when the logic underlying the AEC is clear (say, $\mathcal{K} = \text{Mod}(\psi)$, with $\psi \in L_{\omega_1, \omega}$). Recall that \mathcal{K} is said to be λ -Galois stable if for every $M \in \mathcal{K}_{\lambda}$, $|\text{ga-S}(M)| = \lambda$. Moreover, for any $M, a \in \mathbb{C}$, and $N \prec_{\mathcal{K}} M$, we define the restriction of $\text{ga-tp}(a/M)$ to N , denoted

$\text{ga-tp}(a/M) \upharpoonright N$, to be the orbit of a under $\text{Aut}_N(\mathbb{C})$, and we say that M realizes a type p over $N \prec_{\mathcal{K}} M$ if there is an element $a \in M$ such that $\text{ga-tp}(a/M) \upharpoonright N = p$. Naturally, we say that a model M is λ -Galois saturated if for every $N \prec_{\mathcal{K}} M$ with $|N| < \lambda$ and every $p \in \text{ga-S}(N)$, p is realized in M .

Henceforth, the word “type” should be understood to mean “Galois type.” Moreover, when there is no risk of confusion, we will omit the word “Galois” altogether, speaking simply of types, λ -stability, and λ -saturation.

Initial attempts at establishing a classification theory for AECs have focused on classes satisfying a variety of broad structural conditions: for example, excellence (as described in Grossberg and Kolesnikov [4]) and the existence of good or semigood frames (as considered in Shelah [14] and Jarden and Shelah [7]), which echo classical notions of simplicity, stability, and superstability, respectively. We here concern ourselves primarily with tameness, a property that has no particularly natural classical analogue. Roughly speaking, tameness (of the form that we will consider) means that types are completely determined by their restrictions to small submodels, a condition slightly reminiscent of the locality properties of syntactic types. A possible intuition is the following: we may regard types over small models as playing a role analogous to formulas in first-order model theory, in which case tameness guarantees that types are determined by their constituent “formulas.” In this analogy, tameness of Galois types may in fact be most closely identified with completeness in the realm of syntactic types.

We consider two degrees of tameness.

Definition 2.2

- (1) We say that \mathcal{K} is χ -tame if for every $M \in \mathcal{K}$, if p and p' are distinct types over M , then there is an $N \in \text{Sub}_{\leq \chi}(M)$, such that $p \upharpoonright N \neq p' \upharpoonright N$.
- (2) We say that \mathcal{K} is weakly χ -tame if for every saturated $M \in \mathcal{K}$, if p and p' are distinct types over M , then there is an $N \in \text{Sub}_{\leq \chi}(M)$, such that $p \upharpoonright N \neq p' \upharpoonright N$.

We close the preliminaries with a brief topological aside (a more detailed treatment can be found in Lieberman [9]) which, although it will not necessarily be emphasized in the continuation, provides essential motivation for the definitions that follow. Recall the identification that formed the basis of our intuition concerning tameness of Galois types—types over small submodels as “formulas”—and fix our notion of “small” as “of cardinality less than or equal to λ ,” with $\lambda \geq \text{LS}(\mathcal{K})$. Just as the Stone space topology on syntactic types is generated from a basis of open—indeed, clopen—sets, each one consisting of all complete types containing a given formula, we here consider sets of Galois types extending given “formulas”; that is, for each model $M \in \mathcal{K}$, we consider all sets of the form

$$U_{p,N} = \{q \in \text{ga-S}(M) : q \upharpoonright N = p\},$$

where $N \in \text{Sub}_{\leq \lambda}(M)$ and $p \in \text{ga-S}(N)$. It is an easy exercise to show that the $U_{p,N}$ form a basis for a topology on $\text{ga-S}(M)$ —we denote the resulting space by X_M^λ —and, moreover, that they are clopen sets, analogous to the first-order case. In fact, if \mathcal{K} is an elementary class (in which case Galois types may be identified with complete first-order types), we see that the topologies thus obtained refine the standard syntactic topology.

In fact, what we obtain is a spectrum of spaces X_M^λ associated with each $M \in \mathcal{K}$, with one space for each $\lambda \geq \text{LS}(\mathcal{K})$, and a tight correspondence between topological properties of the spaces in the various spectra and model-theoretic properties of the AEC (see [9] for a complete account). In particular, χ -tameness of an AEC means precisely that any two distinct types over a model $M \in \mathcal{K}$ can be distinguished by their restrictions to a submodel of M of size at most χ ; that is, they can be separated by basic open sets in each of the spaces X_M^λ for $\lambda \geq \chi$. It should come as no surprise, then, that we have the following proposition.

Proposition 2.3 *If \mathcal{K} is χ -tame, then X_M^λ is Hausdorff for all $M \in \mathcal{K}$ and $\lambda \geq \chi$.*

Furthermore, in light of the fact that every X_M^λ has a basis of clopens, the spaces in question will be totally disconnected—nearly Stone spaces. As it happens, though, tameness rules out not merely compactness but even countable compactness. The chief complication results from the following.

Proposition 2.4 *For any $M \in \mathcal{K}$ and $\lambda \geq \text{LS}(\mathcal{K})$, the intersection of any λ open subsets of X_M^λ is open.*

Proof It suffices to show that the intersection of λ basic open sets is open. To that end, let $\{U_{p_i, N_i} \mid i < \lambda\}$ be a family of open subsets of X_M^λ , and let $q \in \bigcap_{i < \lambda} U_{p_i, N_i}$. The union of the N_i is a subset of M of cardinality λ , so, by the downward Löwenheim–Skolem property, there is a submodel $N <_{\mathcal{K}} M$ that contains it. By coherence, $N_i <_{\mathcal{K}} N$ for all i . Consider the basic open neighborhood $U_{q \upharpoonright N, N}$. It certainly contains q . For any $q' \in U_{q \upharpoonright N, N}$, moreover,

$$q' \upharpoonright N_i = q' \upharpoonright N \upharpoonright N_i = q \upharpoonright N \upharpoonright N_i = q \upharpoonright N_i = p_i.$$

Hence q' is contained in $\bigcap_{i < \lambda} U_{p_i, N_i}$ and, as this holds for any such q' , $U_{q \upharpoonright N, N} \subseteq \bigcap_{i < \lambda} U_{p_i, N_i}$. \square

As a result, if an AEC is χ -tame, the spaces X_M^λ for $\lambda \geq \chi$ exhibit an extraordinary degree of separation. Although we leave the proofs to [9], the following results (Propositions 6.5 and 7.1 in the aforementioned paper) convey the essential flavor.

Proposition 2.5 *Let \mathcal{K} be χ -tame, and let $\lambda \geq \chi$. Any set $S \subseteq X_M^\lambda$ with $|S| \leq \lambda$ is closed and discrete.*

Thus there will be a number of infinite sets with no accumulation point, meaning that, as claimed above, the spaces cannot be countably compact. We close with a closely related characterization of what it means to be an accumulation point in X_M^λ .

Proposition 2.6 *Let \mathcal{K} be χ -tame, and let $\lambda \geq \chi$. A type $q \in X_M^\lambda$ is an accumulation point of a set $S \subseteq X_M^\lambda$ if and only if for every neighborhood U of q , $|U \cap S| > \lambda$.*

3 Ranks for Galois Types

We have just seen that if an AEC \mathcal{K} is χ -tame, then for any $\lambda \geq \chi$ (and, of course, $\lambda \geq \text{LS}(\mathcal{K})$), the criterion for a type $q \in \text{ga-S}(M)$ to be an accumulation point of a family of types in X_M^λ is quite stringent: every neighborhood must contain more than λ types in the given family. This condition inspires the definition of the following family of ranks.

Definition 3.1 (R^λ) Assume that \mathcal{K} is χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$. For $\lambda \geq \chi$, we define R^λ by the following induction. For any $q \in \text{ga-S}(M)$ with $|M| \leq \lambda$,

- (1) $R^\lambda[q] \geq 0$;
- (2) $R^\lambda[q] \geq \alpha$ for limit α if $R^\lambda[q] \geq \beta$ for all $\beta < \alpha$;
- (3) $R^\lambda[q] \geq \alpha + 1$ if there exists a structure $M' \succ_{\mathcal{K}} M$ such that q has strictly more than λ extensions to types q' over M' with the property that

$$R^\lambda[q' \upharpoonright N] \geq \alpha \quad \text{for all } N \in \text{Sub}_{\leq \lambda}(M').$$

(\dagger) For types q over M of arbitrary size, we define

$$R^\lambda[q] = \min\{R^\lambda[q \upharpoonright N] : N \in \text{Sub}_{\leq \lambda}(M)\}.$$

Before we proceed, a bit of motivation for this long-winded definition: in topologizing $\text{ga-S}(M)$ as X_M^λ (and in developing our intuition with regard to tameness), we essentially allow the types over substructures of size at most λ to play a role analogous to that ordinarily played by formulas. In defining the ranks R^λ , we first define the rank of these “formulas” and subsequently define the rank on types extending them as the minimum of the ranks of their constituent “formulas,” just as in the definition of Morley rank in classical model theory.

There is something to check here, though. The types over models of cardinality at most λ whose ranks were defined by clauses (1)–(3) are also covered by (\dagger). We must ensure that there is no possibility of conflict between the two. Let R_*^λ denote the ranks assigned to such types using only the first three clauses.

Lemma 3.2 *The ranks R_*^λ are monotone: for any $N, N' \in \mathcal{K}_{\leq \lambda}$, $N' \prec_{\mathcal{K}} N$, and any $q \in \text{ga-S}(N)$, $R_*^\lambda[q] \leq R_*^\lambda[q \upharpoonright N']$.*

Proof We show that for any ordinal α , $R_*^\lambda[q] \geq \alpha$ implies $R_*^\lambda[q \upharpoonright N'] \geq \alpha$. We proceed by an easy induction on α . The zero case is trivial, and the limit cases follow from the induction hypothesis. For the successor case, notice that if $R_*^\lambda[q] \geq \alpha + 1$, the extension $M \succ_{\mathcal{K}} N$ that witnesses this fact is also an extension of N' and witnesses that $R_*^\lambda[q \upharpoonright N'] \geq \alpha + 1$. \square

So we need not worry that the rank assigned to a type $q \in \text{ga-S}(M)$ with $M \in \mathcal{K}_{\leq \lambda}$ under (\dagger), that is, $R^\lambda[q] = \min\{R^\lambda[q \upharpoonright N] : N \prec_{\mathcal{K}} M, |N| \leq \lambda\}$, will differ from the rank of q under the first three clauses. Hence the ranks R^λ are well defined. In fact, we can now restate clause (3) in a more compact form: for $q \in \text{ga-S}(M)$ with $M \in \mathcal{K}_{\leq \lambda}$,

- (3') $R^\lambda[q] \geq \alpha + 1$ if there exists a structure $M' \succ_{\mathcal{K}} M$ such that q has strictly more than λ many extensions to types q' over M' with $R^\lambda[q'] \geq \alpha$.

We will invariably use this formulation in the sequel.

We now consider the basic properties of the ranks R^λ , beginning with an extension of our partial monotonicity result, Lemma 3.2.

Proposition 3.3 (Monotonicity) *If $M \prec_{\mathcal{K}} M'$ and $q \in \text{ga-S}(M')$, then $R^\lambda[q] \leq R^\lambda[q \upharpoonright M]$.*

Proof The proof is trivial, from the second clause of the definition. \square

Bear in mind that “ p is a restriction of q ” corresponds to the first-order “ $q \upharpoonright p$,” so this is an analogue of the monotonicity result for, say, the classical Morley rank.

Proposition 3.4 (Invariance) *Let f be an automorphism of the monster model \mathfrak{C} , let $q \in \text{ga-S}(M)$, and let $M' = f[M]$. Then the type $f[q] \in \text{ga-S}(M')$ satisfies $R^\lambda[f[q]] = R^\lambda[q]$.*

Proof We again proceed by induction, and once again the zero and limit cases are trivial. It remains to show that whenever $R^\lambda[q] \geq \alpha + 1$, $R^\lambda[f[q]] \geq \alpha + 1$, with the converse following by replaying the argument with f^{-1} in place of f . As before, if $R^\lambda[q] \geq \alpha + 1$, then each $N \in \text{Sub}_{\leq \lambda}(M)$ has an extension M_N over which $q \upharpoonright N$ has more than λ many extensions of rank at least α . Notice that for any $N' \in \text{Sub}_{\leq \lambda}(M')$, $N' = f[N]$ for some $N \in \text{Sub}_{\leq \lambda}(M)$ and, moreover, that

$$f[q] \upharpoonright N' = f[q] \upharpoonright f[N] = f[q \upharpoonright N].$$

The extensions of $q \upharpoonright N$ to M_N of rank at least α map bijectively under the automorphism f to extensions of $f[q] \upharpoonright N'$ to types over $f[M_N]$ and, by the induction hypothesis, these image types are also of rank at least α . As a result, for any $N' \in \text{Sub}_{\leq \lambda}(M')$, $R^\lambda[f[q] \upharpoonright N'] \geq \alpha + 1$. Thus $R^\lambda[q] \geq \alpha + 1$, as desired. \square

Trivially, we have the following.

Proposition 3.5 ($\leq \lambda$ -local character) *For any $q \in \text{ga-S}(M)$, there is an $N \in \text{Sub}_{\leq \lambda}(M)$ with $R^\lambda[q \upharpoonright N] = R^\lambda[q]$.*

Slightly less trivially, we have the following.

Proposition 3.6 (Relations between R^λ) *Whenever $\lambda \leq \mu$, $R^\mu[q] \leq R^\lambda[q]$.*

Proof The proof is by induction, with the zero and limit cases still trivial. If $R^\mu[q] \geq \alpha + 1$, then for every $N \in \text{Sub}_{\leq \mu}(M)$, $q \upharpoonright N$ has more than μ extensions to types of R^μ -rank at least α over some $M_N \succ_{\mathcal{X}} N$. In particular, there are more than λ such extensions and, by the induction hypothesis, they are all of R^λ -rank at least α . Noticing that $\text{Sub}_{\leq \lambda}(M) \subseteq \text{Sub}_{\leq \mu}(M)$, one sees that the same holds for all $N \in \text{Sub}_{\leq \lambda}(M)$, meaning that, ultimately, $R^\lambda[q] \geq \alpha + 1$. \square

The ranks $R^\lambda[-]$ also have the following attractive property: letting $\text{CB}^\lambda[q]$ denote the topological Cantor–Bendixson rank of a type $q \in X_M^\lambda$, where M is the domain of q , we have the following proposition.

Proposition 3.7 *For any q (say, $q \in \text{ga-S}(M)$), $\text{CB}^\lambda[q] \leq R^\lambda[q]$.*

Proof We show by induction that $\text{CB}^\lambda[q] \geq \alpha$ implies $R^\lambda[q] \geq \alpha$. The zero and limit cases are not interesting. If $\text{CB}^\lambda[q] \geq \alpha + 1$, q is an accumulation point of types of CB^λ -rank at least α . This means, by Proposition 2.6, that every basic open neighborhood of q contains more than λ such types, which is to say that each $q \upharpoonright N$ with $N \in \text{Sub}_{\leq \lambda}(M)$ has more than λ many extensions to types over M of CB^λ -rank at least α . By the induction hypothesis, these types are of R^λ -rank at least α , and for each $N \in \text{Sub}_{\leq \lambda}(M)$ witness the fact that $R^\lambda[q \upharpoonright N] \geq \alpha + 1$. Naturally, it follows that $R^\lambda[q] \geq \alpha + 1$. \square

We have a quick fact about CB^λ , incidentally, which we obtain in the usual way.

Proposition 3.8 *If every $q \in X_M^\lambda$ has ordinal CB^λ -rank, isolated points are dense in X_M^λ .*

In particular, then, if all of the types over a model $M \in \mathcal{K}$ have ordinal R^λ -rank, isolated points are dense in X_M^λ .

One of the great virtues of the classical Morley rank is that types over models have unique extensions of the same Morley rank: if $p \in S(A)$ has $R[p] = \alpha$ and $B \supseteq A$, then there is at most one type $q \in S(B)$ that extends p and has $R[q] = \alpha$. While we do not have a unique extension property for the ranks R^λ , we do have a very close approximation, Proposition 3.10 below. First, we notice the following.

Lemma 3.9 *Let $M \prec_{\mathcal{K}} M'$, $q \in \text{ga-S}(M)$ with an ordinal R^λ -rank. There are at most λ extensions $q' \in \text{ga-S}(M')$ with $R^\lambda[q'] = R^\lambda[q]$.*

Proof For the sake of notational convenience, say $R^\lambda[q] = \alpha$. Let $S = \{q' \in \text{ga-S}(M') \mid q' \upharpoonright M = q, \text{ and } R^\lambda[q'] = \alpha\}$, and suppose that $|S| > \lambda$. For any $N \in \text{Sub}_{\leq \lambda}(M)$, each q' is an extension of $q \upharpoonright N$, meaning that $R^\lambda[q \upharpoonright N] \geq \alpha + 1$ for all such N . But then $R^\lambda[q] \geq \alpha + 1$, a contradiction. \square

It is worth noting here that we have made no use of tameness in the discussion above—neither in the definition of the ranks R^λ nor in the proofs of the properties thereof—and have merely made use of the fact that $\lambda \geq \text{LS}(\mathcal{K})$. From now on, however, it will often be critically important that we have the ability to separate distinct types, leading us to restrict attention to those R^λ with $\lambda \geq \chi$, the tameness cardinal of the AEC at hand. We will be very explicit in making this assumption whenever it is necessary, beginning with the following essential property of the R^λ .

Proposition 3.10 (Quasi-unique extension) *Let \mathcal{K} be χ -tame with $\chi \geq \text{LS}(\mathcal{K})$, and let $\lambda \geq \chi$. Let $M \prec_{\mathcal{K}} M'$, $q \in \text{ga-S}(M)$, and say that $R^\lambda[q] = \alpha$. Given any rank α extension q' of q to a type over M' , there is an intermediate structure M'' , $M \prec_{\mathcal{K}} M'' \prec_{\mathcal{K}} M'$, $|M''| \leq |M| + \lambda$, and a rank α extension $p \in \text{ga-S}(M'')$ of q' with $p \in \text{ga-S}(M')$ as its unique rank α extension.*

Proof Again, let $S = \{q'' \in \text{ga-S}(M') \mid q'' \upharpoonright M = q, \text{ and } R^\lambda[q''] = \alpha\}$. From Lemma 3.9, $|S| \leq \lambda$, meaning that S is discrete (by tameness and Proposition 2.5). Thus there is some $N \in \text{Sub}_{\leq \lambda}(M')$ with the property that $q' \upharpoonright N$ satisfies $U_{q' \upharpoonright N, N} \cap S = \{q'\}$. Take $M'' \in \mathcal{K}$ containing both M and N and with $|M''| \leq |M| + |N| + \text{LS}(\mathcal{K}) \leq |M| + \lambda$, and set $p = q' \upharpoonright M''$. Notice that, by monotonicity,

$$\alpha = R^\lambda[q'] \leq R^\lambda[p] \leq R^\lambda[q] = \alpha,$$

so $R^\lambda[p] = \alpha$. Naturally, q' is a rank α extension of p . Any such extension q'' of p is also a rank α extension of q , hence in S , and also an extension of $q' \upharpoonright N$. By choice of N , then, we must have $q'' = q'$. \square

A possible gloss for this complicated-looking result: while a type $p \in \text{ga-S}(M)$ of R^λ -rank α may have many extensions of rank α over a model $M' \succ_{\mathcal{K}} M$, we need only expand its domain ever so slightly (adding at most λ elements of M') to guarantee that the rank α extension is unique. As innocuous as this proposition may seem, it is the linchpin of the analysis of stability in this framework and will see extensive use in Sections 5 and 6. Of course, quasi-unique extension applies only to types with ordinal R^λ -rank. To take the fullest possible advantage of this property, then, we would be wise to confine our attention to AECs where R^λ is ordinal valued on all types associated with the class.

4 Stability and λ -Total Transcendence

In classical model theory, we call a first-order theory *totally transcendental* if all types over subsets of the monster model are assigned ordinal values by the rank. We now define analogous notions for Galois types in AECs, one for each rank R^λ .

Definition 4.1 We say that \mathcal{K} is λ -*totally transcendental* if for every $M \in \mathcal{K}$ and $q \in \text{ga-S}(M)$, $R^\lambda[q]$ is an ordinal.

Proposition 4.2 *If \mathcal{K} is λ -totally transcendental, then it is μ -totally transcendental for all $\mu \geq \lambda$.*

Proof See Proposition 3.6, the result concerning the relationship between the various ranks. \square

Putting this proposition together with Propositions 3.7 and 3.8, we have the following.

Proposition 4.3 *If \mathcal{K} is λ -totally transcendental, then for any M and $\mu \geq \lambda$, isolated points are dense in X_M^μ .*

As a nice special case, if $\text{LS}(\mathcal{K}) = \aleph_0$, and \mathcal{K} is \aleph_0 -tame and \aleph_0 -totally transcendental, then isolated points are dense in each space $X_M^{\aleph_0}$. This bears a certain resemblance to the classical result linking total transcendence to the density of isolated points in the spaces of syntactic types.

As interesting as this result may be, the true power of the notion of λ -total transcendence lies in the leverage it provides in rather a different area: bounding the number of types over models. We will take advantage of this in Sections 5 and 6 below to prove a variety of results related to Galois stability. Before we give any further thought to λ -totally transcendental AECs, though, it seems natural to inquire whether the notion is, in fact, a meaningful one, that is, whether one can actually find λ -totally transcendental AECs. One can, as the following proposition suggests.

Theorem 4.4 *If \mathcal{K} is χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$ and is λ -stable with $\lambda \geq \chi$ and $\lambda^{\aleph_0} > \lambda$, then \mathcal{K} is λ -totally transcendental.*

Proof Suppose that R^λ is unbounded; that is, suppose that there is a type p over some $N \prec_{\mathcal{K}} \mathbb{C}$ with $R^\lambda[p] = \infty$. Indeed, we may assume that $|N| \leq \lambda$. (If there is a type of rank ∞ over a larger structure, consider one of its restrictions.) We proceed to construct a \mathcal{K} -structure \bar{N} of size λ over which there are λ^{\aleph_0} many types, thereby establishing that \mathcal{K} is not λ -stable. We first need the following.

Claim 4.5 *For any λ , there exists an ordinal α such that for any q over $M \prec_{\mathcal{K}} \mathbb{C}$, $R^\lambda[q] = \infty$ just in case $R^\lambda[q] \geq \alpha$.*

Proof By local character and monotonicity, any ordinal rank attained over a model in \mathcal{K} is attained over a model of size at most λ , meaning that it suffices to establish the claim for types over models in $\mathcal{K}_{\leq \lambda}$ (i.e., “formulas”). There are at most 2^λ isomorphism classes of such models. Suppose that $M, M' \in \mathcal{K}_{\leq \lambda}$ and that $\varphi : M \xrightarrow{\mathcal{K}} M'$ is an isomorphism. By strong model-homogeneity of \mathbb{C} , φ extends to an automorphism α of \mathbb{C} . Given that R^λ is invariant under such automorphisms, the set of ordinal ranks attained by types over M (of which there are at most 2^λ) is the same as

the corresponding set over M' —in fact, this is an invariant on isomorphism classes. It follows that

$$\begin{aligned} & |\{R^\lambda[q] \mid q \in \text{ga-S}(M), M \in \mathcal{K}_{\leq \lambda}, R^\lambda[q] < \infty\}| \\ & \leq |\{\text{isomorphism classes in } \mathcal{K}_{\leq \lambda}\}| \cdot 2^\lambda \leq 2^\lambda. \end{aligned}$$

Hence the set of all ordinal ranks attained cannot be cofinite in the class of ordinals, and we may take

$$\alpha = \sup\{R^\lambda[q] \mid q \in \text{ga-S}(M), M \in \mathcal{K}_{\leq \lambda}, R^\lambda[q] < \infty\}. \quad \square$$

We construct \bar{N} as follows.

Step 1: Set $p_\emptyset = p$. Since p_\emptyset satisfies $R^\lambda[p_\emptyset] = \infty$, $R^\lambda[p_\emptyset] > \alpha$. Hence there is an extension $M \succ_{\mathcal{K}} N$ over which p_\emptyset has more than λ many extensions of rank at least α and thus of rank ∞ . Take any λ many of them, say, $\{q_i \in \text{ga-S}(M) \mid i < \lambda\}$. By Proposition 2.5, this set is discrete in X_M^λ , meaning that for each $i < \lambda$, there is an $N'_i \in \text{Sub}_{\leq \lambda}(M)$ with the property that $q_j \upharpoonright N'_i \neq q_i \upharpoonright N'_i$ whenever $j \neq i$. Let N_\emptyset be a \mathcal{K} -substructure of M containing $N \cup (\bigcup_{i < \lambda} N_i)$, $|N_\emptyset| = \lambda$. For each $i < \lambda$, define $p_i = q_i \upharpoonright N_\emptyset$. Notice that

- $R^\lambda[p_i] \geq R^\lambda[q_i] = \infty$;
- we have $p_i \upharpoonright N = (q_i \upharpoonright N_\emptyset) \upharpoonright N = q_i \upharpoonright N = p_\emptyset$;
- if $p_i = p_j$, then

$$q_i \upharpoonright N_i = (q_i \upharpoonright N_\emptyset) \upharpoonright N_i = p_i \upharpoonright N_i = p_j \upharpoonright N_i = (q_j \upharpoonright N_\emptyset) \upharpoonright N_i = q_j \upharpoonright N_i,$$

in which case we must have $i = j$. That is, the p_i are distinct.

So we have a family $\{p_i \in \text{ga-S}(N_\emptyset) \mid i < \lambda\}$ of distinct extensions of p_\emptyset , all of rank ∞ .

Step $n + 1$: For each $\sigma \in \lambda^n$, we have a structure N_σ of size at most λ and a family of distinct types $\{p_{\sigma i} \in \text{ga-S}(N_\sigma) \mid i < \lambda\}$, all of rank ∞ . For each $i < \lambda$, apply the process of step 1 to each $p_{\sigma i}$. By this method, we obtain (for each $i < \lambda$) an extension $N_{\sigma i} \succ_{\mathcal{K}} N_\sigma$ with $|N_{\sigma i}| = \lambda$ and a family $\{p_{\sigma ij} \in \text{ga-S}(N_{\sigma i}) \mid j < \lambda\}$ of distinct extensions of $p_{\sigma i}$, all of rank ∞ .

Step ω : Notice that for each $\tau \in \lambda^\omega$, the sequence $N_\emptyset, N_{\tau \upharpoonright 1}, \dots, N_{\tau \upharpoonright n}, \dots$ is increasing and, moreover, $\prec_{\mathcal{K}}$ -increasing (by coherence and the fact that everything embeds strongly in \mathbb{C}). It follows that $N_\tau := \bigcup_{n < \omega} N_{\tau \upharpoonright n}$ is in \mathcal{K} . Notice also that the $p_{\tau \upharpoonright (n+1)} \in \text{ga-S}(N_{\tau \upharpoonright n})$ for $n < \omega$ form an increasing sequence of types over the structures in this union. By (ω, ∞) -compactness of AECs (see [1, Theorem 11.1]), there is a type $p_\tau \in \text{ga-S}(N_\tau)$ with the property that for all $n < \omega$, $p_\tau \upharpoonright N_{\tau \upharpoonright n} = p_{\tau \upharpoonright (n+1)}$. Let \bar{N} be a structure of size λ containing the union

$$\bigcup_{\sigma \in \lambda^{< \omega}} N_\sigma$$

(which is possible, since the union is of size at most $\lambda \cdot \lambda^{< \omega} = \lambda \cdot \lambda = \lambda$). For each $\tau \in \lambda^\omega$, let q_τ be an extension of p_τ to a type over \bar{N} . There are λ^{\aleph_0} many such types, all over \bar{N} —it remains only to show that they are distinct. To that end, suppose that $\tau, \tau' \in \lambda^\omega$, $\tau \neq \tau'$. It must be the case that for some $\sigma \in \lambda^{< \omega}$, $\tau = \sigma i \dots$ and $\tau' = \sigma j \dots$ with $i \neq j$. Then

$$q_\tau \upharpoonright N_\sigma = (q_\tau \upharpoonright N_\tau) \upharpoonright N_\sigma = p_\tau \upharpoonright N_\sigma = p_{\sigma i}.$$

Similarly, we have

$$q_{\tau'} \upharpoonright N_\sigma = (q_{\tau'} \upharpoonright N_{\tau'}) \upharpoonright N_\sigma = p_{\tau'} \upharpoonright N_\sigma = p_{\sigma j}.$$

Since $p_{\sigma i}$ and $p_{\sigma j}$ are distinct by construction, $q_\tau \neq q_{\tau'}$, and the types are distinct as claimed. By our assumption that $\lambda^{\aleph_0} > \lambda$, then, \mathcal{K} is not stable in λ . \square

So the notion of λ -total transcendence is far from vacuous: totally transcendental AECs do exist.

5 Transfer Results

As promised, we will now employ λ -total transcendence as a tool to analyze the proliferation of types over models in AECs. Of the two methods for bounding the number of types over large models in totally transcendental AECs presented in Lieberman [8] (namely, Theorems 4.17 and 4.21), we here concern ourselves only with the latter, which is the crucial ingredient in the generalization of the upward stability transfer result from [2].

Theorem 5.1 *Let \mathcal{K} be χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$, and let it be λ -totally transcendental with $\lambda \geq \chi$. For any $M \in \mathcal{K}$ with $\text{cf}(|M|) > \lambda$,*

$$|\text{ga-S}(M)| \leq |M| \cdot \sup\{|\text{ga-S}(N)| \mid N \prec_{\mathcal{K}} M, |N| < |M|\}.$$

Proof Take a filtration of M ,

$$M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} \cdots \prec_{\mathcal{K}} M_\alpha \prec_{\mathcal{K}} \cdots \prec_{\mathcal{K}} M,$$

for $\alpha < |M|$, with $|M_\alpha| < |M|$ for all α and $M = \bigcup_{\alpha < |M|} M_\alpha$. Let $q \in \text{ga-S}(M)$. By λ -total transcendence, $\text{R}^\lambda[q] = \beta$ for some ordinal β and, moreover, this is witnessed by a restriction to a small submodel of M . That is, there is a submodel $N \in \text{Sub}_{\leq \lambda}(M)$ with $\text{R}^\lambda[q \upharpoonright N] = \beta$. By Proposition 3.10, there is an intermediate extension $N \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} M$ with $|N'| = \lambda$ such that $q \upharpoonright N'$ has unique rank β extension q over M . Since $\text{cf}(|M|) > \lambda$, $N' \subseteq M_\alpha$ for some α (and, by coherence, $N' \prec_{\mathcal{K}} M_\alpha$). Clearly, the type $q \upharpoonright M_\alpha$ has q as its only rank β extension over M . Hence

$$\left| \{q \in \text{ga-S}(M) \mid \text{R}^\lambda[q] = \beta\} \right| \leq \left| \bigcup_{\alpha < |M|} \{p \in \text{ga-S}(M_\alpha) \mid \text{R}^\lambda[p] = \beta\} \right|,$$

and thus

$$\begin{aligned} |\text{ga-S}(M)| &\leq \left| \bigcup_{\alpha < |M|} \text{ga-S}(M_\alpha) \right| \\ &\leq \sum_{\alpha < |M|} |\text{ga-S}(M_\alpha)| \\ &\leq |M| \cdot \sup\{|\text{ga-S}(M_\alpha)| \mid \alpha < |M|\}. \end{aligned}$$

The inequality in the statement of the theorem is an easy consequence. \square

In essence, the theorem asserts that for models M of cardinality μ with $\text{cf}(\mu) > \lambda$, the equality $|\text{ga-S}(M)| = \mu$ fails only if there is a submodel $N \prec_{\mathcal{K}} M$ with $|N| < \mu$ and $|\text{ga-S}(N)| > \mu$. To ensure that this does not occur—that we have, in short, stability in μ —we need considerably less than full stability in the cardinals below μ . To be precise, we have the following theorem.

Theorem 5.2 *Let \mathcal{K} be χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$ and λ -totally transcendental with $\lambda \geq \chi$, and let μ satisfy $\text{cf}(\mu) > \lambda$. If for every $M \in \mathcal{K}_{<\mu}$, $|\text{ga-S}(M)| \leq \mu$, \mathcal{K} is μ -stable.*

Proof For any $M \in \mathcal{K}_\mu$,

$$|\text{ga-S}(M)| \leq |M| \cdot \sup\{|\text{ga-S}(N)| \mid N \prec_{\mathcal{K}} M, |N| < |M|\} \leq |M| \cdot |M| = |M|.$$

□

We may tighten the statement up a bit, once we notice the following.

Lemma 5.3 *If there is a set S of cardinals cofinal in an interval $[\kappa, \mu)$ which has the property that for every $M \in \mathcal{K}$ with $|M| \in S$, $|\text{ga-S}(M)| \leq \mu$, then $|\text{ga-S}(M)| \leq \mu$ for every $M \in \mathcal{K}_{<\mu}$.*

Proof Let $M \in \mathcal{K}_{<\mu}$. If $|M| \in S$, we are done. If not, take $\nu \in S$ with $\nu > |M|$, and let M' be a strong extension of cardinality ν , $M \hookrightarrow_{\mathcal{K}} M' \hookrightarrow_{\mathcal{K}} \mathbb{C}$. By assumption, $|\text{ga-S}(M')| \leq \mu$. Every type over M has an extension to a type over M' , and, of course, distinct types must have distinct extensions. Hence

$$|\text{ga-S}(M)| \leq |\text{ga-S}(M')| \leq \mu.$$

□

So Theorem 5.2 becomes the following.

Theorem 5.4 *Let \mathcal{K} be χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$ and λ -totally transcendental with $\lambda \geq \chi$, and let μ satisfy $\text{cf}(\mu) > \lambda$. If there is a set S of cardinals cofinal in an interval $[\kappa, \mu)$ which has the property that for every $M \in \mathcal{K}$ with $|M| \in S$, $|\text{ga-S}(M)| \leq \mu$, \mathcal{K} is μ -stable.*

The assumption on the number of types over models of cardinality less than μ in the theorem above is very weak—it is certainly more than sufficient to assume that \mathcal{K} is ν -stable for each ν in a cofinal set below μ . In particular, we have the following.

Corollary 5.5 *If \mathcal{K} is χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$ and λ -totally transcendental with $\lambda \geq \chi$, μ satisfies $\text{cf}(\mu) > \lambda$, and \mathcal{K} is stable on a set of cardinals cofinal in an interval $[\kappa, \mu)$, then \mathcal{K} is μ -stable.*

Using Theorem 4.4 again, we may recast the corollary as a stability transfer result that makes no mention of total transcendence. Though the product of this rewriting is a trivial consequence of the foregoing discussion (and, indeed, an extremely special case of Theorem 5.4), it nonetheless generalizes a state-of-the-art result, [2, Theorem 2.1]. First, we have the statement of the theorem.

Theorem 5.6 *If \mathcal{K} is χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$ and λ -stable with $\lambda \geq \chi$ and $\lambda^{\aleph_0} > \lambda$, then for any μ with $\text{cf}(\mu) > \lambda$, if \mathcal{K} is stable on a set of cardinals cofinal in an interval $[\kappa, \mu)$, \mathcal{K} is μ -stable.*

If we restrict our attention to the special case in which the AEC is \aleph_0 -stable and assume more stability below μ than we need, strictly speaking, we have the following corollary.

Corollary 5.7 *If \mathcal{K} has $\text{LS}(\mathcal{K}) = \aleph_0$ and is \aleph_0 -tame and \aleph_0 -stable, then for any μ with $\text{cf}(\mu) > \aleph_0$, if \mathcal{K} is κ -stable for all $\kappa < \mu$, \mathcal{K} is μ -stable.*

In other words, in an AEC satisfying the hypotheses of the corollary, a run of stability will never come to an end at a cardinal μ of uncountable cofinality; it will, in fact, include μ , then μ^+ , then μ^{++} , and so on. This remarkable fact is the aforementioned result of [2] (which also appears as [1, Theorem 11.11]). What we have discerned by our method—and distilled in Theorem 5.6—is that it is not stability in \aleph_0 which is critical but rather stability in any cardinal λ satisfying $\lambda^{\aleph_0} > \lambda$. Moreover, it is not necessary to have stability in every cardinality below the cardinal of interest, μ , but rather to have stability (or, indeed, somewhat less than stability) in a cofinal sequence of cardinals below μ .

6 Weakly Tame AECs

We now shift our frame of reference. Thus far we have limited ourselves to AECs that are tame in some cardinal χ . The time has come to consider how the picture may differ in an AEC that is weakly χ -tame, rather than χ -tame. The most important difference is that the argument for Theorem 4.4 no longer works, meaning that we can no longer infer total transcendence from stability. In a sense, then, we must treat total transcendence as a property in its own right, independent from the more conventional properties of AECs. On the other hand, the argument for Theorem 5.1 still goes through, provided the model in question is saturated. That is, the following holds.

Proposition 6.1 *Let \mathcal{K} be weakly χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$ and μ -totally transcendental with $\mu \geq \chi$. If $M \in \mathcal{K}$ is a saturated model with $\text{cf}(|M|) > \mu$, then*

$$|\text{ga-S}(M)| \leq |M| \cdot \sup\{|\text{ga-S}(N)| \mid N \prec_{\mathcal{K}} M, |N| < |M|\}.$$

We can use this to get bounds on the number of types over more general (i.e., not necessarily saturated) models, provided that there are enough small saturated extensions in \mathcal{K} . In particular, if we can replace a model $M \in \mathcal{K}_\lambda$ with a saturated extension $M' \in \mathcal{K}_\lambda$, we have

$$|\text{ga-S}(M)| \leq |\text{ga-S}(M')|,$$

where the latter is governed by the bound in the proposition above. Given an adequate supply of such extensions, then, we can prove results similar to those found above, but now for weakly tame AECs. The existence of saturated extensions also has the following consequence.

Lemma 6.2 *If every model in \mathcal{K}_λ has a saturated extension in \mathcal{K}_λ , then for any $M \in \mathcal{K}_{<\lambda}$, $|\text{ga-S}(M)| \leq \lambda$.*

Proof Take $M \in \mathcal{K}_{<\lambda}$. Let M' be a strong extension of M of cardinality λ . By assumption, M' has a saturated extension \bar{M} of cardinality λ . This model, \bar{M} , realizes all types over submodels of size less than λ and hence realizes all types over the original model, M . It follows that $|\text{ga-S}(M)| \leq |\bar{M}| = \lambda$. \square

Notice that this is precisely the type-counting condition from which we are able to infer stability using the bound in Proposition 6.1. Hence we have the following.

Theorem 6.3 *Let \mathcal{K} be weakly χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$ and μ -totally transcendental with $\mu \geq \chi$. Suppose that λ is a cardinal with $\text{cf}(\lambda) > \mu$ and that every $M \in \mathcal{K}_\lambda$ has a saturated extension $M' \in \mathcal{K}_\lambda$. Then \mathcal{K} is λ -stable.*

Proof Let $M \in \mathcal{K}_\lambda$. By assumption, we may replace M with a saturated extension M' of cardinality λ , over which there can be at most λ types, by Proposition 6.1 and Lemma 6.2. As noted above, $|\text{ga-S}(M)| \leq |\text{ga-S}(M')|$, and we are done. \square

As a simple special case, we have the following.

Corollary 6.4 *Suppose that \mathcal{K} is weakly χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$ and μ -totally transcendental with $\mu \geq \chi$, and suppose that λ is a regular cardinal with $\lambda > \mu$. If \mathcal{K} is stable on an interval $[\kappa, \lambda)$, \mathcal{K} is λ -stable.*

Proof Let $M \in \mathcal{K}$ be of cardinality λ . By the usual union of chains argument, M has a saturated extension $M' \in \mathcal{K}$ which is also of cardinality λ . Using the bound in Proposition 6.1, we get $|\text{ga-S}(M')| = \lambda$, whence $\lambda = |M| \leq |\text{ga-S}(M)| \leq |\text{ga-S}(M')| = \lambda$, and thus $|\text{ga-S}(M)| = \lambda$. The result follows. \square

In particular, we have the following.

Corollary 6.5 *Suppose that \mathcal{K} is weakly χ -tame for some $\chi \geq \text{LS}(\mathcal{K})$, μ -totally transcendental with $\mu \geq \chi$, and λ -stable for some $\lambda \geq \mu$. Then \mathcal{K} is λ^+ -stable.*

This is weaker than a result of [2], which infers λ^+ -stability from λ -stability in any AEC that is weakly tame in $\chi \leq \lambda$, with no additional assumptions. The methods of [2]—splitting, limit models—are better adapted to the task of proving local transfer results of this form in the context of weakly tame AECs. The machinery of ranks and total transcendence provides us with leverage of a different sort, well suited to the task of producing partial spectrum results of a more global nature.

In particular, we saw in Theorem 6.3 that in a μ -totally transcendental AEC, even if merely weakly tame, we may infer stability in λ given an adequate supply of saturated extensions of cardinality λ . As it happens, the category-theoretic notion of weak λ -stability—introduced in Rosický [11], and analyzed in the context of AECs in Lieberman [10]—ensures that this requirement is met. The implications of this fact for the Galois stability spectra of weakly tame AECs are examined in [10].

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