Notre Dame Journal of Formal Logic
Volume 54, Number 2, 2013

# Complementation in Representable Theories of Region-Based Space

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**Abstract** Through contact algebras we study theories of mereotopology in a uniform way that clearly separates mereological from topological concepts. We identify and axiomatize an important subclass of closure mereotopologies (CMT) called *unique closure mereotopologies* (UCMTs) whose models always have orthocomplemented contact algebras (OCAs), an algebraic counterpart. The notion of MT-representability, a weak form of spatial representability but stronger than topological representability, suffices to prove that spatially representable complete OCAs are pseudocomplemented and satisfy the Stone identity. Within the resulting class of contact algebras the strength of the algebraic complementation delineates two classes of mereotopology according to the key ontological choice between mereological and topological closure operations. All closure operations are defined mereologically if and only if the corresponding contact algebras are uniquely complemented while topological closure operations highly restrict the contact relation but allow not uniquely complemented and nondistributive contact algebras. Each class contains a single ontologically coherent theory that admits discrete models.

## 1 Introduction

Qualitative spatial reasoning (QSR) studies how the space surrounding us can be described using only qualitative aspects, that is, without reference to some metric, and how we can efficiently automate reasoning with such descriptions. QSR has been a very active area of interdisciplinary research with interest from artificial intelligence (AI), cognitive science, formal logic, geographical information systems, and spatial databases—just to name a few. Extensive introductions and overviews of QSR can be found, for example, in Cohn and Hazarika [9], Cohn and Renz [10], and Vieu [46]. Mereotopologies theories—which model only topological (of "connection") and mereological (of "parthood") aspects of space—are foundational

Received August 26, 2011; accepted November 8, 2011
2010 Mathematics Subject Classification: Primary 68T27; Secondary 54H10, 06C15
Keywords: mereotopology, contact algebras, spatial representability, complementation
© 2013 by University of Notre Dame 10.1215/00294527-1731344

within QSR. In the last two decades many first-order theories of mereotopology have been proposed for QSR, which has in turn led to fruitful systematic analyses exploring the ontological assumptions and the entailed logical properties of different sets of axioms for mereotopological theories (see Casati and Varzi [7], Eschenbach [19]). Closure mereotopology (**CMT**; see [7]) is widely accepted as the most restricted mereotopology that does not contain any controversial ontological assumptions. Though some specific extensions of **CMT** have been studied in great detail, the question of what constitutes a mereotopology that adequately represents physical space has been largely neglected. In particular, the existing work on specific mereotopologies suggests that still new combinations of axioms could yield yet unexplored theories of closure mereotopology. We give strong evidence why this is not the case. We do so by focusing on the spatial representability of the models of a mereotopology. Though many concrete embeddings of mereotopological models in topological spaces have been constructed (see Bennett and Düntsch [3], Dimov and Vakarelov [11], Düntsch and Winter [15], [16], Düntsch et al. [12], [13], Düntsch and Vakarelov [14], Vakarelov [44]), the question of whether these topological representations adequately reflect the intended structure of physical space has not been addressed.[1] As it turns out, the key in this pursuit is the necessary strength of the complementation operation. We show that assuming the existence of some kind of uniquely defined complements and requiring a weak form of spatial representability restrict the algebraic structure arising from mereotopologies to an extent that only a few particular theories remain. Only two distinct minimal classes of ontologically coherent mereotopologies (we define C-closure in that regard) are conceivable—distinguished by the presence or absence of unique complements. Our analysis further identifies the algebraic properties that correspond to the various closure operations and other ontological assumptions of the mereotopologies.

For our investigation we treat mereotopology algebraically as first proposed by Stell and Worboys [41], Stell [39], and Düntsch and Winter [15], [17]. The systematic studies of algebraic counterparts[2] of mereotopologies in [44] and Hahmann and Grüninger [28] offer many insights that help us understand the different mereotopological theories and the relationships among them. The study of algebraic theories of mereotopology is, for example, most convincing in separating the mereological component from the topological component as pointed out by Li and Ying [32]. We are particularly interested in *unique closure mereotopology* (**UCMT**) and *unique infinitary closure mereotopology* (**UGMT**), subclasses of **CMT** of which *all* models have algebraic counterparts. The class **UCMT** includes many prominent mereotopologies such as the theories of Whitehead [47], of Clarke [8], the region-connection calculus (RCC; see Randell, Cui, and Cohn [36], Gotts, Gooday, and Cohn [25]), and its generalization (GRCC; see [32]) which admits discrete models. The theory of **UCMT**s is introduced in Section 2; it assumes closure under (binary) sums and intersections just as **CMT** does, but additionally assumes closure under complementation with respect to a universal region and that all these closure operations are unique. In Section 3 we show that the algebraic counterparts of **UCMT** are orthocomplemented contact algebras (OCAs). Thus, the spatial representability of **UCMT**s can be studied through the spatial representability of OCAs—a task we are much more comfortable with. In Section 4 we look at spatially representable OCAs, but lacking a complete definition of spatial representability we resort to a weaker form thereof, MT-representability. We can show that every MT-representable complete OCA is pseudocomplemented and satisfies the Stone identity, that is, is a SPOCA. For this result, we rely on the lattices being complete. However, this is only a minor restriction since we can reasonably expect all spatially representable contact algebras to be complete. For discrete MT-representable mereotopologies, it is no restriction at all.

Section 5 contains our key contribution. We identify algebraic conditions that are necessary and sufficient for closure operations sums, intersections, complements, and universal to be defined mereologically or topologically in SPOCAs. In particular, we show that the ontological choice between a mereological or topological complementation in a mereotopology is reflected in the algebraic structure: The algebras of mereologically closed mereotopological models are uniquely complemented and thus distributive, while those of topologically closed models are only pseudo- and orthocomplemented but potentially nondistributive. This confirms how central complementation is in mereotopology as suggested by Stell [40].

We identify the two minimal classes that emerge as MT-representable and ontologically coherent (a notion defined later) algebraic structures from those two classes of SPOCAs in Section 6. The first class, namely, weak Boolean contact algebras (WBCA), defines all closure operations mereologically; though only the more restricted generalized Boolean contact algebras (GBCAs) are guaranteed to have intuitive spatial representations. The second class, namely, SPOCAs with contact defined as $x\mathbf{C}y \Leftrightarrow x \not\leq y^{\perp}$ or as $x\mathbf{C}y \leftrightarrow x \not< y^{\perp}$, defines all closure operations topologically. These two classes are also the weakest ones that could axiomatize space as intended by Whitehead [47]. However, neither of them satisfies all conditions discussed by Whitehead. As a further consequence of our work, we can verify algebraically that the assumptions of Whiteheadean mereotopology as outlined in Forrest [20] and Mormann [35] are not compatible with the connectivity axiom (Con) $\forall x C(x, -x)$. Ways to overcome this problem are also discussed in Section 7. Furthermore, we prove that no "true mereotopology," that is, no MT-representable MT-closed mereotopology, with atoms can exist. Only if we allow coherently closed (C-closed) instead of MT-closed mereotopologies, exactly two theories (among all combinations of mereological and topological definitions of each of the closure operation sum, intersection, complement, and universal), namely, the GBCAs and the SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$, admit both continuous and discrete models.

On a different note, our work demonstrates that the duality between algebraic structures and topological spaces is not a mere theoretical exercise only of mathematical interest but helps us understand the diversity of theories of qualitative space and select an axiomatization according to any given set of desirable ontological assumptions. Our methodology is outlined in Figure 1. We leverage the knowledge about duality between certain lattices and topological spaces to the understanding of mereotopology. The models of all mereotopologies satisfying the discussed closure assumptions can be represented algebraically in a straightforward manner. With the introduced notion of MT-representability we are then able to reduce the contact algebras resulting from **UCMT**s to a much more restricted set of contact algebras, namely, SPOCAs, that includes all spatially representable and ontologically coherent algebraic counterparts to models of **UCMT**. Two examples of such contact algebras arising from **UCMT**s, which are representative of the only two C-closed MT-representable contact algebras with discrete models, are given in Figure 1. The figure also describes their logical counterparts as well as the common spatial interpretation of their models.

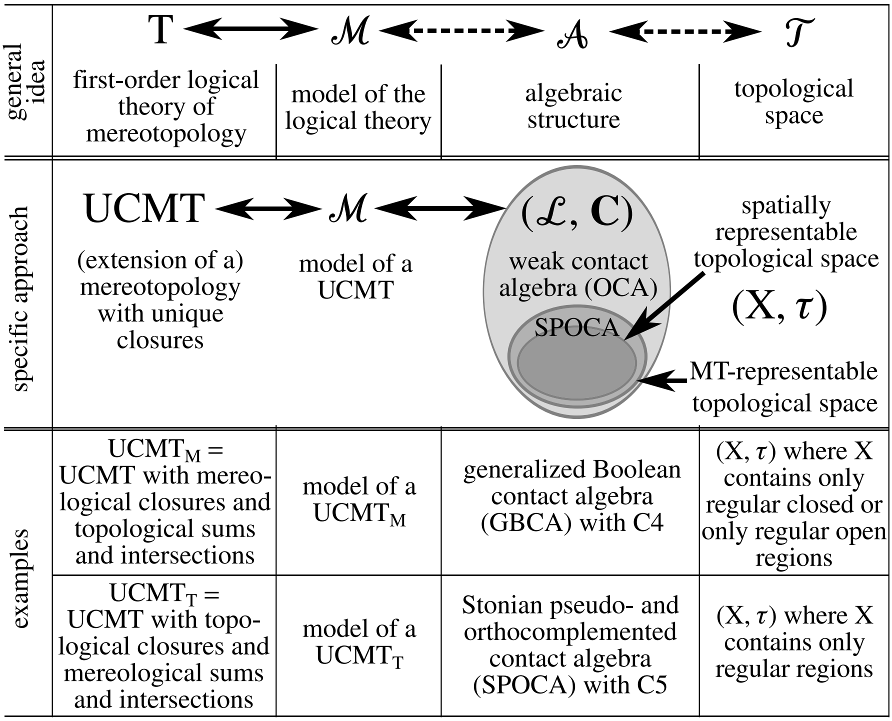



**Figure 1** An overview of our approach. The correspondence between a logical theory of mereotopology and its model on the left-hand side are fairly standard. Representation results between some classes of algebraic structures, specifically lattices, and topological spaces as indicated on the right-hand side are known for some specific cases. In order to establish a subset of the logical theories of mereotopologies that have spatially representable topological interpretations, we need to, first and foremost, establish a correspondence between the models of **CMT**s and contact algebras. Since we cannot achieve this in general (indicated by the dashed arrow), we resort to the restriction of **CMT**s to **UCMT**s as shown in the second row. Every model of a **UCMT** has an algebraic counterpart in the class of orthocomplemented contact algebras (OCAs) as indicated by the solid arrow in the middle. As the second crucial step (the right dashed arrow in the top row), we try to reduce the class of OCAs to a smaller class that still includes all spatially representable contact algebras. This is a subclass of the Stonian pseudo- and orthocomplemented contact algebras (SPOCAs), which are at least MT-representable. However, as indicated by the unidirected solid arrow on the right, not all SPOCAs are spatially representable or even topologically representable.

## 2 Mereotopologies with Complements

We only consider so-called equidimensional mereotopologies, that is, unsorted mereotopological theories whose domain elements can be interpreted as being of a single uniform dimension. For example, the domain elements could be interpreted all as 1D regions (such as time intervals or intervals on a line) as in Allen's interval algebra (see [1]), or as spatial regions which are all 2D or all 3D, or as spatiotemporal regions of either all 3D or all 4D. Explicit multidimensional mereotopologies, that is, mereotopologies with multiple sorts or explicit dimensions such as proposed in Galton [21], [23], Gotts [24], and Hahmann and Grüninger [29], are beyond our scope here.

All equidimensional mereotopological theories consist of a single parthood and a single contact relation that satisfy the axioms (P.1)–(P.3) and (C.1)–(C.3) below (see Varzi [45]). Such theories are commonly referred to as *ground mereotopologies* (**MT**) (see Casati and Varzi [7]). If C and/or P are not explicitly present or are not primitive relations, they still form an alternative set of primitives in a logically equivalent mereotopology. Throughout the paper we assume that any two regions with identical extensions of parthood *and* contact are identical. This follows immediately in our restriction to a single class of regions of equal dimensions.

Throughout the paper we assume standard first-order logic with equality and all logical sentences as implicitly universally quantified.

(P.1) $P(x, x)$ (P reflexive),

(P.2) $P(x, y) \wedge P(y, x) \rightarrow x = y$ (P antisymmetric),

(P.3) $P(x, y) \wedge P(y, z) \rightarrow P(x, z)$ (P transitive),

(C.1) $C(x, x)$ (C reflexive),

(C.2) $C(x, y) \rightarrow C(y, x)$ (C symmetric),

(C.3) $C(z, x) \wedge P(x, y) \rightarrow C(y, z)$ (C monotone with respect to P).

Equivalent to (C.3) is the following axiom:

$$P(x, y) \rightarrow \forall z(C(z, x) \rightarrow C(z, y)).$$

Any such ground mereotopology allows defining the concepts of "overlap" O, "underlap" U, and "proper part" PP in the following natural way:

(O) $O(x, y) \leftrightarrow \exists z[P(z, x) \wedge P(z, y)]$ (overlap),

(U) $U(x, y) \leftrightarrow \exists z[P(x, z) \wedge P(y, z)]$ (underlap),

(PP) $PP(x, y) \leftrightarrow P(x, y) \wedge \neg P(y, x)$ (proper part).

In the sequel, we take these definitions for granted in any mereotopological theory.

**2.1 Closure mereotopology with unique closures (UCMT)** Common requirements for mereotopological theories are closure operations. These require an intersection for any two overlapping entities and a sum for any two underlapping entities; compare, for example, the closure mereotopology (**CMT**; see [7]). Here, we go beyond **CMT** in three ways in order to define *unique closure mereotopology* (**UCMT**).

First, we require a greatest entity to exist, that is, something that everything else is a part of (UCMT.4). The existence of such a universal entity is plausible in any restricted domain of interest, such as the earth, a specific country, building, or an even smaller experimental domain (such as a closed "blocks world" consisting of a finite number of blocks):

(UCMT.4) $\forall x P(x, u)$ (unique universal entity).

Second, we require all closure operations to be uniquely defined. The universal must be always unique by (P.2), we denote it by a constant $u$. If sums and intersections are unique for all pairs of entities, we can denote them by function symbols, namely, $\oplus$ and $\odot$. Moreover, we need to ensure that the sum $x \oplus y$ is the smallest element which has both $x$ and $y$ as parts (*supremum*) and that anything that overlaps the sum must also overlap either $x$ or $y$. Likewise, the intersection $x \odot y$ is the greatest element which is both part of $x$ and $y$ (*infimum*) if $x$ and $y$ overlap at all. If $x$ and $y$ do not overlap, then the intersection $x \odot y$ is meaningless and may be assigned an arbitrary entity without further logical consequences. These conditions are reflected in the axioms UCMT.1 and UCMT.2, which entail $\forall x, y\ P(x, x \oplus y)$ and $\forall x, y\ P(x \odot y, x)$. It follows that every pair of elements has a sum and intersection so that $P(x_1, x_2)$ and $P(y_1, y_2)$ together imply $P(x_1 \oplus y_1, x_2 \oplus y_2)$ and $P(x_1 \odot y_1, x_2 \odot y_2)$, the latter only if $O(x_1, y_1)$. In the presence of (UCMT.4) any two entities underlap; thus we do not require a conditional in (UCMT.1).

(UCMT.1) $\forall z[(O(x,z) \vee O(y,z)) \leftrightarrow O(x \oplus y, z)]$ (sum is supremum),
(UCMT.2) $O(x,y) \rightarrow \forall z[(P(z,x) \wedge P(z,y)) \leftrightarrow P(z, x \odot y)]$
(intersection is infimum).

Finally, we require models not only to be closed under intersections and sums but also to be closed under complementation. Given that a universal entity exists, complements are a natural concept motivated by human perception of physical space: If we are given a restricted physical space, we can easily identify the complement with respect to the universal entity. Again, the complement shall be uniquely defined; hence we denote it by a function, namely, $\ominus$. Note, however, that the complement of the universal $u$ is not meaningful because in a moment we will specifically prohibit a null (empty) entity to exist. We can choose, for example, $\ominus u = u$. The complement function shall be involutary (UCMT.5)—a reasonable assumption for uniquely defined complements. Additionally, (UCMT.6) and (UCMT.7) ensure that entities and their complement interact correctly with respect to sums and intersections (overlap). Though $\ominus$ is a total function, the universal's complement is not meaningful. For this reason, UCMT.5, UCMT.6, and UCMT.7 explicitly do not apply to the universal $u$.

(UCMT.5) $x \neq u \rightarrow x = \ominus(\ominus x)$ (complements involutary),
(UCMT.6) $x \neq u \rightarrow x \oplus (\ominus x) = u$ (sum of complements),
(UCMT.7) $x \neq u \rightarrow \neg O(x, \ominus x)$ (complements nonoverlapping).

We do not restrict $\oplus$, $\odot$, and $\ominus$ any further at this point. Instead, we consider in Section 5 two plausible definitions, a mereological and a topological one, of each of these functions.

Contrary to the existence of a universal entity, a null entity is cognitively undesirable. The null entity would be part of every entity and thus be in contact to every other entity, but on the other side not really existent, that is, not in contact to anything at all. Therefore we postulate the following to ensure the cognitive adequacy of the mereotopological theories.

(UCMT.3) $\forall x \exists y \neg P(x,y)$ (no null entity).

However, it is not an essential assumption in our work because the algebraic counterparts of these mereotopologies explicitly introduce a null entity. Hence our analysis extends to mereotopologies with unique closures that allow or require a null entity such as in Roeper [37].

Throughout this paper we use the term **UCMT** in the following broad sense.

**Definition 2.1** Let $\mathfrak{MT}$ be a consistent, unsorted first-order theory with two distinguished binary predicates C and P, two binary functions $\oplus$, $\odot$, a unary function $\ominus$, and a constant $u$. If $\mathfrak{MT}$ entails the sentences (P.1)–(P.3), (C.1)–(C.3), and (UCMT.1)–(UCMT.7) with the definitions (O), (U), (PP), we call $\mathfrak{MT}$ a **UCMT**.

The domain elements in a model of **UCMT** are often called *regions*.

Any **UCMT** has a mereological component that is restricted to a closed mereology **CM** where sums, intersections, complements, and the universal are unique but is noncommittal with respect to other mereotopological principles. These mereotopological principles, their corresponding axioms, and the properties of the resulting logical theories have been studied in much detail in [7] and [19]. We will show later that the requirement of unique closures including unique complements does not leave many choices with respect to other mereotopological principles if we require spatial representability and ontological coherence.

**2.2 General mereotopology with unique infinitary closures (UGMT)** Many mereotopologies go beyond **CMT** by requiring sums and intersections of arbitrarily many—possibly infinitely many—entities to exist. Axioms postulating such infinitary closures or unrestricted fusions either require axiom schemas or sets or classes, (cf. [7]). For better readability we use a set notation here; $\mathcal{X}$ denotes an arbitrary set of domain entities:

(UGMT.1) $\forall \mathcal{X}[\forall z[\exists x \in \mathcal{X}(O(x,z)) \leftrightarrow O(\bigoplus \mathcal{X}, z)]]$ (unrestricted sum),

(UGMT.2) $\forall \mathcal{X}[\exists z[\forall x \in \mathcal{X}(P(z,x))] \rightarrow \forall z[\forall x \in \mathcal{X}(P(z,x)) \leftrightarrow P(z, \bigodot \mathcal{X})]]$ (unrestricted intersection).

A **UCMT** that satisfies these axioms is a *general mereotopology* (**GMT**) with unique infinitary closures (including complements).

**Definition 2.2** A **UGMT** is a **UCMT** that satisfies (UGMT.1) and (UGMT.2).

In Section 3.2 we will briefly discuss the subclass **UGMT** and how their algebraic counterparts yield complete lattices.

## 3 The Algebraic Structures Arising from Models of UCMT

We now introduce a class of algebraic structures called *contact algebras* and show that the models of **UCMT** correspond to orthocomplemented contact algebras (OCA) while the models of **UGMT** correspond to complete OCAs. First let us define what we mean by a contact algebra. Contact algebras are not a new concept; various classes thereof have been studied as algebraic counterparts of specific mereotopological theories, for example, by Bennett [3], Düntsch and Winter [15], [17], Stell and Worboys [41], Stell [39], and Vakarelov [44]. Our definition here encompasses the weakest common properties.

**Definition 3.1** A contact algebra $(\mathcal{L}, \mathbf{C})$ consists of a bounded lattice $\mathcal{L}$ which defines a partial order $\leq$ and a contact relation $\mathbf{C}$ that satisfies the following axioms:

(C0) $0 - \mathbf{C} x$ (null disconnectedness),

(C1) $x \neq 0 \rightarrow x\mathbf{C}x$ (reflexivity of C),

(C2) $x\mathbf{C}y \leftrightarrow y\mathbf{C}x$ (symmetry of C),

(C3) $x\mathbf{C}y \wedge y \leq z \rightarrow x\mathbf{C}z$ (monotonicity).

Thus, the contact relation must satisfy the axioms of a ground mereotopology. The axioms (C1)–(C3) are algebraic versions of the axioms (C.1)–(C.3) of **MT**s while (C0) deals with the newly introduced smallest element zero that is necessary to construct a lattice from a mereotopological model. The assumption that zero is not connected to any other entities is merely a convenient choice without deeper implications. To distinguish the contact relation in a mereotopological theory from the

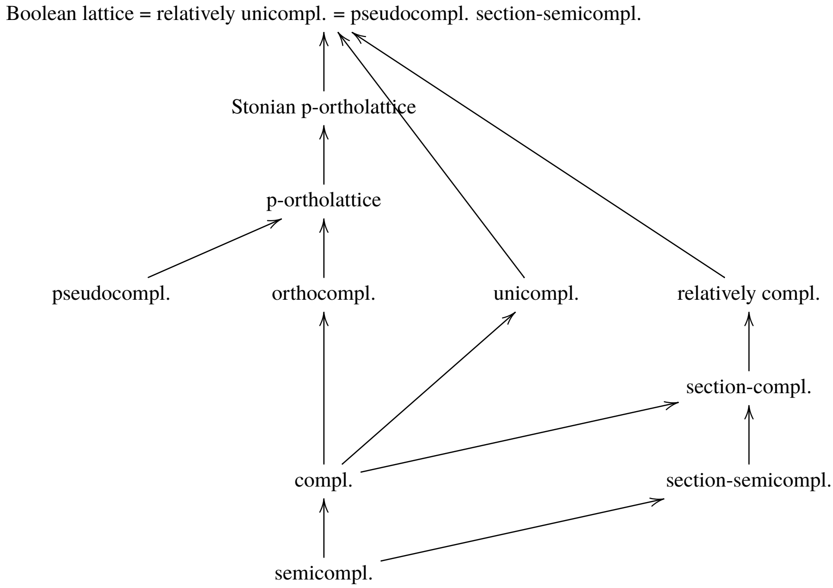



**Figure 2** Relationships between bounded lattices with varying kinds of complementation; adapted from Hahmann [27] and Stern [42]. The arrows indicate refinement; for example, every p-ortholattice is also a pseudocomplemented and an orthocomplemented lattice. These refinements are transitive. In the case of distributive bounded lattices many of these classes of lattices collapse.

contact relation in its algebraic counterpart, we write $C(x, y)$ to refer to the former and $x\mathbf{C}y$ to refer to the latter.

**3.1 Relevant classes of lattices** Before we show how to construct the algebraic counterparts of **UCMT**s, we review the various classes of lattices necessary in our discussions throughout the remainder of the paper. These are used to define more restricted classes of contact algebras. Most of these classes of lattices are defined in standard references such as Blyth [6] and Grätzer [26], while more specialized classes are covered in [42]. Each class allows nondistributive models unless they are explicitly ruled out. The relations between these classes of bounded lattices are illustrated in Figure 2.

One remark up front: Any lattice can be treated as an algebraic structure $\langle L, \cdot, +\rangle$ as well as a partially ordered set $\langle L, \leq\rangle$ with unique supremum $+$ and unique infimum $\cdot$ for any pairs of entities. We can define $x \leq y \leftrightarrow x \cdot y = x$ for any $x, y \in L$. Throughout the paper, we depict lattices as Hasse diagrams which are transitive reductions of the partial order of the lattice. That means only the direct, that is, covering, order relations are depicted, while transitive closure is implied; $x \leq y$ holds if and only if there is a path consisting of one or multiple line segments strictly leading upwards from $x$ to $y$.

**Definition 3.2** A *bounded lattice* is a structure $\langle L, \cdot, +, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ such that

(B0) $\langle L, \cdot, + \rangle$ is a lattice, that is, $a + b$ and $a \cdot b$ are uniquely defined for all $a, b \in L$;
(B1) there exists an element $1 \in L$ so that $1 \cdot a = a$ (and $1 + a = 1$) for all $a \in L$;
(B2) there exists an element $0 \in L$ so that $0 \cdot a = 0$ (and $0 + a = a$) for all $a \in L$.

**Definition 3.3** A *bounded distributive lattice* is a structure $\langle L, \cdot, +, 0, 1 \rangle$ such that

(D0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice;
(D1) the distributive law holds, that is, $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in L$.

The structures in Definitions 3.4 to 3.8 are all of type $\langle 2, 2, 1, 0, 0 \rangle$ equipped with a unary function of complementation or pseudocomplementation.

**Definition 3.4** A *complemented lattice* is a structure $\langle L, \cdot, +, ', 0, 1 \rangle$ such that

(O0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice;
(O1) $a'$ is a complement of $a$, that is, $a + a' = 1$ and $a \cdot a' = 0$.

**Definition 3.5** An *orthocomplemented lattice* (for short: *ortholattice*) is a structure $\langle L, \cdot, +, ^{\perp}, 0, 1 \rangle$ such that

(O0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice;
(O1) $a^{\perp}$ is an orthocomplement of $a$, that is, for all $a, b \in L$ we have
 (a) $a^{\perp\perp} = a$,
 (b) $a \cdot a^{\perp} = 0$,
 (c) $a \leq b$ implies $b^{\perp} \leq a^{\perp}$.

Notice that orthocomplemented lattices are complemented.

**Definition 3.6** A *pseudocomplemented lattice* is a structure $\langle L, \cdot, +, ^*, 0, 1 \rangle$ such that

(P0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice;
(P1) $a^*$ is the pseudocomplement of $a$, that is, for all $b \in L$, $a \cdot b = 0 \iff b \leq a^*$.

**Definition 3.7** A *quasicomplemented lattice* is a structure $\langle L, \cdot, +, ^+, 0, 1 \rangle$ such that

(Q0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice;
(Q1) $a^+$ is the quasicomplement of $a$, that is, for all $b \in L$, $a + b = 1 \iff b \geq a^+$.

Quasicomplemented lattices are also known as *dually pseudocomplemented lattices*.

**Definition 3.8** A *uniquely complemented lattice* (for short: *unicomplemented lattice*) is a structure $\langle L, \cdot, +, ', 0, 1 \rangle$ such that

(U0) $\langle L, \cdot, +, ', 0, 1 \rangle$ is a complemented lattice;
(U1) $a'$ is the unique complement of $a$, that is, for all $b \in L$, $b + a = 1$ and $b \cdot a = 0$ imply $b = a'$.

Clearly, every uniquely complemented lattice is orthocomplemented but not necessarily pseudocomplemented or quasicomplemented. On the other side, Figure 3 gives an example of an orthocomplemented, pseudocomplemented, and quasicomplemented lattice which is not unicomplemented. Pseudo- or quasicomplemented

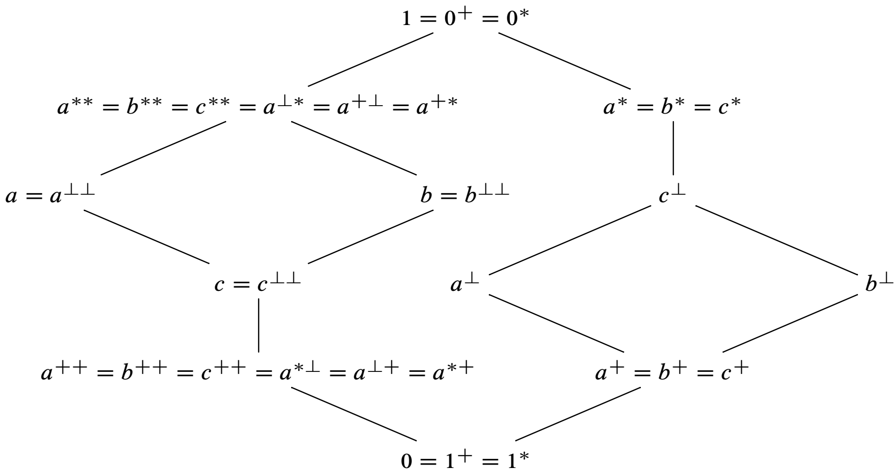



**Figure 3** A simple p-ortholattice with orthocomplements, pseudocomplements, and quasicomplements indicated.

lattices do not even have to be complemented. Lattices that are both pseudocomplemented and orthocomplemented (and thus also complemented and quasicomplemented) but not unicomplemented were introduced in Hahmann, Winter, and Grüninger [30] as p-ortholattices.

**Definition 3.9** A *p-ortholattice* is a structure $\langle L, \cdot, +, ^+, ^\perp, 0, 1\rangle$ such that

(PO0) $\langle L, \cdot, +, ^+, 0, 1\rangle$ is a quasicomplemented lattice;
(PO1) $\langle L, \cdot, +, ^\perp, 0, 1\rangle$ is an ortholattice.

An ortholattice is pseudocomplemented if and only if it is quasicomplemented. For a given p-ortholattice $\langle L, \cdot, +, ^+, ^\perp, 0, 1\rangle$, the structure $\langle L, \cdot, +, ^*, 0, 1\rangle$ is a pseudocomplemented lattice if we define $x^* = x^{\perp+\perp}$. P-ortholattices in which the orthocomplementation and pseudocomplementation operations coincide (unlike Figure 3) are unicomplemented. Unicomplemented ortholattices are Boolean (see Birkhoff [5]).

**Definition 3.10** A *Boolean lattice* is a structure $\langle L, \cdot, +, ', 0, 1\rangle$ such that

(BO0) $\langle L, \cdot, +, ', 0, 1\rangle$ is an orthocomplemented lattice;
(BO1) the distributive law holds; that is, $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in L$.

But there are other interesting subclasses of p-ortholattices that are not distributive and thus not Boolean. Stonian p-ortholattices were introduced in [30] to algebraically capture the structure of the mereotopology of Asher and Vieu [2]. A Stonian p-ortholattice is a p-ortholattice that satisfies the Stone identity (SPO1). Figure 4 illustrates that not all p-ortholattices are Stonian.

**Definition 3.11** A *Stonian p-ortholattice* is a structure $\langle L, \cdot, +, ^+, ^\perp, 0, 1\rangle$ such that

(SPO0) $\langle L, \cdot, +, ^+, ^\perp, 0, 1\rangle$ is a p-ortholattice;
(SPO1) the Stone identity holds; that is, $(a + b)^+ = a^+ \cdot b^+$ for all $a, b \in L$.

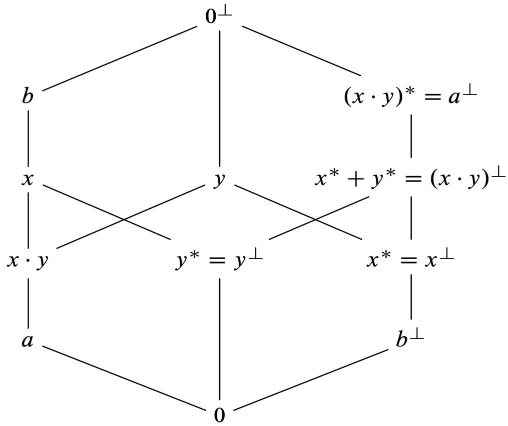



**Figure 4** A p-ortholattice that violates $(x \cdot y)^* = x^* + y^*$ and is therefore not Stonian (see [30]).

Again, a Stonian p-ortholattice $\langle L, \cdot, +, ^+, ^\perp, 0, 1\rangle$ can also be defined equivalently as $\langle L, \cdot, +, ^*, ^\perp, 0, 1\rangle$ using the pseudocomplementation operation if we choose $x^* = x^{\perp+\perp}$. We use both structures interchangeably. The Stone identity was originally proposed by Marshall Stone as an immediate generalization of Boolean algebras to so-called Stone lattices—pseudocomplemented distributive lattices which satisfy the Stone identity. Several other ways of stating the Stone identity are known, among them $a^* + a^{**} = 1$ and $(a \cdot b)^{++} = a^{++} \cdot b^{++}$. But one version of the Stone identity that is in distributive lattices also equivalent to those, namely, $a^* + a^{**} = 1$, is inadequate here since it holds for all p-ortholattices. Stonian p-ortholattices generalize the (distributive) Stone lattices to nondistributive lattices.

Notice that the dual of (SPO1), $(a \cdot b)^+ = a^+ + b^+$, holds for all quasi-complemented lattices and, equally, $(a + b)^* = a^* \cdot b^*$ holds for all pseudocomplemented lattices. Moreover, (SPO1) and its dual hold for orthocomplements in ortholattices; that is, $(a + b)^\perp = a^\perp \cdot b^\perp$ and $(a \cdot b)^\perp = a^\perp + b^\perp$ for all $a, b \in L$ if $L$ is orthocomplemented (see [30]). Finally, it is easily verifiable that Boolean lattices are Stonian p-ortholattices.

**3.2 Orthocomplemented contact algebras (OCAs)** We now show that all the models of a **UCMT** can be viewed algebraically as contact algebras in which the lattice is orthocomplemented.[3]

**Definition 3.12** An *orthocomplemented contact algebra* (OCA) is an algebraic structure $\mathcal{A} = (\mathcal{L}, \mathbf{C})$ consisting of an ortholattice $\mathcal{L} = \langle L, \cdot, +, ^\perp, 0, 1\rangle$ equipped with a contact relation $\mathbf{C}$ satisfying (C0)–(C3).

The theory $T_{\text{OCA}}$ = {L2$^\vee$–L6$^\vee$, L2$^\wedge$–L4$^\wedge$, O1′–O3′, C0–C3} axiomatizes OCAs (see Appendix A for the axioms). Notice that OCAs are not necessarily distributive. We only consider nontrivial OCAs which contain an element apart from 0 and 1. Now we show how to construct an OCA from an arbitrary model of **UCMT**.

**Theorem 3.1** *Let $\mathcal{M}$ be a model of* **UCMT** *with domain* **M**.

*Then $\mathcal{M}$ with the extended domain $L = M \cup \{0\}$ where $0 \notin M$ defines a structure $\mathcal{A} = (\mathcal{L}, \mathbf{C}) = (\langle L, \cdot, +, ^\perp, 0, 1\rangle, \mathbf{C})$ that is an OCA.*

**Proof** Define the mapping $g$ from $\mathcal{M}$ with domain $M$ into the algebraic structure $\mathcal{A} = (\mathcal{L}, \mathbf{C}) = (\langle L, +, \cdot, ^\perp, 0, 1\rangle, \mathbf{C})$ with $L = M \cup \{0\}$ and $0 \notin M$ as follows:

1. $g(x) = x$;
2. $g(\ominus x) = x^\perp$ for all $x \neq u$;
3. $g(x \odot y) = x \cdot y$ iff $x\mathbf{O}y$;
4. $g(x \oplus y) = x + y$;
5. $g(u) = 1$;
6. $x\mathbf{C}y$ iff $C(x, y)$;
7. $1^\perp = 0$ and $0^\perp = 1$;
8. $g(x \odot y) = 0$ if $x\!-\!\mathbf{O}y$;
9. $0 + x = x$ and $0 \cdot x = 0$;
10. $0\!-\!\mathbf{C}x$.

We now need to show that

(i) $g$ is a homomorphism (structure preserving);
(ii) $\mathcal{L}$ is an ortholattice; and
(iii) $\mathbf{C}$ satisfies (C0) to (C3).

**(i):** It is easy to see that $g$ is an injective function. It is a homomorphism because the operations $\ominus, \odot, \oplus$, directly correspond to $^\perp, \cdot, +$ for all elements in $M$.

**(ii):** Since $\oplus$ and $\odot$ define supremum and infimum for every pair of elements (infimum is defined as 0 for all nonoverlapping pairs), $\mathcal{L} = (M \cup \{0\}, +, \cdot, ', 0, 1)$ is a lattice with $x \leq y \leftrightarrow x \cdot y = x$ and thus,

11. $x \leq y \iff P(x, y)$ or $x = 0$.

By (8) and (9), the lattice has in 0 a lower bound. By (5) and (UCMT.4), it has in 1 an upper bound. Thus, $\mathcal{L}$ is a bounded lattice.

$\mathcal{L}$ further satisfies the properties O1(a)–O1(c) of ortholattices:

**O1(a):** follows because $^\perp$ is involutary (by UCMT.5) and (7);

**O1(b):** follows from $x \cdot x^\perp = 0$ by (UCMT.7) and (8);

**O1(c):** by (11) and (2) it suffices to prove $P(x, y) \rightarrow P(\ominus y, \ominus x)$. For $x = 0$ or $y = 0$, it holds trivially by (7), (9), (5), and (UCMT.4). Now suppose for two elements $x, y \in M \setminus \{0\}$, $P(x, y)$ but not $P(\ominus y, \ominus x)$. Then we get a contradiction from the following derivation:

$\exists z[P(z, \ominus y) \wedge \neg P(z, \ominus x)]$ P transitive and antisymmetric (P.2), (P.3),
$\Rightarrow \exists z[P(z, \ominus y) \wedge O(z, x)]$ (UCMT.1), (UCMT.6),
$\Rightarrow \exists z, v[P(z, \ominus y) \wedge P(v, z) \wedge P(v, x)]$ definition of O (O),
$\Rightarrow \exists v[P(v, \ominus y) \wedge P(v, x)]$ P transitive (P.3),
$\Rightarrow \exists v[\neg P(v, y) \wedge P(v, x)]$ (UCMT.7), definition of O (O),
$\Rightarrow \neg P(x, y)$ P transitive (P.3).

**(iii):** The contact relation $\mathbf{C}$ satisfies (C0) by definition, and (C1)–(C3) follow directly from (C.1)–(C.3) of a **UCMT**.

Thus, the structure $\mathcal{A} = (\mathcal{L}, \mathbf{C}) = (\langle M \cup \{0\}, +, \cdot, ^\perp, 0, 1\rangle, \mathbf{C})$ is an OCA, and $g$ is a homomorphism from $\mathcal{M}$ into $\mathcal{A}$. □

We can obtain an analogous result for **UGMT** in terms of complete OCAs. First, we define what it means for a lattice to be complete—a second-order property similar to the fusion operator in **UGMT**.

**Definition 3.13** Let $\langle L, \cdot, +, 0, 1\rangle$ be a lattice. It is complete if and only if it is closed under arbitrary meets; that is,

$$\forall S \subseteq L\ \exists x \in L : x = \prod S.$$

A complete lattice is also complete under arbitrary joins; that is,

$$\forall S \subseteq L\ \exists x \in L : x = \sum S.$$

These so-called fusion operators $\sum$ and $\prod$ are often alternatively denoted as $\bigvee$ and $\bigwedge$, respectively. We call a contact algebra *complete* if its underlying lattice is complete.

Then, the next corollary immediately follows.

**Corollary 3.2** *Let $\mathcal{M}$ be a model of* **UGMT** *with domain* **M**.

*Then $\mathcal{M}$ with the extended domain $L = M \cup \{0\}$ where $0 \notin M$ defines a structure $\mathcal{A} = (\mathcal{L}, \mathbf{C}) = (\langle L, \cdot, +, ^{\perp}, 0, 1\rangle, \mathbf{C})$ that is a complete OCA.*

**Proof** With $\mathcal{M}$ being a model of **UGMT**, it is also a model of **UCMT** which defines some OCA. By (UGMT.1) and (UGMT.2) it is complete. □

This enables us to focus on the topological representability or embeddability of (complete) OCAs in order to study representability of all the models of **UCMT** and of **UGMT**.

## 4 Mereotopologically Representable Complete OCAs

The study of topological representability of algebraic structures has a long tradition established in the seminal work by Stone [43] on the duality between Boolean algebras and the topological spaces now known as *Stone spaces*. Since then, many generalizations thereof have been found. Here we are not interested in full duality but rather in embeddings of OCAs (with lattices as core) in a topological space in a way that preserves the mereotopological structure, that is, gives point-set interpretations to all lattice elements so that parthood and contact also have point-set interpretations that reflect their intended spatial meaning. If an OCA has such a topological representation or embedding, we call it *spatially representable*. But instead of giving a complete definition of spatial representability, we only partially define it by giving a few necessary conditions that must hold in a spatially representable OCA. Every OCA that satisfies these conditions is called *mereotopologically representable* (MT-representable). Then we have for all OCAs

$$\text{spatially representable} \Rightarrow \text{MT-representable}$$

but not its converse, that is,

$$\text{MT-representable} \not\Rightarrow \text{spatially representable}.$$

Nevertheless, by showing that MT-representable complete OCAs are pseudocomplemented and satisfy the Stone identity we can conclude the same for spatially representable complete OCAs. Thus, MT-representability restricts the behavior of complementation in the lattice structure of the algebraic counterparts resulting from models of **UGMT**. Translated into the realm of the logical theories, we essentially show that all models of **UGMT**s that have some spatial representation must have an algebraic structure whose lattice is a Stonian p-ortholattice. This defines a weakest class of equidimensional mereotopologies with unique closures under arbitrary sums, arbitrary joins, and under complementation.

For this section, we assume a basic familiarity with topological spaces. A few words on our notation: A topological space $\langle X, \tau\rangle$ is defined by its universe $X$ and its topology $\tau$, the set of all open subsets of $X$. Sets are denoted by capital letters to distinguish them from lattice elements; $h(a)$ denotes the set that a lattice element $a$ is represented by. The interior, closure, and complement (with respect to $X$) of a set $A$ are denoted by $\text{int}(A)$, $\text{cl}(A)$, and $\bar{A}$. Set intersection, union, and inclusion are denoted by $\cap$, $\cup$, and $\subseteq$. The following set-theoretic equivalences in topological spaces are used without further mentioning.

**Lemma 4.1** *Let $\langle X, \tau\rangle$ be a topological space. Then for all sets $S = \{A : A \subseteq X\}$,*

$$\text{int}\Big(\bigcap_{A\in S} A\Big) = \bigcap_{A\in S} \text{int}(A) \qquad \textit{and} \qquad \text{cl}\Big(\bigcup_{A\in S} A\Big) = \bigcup_{A\in S} \text{cl}(A),$$
$$\text{int}\Big(\bigcup_{A\in S} A\Big) \supseteq \bigcup_{A\in S} \text{int}(A) \qquad \textit{and} \qquad \text{cl}\Big(\bigcap_{A\in S} A\Big) \subseteq \bigcap_{A\in S} \text{cl}(A).$$

**4.1 MT-representability** For an OCA to be spatially representable, we require that a lattice homomorphism into a set of subsets of $X$ of a topological space $(X, \tau)$ exist. The lattice operations $\cdot$ and $+$ correspond to operations $\sqcap$ and $\sqcup$ defined over the subsets of $X$. They may map to standard set intersection $\cap$ and union $\cup$ in the topological space, though this is not required. The infinitary versions of $\cdot$ and $+$ that must exist in complete lattices then map to infinitary versions of $\sqcap$ and $\sqcup$, which we denote as $\bigsqcap$ and $\bigsqcup$. Notice that as a lattice homomorphism, $h$ must preserve joins and meet; that is, $h(x\cdot y) = h(x) \sqcap h(y)$ and $h(x+y) = h(x) \sqcup h(y)$. In particular, we must have $h(x) \subseteq h(y) \iff x\cdot y = y$; in other words, the lattice order $\leq$ and thus the parthood order $P$ is preserved as subset inclusion $\subseteq$ in the representing topological space.

We are now ready to define MT-representability of a complete OCA.

**Definition 4.1** Let $\mathcal{A} = (\langle L, +, \cdot, ^{\perp}, 0, 1\rangle, \mathbf{C})$ be a complete OCA.

It is called *MT-representable* if there is some topological space $\langle X, \tau\rangle$ and an injective lattice homomorphism $h$ from $L$ into the structure $\langle \mathcal{T}, \sqcap, \sqcup\rangle$ where $T \subseteq X$ for each $T \in \mathcal{T}$ and the following conditions are satisfied:

1. $h(1) = X$ and $h(0) = \emptyset$;
2. for all sets $S \subseteq L$ we have
$$\bigcap_{x\in S} \text{int}\big(h(x)\big) \subseteq h\Big(\prod S\Big) = \bigsqcap_{x\in S} h(x) \subseteq \bigcap_{x\in S} \text{cl}\big(h(x)\big) \qquad \text{and}$$
$$\bigcup_{x\in S} \text{int}\big(h(x)\big) \subseteq h\Big(\sum S\Big) = \bigsqcup_{x\in S} h(x) \subseteq \bigcup_{x\in S} \text{cl}\big(h(x)\big);$$
3. any $x \in L$ is regular, that is, satisfies $\text{int}(x) = \text{int}(\text{cl}(x))$ and $\text{cl}(\text{int}(x)) = \text{cl}(x)$;
4. for all $x, y \in L$, $\text{int}(h(x)) \cap \text{int}(h(y)) \neq 0 \Rightarrow x\mathbf{C}y$ for all $x \in L$;
5. for all $x, y \in L$, $\text{cl}(h(x)) \cap \text{cl}(h(y)) = 0 \Rightarrow x{-}\mathbf{C}y$ for all $x \in L$.

Condition (1) ensures that the space is not larger than necessary, while condition (2) ensures that the set that represents the meet (or join) of two entities differs only in boundaries from the point-set intersection (or union) of their representing sets. More specifically, the representation of the meet of two entities is not smaller than the intersection of the interiors of their representations and not larger than the intersection of the closures of their representations. Condition (3) ensures that everything apart from the zero entity has a nonempty interior, that is, we also have

3′. for all $x \in L$, $\mathrm{int}(h(x)) = \emptyset$ if and only if $x = 0$.

Conditions (4) and (5) ensure that contact is adequately interpreted so that two entities whose representations share a point are indeed in contact, while if the closures of their representations do not share a point, they are not in contact. Finally, if $x \cdot y = 0$ and $x + y = 1$, then $h(x) \sqcap h(y) = \emptyset$ and $h(x) \sqcup h(y) = X$. Then from conditions (2) and (3) of Definition 4.1 we deduce the following additional condition:

6. for all $x, y \in L$, if $x \cdot y = 0$ and $x + y = 1$, then $\mathrm{int}(\overline{h(x)}) \subseteq h(y) \subseteq \mathrm{cl}(\overline{h(x)})$.

Special versions of MT-representability are representable by regular closed (or regular open) sets or by regular sets as for the Boolean contact algebras (BCAs) with $x{-}\mathbf{C}y \leftrightarrow x < y'$ or the Stonian p-ortholattices with $x{-}\mathbf{C}y \leftrightarrow x \leq y^{\perp}$. In other words, lattices representable by regular closed sets of a topological space, such as BCAs, satisfy all conditions of Definition 4.1. Key here is that conditions (2) and (4), (5) are satisfied if we use $\cap$ as $\sqcap$ and have $\mathrm{cl}(x) = x$ for all $x \in L$; (2) then simplifies to $\mathrm{int}(h(x)) \cap \mathrm{int}(h(y)) \subseteq h(x) \cap h(y) \subseteq h(x) \cap h(y)$, which is trivially true, while (4) and (5) amount to $\mathrm{cl}(h(x)) \cap \mathrm{cl}(h(y)) \neq 0 \Leftrightarrow x\mathbf{C}y$, which is satisfied once we define $x{-}\mathbf{C}y \leftrightarrow x < y$ as in BCAs (cf. [16]). For the representation of Stonian p-ortholattices by regular sets, we can choose $x \sqcap y = x \cap y \cap \mathrm{int}(\mathrm{cl}(x \cap y))$ to satisfy condition (2), while conditions (4) and (5) are satisfied if we define $h(x) \cap h(y) \neq 0 \Leftrightarrow x\mathbf{C}y$ (cf. [2]). Now we can prove the first property of MT-representable complete OCAs.

**Theorem 4.2** *An MT-representable complete OCA is pseudocomplemented.*

**Proof** Suppose $\mathcal{A} = (\langle L, +, \cdot, ^{\perp}, 0, 1\rangle, \mathbf{C})$ is an MT-representable complete OCA. Let $x \in L$ be an arbitrary lattice element. We will show that it must have a pseudocomplement in $L$.

Let $S_x = \{x_i^* : x_i^* \in L \text{ and } x \cdot x_i^* = 0\} \subseteq L$ denote the set of meet-complements of $x$ in $L$. Because $\mathcal{A}$ is a complete lattice, we have $\sum S_x \in L$. We will now show that $x \cdot \sum S_x = 0$ and thus $\sum S_x \in S_x$. Note that all $x_i^* \in S_x$ not only satisfy $x \cdot x_i^* = 0$ but also $x + x_i^* \geq x + x^{\perp} = 1$, allowing us to utilize Definition 4.1(6) in the following computation:

$$\begin{aligned}
\mathrm{int}\Big(h\Big(x \cdot \sum S_x\Big)\Big) &\subseteq \mathrm{int}\Big(\mathrm{cl}\Big[h(x) \cap h\Big(\sum S_x\Big)\Big]\Big) && \text{Def. 4.1(2)}\\
&\subseteq \mathrm{int}\Big(\mathrm{cl}\Big[h(x) \cap \mathrm{cl}\Big(\bigcup_{y \in S_x} h(y)\Big)\Big]\Big) && \text{Def. 4.1(2)}\\
&\subseteq \mathrm{int}\Big(\mathrm{cl}\Big[h(x) \cap \bigcup_{y \in S_x} \mathrm{cl}\big(h(y)\big)\Big]\Big) &&\\
&\subseteq \mathrm{int}\Big(\mathrm{cl}\Big[h(x) \cap \bigcup_{y \in S_x} \mathrm{cl}\big(\mathrm{cl}(\overline{h(x)})\big)\Big]\Big) && \text{Def. 4.1(6)}
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{int}\big(\operatorname{cl}\big[h(x) \cap \operatorname{cl}[\overline{h(x)}]\big]\big) \quad \operatorname{cl}\big(\operatorname{cl}(A)\big) = \operatorname{cl}(\mathbf{A}) \\
&\subseteq \operatorname{int}\big(\operatorname{cl}(h(x)) \cap \operatorname{cl}[\overline{h(x)}]\big) \\
&= \operatorname{int}\big(\operatorname{cl}(h(x))\big) \cap \operatorname{int}\big(\operatorname{cl}[\overline{h(x)}]\big) \\
&= \operatorname{int}\big(h(x)\big) \cap \operatorname{int}\big(\operatorname{cl}[\overline{h(x)}]\big) \quad \text{Def. 4.1(3)} \\
&= \operatorname{int}\big(h(x)\big) \cap \operatorname{int}\big(\overline{\operatorname{int}(h(x))}\big) \quad \operatorname{cl}(\overline{A}) = \overline{\operatorname{int}(A)} \\
&= \operatorname{int}\big(h(x)\big) \cap \big(\overline{\operatorname{cl}(\operatorname{int}(h(x)))}\big) \quad \operatorname{int}(\overline{A}) = \overline{\operatorname{cl}(A)} \\
&= \operatorname{int}\big(h(x)\big) \cap \big(\overline{\operatorname{cl}(h(x))}\big) \quad \text{Def. 4.1(3)} \\
&= \operatorname{int}\big(h(x)\big) \setminus \big(\operatorname{int}(h(x)) \cap \operatorname{cl}(h(x))\big) \quad A \cap \overline{B} = A \setminus (A \cap B) \\
&= \operatorname{int}\big(h(x)\big) \setminus \operatorname{int}\big(h(x)\big) \\
&= \emptyset.
\end{aligned}
$$

By Definition 4.1(1) and (3′) we conclude that $x \cdot \sum S_x = 0$. Hence, $\sum S_x$ is the pseudocomplement of $x$. Thus any element in $\mathcal{A}$ must have a pseudocomplement. Consequently, $\mathcal{A}$ is pseudocomplemented. □

The restriction to complete lattices essentially shifts the focus from **UCMT** to **UGMT**. Notice, however, that all discrete models of **UCMT** are trivially complete.

Now we prove that in an MT-representable OCA the Stone identity must also hold. First, recall that a pseudocomplemented ortholattice is also quasicomplemented, which also applies to contact algebras defined over those lattices. In the following, we utilize the fact that MT-representable complete OCAs are quasicomplemented to prove that they satisfy the Stone property. We exploit the fact that $h(x) \to h(x^{++})$ is an interior mapping in the topological sense for a quasicomplemented OCA given condition (2) of Definition 4.1 (cf. [30]), that is,

$$h(x^{++}) = \operatorname{int}\big(h(y)\big). \tag{+}$$

This is well known for Boolean lattices which are representable by the regular open sets of a topological space. More generally, it can be justified by considering that by the definition of a quasicomplement, $x^+$ is the smallest entity such that $x + x^+ = 1$. We have then $h(x^+ + x^{++}) = h(x^+ + x^{\perp +}) = h(x^+) \cup h(x^{\perp +}) = h(1) = X$ which is an open set in every topological space.

Analogously, $h(x) \to h(x^{**})$ is a closure mapping in the representation of a pseudocomplemented OCA given condition (2) of Definition 4.1, that is,

$$h(x^{**}) = \operatorname{cl}\big(h(y)\big). \tag{*}$$

We further need the following result from [30].

**Lemma 4.3** *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a p-ortholattice. Then we have*

1. $a^{**} = (a^{++})^{**}$,
2. $a^{++} = (a^{**})^{++}$.

We are now in the position to prove the Stone identity for MT-representable, quasicomplemented OCAs.

**Theorem 4.4** *An MT-representable OCA $\mathcal{A} = \langle (L, +, \cdot, ^\perp, ^+, 0, 1), \mathbf{C}\rangle$ satisfies $(x \cdot y)^{++} = x^{++} \cdot y^{++}$ for all $x, y \in L$.*

**Proof** Suppose $\mathcal{A} = \langle (L, +, \cdot, ^{\perp}, ^{+}, 0, 1), \mathbf{C}\rangle$ is an MT-representable quasicomplemented OCA. Let $x, y \in L$ denote two arbitrary lattice elements. We prove the two directions $(x \cdot y)^{++} \subseteq x^{++} \cdot y^{++}$ and $(x \cdot y)^{++} \supseteq x^{++} \cdot y^{++}$ individually.

First, $(x \cdot y)^{++} \subseteq x^{++} \cdot y^{++}$ follows from

$$\begin{aligned}
h\big((x \cdot y)^{++}\big) &= \operatorname{int}\big(h(x \cdot y)\big) \quad (+),\\
&\subseteq \operatorname{int}\big(\operatorname{cl}(h(x)) \cap \operatorname{cl}(h(y))\big) \quad \text{Definition 4.1(2)},\\
&= \operatorname{int}\big(h(x^{**}) \cap h(y^{**})\big) \quad (*),\\
&= \operatorname{int}\big(\operatorname{int}(h(x^{**}) \cap h(y^{**}))\big) \quad \operatorname{int}\big(\operatorname{int}(A)\big) = \operatorname{int}(A),\\
&= \operatorname{int}\big(\operatorname{int}(h(x^{**})) \cap \operatorname{int}(h(y^{**}))\big) \quad \text{Lemma 4.1},\\
&= \operatorname{int}\big(h(x^{**++}) \cap h(y^{**++})\big) \quad (+),\\
&\subseteq h(x^{**++} \cdot y^{**++}) \quad \text{Definition 4.1(2)},\\
&= h(x^{++} \cdot y^{++}) \quad \text{Lemma 4.3}.
\end{aligned}$$

For the other direction, $(x \cdot y)^{++} \supseteq x^{++} \cdot y^{++}$, suppose $(x \cdot y)^{++} \not\supseteq x^{++} \cdot y^{++}$. Then $h((x \cdot y)^{++}) \not\supseteq h(x^{++} \cdot y^{++})$, and there must exist a nonempty set $z$ so that $h(z) \subseteq h(x^{++} \cdot y^{++})$ but $h(z) \not\subseteq h((x \cdot y)^{++})$. By Definition 4.1(3), we know that $\operatorname{int}(h(z))$ is nonempty; hence we assume that

$$\operatorname{int}\big(h(z)\big) \subseteq \operatorname{int}\big(h(x^{++} \cdot y^{++})\big) \quad \text{(assumption)},$$

while $\operatorname{int}(h(z)) \not\subseteq \operatorname{int}(h((x \cdot y)^{++}))$ is contradicted by the following computation:

$$\begin{aligned}
\operatorname{int}\big(h(z)\big) &\subseteq \operatorname{int}\big(h(x^{++} \cdot y^{++})\big) \quad \text{(assumption,)}\\
&\subseteq \operatorname{int}\big(\operatorname{cl}(h(x^{++}) \cap h(y^{++}))\big) \quad \text{Definition 4.1(2)},\\
&= \operatorname{int}\big(\operatorname{cl}(\operatorname{int}(h(x)) \cap \operatorname{int}(h(y)))\big) \quad (+),\\
&\subseteq \operatorname{int}\big(\operatorname{cl}(h(x \cdot y))\big) \quad \text{Definition 4.1(2)},\\
&= \operatorname{int}\big(\operatorname{int}(\operatorname{cl}(h(x \cdot y)))\big) \quad \operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A),\\
&= \operatorname{int}\big(\operatorname{int}(h((x \cdot y)^{**}))\big) \quad (*),\\
&= \operatorname{int}\big(h((x \cdot y)^{**++})\big) \quad (+),\\
&= \operatorname{int}\big(h((x \cdot y)^{++})\big) \quad \text{Lemma 4.3}.
\end{aligned}$$

With $h$ being an injective lattice homomorphism, we conclude that $(x \cdot y)^{++} = x^{++} \cdot y^{++}$. □

This leads us to the definition of SPOCAs as a subclass of OCAs which contains all complete OCAs that are MT-representable. SPOCAs can be axiomatized algebraically by the theory $T_{\text{SPOCA}} \cup$ {L2$^\vee$–L6$^\vee$, L2$^\wedge$–L4$^\wedge$, O1′, O2′, O3′, PC1, PC2′, and PC2″, S, C0–C3} (see Appendix A for the axioms, and see Winter, Hahmann, and Grüninger [48] for more explanations and a reduction of this nonminimal theory).

**Definition 4.2** A *Stonian pseudocomplemented and orthocomplemented contact algebra* (SPOCA) is a structure $(\langle L, \cdot, +, ^{\perp}, ^{+}, 0, 1\rangle, \mathbf{C})$ such that

1. $\langle L; \cdot, +, ^{\perp}, 0, 1\rangle$ is an ortholattice;
2. $\langle L, \cdot, +, ^{+}, 0, 1\rangle$ is a quasicomplemented lattice;

3. $(a+b)^+ = a^+ \cdot b^+$ for all $a, b \in L$;
4. **C** satisfies (C0) to (C3).

The following corollary summarizes our result for this section.

**Corollary 4.5** *An MT-representable complete OCA is a complete SPOCA.*

As a consequence, from now on we can focus our attention to SPOCAs without worrying that other spatially representable classes of contact algebras may be overlooked. The only case we have not accounted for are strictly noncomplete lattices. It is, however, unlikely that any such class is of relevance for a spatially representable mereotopology.

## 5 Closure Operations in SPOCAs

In this section, we give a mereological and a topological definition of each of the closure operations sum, intersection, complement, and universal, closely adhering to the definitions presented in [7]. We investigate whether each of the four closure operations are defined in either (or in both) ways in general SPOCAs. For those mereological or topological closure operations that are not entailed, we identify equivalent algebraic properties. Surprisingly, very few such additional properties are necessary; the necessary ones primarily arise from complements being defined mereologically or topologically. If we define complements mereologically, the arising SPOCAs are distributive, while defining complements topologically allows SPOCAs whose underlying Stonian p-ortholattices are nondistributive. In the later case, the contact relation must be more restricted. The resulting two main types of SPOCAs are explored in detail in Section 6.

Generally, we expect each of the closure operations to be defined at least mereologically or topologically. But from an ontologically sound theory of mereotopology, we expect further that all closure operations are defined consistently; for example, either all are defined mereologically or all are defined topologically. We use the following terminology; the axioms follow shortly.

**Definition 5.1** A **UCMT** is *M-closed* if and only if it satisfies M-$\text{I}_{\text{UCMT}}$, M-$\text{S}_{\text{UCMT}}$, and M-$\text{C}_{\text{UCMT}}$.

**Definition 5.2** A **UCMT** is *T-closed* if and only if it satisfies T-$\text{I}_{\text{UCMT}}$, T-$\text{S}_{\text{UCMT}}$, and T-$\text{C}_{\text{UCMT}}$.

**Definition 5.3** A **UCMT** is *$T'$-closed* if and only if it satisfies T-$\text{I}_{\text{UCMT}}$, T-$\text{S}_{\text{UCMT}}$, and T-$\text{C}'_{\text{UCMT}}$.

A **UCMT** is then *coherently closed* (C-closed) if it is defined in one of those three ways.

**Definition 5.4** A **UCMT** is *C-closed* if and only if it is M-closed, T-closed, or T′-closed.

Ideally, the closure operations can be defined mereologically and topologically at the same time. Then we call it *MT-closed*.

**Definition 5.5** A **UCMT** is *MT-closed* if and only if it is

1. M-closed, and
2. T-closed or T′-closed.

We use all these properties both for the logical theories as well as for their corresponding algebraic theories.

All lemmas throughout this and the subsequent section have been proved using the automated theorem prover Prover9 (see McCune [34]) unless otherwise stated. Most proofs are omitted since they contribute only little insight; proof inputs and outputs can be found at www.cs.toronto.edu/~torsten/CA/.

**5.1 Mereological closure operations** The closure operations intersection, sum, and complementation can be defined mereologically as follows. It is easily verified that these are consistent with (UCMT.6) and (UCMT.7).

(M-I$_{\text{UCMT}}$) $\forall w[P(w, x \odot y) \leftrightarrow (P(w, x) \wedge P(w, y))]$ (intersection),
(M-S$_{\text{UCMT}}$) $\forall w[O(w, x \oplus y) \leftrightarrow (O(w, x) \vee O(w, y))]$ (sum),
(M-C$_{\text{UCMT}}$) $\forall w[O(w, \ominus x) \leftrightarrow \neg P(w, x)]$ (complement).

In the sequel we will exclusively use the algebraic equivalents of these axioms as found in Appendix A. These differ only slightly from the above axioms to account for the additional bottom element 0 in a contact algebra (see Lemma B.1 in Appendix B for the proof of the equivalence of the two versions). In the algebraic versions of the axioms, $\wedge$ and $\vee$ denote meet and join, while the logical connectives are written as & and |.

Notice that the universal $u$ is always defined mereologically as $\forall x P(x, u)$. Moreover, we can easily prove that the algebraic equivalents of (M-I$_{\text{UCMT}}$) and (M-S$_{\text{UCMT}}$), that is, (M-I) and (M-S), are theorems in SPOCAs.

**Lemma 5.1** *We have* $T_{\text{SPOCA}} \models$ M-I.

**Lemma 5.2** *We have* $T_{\text{SPOCA}} \models$ M-S.

(M-C) does not necessarily hold in SPOCAs. Defining complementation mereologically requires the SPOCA to be uniquely complemented and thus distributive and Boolean. The necessary axiom (Uni) postulating unique complementation can be found in Appendix A.

**Lemma 5.3** *We have* $T_{\text{SPOCA}} \models$ M-C $\leftrightarrow$ Uni.

**Proof** Since unicomplemented ortholattices are Boolean, and vice versa, it suffices to show that a unicomplemented SPOCA satisfies the algebraic equivalence of (M-C), $z \cdot x^{\perp} \neq 0 \leftrightarrow z \not\leq x$, and that a SPOCA satisfying this property is unicomplemented. This has been done using the automated theorem prover. □

For the sums and complements to be unique, we further need extensionality of O postulated as follows. Recall that $\neg O(x, y) \iff x \wedge y = 0$.

(O-Ext) $\forall z(z \wedge x = 0 \leftrightarrow z \wedge y = 0) \leftrightarrow x = y$ (O-extensionality).

But from (M-C) we can already prove extensionality of O.

**Lemma 5.4** *We have* $T_{\text{SPOCA}} \cup$ M-C $\models$ O-Ext.

We obtain the following corollary on the effects of mereological closures in SPOCAs.

**Corollary 5.5** *A SPOCA is M-closed if and only if it is unicomplemented. An M-closed SPOCA is O-extensional.*

**5.2 Topological closure operations** The closure operations intersection, sum, and complementation can be defined topologically as follows. Again, their algebraic versions are found in Appendix A, with Lemma B.2 in Appendix B proving the equivalence of both versions. It is easily verified that these are consistent with (UCMT.6) and (UCMT.7). There are two slightly distinct ways of defining topological complements, denoted by (T-C) and (T-C′):

(T-I$_{\text{UCMT}}$) $\forall w[C(w, x \odot y) \rightarrow (C(w,x) \wedge C(w,y))]$ (intersection),
(T-S$_{\text{UCMT}}$) $\forall w[C(w, x \oplus y) \leftrightarrow (C(w,x) \vee C(w,y))]$ (sum),
(T-C$_{\text{UCMT}}$) $\forall w[P(w, \ominus x) \leftrightarrow \neg C(w,x)]$ (complement),
(T-C′$_{\text{UCMT}}$) $\forall w[PP(w, \ominus x) \leftrightarrow \neg C(w,x)]$ (alternative complement).

Notice that since the universal is always defined mereologically as $\forall x P(x,u)$, it is also automatically defined topologically as $\forall x C(x,u)$. However, this does not guarantee the topological uniqueness of the universal, that is, that $\forall y[(\forall x C(x,y)) \rightarrow y = u]$ holds. Therefore, we introduce (Dis) which has been previously used to study contact algebras.

(Dis) $\forall x[x \neq 1 \rightarrow \exists y(y \neq 0 \,\&\, x - \mathbf{C} y)]$ (only 1 is connected to all entities).

Intersections are always defined topologically in SPOCAs. Notice however that (T-I) only contains a simple implication and not a biconditional. The reverse direction is not desirable as Figure 5 illustrates.

**Lemma 5.6** *We have* $T_{\text{SPOCA}} \models$ T-I.

**Proof** This follows directly from (C3). □

Moreover, SPOCAs satisfy one direction of the implication in the axiom (T-S).

**Lemma 5.7** *We have* $T_{\text{SPOCA}} \models \forall x[x\mathbf{C}(y+z) \leftarrow (x\mathbf{C}y \mid x\mathbf{C}z)]$.

**Proof** This follows directly from (C3). □

Since the reverse direction of (T-S) does not always hold, we can use (C4) to guarantee that sums are defined topologically in SPOCAs; that is, if an element $x$ is connected to another element $z$, it is also connected to one of the parts of $z$ that make up $z$:

(C4) $x\mathbf{C}(y+z) \rightarrow x\mathbf{C}y \mid x\mathbf{C}z$ (topological sum).

**Lemma 5.8** *We have* $T_{\text{SPOCA}} \cup \text{C4} \models$ T-S.

**Proof** This follows immediately from Lemma 5.7. □

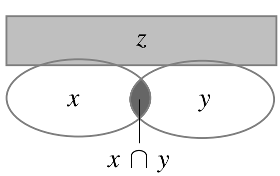



**Figure 5** Two regions $x$ and $y$ connected to $z$ whose set intersection is not connected to $z$ due to the nontransitive nature of contact.

*5.2.1 Topological complement operation* Now we turn to the complement. We have two options, using either (T-$C_{\mathrm{UCMT}}$) or (T-$C'_{\mathrm{UCMT}}$). We first study (T-$C_{\mathrm{UCMT}}$) and then proceed with (T-$C'_{\mathrm{UCMT}}$). In SPOCAs, (T-$C_{\mathrm{UCMT}}$) is captured algebraically by (C5) which requires an element to be in contact to all elements that are not parts of its orthocomplement. In particular, for any $x$, $x{-}\mathbf{C}x^{\perp}$,

(C5) $z\mathbf{C}x \leftrightarrow z \not\leq x^{\perp}$ (topological complement).

Interestingly, (C5) alone is sufficient to ensure that (T-S) holds and that C is extensional; that is, (C4) and (C-Ext) are satisfied in SPOCAs which satisfy (C5). (C-Ext) expresses extensionality of C; that is, two elements are considered identical if they are in contact to exactly the same elements. C-extensionality is equivalent to requiring that a mereotopology can be reconstructed from contact as the only primitive relation. It further ensures topological uniqueness of the universal (Dis):

(C-Ext) $\forall z(z\mathbf{C}x \leftrightarrow z\mathbf{C}y) \leftrightarrow x = y$ (C-extensionality).

**Lemma 5.9** *We have* $T_{\mathrm{SPOCA}} \cup \mathrm{C5} \models \mathrm{C4}$.

**Lemma 5.10** *We have* $T_{\mathrm{SPOCA}} \cup \mathrm{C5} \models$ C-Ext, Dis.

Moreover, (Int) must hold in SPOCAs satisfying (C5). This seems, however, coincidental and owed to the fact that elements are disconnected from their complements, that is, (¬Con) holds. Despite its name, (¬Con) is not the negation of (Con) but the exact opposite assumption. (Con) is inconsistent with a nontrivial SPOCA satisfying (C5). (Int) and (Con) have previously only been used in the context of contact algebras with Boolean lattices but easily generalize to SPOCAs. In our study we include (Int) only for completeness purposes, it is not motivated by or directly related to the closure operations:

(Con) $\forall x \neq 0, 1[x\mathbf{C}x^{\perp}]$ (connected complements),
(¬Con) $\forall x[x{-}\mathbf{C}x^{\perp}]$ (disconnected complements),
(Int) $\forall x, y[x{-}\mathbf{C}y \rightarrow \exists z(x{-}\mathbf{C}z \;\&\; y{-}\mathbf{C}z^{\perp})]$ (interpolation).

**Lemma 5.11** *We have* $T_{\mathrm{SPOCA}} \cup \mathrm{C5} \models \neg\mathrm{Con}$.

**Proof** Choose $y = x^{\perp}$ in (C5) to obtain $x{-}\mathbf{C}y$. □

**Lemma 5.12** *We have* $T_{\mathrm{SPOCA}} \cup \mathrm{C5} \models \mathrm{Int}$.

**Proof** We show that choosing $z = x^{\perp}$ in (Int) always evaluates to true. We obtain $x{-}\mathbf{C}y \rightarrow (x{-}\mathbf{C}x^{\perp} \;\&\; y{-}\mathbf{C}x^{\perp\perp})$. By Lemma 5.11 it is sufficient to prove $\forall x, y[x{-}\mathbf{C}y \rightarrow y{-}\mathbf{C}x^{\perp\perp}]$, which is with $x = x^{\perp\perp}$ the trivially true inverse of (C2). □

We obtain the following corollary on the effect of topological closures in SPOCAs.

**Corollary 5.13** *A SPOCA is T-closed if and only if it satisfies (C5). A T-closed SPOCA is C-extensional and satisfies (C4), (¬Con), and (Int).*

Finally, we verify that (C5) and (Uni) are independent of one another, that is, that there exist SPOCAs that satisfy (C5) but are not uniquely complemented and that there exist SPOCAs with a Boolean lattice that do not satisfy (C5). Both results are not very surprising.

**Lemma 5.14** *We have* $T_{\mathrm{SPOCA}} \cup \mathrm{Uni} \not\models \mathrm{C5}$.

**Lemma 5.15** *We have* $T_{\mathrm{SPOCA}} \cup \mathrm{C5} \not\models \mathrm{Uni}$.

*5.2.2 Quasitopological complement operation* Now we turn to (T-$C'_{\text{UCMT}}$) as an alternative to the axiom (T-$C_{\text{UCMT}}$) for defining complements topologically. (T-$C'_{\text{UCMT}}$) is captured algebraically by (C5′):

(C5′) $(x \neq 0 \mid z \neq 1) \,\&\, (x \neq 1 \mid z \neq 0) \rightarrow [z\mathbf{C}x \leftrightarrow z \not\leq x^{\perp}]$
(alternative topological complement).

Obviously, (C5) and (C5′) are mutually inconsistent but what are the consequences of using (C5′) instead of (C5) to define complements? First, C5′ is in SPOCAs not sufficient to entail (C4).

**Lemma 5.16** *We have* SPOCA $\cup$ C5′ $\not\models$ C4.

Subsequently, we will focus on SPOCA together with (C4) and (C5′). (C5′) requires an element to be connected to all other elements that are not proper parts of its (ortho-)complement; that is, (Con) is a theorem.

**Lemma 5.17** *We have* $T_{\text{SPOCA}} \cup$ C5′ $\models$ Con.

By Lemma 5.17 (C5′) is not really a topological definition of complementation since complements are connected, that is, $x\mathbf{C}x^{\perp}$. Truly topological complements are complementary with respect to C. In a SPOCA that satisfies (C5′), none of (C-Ext), (Dis), (Int), or (Uni) necessarily hold. Let us start with (C-Ext): we can have models in which $\exists x, y[x \neq y \,\&\, \forall z(x\mathbf{C}z \,\&\, y\mathbf{C}z)]$. Then, the universal is no longer topologically unique; this requires (Dis) in addition. For that reason, we refer to (C5′) as a *quasitopological complement*.

**Lemma 5.18** *We have* SPOCA $\cup$ {C4, C5′} $\not\models$ C-Ext.

In the presence of (C5′), (Int) is a also theorem of SPOCAs.

**Lemma 5.19** *We have* SPOCA $\cup$ {C4, C5′} $\not\models$ Int.

Finally, neither (C5′) together with (C4) entails (Uni) in SPOCAs, nor vice versa.

**Lemma 5.20** *We have* SPOCA $\cup$ {C4, C5′} $\not\models$ Uni.

**Lemma 5.21** *We have* SPOCA $\cup$ C5′ $\not\models$ Uni.

Therefore the class of SPOCAs satisfying (C5′) does not necessarily have a Boolean lattice structure. Those that additionally satisfy (C4) have all closure operations defined mereologically and topologically except for the complement, which is defined mereologically but only quasitopologically. The following corollary summarizes the effect of quasitopological closures in SPOCAs.

**Corollary 5.22** *A SPOCA is $T'$-closed if and only if it satisfies (C4) and (C5′). A $T'$-closed SPOCA satisfies (Con).*

## 6 Coherently Closed MT-Representable UCMTs

We already mentioned that a **UCMT** is only ontologically coherent if it is M-closed, T-closed, or T′-closed. Now we can use Corollaries 5.5, 5.13, and 5.22 to identify the two weakest theories of C-closed MT-representable **UCMT**s and explore the theories with stronger topological or mereological closure conditions. A particular emphasis will be on theories that admit discrete models, that is, theories allowing models that contain atomic entities.

**6.1 M-closed MT-representable UCMTs** Because M-closed SPOCAs are unicomplemented, they must have a Boolean lattice.

**Corollary 6.1** *The algebraic counterpart of an M-closed **UCMT** has a Boolean lattice.*

**Proof** This follows from unicomplemented ortholattices being Boolean (see Birkhoff [5]). □

Many of the contact algebras previously studied in the literature have Boolean lattices and satisfy (C0)–(C3) (see [15], [32], [39]). The most important ones are the following.

**Definition 6.1** A contact algebra $(\mathcal{L}, \mathbf{C})$ in which $\mathcal{L}$ is a Boolean lattice is a

1. *generalized Boolean contact algebra* (GBCA) if $\mathbf{C}$ satisfies (C4);
2. *Boolean contact algebra* (BCA) if $\mathbf{C}$ satisfies (C4) and (C-Ext);
3. *RCC algebra* (RBCA) if $\mathbf{C}$ satisfies (C4), (C-Ext), and (Con);
4. *Proximity BCA* (PBCA) if $\mathbf{C}$ satisfies (C4), (C-Ext), and (Int).

For a more comprehensive overview of the different classes of contact algebras and their relationships to each other we refer to [28]. Contact algebras that have Boolean lattices but do not satisfy (C4) are even weaker than GBCAs; we call them *weak Boolean contact algebras* (WBCAs).

**Definition 6.2** A *weak Boolean contact algebra* (WBCA) is a contact algebra $(\mathcal{L}, \mathbf{C})$ in which $\mathcal{L}$ is a Boolean lattice $\mathcal{L}$.

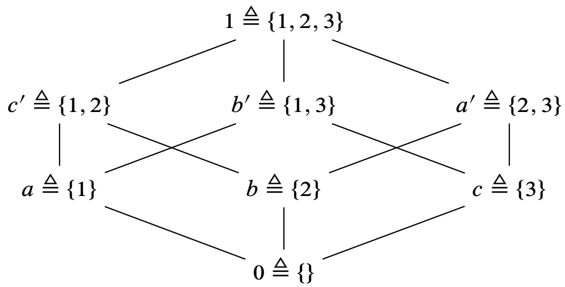
As illustrated by the model in Figure 6(a), there do exist WBCAs that satisfy neither (C4) nor (C-Ext). Thus, the class of WBCAs is strictly more general than both EWBCAs (to be introduced shortly) and GBCAs. WBCAs are the weakest algebraic structures resulting from an MT-representable **UCMT** that is M-closed. WBCAs admit atoms and, in particular, finite models as Figure 6(a) shows.

**Theorem 6.2** *An M-closed MT-representable **UCMT** has an algebraic structure* $(\mathcal{L}, \mathbf{C})$ *whose lattice* $\mathcal{L}$ *is Boolean and whose contact relation satisfies (C0) to (C3).*

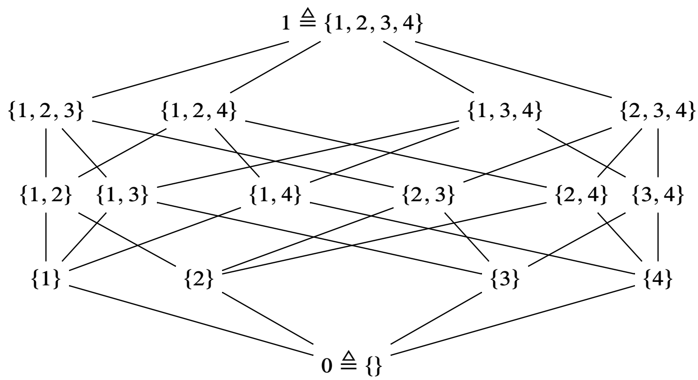
This is a more general perspective of the results from Düntsch and Winter [18] in which the different contact relations definable on Boolean algebras have been studied. The weakest contact relation in [18] already satisfies (C4), while Figure 6(a) shows that there are weaker contact relations definable on a contact algebra with a Boolean lattice which may arise from M-closed **UCMT**s whose sums are not topologically closed, that is, which violate (C4). We do not argue for the usefulness of these structures; in practice (C4) seems like a reasonable assumption. We only explore weaker M-closed contact algebras by showing what other contact relations are theoretically definable on a Boolean lattice (see Figure 7).

WBCAs can be extended by (C-Ext) to obtain *extensional weak Boolean contact algebras* (EWBCAs) or by (C4) to obtain the already defined GBCAs. EWBCAs are axiomatizable by the theory $T_{\mathrm{EWBCA}} = T_{\mathrm{SPOCA}} \cup \{\mathrm{Uni}, C\text{-}\mathrm{Ext}\}$ (see Appendix A for the axioms).

**Definition 6.3** An *extensional weak Boolean contact algebra* (EWBCA) is a WBCA $(\mathcal{L}, \mathbf{C})$ in which the contact relation $\mathbf{C}$ satisfies (C-Ext).



(a) The Boolean lattice $\boldsymbol{B}_3$ with 3 atoms.



(b) The Boolean lattice $\boldsymbol{B}_4$ with 4 atoms.

**Figure 6** (a) $B_3$ with $\{\{(0, x) \mid x \in L\} \cup \{(a, b), (a, c), (b, c)\}\} \notin \mathbf{C}$ (and symmetric tuples) defining disconnection is a WBCA which does not satisfy (C-Ext) or (C4). $B_3$ with $\{\{(0, x) \mid x \in L\} \cup \{(a, c)\}\} \notin \mathbf{C}$ (and symmetric tuples) is a nonextensional GBCA. The elements $a'$, $b'$, $c'$, and 1 are indistinguishable with respect to the contact relation.
(b) $B_4$ with $x < y' \to x{-}\mathbf{C}y$ defining disconnection except for $\{\{1, 2\}, \{3, 4\}\} \in \mathbf{C}$ results in an EWBCA not satisfying (C4).

Again, there exist EWBCAs whose contact relations do not satisfy (C4) (cf. Figure 6(a)). However, in the following we show that in all EWBCAs not satisfying (C4), $x\mathbf{C}x'$ holds for some elements, while for atoms it cannot hold. In other words, the theory of EWBCAs extended by the negation of (C4) and (Atom) is inconsistent with either of (¬Con) and (Con). For the proof we rely on the following result from [17] stating that (Dis) implies (C-Ext) in contact algebras and thus in WBCAs. This result extends to SPOCAs:

(¬C4) $\exists x, y, z[x\mathbf{C}(y \vee z) \,\&\, x{-}\mathbf{C}y \,\&\, x{-}\mathbf{C}z]$
(some $y$ is connected to $y \vee z$ but neither to $y$ nor to $z$)

(¬Triv) $\exists y[y \neq 1 \,\&\, y \neq 0]$ (some entity besides 0 and 1 exists)

(Atom) $\exists a[a \neq 0 \,\&\, \forall x(x = 0 \mid x = a \mid x \cdot a \neq x)]$ (existence of an atom)

**Lemma 6.3** *We have* $T_{\mathrm{SPOCA}} \models \text{C-Ext} \to \text{Dis}$.

**Lemma 6.4** *We have* $T_{\mathrm{EWBCA}} \cup \neg\text{C4} \cup \neg\,\text{Con} \cup \neg\,\text{Triv} \models \bot$.

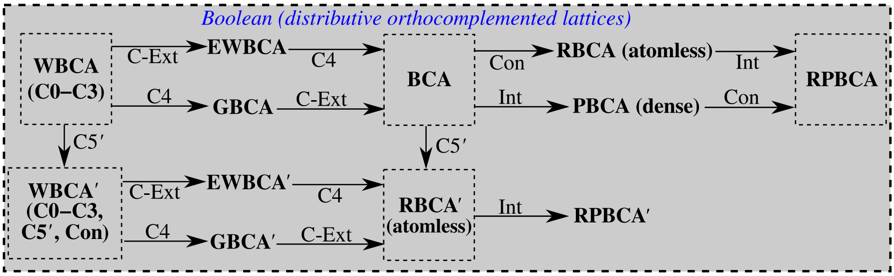



**Figure 7** The classes of M-closed MT-representable contact algebras and their extension relations among them indicated by arrows. For each class more than a single contact relation may exist. For example, on BCAs contact defined as overlap $x\mathbf{C}y \leftrightarrow x \cdot y \neq 0$ or as standard contact $x{-}\mathbf{C}y \leftrightarrow x < y'$ are two distinct extensional contact relations. On the other side, there are Boolean algebras that only allow strictly nonweak and/or extensional contact relations.

**Proof** We give an automatic proof showing that $T_{\text{SPOCA}} \cup \{\text{Uni}, \text{Dis}\} \cup \neg\text{C4} \cup \neg\text{Con} \models \bot$ for all nontrivial models. Since by Lemma 6.3 $T_{\text{SPOCA}} \cup \{\text{Uni}, \text{Dis}\}$ is strictly weaker than $T_{\text{EWBCA}}$, ($\neg$Con) is inconsistent with any nontrivial EWBCA that does not satisfy (C4). □

That does not mean that EWBCA $\cup$ {$\neg$C4, $\neg$Triv} entails (Con) because ($\neg$Con) is not the simple negation of (Con) but states that *all* entities are disconnected from their complement. EWBCAs that contain an atom are inconsistent with (Con) because the atom must be connected to its complement. This is generally true for all SPOCAs that satisfy (Atom) and (Con).

**Lemma 6.5** *We have* $T_{\text{SPOCA}} \cup$ {Dis, Atom, Con} $\models \bot$.

**Proof** Let $a$ be an atom in $L$. Then $a^\perp$ is a dual atom, that is, 1 is the only element greater than $a^\perp$. By overlap, $a^\perp$ is in contact to all elements except for $a$ and 0. Suppose (Con) would hold; then $a\mathbf{C}a^\perp$ holds and $\forall y[a^\perp\mathbf{C}y \leftrightarrow 1\mathbf{C}y]$ but $1 \neq a^\perp$, a violation of (Dis). This does not hold for a trivial model in which 1 is the only atom. □

It immediately follows that EWBCAs with atoms cannot satisfy (Con).

**Lemma 6.6** *We have* $T_{\text{EWBCA}} \cup \text{Atom} \cup \text{Con} \models \bot$.

**Proof** By Lemma 6.3, EWBCA $\models$ Dis and by Lemma 6.6 EWBCA $\cup$ {Atom, Con} $\models \bot$ follows. □

Therefore, all models of EWBCAs which do not satisfy (C4) but contain an atom suffer from this nonuniform interpretation of the contact relation—in particular all atomic, all atomistic, and all finite models of EWBCAs and, more generally, of WBCAs with (Dis). That $x\mathbf{C}x^\perp$ for atoms $x$ is inconsistent with extensionality has been observed for BCAs in Roy and Stell [38]. Our proofs are slightly stronger and show that this problem persists in the weaker-theory WBCAs extended by (Dis) requiring a topologically unique universal element. The failure of $x\mathbf{C}x^\perp$ for some elements is not by itself a concern; in a disconnected model one element may be isolated from the remaining space. However, the failure of $x\mathbf{C}x^{\perp}$ for *all* atoms is a serious issue hinting at a weakness in the theory. Although it can be overcome by enforcing (C4), this only creates other problems since (C4) and (C-Ext) together disallow any discrete models unless contact is reduced to overlap (which in turn reduces the theory to a pure mereology). The problem does not persist in WBCAs; for those we can prove that (Con) is consistent.

**Lemma 6.7** *We have* $T_{\mathrm{WBCA}} \cup \neg\mathrm{C4} \cup \mathrm{Atom} \cup \mathrm{Con} \not\models \bot$.

**Proof** Figure 6(a) provides a counterexample. □

What extensions of WBCAs are obtained if some of the closure operations are also defined topologically? Intersections are already defined topologically in WBCAs. If sums are defined topologically, we require (C4) and obtain GBCAs. If we define complements topologically by (C5), we obtain PBCAs. Its discrete models again reduce contact to overlap. Finally, if we require neither sums nor complements to be defined topologically but instead enforce C-extensionality, we obtain BCAs whose discrete models have overlap as the only feasible contact relation. Hence, among the different strengths of closure operations, the two classes WBCAs and GBCAs are the only algebraic theories of M-closed MT-representable **UCMT**s that admit nonatomless models with a contact relation different from overlap.

The extensions of WBCAs with the quasitopological complements require adding (C5′) and (Con), resulting in MT-representable contact algebras that parallel those without (C5′) (see Figure 7). Those in the classes WBCA′ and GBCA′ that do not satisfy (Dis) admit finite models, but those that satisfy (Dis) and, in particular (C-Ext), do not admit any models with atoms. In any of those models (Con) is satisfied and thereby $C \not\cong O$.

**6.2 T-closed MT-representable UCMTs** T-closed MT-representable **UCMT**s have SPOCAs as algebraic counterparts which may be nondistributive as long as (C5) is satisfied.

**Theorem 6.8** *A T-closed MT-representable* ***UCMT*** *has an algebraic structure* $(\mathcal{L}, \mathbf{C})$ *in which* $\mathcal{L}$ *is a Stonian p-ortholattice and* $\mathbf{C}$ *satisfies (C0)–(C5), (C-Ext), (¬Con), and (Int).*

This nondistributive class of SPOCAs has been studied in depth in [30]; it is the algebraic equivalent of the subtheory $RT^{-}$ of the mereotopology of [2].

Because such T-closed SPOCAs also satisfy (C4), intersections and sums are implicitly defined mereologically as well. The only real extension in terms of additional mereological closure operations requires complements to be mereologically defined, which in turn by Lemma 5.3 makes the lattice Boolean and thus results in a PBCA: (C-Ext) as well as (Int) are already entailed in all T-closed SPOCAs. This also means (C-Ext) extends T-closed SPOCAs nonconservatively, while (Con) is altogether inconsistent with T-closed SPOCAs. We already know that PBCAs are always atomless, hence the theory SPOCA ∪ (C5) is—among all possible extensions of T-closed MT-representable **UCMT**s by additional mereological closure operations—the only theory that admits atoms. Figure 8 gives such a model.

T′-closed MT-representable **UCMT**s also have SPOCAs as algebraic counterparts which may be nondistributive as long as (C5′) is satisfied. They differ from the T-closed ones in that they satisfy (Con) instead of (¬Con) but do not necessarily satisfy

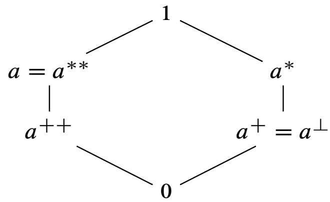



**Figure 8** Let $\forall x, y[x \neq 0 \,\&\, x \leq y \rightarrow x\mathbf{C}y]$ and $a\mathbf{C}a^*$ (and symmetric tuples) define contact. Then the displayed lattice $\mathscr{C}_6$ together with $\mathbf{C}$ defines a SPOCA $(\mathscr{C}_6, \mathbf{C})$ that satisfies (C5) and is thereby T-closed.

(Dis), (C-Ext), or (Int). However, those that have a universal that is topologically defined, i.e., those that satisfy (Dis), are always atomless by Lemma 6.5.

**Theorem 6.9** *A T′-closed MT-representable **UCMT** has an algebraic structure $(\mathscr{L}, \mathbf{C})$ in which $\mathscr{L}$ is an atomless Stonian p-ortholattice and $\mathbf{C}$ satisfies (C0)–(C4), (C5′), and (Con).*

**6.3 MT-closed MT-representable UCMTs** Sections 6.1 and 6.2 let us conclusively answer the question whether MT-closed MT-representable **UCMT**s exist and what their structure is. Such structures must be M-closed and either T-closed or T′-closed. For the first case (an M- and T-closed theory), the intersection of the respective minimal theories, that is, of WBCAs and SPOCAs satisfying (C5), results in PBCAs that satisfy (C5) and which are necessarily atomless. In these structures (¬Con) is entailed; it requires that the contact relation be defined as overlap $x\mathbf{C}y \leftrightarrow x \cdot y \neq 0$ which reduces the theory to a pure mereology. For the second case (a M- and T′-closed theory) we get the RBCA′s as minimal theory, which are RBCAs with contact defined as C5′ and which are also atomless. Hence, we have the following result.

**Theorem 6.10** *Every MT-closed MT-representable **UCMT** has an algebraic structure that is an atomless BCA.*

We also have negative results on the existence of MT-representable **UCMT**s with $C \not\cong O$ or with atoms.

**Theorem 6.11** *No M-closed and T-closed MT-representable **UCMT** with $C \not\cong O$ exists.*

**Theorem 6.12** *No MT-closed MT-representable **UCMT** with atoms exists.*

## 7 Summary and Discussion

Our exploration revealed the three weakest classes of potentially spatially representable complete OCAs that correspond to extensions of **UCMT**. The first are WBCAs, the weakest class in which all closure operations are defined mereologically. The second class are SPOCAs with (C5), the weakest class in which all closure operations are defined topologically. The third class are SPOCAs with (C4), (C5′), and (Dis) (which further implies (C-Ext)). We are not aware of full embedding theorems for these three weakest classes of contact algebras. This remains to be investigated in the future.

**7.1 Spatially representable contact algebras with discrete models** Among the spatially representable OCAs, the classes allowing discrete models are of particular interest. Although space is potentially infinitely divisible according to Aristotle, in practical applications any concrete model of space will have "atoms" at some level; that is, there is some finest granularity. This granularity is usually determined by the precision of available data or measurement devices (think of satellite images vs. microscopic pictures) or the precision we want to reason at (think of a car navigation system vs. the accurate description of surface chemistry). For a generic ontology (in the philosophical sense) of space discrete models might not be that important, but for any specific domain we want to be able to specify models completely, for example, by explicitly listing a finite set of regions and the primitive relations (such as connection and parthood) among them. Such a specification should be consistent with the theory and not a mere approximation thereof. Many mereotopologies, for example, the RCC (corresponding to RBCAs), prevent the existence of atomic regions by including a divisibility axiom, that is, requiring the existence of an interior part for each region. Such theories do not allow us to list all atomic regions of a specific model. Of course, approximations of such models are possible, but these approximations have different model-theoretic properties. This has an important consequence. The construction of and the reasoning with specific models using a theory consistent with discrete, and especially finite, models can be achieved using standard theorem provers, which is not possible for mereotopological theories that only admit infinite models.

Which extensions of the three weakest classes of MT-representable OCAs allow discrete models, that is, are not atomless? We showed that nonatomless WBCAs and EWBCAs have contact relations that behave erratically with regard to contact among complements. While the stronger BCAs and extensions thereof do not suffer from this problem, their discrete models are only of mereological nature, that is, $C \cong O$ (see [16]). Similarly, SPOCAs satisfying (C5′) and (Dis) rule out discrete models by Lemma 6.5. This leaves GBCAs and SPOCAs with $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$ as the only (among all combinations of mereological and topological or quasitopological closure operations) MT-representable OCAs that admit discrete models. These two classes can be characterized as the following:

1. GBCAs in which all closure operations are defined mereologically while sums and intersection are also defined topologically. In general, GBCAs are consistent with either of (Con) or (¬Con). The entities in such algebraic structures are representable by either (1) only regular open, (2) only regular closed, or (3) unrestricted point sets (with point-set intersections, unions, and complements). In the second case (Con) must hold, while in the other cases (¬Con) must hold. The lattices underlying this class are distributive; that is, parthood is distributive with respect to sum and intersections.
2. The subclass of SPOCAs with $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$ as weakest contact algebras defining all closure operations topologically while sums and intersections are also defined mereologically. Due to the topological nature of complements, (¬Con) must hold. The representation of such algebraic structures must include both regular open and regular closed sets, since each regular closed

set has a regular open set as complement and vice versa. In this class, the underlying lattices—and thus the parthood relation—may be nondistributive.

Indeed, GBCAs and SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ exemplify the two ways of constructing discrete mereotopologies discussed in Masolo and Vieu [33]. SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ constitute a C-extensional theory with classical topological operators in which each entity, in particular each atom, is "duplicated" as an open and as a closed set, while GBCAs define an O-extensional theory without classical topological operations, that is, that do not distinguish regions with identical closures.

**7.2 Spatially representable Whiteheadean mereotopology** In [47], Whitehead originally proposed a C-extensional mereotopology and defined atoms as regions without proper parts. We can interpret this as an implicit endorsement of the existence of atoms. Unfortunately, as Theorem 6.12 shows, no MT-closed MT-representable mereotopology with atoms can exist. In fact, the only theory that (1) allows atoms, (2) is C-extensional, and (3) is MT-representable are the SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ defining contact—assuming that this class of SPOCAs can be further strengthened to a class of spatially representable SPOCAs (cf. [48] for work in this direction). From [30] we know that such a theory is also definable by a single mereological primitive $P$ (the partial order relation $\leq$ in the lattices) or by a single topological primitive $C$; it seems to seamlessly bridge the gap between mereology and topology. But at the same time, Whitehead never distinguished sets with identical closures. We can understand this as an implicit condition for representations by closed regions (or, dually, by only open regions); in fact many researchers followed this understanding of Whitehead's intentions. He entices us to believe that the two assumptions, namely, existence of atoms and representability by closed regions, are consistent. However, SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ as the only remaining candidate for true Whiteheadean mereotopology do rely on this difference between interiors and closures. If the distinction between interiors and closures is removed, these models collapse into Boolean contact algebras (cf. Winter, Hahmann, and Grüninger [49]), and thereby prevent a meaningful definition of contact apart from overlap in discrete models. With this stricter requirement of representability by only closed sets, no discrete region-based theory in the intention of Whitehead is definable (see also Forrest [20], Mormann [35]). Further research on theories of qualitative discrete space must therefore concentrate on nontopological, such as graph-based, approaches. Independently, mereotopologies accommodating regions of various dimensions deserve more attention.

There are other ways out of this dilemma, as demonstrated in the literature. If we do not insist on discrete models, RBCAs and the equivalent logical theory RCC provide a truly Whiteheadean account of *continuous* space. One spatial representation thereof is the complemented disk algebra consisting of all simple closed regions of, for example, $\mathbb{R}^2$ as described in detail in Li and Li [31]. If we abandon C-extensionality instead we can rely on GBCAs. Nonextensional theories have also been used for defining multidimensional mereotopologies (see Galton [22], Roy and Stell [38]). The rationale for giving up C-extensionality is simple (see [38]): C-extensionality is a principle that holds in the perfect world where we can always find smaller parts that distinguish two distinct entities. If finite models are considered as models with limited accuracy, that is, as approximations of continuous models, C-extensionality may be violated because the distinction in the contact between two entities may be too small a part so that it is lost in the approximation.

An alternative parsimonious way out of this dilemma is to abandon $\forall x C(x, \ominus x)$ instead. The nondistributive SPOCAs with $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$ allow such a choice. At first sight it seems to be a surprising choice since the well-behavior of lattices is usually associated with distributivity. But as we have shown in [30], the nondistributive lattices in question (Stonian p-ortholattices and restrictions) behave nicely even without distributivity. In particular, these structures also satisfy the DeMorgan laws and stop only short of being Boolean. We thereby are able to answer the question posed in [13] asking what kind of structures should be considered the standard model of a nondistributive contact algebra for the case of spatially representable contact algebras. The standard (and only) models of spatially representable complete nondistributive contact algebras are the regular sets of a topological space.

Notice that there is no need to completely abandon (Con). If we define an additional "attachment" relation $A$ from $C$ as $A(x, y) \leftrightarrow [C(x, y^{**}) \vee C(x^{**}, y)] \wedge \neg C(x, y)$, we can prove $\forall x A(x, \ominus x)$ in a connected space even if $\forall x \neg C(x, \ominus x)$. Attachment is a stronger relation than contact defined in SPOCAs as $\neg C(x, y) \leftrightarrow x \leq y^{\perp}$, but weaker than *weak contact* WCont as defined in [2]. Moreover, $C$ and $A$ make the distinction between the intended interpretations of "sharing a point" and "overlapping neighborhoods" clear.

**7.3 Conclusion** This work treated mereotopology with unique closure operations algebraically and studied the arising contact algebras that may yield spatial representations for all their models. In particular, this is the first time that nondistributive contact algebras are included and studied comprehensively as algebraic counterparts of mereotopologies. We showed that SPOCAs defined over Stonian p-ortholattices with $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$ as contact are a good candidate for an ontologically coherent region-based theory of space. In fact, these are the least constrained algebraic structures that admit discrete C-extensional models among all of the algebraic theories satisfying the conditions of MT-representability which are at the same time necessary conditions for spatial representability. The other candidates for spatially representable contact algebras are SPOCAs with $x\mathbf{C}y \leftrightarrow x \not< y^{\perp}$, BCAs, in particular, its atomless extension RBCA and the weaker GBCAs. The later two correspond to the logical theories RCC and GRCC known from the literature. While RCC models are C-extensional and always continuous, the models of GRCC can be discrete but are not C-extensional. We demonstrated that the main difference between GBCAs and SPOCAs with $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$ or $x\mathbf{C}y \leftrightarrow x \not< y^{\perp}$ is whether complements are defined mereologically or topologically. Mereological complements require distributive contact algebras such as GBCA, BCA, or RBCA, while topological complements allow nondistributive contact algebra based on Stonian p-ortholattices. The remaining closure operations sum, intersections, and universal are in either case defined mereologically; topological sums require (C4), while a topological universal requires (Dis) or (C-Ext). As one of our key contributions, all mereological and topological closure operations are directly attributed to properties of the parthood lattice or the contact relation. Mereological complements manifest themselves in unique complementation in the algebraic counterparts, while topological complements require (C5) which binds the contact relation to the orthocomplementation operation. Contact algebras with topological complements can be nondistributive but are required to satisfy (Con), (C-Ext), and (C4). Thus the ontological choice of defining complements topologically is directly associated with other, more implicit, ontological choices.

**7.4 Outlook** We have established in GBCAs and SPOCAs with (C5) two weakest, potentially spatially representable theories that allow atoms and that define all closure operations either mereologically or topologically. As natural next steps concrete topological embeddings theorems for these two classes of contact algebras need to be established analogously to the topological embeddings for BCAs (see [16]). For the SPOCAs with (C5), we know that nonrepresentable models exist (see [48]). Extending the theory of SPOCAs with axioms that rule out some of the nonrepresentable models (see [48]) is a first step toward such an embedding theorem.

## Appendix A Axioms for Automated Proofs

Axioms as used for the automated proofs in Prover9 (the notation has been slightly changed to make it more readable) are the following: &, |, →, and ↔ denote the logical connectives "and," "or," simple implication, and "if and only if," respectively, while $\wedge$ and $\vee$ denote the lattice operations of meet and join. Universal closure is assumed throughout.

The theories $T_{\mathrm{OCA}} = \{\mathrm{L2}^{\vee}\text{–}\mathrm{L6}^{\vee}, \mathrm{L2}^{\wedge}\text{–}\mathrm{L4}^{\wedge}, \mathrm{C0}\text{–}\mathrm{C3}, \mathrm{O1}', \mathrm{O2}', \mathrm{O3}'\}$ and $T_{\mathrm{SPOCA}} = T_{\mathrm{OCA}} \cup \{\mathrm{PC1}, \mathrm{PC2}', \mathrm{PC2}'', \mathrm{S}\}$ axiomatize OCAs and SPOCAs.

*Lattice: Standard axioms for commutativity, associativity, and absorption*

| | | | |
|---|---|---|---|
| ($\mathrm{L2}^{\wedge}$) | $x \wedge y = y \wedge x$, | ($\mathrm{L2}^{\vee}$) | $x \vee y = y \vee x$, |
| ($\mathrm{L3}^{\wedge}$) | $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, | ($\mathrm{L3}^{\vee}$) | $(x \vee y) \vee z = x \vee (y \vee z)$, |
| ($\mathrm{L4}^{\wedge}$) | $x \vee (x \wedge y) = x$, | ($\mathrm{L4}^{\vee}$) | $x \wedge (x \vee y) = x$. |

*Boundedness: Existence of a null and one (universal) element*

| | | | |
|---|---|---|---|
| ($\mathrm{L5}^{\vee}$) | $0 \vee x = x$, | ($\mathrm{L6}^{\vee}$) | $1 \vee x = 1$. |

*Orthocomplementation and pseudocomplementation*

| | | | |
|---|---|---|---|
| ($\mathrm{O1}'$) | $x^{\perp\perp} = x$, | (PC1) | $x \wedge (x \wedge y)^* = x \wedge y^*$, |
| ($\mathrm{O2}'$) | $x \vee x^{\perp} = 1$, | ($\mathrm{PC2}'$) | $0^* = 1$, |
| ($\mathrm{O3}'$) | $x \wedge y = (x^{\perp} \vee y^{\perp})^{\perp}$, | ($\mathrm{PC2}''$) | $1^* = 0$. |

*The Stone identity*

(S) $(x \vee y)^{**} = x^{**} \vee y^{**}$.

*Contact: Basic axioms of a weak contact algebra*

| | | | |
|---|---|---|---|
| (C0) | $0 - \mathbf{C}x$, | (C1) | $x \neq 0 \rightarrow x\mathbf{C}x$, |
| (C2) | $x\mathbf{C}y \rightarrow y\mathbf{C}x$, | (C3) | $x \wedge y = x \,\&\, z\mathbf{C}x \rightarrow z\mathbf{C}y$. |

*Mereological closures as defined in Section 5.1*

(M-I) $x \wedge y \neq 0 \rightarrow (z \wedge (x \wedge y) = z \leftrightarrow (z \wedge x = z \,\&\, z \wedge y = z))$,

(M-S) $z \wedge (x \vee y) \neq 0 \leftrightarrow (x \wedge z \neq 0 \mid y \wedge z \neq 0)$,

(M-C) $z \wedge x^{\perp} = 0 \leftrightarrow z \wedge x = z$,

(O-Ext) $\forall z (x \wedge z = 0 \leftrightarrow y \wedge z = 0) \leftrightarrow x = y$.

*Topological closures as defined in Section 5.2*

(T-I) $x \wedge y \neq 0 \to (z\mathbf{C}(x \wedge y) \to (z\mathbf{C}x \,\&\, w\mathbf{C}y))$,
(T-S) $x\mathbf{C}(y \vee z) \leftrightarrow x\mathbf{C}y \mid x\mathbf{C}z$,
(C4) $x\mathbf{C}(y \vee z) \to x\mathbf{C}y \mid x\mathbf{C}z$,
(C5) $z \wedge x^{\perp} = z \leftrightarrow z - \mathbf{C}x$ $\simeq$ (T-C),
(C5′) $(x \neq 0 \mid z \neq 1 \mid (x \neq 1 \,\&\, z \neq 0)) \to (z\mathbf{C}x \leftrightarrow (z = x^{\perp} \mid z \wedge x^{\perp} \neq z))$ $\simeq$ (T-C′),
(C-Ext) $\forall z(x\mathbf{C}z \leftrightarrow y\mathbf{C}z) \leftrightarrow x = y$.

*Other axioms of interest*

(Dis) $x \neq 1 \to \exists y(y \neq 0 \,\&\, x{-}\mathbf{C}y)$,
(Int) $x{-}\mathbf{C}y \to \exists z(x{-}\mathbf{C}z \,\&\, y{-}\mathbf{C}z^{\perp})$,
(Con) $x = 0 \mid x = 1 \mid x\mathbf{C}x^{\perp}$,
(¬Con) $x{-}\mathbf{C}x^{\perp}$,
(Uni) $(x \wedge y = 0 \,\&\, x \vee y = 1 \,\&\, x \wedge z = 0 \,\&\, x \vee z = 1) \to y = z$ (unicomplemented),
(¬C4) $\exists x, y, z[x\mathbf{C}(y \vee z) \,\&\, x{-}\mathbf{C}y \,\&\, x{-}\mathbf{C}z]$ (violating (C4)),
(¬Triv) $\exists y[y \neq 1 \,\&\, y \neq 0]$ (nontrivial),
(Atom) $\exists a(a \neq 0 \,\&\, \forall x(x = 0 \mid x = a \mid x \wedge a \neq x))$ (existence of an atom).

## Appendix B Equivalence of Algebraic Axioms

Here we show that the axioms from Section 5 in a UCMT are equivalent to the algebraic versions thereof, that is, the axioms used for the automated proofs as shown in Appendix A. We can mainly rely on Theorem 3.1 but have to show additionally that all cases involving the introduced null element 0 are properly covered. We first show the equivalence for the mereological axioms and then for the topological axioms.

### B.1 Mereological axioms

(M-I$_{\text{UCMT}}$) $\forall w[P(w, x \odot y) \leftrightarrow (P(w, x) \wedge P(w, y))]$ (intersection),
(M-S$_{\text{UCMT}}$) $\forall w[O(w, x \oplus y) \leftrightarrow (O(w, x) \vee O(w, y))]$ (sum),
(M-C$_{\text{UCMT}}$) $\forall w[O(w, \ominus x) \leftrightarrow \neg P(w, x)]$ (complement),

(M-I) $x \wedge y \neq 0 \to [\forall z[z \wedge (x \wedge y) = z \leftrightarrow (z \wedge x = z \,\&\, z \wedge y = z)]]$,
(M-S) $z \wedge (x \vee y) \neq 0 \leftrightarrow (x \wedge z \neq 0 \mid y \wedge z \neq 0)$,
(M-C) $z \wedge x^{\perp} = 0 \leftrightarrow z \wedge x = z$.

**Lemma B.1** *Let $T_{\text{UCMT}}$ be the theory of* ***UCMT*** *that satisfies (P.1)–(P.3), (C.1)–(C.3), (UCMT.1)–(UCMT.7) with the definitions (O), (U), (PP). Let $T_{\text{OCA}}$ be the theory of orthocomplemented contact algebras as constructed in Theorem 3.1. Then*

1. $T_{\text{UCMT}} \models$ M-I$_{\text{UCMT}}$ *iff* $T_{\text{OCA}} \models$ M-I*;*
2. $T_{\text{UCMT}} \models$ M-S$_{\text{UCMT}}$ *iff* $T_{\text{OCA}} \cup$ O-Ext $\models$ M-S*;*
3. $T_{\text{UCMT}} \models$ M-C$_{\text{UCMT}}$ *iff* $T_{\text{OCA}} \cup$ O-Ext $\models$ M-C.

**Proof** Let us define $z = g(w)$ throughout.

1. Assume (M-I$_{\text{UCMT}}$). By definition $P(a,b) \Leftrightarrow a \leq b$, and because of $a \leq b \leftrightarrow a \wedge b = a$ we obtain $z \wedge (x \wedge y) = z$ iff $z \wedge x = z$ and $z \wedge y = z$ for all $x, y, z \neq 0$. If $x = 0$ or $y = 0$, then $x \wedge y = 0$ and (M-I) holds. Otherwise, if $z = 0$, then $z \wedge (x \wedge y) = 0 = z$ and $z \wedge x = 0 = z$ and $z \wedge y = 0 = z$, and thus (M-I) also holds.

If (M-I), then for all $x, y, z \neq 0$ (M-I$_{\text{UCMT}}$) follows from $P(a,b) \Leftrightarrow a \wedge b = a$.

2. Assume (M-S$_{\text{UCMT}}$); then if O is extensional by (O-Ext) and by definition of O we obtain $O(w, x \oplus y) \Leftrightarrow z \wedge (x \vee y) \neq 0$ and thus also (M-S) for all $x, y, z \neq 0$. If $z = 0$, then $z \wedge (x \vee y) = 0$ and $x \wedge z = 0$ and $y \wedge z = 0$. (The same is true if both $x = 0$ and $y = 0$.) If only $x = 0$, then $z \wedge (x \vee y) = z \wedge y$ iff $y \wedge z = 0$ since $x \wedge z = 0$. (The same is true for $y = 0$.)

   Reversely, if (M-S), then for all $x, y, z \neq 0$ (M-S$_{\text{UCMT}}$) directly follows.
3. Note that (M-C) is equivalent to $z \wedge x^{\perp} \neq 0 \leftrightarrow z \wedge x \neq z$.

   Assume (M-C$_{\text{UCMT}}$). Since we already established that the complementation operator $\ominus$ must at least satisfy the properties of an orthocomplementation, we have $O(w, \ominus x) \Rightarrow z \wedge x^{\perp} \neq 0$ in the presence of (O-Ext) and $\neg P(w, x) \Rightarrow z \wedge x \neq z$ which covers all cases of (M-C) in which $x, z \notin \{0, 1\}$. The remaining cases of (M-C) are the following.
   (i) If $z = 0$, then $z \wedge x^{\perp} = 0$ and $z \wedge x = 0 = z$.
   (ii) If $z = 1$, then $1 \wedge x^{\perp} = x^{\perp} \neq 0$ unless $x = 1$ and $z \wedge x = x \neq 1$ unless $x = 1$. The case when $x = 1$ is covered by (4).
   (iii) If $x = 0$, then $z \wedge 0^{\perp} = z \neq 0$ unless $z = 0$ and $z \wedge x = 0 \neq 0$ unless $z = 0$. The case $z = 0$ has already been covered by (1).
   (iv) If $x = 1$, then $z \wedge 1^{\perp} = 0$ and $z \wedge 1 = z$.

   Reversely, if (M-C), then for all $x, z \neq 0$ and $x \neq 1$ (M-C$_{\text{UCMT}}$) directly follows. □

## B.2 Topological axioms

| | | |
|---|---|---|
| (T-I$_{\text{UCMT}}$) | $\forall w[C(w, x \odot y) \rightarrow (C(w, x) \wedge C(w, y))]$ | (intersection), |
| (T-S$_{\text{UCMT}}$) | $\forall w[C(w, x \oplus y) \leftrightarrow (C(w, x) \vee C(w, y))]$ | (sum), |
| (T-C$_{\text{UCMT}}$) | $\forall w[P(w, \ominus x) \leftrightarrow \neg C(w, x)]$ | (complement), |
| (T-C$'_{\text{UCMT}}$) | $\forall w[PP(w, \ominus x) \leftrightarrow \neg C(w, x)]$ | (alternative complement), |

| | |
|---|---|
| (T-I) | $x \wedge y \neq 0 \rightarrow (z\mathbf{C}(x \wedge y) \rightarrow (z\mathbf{C}x \,\&\, w\mathbf{C}y))$, |
| (T-S) | $z\mathbf{C}(x \vee y) \leftrightarrow z\mathbf{C}x \mid z\mathbf{C}y$, |
| (C5) | $z \wedge x^{\perp} = z \leftrightarrow z{-}\mathbf{C}x$, |
| (C5′) | $(x \neq 0 \mid z \neq 1 \mid (x \neq 1 \,\&\, z \neq 0)) \rightarrow (z\mathbf{C}x \leftrightarrow (z = x^{\perp} \mid z \wedge x^{\perp} \neq z))$. |

**Lemma B.2** *Let $T_{\text{UCMT}}$ be the theory of **UCMT** that satisfies (P.1)–(P.3), (C.1)–(C.3), (UCMT.1)–(UCMT.7) with the definitions (O), (U), (PP). Let $T_{\text{OCA}}$ be the theory of orthocomplemented contact algebras as constructed in Theorem 3.1 Then*

1. $T_{\text{UCMT}} \models$ T-I$_{\text{UCMT}}$ *iff* $T_{\text{OCA}} \models$ T-I;
2. $T_{\text{UCMT}} \models$ T-S$_{\text{UCMT}}$ *iff* $T_{\text{OCA}} \models$ T-S;
3. $T_{\text{UCMT}} \models$ T-C$_{\text{UCMT}}$ *iff* $T_{\text{OCA}} \models$ C5;
4. $T_{\text{UCMT}} \models$ T-C$'_{\text{UCMT}}$ *iff* $T_{\text{OCA}} \models$ C5′.

**Proof** Notice that we again define $z = g(w)$ throughout.

1. Assume (T-I$_{\text{UCMT}}$); then (T-I) for all $x, y, z \neq 0$. If $z = 0$, then $\forall v[\neg C(w, v)]$ and thus (T-I). Otherwise, if $x = 0$ (or $y = 0$), then $x \wedge y = 0$ and thus also (T-I).
   Reversely, if (T-I), then for all $x, y, z \neq 0$ (T-I$_{\text{UCMT}}$) directly follows.

2. Assume (T-$S_{UCMT}$); then (T-S) for all $x, y, z \neq 0$. If $z = 0$, then for all v $\neg C(w, v)$ and thus (T-S) holds. (The same is true if $x = 0$ and $y = 0$.) If only $x = 0$ then $C(z, y)$ if and only if $C(z, y)$. (The same is true if $y = 0$.) Reversely, if (T-S), then for all $x, y, z \neq 0$ (T-$S_{UCMT}$) directly follows.
3. Assume (T-$C_{UCMT}$). Since $\ominus$ must at least satisfy the properties of an orthocomplementation, $P(w, \ominus x) \Leftrightarrow z \wedge x^{\perp} = z$. The remaining cases are as follows.
   (i) If $z = 0$, then $z \wedge x^{\perp} = 0 = z$ and $\neg C(0, x)$.
   (ii) If $z = 1$, then $1 \wedge x^{\perp} = x^{\perp} \neq z$ unless $x = 0$ and $C(w, x)$ unless $x = 0$. The case when $x = 0$ is covered by (3).
   (iii) If $x = 0$, then $z \wedge 0^{\perp} = z$ and $\neg C(w, 0)$.
   (iv) If $x = 1$, then $z \wedge 1^{\perp} = 0 \neq z$ unless $z = 0$ and $C(z, 1)$ unless $z = 0$. The case when $z = 0$ is covered by (1).

   Reversely, if (T-C), then for all $x, z \neq 0$ and $x \neq 1$ (T-$C_{UCMT}$) directly follows.
4. We can rewrite (T-$C'_{UCMT}$) as $\forall w[\neg PP(w, \ominus x) \leftrightarrow C(w, x)]$. Assume (T-$C'_{UCMT}$). Then $\neg PP(w, \ominus x) \Leftrightarrow z = x^{\perp}$ or $z \wedge x^{\perp} \neq z$ since $\ominus$ must at least satisfy the properties of an orthocomplementation. Then for all $x, z \notin \{0, 1\}$, (C5$'$) holds. Trivially, (C5$'$) holds if $x = 0$ or $z = 1$ or $x = 1$ and $z = 0$. For the remaining cases:
   (i) If $z = 0$, then $\neg C(0, x)$ and $0 \neq x^{\perp}$ unless $x = 1$ and $0 \wedge x^{\perp} = 0 = z$; the case $x = 1$ is covered by the precondition of (C5$'$).
   (ii) If $x = 1$, then $C(z, 1)$ unless $z = 0$ and $z \neq 1^{\perp}$ unless $z = 0$ and $z \wedge 1^{\perp} = 0 \neq z$ unless $z = 0$; the case $z = 0$ is covered by the precondition of (C5$'$).

   Reversely, if (T-C$'$), then for all $x, z \neq 0$ and $x \neq 1$ (T-$C'_{UCMT}$) directly follows. □

## Notes

1. Our notion of *spatial representability* deviates from standard topological representations in the sense that we are interested in whether all regions of an algebraic theory of mereotopology can be represented by adequately sized point sets so that notions such as contact (sharing a point), overlap (sharing a region), and complementation have intuitive spatial semantics. This understanding of spatial representations is similar to what are called "faithful interpretations" by [20] and [35]. It is more stringent than the standard notion of topological representability of algebraic structures in pure mathematics.

2. "Algebraic counterpart" refers to the class of contact algebras that can be constructed according to Theorem 3.1. Equally, a mereotopology whose models can be mapped to structures of a certain class of contact algebras is referred to as the "logical counterpart" of the class of contact algebras.

3. Orthocomplemented lattices have already been used in Biacino and Gerla [4] as an algebraic theory of Clarke's axiomatization of mereotopology.

## References


[1] Allen, J. F., "Maintaining knowledge about temporal intervals," *Communications of the ACM*, vol. 26 (1983), pp. 832–43. Zbl 0519.68079. 181

[2] Asher, N., and L. Vieu, "Toward a geometry of common sense: A semantics and a complete axiomatization for mereotopology," pp. 846–52 in *International Joint Conference on Artificial Intelligence (IJCAI-95)*, Morgan Kaufmann, San Francisco, 1995. MR 1464505. 186, 191, 202, 206

[3] Bennett, B., and I. Düntsch, "Axioms, algebras and topology," pp. 99–159 in *Handbook of Spatial Logics*, edited by M. Aiello, I. Pratt-Hartmann, and J. van Benthem, Springer, Dordrecht, 2007. MR 2393887. 178, 183

[4] Biacino, L., and G. Gerla, "Connection structures," *Notre Dame Journal of Formal Logic*, vol. 32 (1991), pp. 242–47. MR 1122997. 210

[5] Birkhoff, G., *Lattice Theory*, 3rd edition, vol. 25 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, 1967. MR 0227053. 186, 199

[6] Blyth, T. S., *Lattices and Ordered Algebraic Structures*, Universitext, Springer, London, 2005. Zbl 1073.06001. MR 2126425. 184

[7] Casati, R., and A. C. Varzi, *Parts and Places: The Structures of Spatial Representation*, MIT Press, Cambridge, Mass., 1999. 178, 181, 183, 194

[8] Clarke, B. L., "A calculus of individuals based on 'connection'," *Notre Dame Journal of Formal Logic*, vol. 22 (1981), pp. 204–18. Zbl 0438.03032. MR 0614118. 178

[9] Cohn, A. G., and S. M. Hazarika, "Qualitative spatial representation and reasoning: An overview," *Fundamenta Informaticae*, vol. 46 (2001), pp. 1–29. MR 2009798. 177

[10] Cohn, A. G., and J. Renz, "Qualitative spatial representation and reasoning," pp. 551–96 in *Handbook of Knowledge Representation*, edited by F. van Harmelen, V. Lifschitz, and B. Porter, vol. 3 of *Foundations of Artificial Intelligence*, Elsevier, Amsterdam, 2008. 177

[11] Dimov, G., and D. Vakarelov, "Contact algebras and region-based theory of space: A proximity approach, I," *Fundamenta Informaticae*, vol. 74 (2006), pp. 209–49. MR 2284194. 178

[12] Düntsch, I., W. MacCaull, D. Vakarelov, and M. Winter, "Topological representations of contact lattices," pp. 135–47 in *Relations and Kleene Algebra in Computer Science (RelMiCS-9)*, vol. 4136 of *Lecture Notes in Computer Science*, Springer, Berlin, 2006. MR 2281597. 178

[13] Düntsch, I., W. MacCaull, D. Vakarelov, and M. Winter, "Distributive contact lattices: Topological representations," *Journal of Logic and Algebraic Programming*, vol. 76 (2008), pp. 18–34. MR 2423465. 178, 206

[14] Düntsch, I., and D. Vakarelov, "Region-based theory of discrete spaces: A proximity approach," *Annals of Mathematics and Artificial Intelligence*, vol. 49 (2007), pp. 5–14. MR 2348380. 178

[15] Düntsch, I., and M. Winter, "Algebraization and representation of mereotopological structures," *Journal of Relational Methods in Computer Science*, vol. 1 (2004), pp. 161–80. 178, 183, 199

[16] Düntsch, I., and M. Winter, "A representation theorem for Boolean contact algebras," *Theoretical Computer Science*, vol. 347 (2005), pp. 498–512. MR 2187916. 178, 191, 204, 207

[17] Düntsch, I., and M. Winter, "Weak contact structures," pp. 73–82 in *Relational Methods in Computer Science*, vol. 3929 of *Lecture Notes in Computer Science*, Springer, Berlin, 2005. MR 2251975. 178, 183, 200

[18] Düntsch, I., and M. Winter, "The lattice of contact relations on a Boolean algebra," pp. 99–109 in *Relations and Kleene Algebra in Computer Science (RelMiCS-10)*,

vol. 4988 in *Lecture Notes in Computer Science*, Springer, Berlin, 2008. MR 2497622. 199

[19] Eschenbach, C., "A comparison of calculi of mereotopology," unpublished manuscript, 2007. 178, 183

[20] Forrest, P., "From ontology to topology in the theory of regions, *The Monist*, vol. 79 (1996), pp. 34–50. 179, 205, 210

[21] Galton, A., "Taking dimension seriously in qualitative spatial reasoning," pp. 501–5 in *European Conference on Artificial Intelligence (ECAI-96)*, Wiley, Chichester, UK, 1996. 181

[22] Galton, A., "The mereotopology of discrete space," pp. 251–66 in *Conference on Spatial Information Theory (COSIT-99)*, vol. 1661 in *Lecture Notes in Computer Science*, Springer, Berlin, 1999. 205

[23] Galton, A., "Multidimensional mereotopology," pp. 45–54 in *Principles of Knowledge Representation and Reasoning (KR'04)*, 2004. 181

[24] Gotts, N. M., "Formalizing commonsense topology: the INCH calculus," pp. 72–75 in *International Symposium on Artificial Intelligence and Mathematics*, 1996. 181

[25] Gotts, N. M., J. M. Gooday, and A. G. Cohn, "A connection based approach to commonsense topological description and reasoning," *The Monist*, vol. 79 (1996), pp. 51–75. 178

[26] Grätzer, G., *General Lattice Theory*, 2nd edition, Birkhäuser, Basel, 1998. MR 1670580. 184

[27] Hahmann, T., *Model-Theoretic Analysis of Asher and Vieu's Mereotopology*, Master's thesis, University of Toronto, 2008. 184

[28] Hahmann, T., and M. Grüninger, "Region-based Theories of Space: Mereotopology and Beyond," pp. 1–62 in *Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions*, edited by S. Hazarika, IGI Publishing, 2012. 178, 199

[29] Hahmann, T., and M. Grüninger, "A naïve theory of dimension for qualitative spatial relations," pp. 42–48 in *Symposium on Logical Formalizations of Commonsense Reasoning (CommonSense 2011)*, AAAI Press, Palo Alto, Calif., 2011. 181

[30] Hahmann, T., M. Winter, and M. Grüninger, "Stonian p-ortholattices: A new approach to the mereotopology $RT_0$," *Artificial Intelligence*, vol. 173 (2009), pp. 1424–40. Zbl 1190.68061. MR 2569117. 186, 187, 192, 202, 205, 206

[31] Li, S., and Y. Li, "On the complemented disk algebra," *Journal of Logic and Algebraic Programming*, vol. 66 (2006), pp. 195–211. MR 2196915. 205

[32] Li, S., and M. Ying, "Generalized region connection calculus," *Artificial Intelligence*, vol. 160 (2004), pp. 1–34. MR 2104083. 178, 199

[33] Masolo, C., and L. Vieu, "Atomicity vs. infinite divisibility of space," pp. 235–50 in *Conference on Spatial Information Theory (COSIT-99)*, vol. 1661 of *Lecture Notes in Computer Science*, Springer, Berlin, 1999. 205

[34] McCune, W., "Prover9," 2010, http://www.cs.unm.edu/~mccune/prover9/. 195

[35] Mormann, T., "Continuous lattices and Whiteheadian theory of space," pp. 35–54 in *Logic and Logical Philosophy (Żagań, Poland, 1998)*, vol. 6 of *Logic and Logical Philosophy*, Nicolaus Copernicus University Science Publishing House, Torún, Poland, 1998. Zbl 0974.03011. MR 1796917. 179, 205, 210

[36] Randell, D. A., Z. Cui, and A. G. Cohn, "A spatial logic based on regions and connection," pp. 165–76 in *Principles of Knowledge Representation and Reasoning (KR'92)*, Morgan Kaufmann, San Francisco, 1992. 178

[37] Roeper, P., "Region-based topology," *Journal of Philosophical Logic*, vol. 26 (1997), pp. 251–309. Zbl 0873.54001. MR 1456613. 182

[38] Roy, A. J., and J. G. Stell, "A qualitative account of discrete space," pp. 276–90 in *Conference on Geographic Information Science (GIScience-02)*, vol. 2478 of *Lecture Notes in Computer Science*, Springer, Berlin, 2002. 201, 205

[39] Stell, J. G., “Boolean connection algebras: A new approach to the region-connection calculus,” *Artificial Intelligence*, vol. 122 (2000), pp. 111–36. Zbl 0948.68142. MR 1785701. 178, 183, 199

[40] Stell, J. G., “Part and complement: Fundamental concepts in spatial relations,” *Annals of Mathematics and Artificial Intelligence*, vol. 41 (2004), pp. 1–17. Zbl 1062.68121. MR 2050066. 179

[41] Stell, J. G., and M. F. Worboys, “The algebraic structure of sets of regions,” pp. 163-74 in *Conference on Spatial Information Theory (COSIT-97)*, vol. 1329 of *Lecture Notes in Computer Science*, Springer, Berlin, 1997. 178, 183

[42] Stern, M., *Semimodular Lattices: Theory and Applications*, vol. 73 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, 1999. MR 1695504. 184

[43] Stone, M. H., “The theory of representations for Boolean algebras,” *Transactions of the American Mathematical Society*, vol. 40 (1936), pp. 37–111. MR 1501865. 189

[44] Vakarelov, D., “Region-based theory of space: Algebras of regions, representation theory, and logics,” pp. 267–348 in *Mathematical Problems from Applied Logic, II*, edited by D. Gabbay, S. Goncharov, and M. Zakharyaschev, Springer, New York, 2007. MR 2344365. 178, 183

[45] Varzi, A. C., “Basic problems of mereotopology,” pp. 29–38 in *Formal Ontology in Information Systems (FOIS-98)*, IOS Press, Amsterdam, 1998. 181

[46] Vieu, L., “Spatial representation and reasoning in artificial intelligence,” pp. 5–42 in *Spatial and Temporal Reasoning*, edited by O. Stock, Kluwer, Norwell, Mass., 1997. 177

[47] Whitehead, A. N., *Process and Reality*, MacMillan, New York, 1929. Zbl 55.0035.03. 178, 179, 205

[48] Winter, M., T. Hahmann, and M. Grüninger, “On the algebra of regular sets–Properties of representable Stonian $p$-ortholattices,” *Annals of Mathematics and Artificial Intelligence*, vol. 65 (2012), pp. 25–60. 193, 205, 207

[49] Winter, M., T. Hahmann, and M. Grüninger, “On the skeleton of Stonian p-ortholattices,” pp. 351–65 in *Relations and Kleene Algebra in Computer Science (RelMiCS-11)*, vol. 5827 of *Lecture Notes in Computer Science*, Springer, Berlin, 2009. MR 2781245. 205

## Acknowledgments

We are deeply grateful to Michael Winter for our extended discussions of the material in Section 4 and to Carola Eschenbach for her constructive comments on an earlier version of this paper. Their comments helped to significantly improve the clarity of our presentation here. We also thank Sanjiang Li and two anonymous reviewers for their valuable comments.

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