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Minimally Congruential Contexts: Observations and Questions on Embedding E in K

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Abstract Recently, an improvement in respect of simplicity was found by Rohan French over extant translations faithfully embedding the smallest congruential modal logic (E) in the smallest normal modal logic (K). After some preliminaries, we explore the possibility of further simplifying the translation, with various negative findings (but no positive solution). This line of inquiry leads, via a consideration of one candidate simpler translation whose status was left open earlier, to isolating the concept of a minimally congruential context. This amounts, roughly speaking, to a context exhibiting no logical properties beyond those following from its being congruential (i.e., from its yielding provably equivalent results when provably equivalent formulas are inserted into the context). On investigation, it turns out that a context inducing a translation embedding E faithfully in K need not be minimally congruential in K. Several related minimality conditions are noted in passing, some of them of considerable interest in their own right (in particular, minimal normality). The paper is exploratory, raising more questions than it settles; it ends with a list of open problems.

1 Background and Terminology

In nomenclature and terminology derived from Chellas [2] and Segerberg [14], E, EM, and K (or more explicitly EMCN) are respectively the smallest congruential, monotone, and normal monomodal logics; the terminology here (*congruential*, etc.) is also spelled out at the end of this section. We are concerned here with faithful embeddings of one modal logic S_1 (the *source* of the embedding), in another, S_2 (the *target*), via a translation, τ, mapping formulas to formulas and satisfying the condition that for all formulas A,

$$A \in \mathsf{S}_1 \text{ if and only if } \tau(A) \in \mathsf{S}_2. \tag{1.1}$$

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The adjective *faithful* alludes to the "if" direction of (1.1); from now on, we take this as understood whenever embeddings are under discussion and make it explicit only for occasional emphasis. In future, rather than writing such things as $A \in \mathsf{S}_1$ we write $\vdash_{\mathsf{S}_1} A$ and describe A in this case as S_1-provable. More specifically, we consider only such τ as satisfy the further conditions

(i) that τ maps each propositional variable or sentence letter[1] to itself and
(ii) that for every primitive Boolean connective # of arity k the requirement that $\tau(\#(A_1, \ldots, A_k)) = \#(\tau(A_1), \ldots, \tau(A_k))$ for all formulas $A_1, \ldots, A_k$, and that for some formula $C(p)$ in which at most the propositional variable p occurs, $\tau(\Box A) = C(\tau(A))$, the latter being the result of substituting $\tau(A)$ uniformly for p in $C(p)$.[2] In this case we call τ the translation *induced by* the context $C(p)$.

Gasquet and Herzig [6] (developing ideas from their earlier work in [5]) note that a τ along these lines, with $C(p) = \Diamond\Box p$, the latter abbreviating the formula $\neg\Box\neg\Box p$, embeds EM in K, correcting—without actually mentioning—an erroneous claim in Brown [1] to the effect that this translation embedded E in K. (Because only faithful translational embeddings are at issue and $\mathsf{E} \subsetneq \mathsf{EM}$, these claims are not consistent. The error in [1] is noted in French [3]. In [4, Theorem 6.1.15], French shows that the translation induced by the context $C(p) = Xp$ embeds EM in K, where X is any "mixed" affirmative modality: that is, $X = O_1 O_2 \cdots O_k$ with each O_i being $\Box$ or $\Diamond$, and each of $\Box$, $\Diamond$, appearing at least once in the sequence $O_1 O_2 \cdots O_k$.) This leaves open the question of whether a translation of the present kind can be found which embeds E in K. In [6], Gasquet and Herzig prove that there is a simple embedding of E into the smallest normal trimodal logic—variously known as K^3, K_3, among other things—namely, by setting

$$\tau(\Box A) = \Diamond_1\big(\Box_2\tau(A) \wedge \Box_3\neg\tau(A)\big),$$

and they remark that we change the target of the embedding from trimodal K to bimodal K by putting $\Box_1$ for the occurrence of $\Box_3$ here.[3] The question of whether we can improve this and embed E in monomodal K itself was answered affirmatively in French [3], where τ is defined for $\Box$-formulas by

$$\tau(\Box A) = \Diamond\big(\Diamond(\Box\tau(A) \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond(\Box\neg\tau(A) \wedge \Box\Diamond\top) \wedge \Diamond\Diamond\Box\bot)\big).$$

French writes $\Box' A$ for the formula $\Diamond(\Diamond(\Box A \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond(\Box\neg A \wedge \Box\Diamond\top) \wedge \Diamond\Diamond\Box\bot))$; thus the current translation τ is that induced by the context $C(p) = \Box' p$. Evidently this τ produces formulas of considerable complexity, and although, as French notes, it is simpler than other candidates in—or derivable on the basis of—the published literature, one wonders if something still simpler may be possible. Two measures are of interest in connection with the extent to which a translation modally complicates what it translates: the *modal degree* of A, by which is meant the maximal depth of embedding of $\Box$ in A, and the *modal complexity* of A, meaning the number of occurrences of $\Box$ in A (taking $\Diamond$ as $\neg\Box\neg$). Abbreviating these to $\mathrm{md}(A)$ and $\mathrm{mc}(A)$, respectively, we have the following table relating $\mathrm{md}(A)$ and $\mathrm{md}(\tau(A))$ for

French's τ (called $\tau_{\Box'}$ in [3]):

md(A)	md$\big(\tau(A)\big)$
0	0
1	5
2	9
3	13
4	17
⋮	⋮

The early discontinuity in the right-hand column—a jump of 5 rather than the eventually steady increase of 4 in the modal degree of the translation—is occasioned by the fact that at this stage the modal degree of the "marker" subformulas involving $\Diamond\top$ and $\Box\bot$[4] is not yet swamped by the recursive effect of τ. A similar table for mc(A) cannot be provided since mc($\tau(A)$) is not fixed, given mc(A); for example, $\Box p \wedge \Box q$ and $\Box\Box p$ are both of modal complexity 2, while their τ-translations have complexity 28 and 42, respectively. But we can register the complicating effect of τ in this respect by comparing the result of adding an initial $\Box$ to the formula to be translated, and note that if $\text{mc}(\tau(A)) = n$, then $\text{mc}(\tau(\Box A)) = 2n + 14$.[5]

In Section 2, we will explore some possibilities for simplifying the translation in these (md and mc) respects (especially the latter, by getting rid of the "marker" formulas), while retaining the status of the simplified version as an embedding of E in K. French's own translation itself represents a simplification (as the title of [3] suggests) of another translation embedding E in K that he derives in [3, Section 2] by combining a translation from Gasquet and Herzig which embeds E in the smallest normal bimodal logic—that induced by the context $\Diamond_1(\Box_1 p \wedge \Box_2\neg p)$[6]—with a translation embedding normal bimodal logics into monomodal K. This goes back to work by S. K. Thomason via Kracht and Wolter [10] and other papers by the latter authors listed in the bibliography of [3] (to which we may add the further reference: Kracht [9, Section 4]). The upshot is a translation which maps $\Box p$ to a formula of modal complexity 17 and modal degree 5 (see the end of [3, Section 2]).[7]

We close this section with the promised terminological explanations. First we review some established terminology, itself most conveniently expounded with the aid of an abbreviative device that reduces the clutter of Boolean connectives somewhat: we write $A_1, \ldots, A_m \vdash_{\mathsf{S}} B_1, \ldots, B_n$ to indicate the provability in S of the implication with the conjunction of the A_i as antecedent and the disjunction of the B_j, identifying that conjunction with $\top$ when $m = 0$ (which then amounts to the provability of the disjunction of the B_j) and identifying that disjunction with $\bot$ when $n = 0$. We further abbreviate "$A \vdash_{\mathsf{S}} B$ and $B \vdash_{\mathsf{S}} A$" to $A \dashv\vdash_{\mathsf{S}} B$. Given the restriction to translations τ satisfying the conditions set down above, since the only point at which τ differs from the identity map is on formulas of the form $\Box A$, the condition (1.1) above as to what it takes for such a translation to embed S_1 (faithfully) in S_2 can be formulated equivalently in the following terms:

$$A_1, \ldots, A_m \vdash_{\mathsf{S}_1} B_1, \ldots, B_n \Leftrightarrow \tau(A_1), \ldots, \tau(A_m) \vdash_{\mathsf{S}_2} \tau(B_1), \ldots, \tau(B_n). \quad (1.2)$$

The (standard) terminology of the opening paragraph above can now be explained as follows. A 1-ary context $C = C(p)$—not necessarily simple (in the sense of note 2)—is *congruential* in S if $A \dashv\vdash_{\mathsf{S}} B$ implies $C(A) \dashv\vdash_{\mathsf{S}} C(B)$ (for all A, B),[8] is *monotone* in S if $A \vdash_{S} B$ implies $C(A) \vdash_{\mathsf{S}} C(B)$ (for all A, B), and is *normal* in S

if $A_1, \ldots, A_m \vdash_{\mathsf{S}} B$ implies $C(A_1), \ldots, C(A_m) \vdash_{\mathsf{S}} C(B)$ (for all $A_1, \ldots, A_m, B$). The a 1-ary connective # is congruential, monotone, or normal, respectively, in S if the context $C(p) = \#p$ is congruential, monotone, or normal in S, and if S is a monomodal logic, then S is congruential, monotone, or normal, respectively, according as $\Box$ is congruential, monotone, or normal in S. In Section 3 we will introduce the idea of a context's being "minimally" congruential (monotone, normal) in S; this elaboration of the terminology is not needed for the discussion in Section 2, which, however, ends with an example motivating its introduction in Section 3 (see Example 3.5 there).

2 Gasquet–Herzig Translations

Inspired by [6], but concentrating on the monomodal case, let us define a translation τ to be a *Gasquet–Herzig translation* if there are affirmative modalities X, Y for which τ is the translation induced by the context $C(p) = \Diamond(Xp \wedge Y\neg p)$. Since $C(p)$ uniquely determines (and is determined by) τ, we will say that $C(p)$ succeeds in embedding (or fails to embed) E in K, if τ embeds (or does not embed) E faithfully in K. We sometimes just say that $C(p)$, or τ, succeeds or fails, omitting the explicit reference to E and K.

Note that since $\Box$ and $\Diamond$ have the same logical properties in E—namely, precisely such properties as follow from congruentiality—$\Diamond(Xp \wedge Y\neg p)$ succeeds just in the case of the dual formula $\Box(\widetilde{X}p \vee \widetilde{Y}\neg p)$, in which $\widetilde{X}$ is the dual of the modality X (i.e., the result of replacing all $\Diamond$'s with $\Box$'s in X, and vice versa) and likewise with Y, $\widetilde{Y}$. But we continue to work with Gasquet–Herzig formulas of the original form, as they are more perspicuously connected with the transformation of neighborhood models into Kripke models underlying the other embeddings mentioned in Section 1 (see [1], [3], [6]). A few words are in order on the general shape of such formulas. We can think of the $Y\neg p$ conjunct as intended to prevent the context $C(p)$ from being monotone, needed because $\Box$ is not monotone in E, and the first "positive" conjunct Xp as needed because we do not want $C(p)$ to be antitone either.[9] The outer $\Diamond$ is needed because in its absence we have the conjunction of a monotone and an antitone context, all of which enjoy the following *convexity* property in K, expressed using "rule notation," which means that S has the property that whenever what is above the horizontal line holds for $\vdash = \vdash_{\mathsf{S}}$, then so does what is below the line; the context $C(p)$ is replaced here by Op, thinking of O as a primitive or derived 1-ary connective

$$\frac{A \vdash B \qquad B \vdash C}{OA, OC \vdash OB}.$$

Now $\Box$ does not satisfy this condition in E, as one sees from, for example, the fact that $\Box(p \wedge (q \wedge r)), \Box r \nvdash_{\mathsf{E}} \Box(q \wedge r)$. (Note incidentally that the *disjunction* of a monotone with an antitone operator yields the dual "coconvexity" rule

$$\frac{A \vdash B \qquad B \vdash C}{OB \vdash OA, OC},$$

which is equally unwanted for $\Box$ as O, when $\vdash$ is taken as $\vdash_{\mathsf{E}}$.)

With these motivating generalities out of the way we proceed to consider some specific Gasquet–Herzig formulas $\Diamond(Xp \wedge Y\neg p)$ which may, because of more or less obviously infelicitous choices of X and Y, still turn out to fail to embed E in K.

A simple example of failure along these lines is mentioned in [3]: $\Diamond(\Box p \wedge \Box\neg p)$, concerning which French remarks that it fails to embed E in K because the translation of $\Box p \leftrightarrow \Box\neg p$ is now K-provable, although that formula itself is not E-provable (not even being K-provable). This candidate $C(p)$ is even in worse shape; in fact, $C(p)$ is "essentially nullary" in the sense that for all A, B, $C(A)$ is K-provably equivalent to $C(B)$ (each being equivalent to $\Diamond\Box\bot$). (Many a respectable modal notion, represented by a 1-ary operator O, does enjoy the behavior in question here, $Op \leftrightarrow O\neg p$ provable in K, without being essentially nullary: contingency and noncontingency being prominent examples, taking Op in the former case to be $\Diamond p \wedge \Diamond\neg p$, and its negation in the latter.) But the problem with $\Diamond(\Box p \wedge \Box\neg p)$ highlighted by French's discussion is more general, since it obviously arises for any Gasquet–Herzig formula $\Diamond(Xp \wedge Y\neg p)$ in which $X = Y$. No such formula can succeed in embedding E in K, for the reason given by French. Similarly, we may add, there is no successful Gasquet–Herzig formula $\Diamond(Xp \wedge Y\neg p)$ in which $Y = \widetilde{X}$, since faithfulness here would require $\vdash_{\mathsf{E}} \neg\Box p$, as $\vdash_{\mathsf{K}} \neg\Diamond(Xp \wedge Y\neg p)$ in this case. Aside from the generalization already mentioned of the failing $\Diamond(\Box p \wedge \Box\neg p)$, with $\Box$ replaced by an arbitrary modality, we can generalize in a different direction, showing that for a successful Gasquet–Herzig formula $\Diamond(Xp \wedge Y\neg p)$, neither X nor Y can be just plain $\Box$. (Thus, in particular, a "minimal mutilation," $\Diamond(\Box p \wedge \Box\Box\neg p)$, of French's example, $\Diamond(\Box p \wedge \Box\neg p)$, is also seen to fail.)

In the proof of Proposition 2.1 and elsewhere below, we make use of the following convention. When a particular translation τ is under discussion, we write $A \mapsto B$ to mean that $\tau(A) = B$, or even that $\tau(A) = B'$, for some B' truth-functionally equivalent to B (e.g., having $\top$ in place of $\neg\bot$).

Proposition 2.1

(i) *For no Y does $\Diamond(\Box p \wedge Y\neg p)$ succeed in embedding* E *in* K.
(ii) *For no X does $\Diamond(Xp \wedge \Box\neg p)$ succeed in embedding* E *in* K.

Proof We do the proof for (i), the case of (ii) being essentially similar. Suppose that $\Diamond(\Box p \wedge Y\neg p)$ is successful (in embedding E in K). Three subcases arise according as the leftmost modal operator in Y is $\Box$ or $\Diamond$ or nonexistent (because Y is the null modality). Take the first possibility first. Write Y as $\Box Y_0$ for the current case. Note that

$$\Box\top \mapsto \Diamond(\Box\top \wedge \Box Y_0\bot) \quad \text{while} \quad \Box\bot \mapsto \Diamond(\Box\bot \wedge \Box Y_0\top).$$

Noting that the target formulas here are K-equivalent, respectively, to

$$\Diamond\Box Y_0\bot \quad \text{and} \quad \Diamond\Box\bot,$$

we see that the second provably implies the first in K. Since this is not so for the corresponding source formulas $\Box\bot$ and $\Box\top$, the formula $\Diamond(\Box p \wedge Y\neg p)$ fails to embed E in K when Y has the form $\Box Y_0$. We turn to the second case, in which Y has the form $\Diamond Y_0$. Note that here we have $\Box\bot \mapsto \Diamond(\Box\bot \wedge \Diamond Y_0\top)$. As with any formula of the form $\Box\bot \wedge \Diamond A$, the negation of $\Box\bot \wedge \Diamond Y_0\top$, and therefore also the negation of the result of prefixing a $\Diamond$ to it, is K-provable. But $\nvdash_{\mathsf{E}} \neg\Box\bot$, so again the translation fails. Finally, if Y is the null modality, we have $\Box\top \mapsto \Box\top \wedge \bot$, giving something whose negation is K-provable, conflicting with the fact that $\nvdash_{\mathsf{E}} \neg\Box\top$. $\Box$

The particular proof given here yields the following generalization.

Corollary 2.2

(i) *For no* Y *does* $\Diamond(\Box\Diamond^n p \wedge Y\neg p)$ *succeed in embedding* E *in* K*, for any* $n \geq 0$.

(ii) *For no* X *does* $\Diamond(Xp \wedge \Box\Diamond^n \neg p)$ *succeed in embedding* E *in* K*, for any* $n \geq 0$.

Proof Again we work part (i). With the new translation, for arbitrary n, we have, for Y with initial $\Box$:

$$\Box\top \mapsto \Diamond(\Box\Diamond^n\top \wedge \Box Y_0\bot) \qquad \text{and} \qquad \Box\bot \mapsto \Diamond(\Box\Diamond^n\bot \wedge \Box Y_0\top).$$

Since $\Box\Diamond^n\bot$ is K-equivalent to $\Box\bot$, the second target formula simplifies again to $\Diamond\Box\bot$. This time the first target formula does not permit a similar simplification, but since $\Diamond\Box\bot$ K-implies $\Diamond(\Box A \wedge \Box B)$, for any A and B it still implies the first target formula, so the translation fails since $\Box\bot$ does not E-imply $\Box\top$. For the case of $Y = \Diamond Y_0$, we have $\Box\bot \mapsto \Diamond(\Box\Diamond^n\bot \wedge \Diamond Y_0\top)$, and the already-noted K-equivalence of $\Box\Diamond^n\bot$ with $\Box\bot$ returns us to the corresponding point in the second case treated in the proof of Proposition 2.1. Finally, there is also the case of Y as the null modality, which again is treated along the lines of the corresponding case at the end of that proof. □

Let us now calculate the minimal lengths for X, Y in a successful Gasquet–Herzig formula $\Diamond(Xp \wedge Y\neg p)$. Neither of these can be zero, since substituting $\bot$ for p when X is the null modality, and substituting $\top$ for p when Y is null (as at the end of the proof of Proposition 2.1), gives us a K-refutable formula, whereas the formulas whose translations these would be (resp., $\Box\bot$ and $\Box\top$) are not E-refutable. Nor can either X or Y be of length 1, since Proposition 2.1 rules out the possibility that X or Y is $\Box$, and we can exclude the possibility that either of them is $\Diamond$ by the substitutions already cited. More explicitly, if we suppose that $X = \Diamond$, then substituting $\bot$ for p makes the translation of $\Box\bot$, namely, $\Diamond(\Diamond\bot \wedge Y\neg\bot)$, K-refutable because of the first inner conjunct, while if we suppose that $Y = \Diamond$, then substituting $\top$ for p makes the translation of $\Box\top$ be $\Diamond(\Diamond\top \wedge \Diamond\bot)$, which is K-refutable because of the second inner conjunct. Thus the lengths of X and Y must be at least 2. Can they both be precisely 2?

Of the four affirmative modalities of length 2, namely, $\Diamond\Diamond$, $\Box\Diamond$, $\Diamond\Box$, and $\Box\Box$, we can rule out the first as an option for either X or Y because the argument just given against taking either X or Y as $\Diamond$ works equally well against $\Diamond\Diamond$.[10] And neither X nor Y can be $\Box\Diamond$, by Corollary 2.2. Nor can X and Y be the same, so this leaves only two possible cases:

(1) X is $\Diamond\Box$ and Y is $\Box\Box$, and

(2) X is $\Box\Box$ and Y is $\Diamond\Box$.

But a consideration of pure formulas again rules out each of these possibilities. On option (1) we have

$$\Box\top \mapsto \Diamond(\Diamond\Box\top \wedge \Box\Box\bot) \quad \text{while} \quad \Box\bot \mapsto \Diamond(\Diamond\Box\bot \wedge \Box\Box\top),$$

and the target formulas here simplify in K, respectively, to

$$\Diamond(\Diamond\top \wedge \Box\Box\bot) \qquad \text{and} \qquad \Diamond\Diamond\Box\bot.$$

But the first of these provably implies the second in K, so we have an unwanted formula showing up in the target of the embedding, namely, the translation of the

E-unprovable $\Box\top \to \Box\bot$. Similar reasoning in the case of option (2) shows that here we have the equally unwanted converse of that implication with a K-provable translation. Summarizing these findings, we have the following proposition.

Proposition 2.3 *For a Gasquet–Herzig formula $\Diamond(Xp \wedge Y\neg p)$ to succeed in embedding E in K the length of each of X, Y must be at least* 2*, and the length of at least one of them must be at least* 3.

Thus the simplest possible successful Gasquet–Herzig formula, compatibly with what has been shown thus far, will be one in which one of X, Y is of length 2 and the other of length 3. A candidate is presented in Example 2.5, whose success or failure for inducing a translation embedding E in K is unclear, though it will figure again in the discussion in Section 3. First we begin here with a very similar—but more evidently unsuccessful—example, minimally satisfying the length constraints of Proposition 2.3 (as well as Corollary 2.2).

Example 2.4 Consider the Gasquet–Herzig formula with $X = \Diamond\Box$, $Y = \Diamond\Box\Box$. For the induced translation we have

$$\Box\top \mapsto \Diamond(\Diamond\Box\top \wedge \Diamond\Box\Box\bot) \qquad \text{and} \qquad \Box\bot \mapsto \Diamond(\Diamond\Box\bot \wedge \Diamond\Box\Box\top),$$

with the target formulas simplifying to $\Diamond\Diamond\Box\Box\bot$ and $\Diamond\Diamond\Box\bot$, respectively. Since the second of these provably implies the first in K, we get the translation of the E-unprovable $\Box\bot \to \Box\top$ provable in K, and the translation fails.

The next example is of a slight variation on that just given; as already mentioned, its status as inducing a successful Gasquet–Herzig translation is not known (to the author). It will reappear in a related setting in Example 3.5.

Example 2.5 If we tweak the Gasquet–Herzig formula of Example 2.4 by retaining X as $\Diamond\Box$ but changing the central $\Box$ of its Y to $\Diamond$, so that the new Y is $\Diamond\Diamond\Box$, we obtain a context inducing a translation τ for which $\tau(\Box A)$ is $\Diamond(\Diamond\Box\tau(A) \wedge \Diamond\Diamond\Box\neg\tau(A))$. If this translation does indeed embed E in K, then it can be regarded as simplifying that of [3] by discarding the $\top$- and $\bot$-involving "marker" subformulas. It results in a steady increase in modal degree of 4 (the entries for the right-hand column of an md-table like that given for French's translation in Section 1 running: $0, 4, 8, 12, \ldots$), and for modal complexity we have $\mathrm{mc}(\tau(A)) = 2n + 6$, where $\mathrm{mc}(A) = n$.

3 Minimal Congruentiality and Related Concepts

Each of the conditions of congruentiality, monotony, and normality from the end of Section 1, and many other conditions in the same vein, says that certain basic logical relations—this terminology to be clarified in the following paragraph—hold among formulas $\Box D_i$ whenever certain (not necessarily basic) logical relations hold among D_i. In the case of the monotone condition, for instance, what is required for this to be satisfied by the context $C(p)$ in S is that whenever the binary relation "S-provably implies" holds between A and B, this same relation holds between $C(A)$ and $C(B)$, while the antitone condition (see note 9) says instead that the converse of this relation holds these formulas. Sticking with this example to illustrate the minimality theme, but specializing $C(p)$ to $\Box p$ for convenience, we are interested in spelling out the idea that for any k the *only* conditions under which a k-ary logical relation holds $\Box$-formulas $\Box D_1, \ldots, \Box D_k$

(in S) are those consequential on $\Box$'s being monotone; that is, for any m, n, for which $k = m + n$ and $\{D_1, \ldots, D_k\} = \{A_1, \ldots, A_m, B_1, \ldots, B_n\}$, whenever $\Box A_1, \ldots, \Box A_m \vdash_S \Box B_1, \ldots, \Box B_n$, we have $A_i \vdash B_j$ for some i, j. So the *only* logical relations holding among $\Box$-formulas are those which do so by virtue of the monotone condition. This we will express by saying that $\Box$ is "minimally monotone" in S, in the precise definition of this and kindred notions below. Note, apropos of the phrase "consequential on $\Box$'s being monotone" just used, that we cannot simply look at the monotone condition and reverse it by saying that whenever $\Box A_1, \ldots, \Box A_m \vdash_S \Box B_1, \ldots, \Box B_n$, we have $m = n = 1$ and $A_1 \vdash_S B_1$, since any $\vdash_S$ can be weakened by the addition of arbitrary formulas, including arbitrary $\Box$-formulas, on the left and right. Similarly, in the case of something like the convexity condition from Section 1, we cannot simply say, for the associated minimality condition, that whenever $\Box A_1, \ldots, \Box A_m \vdash_S \Box B_1, \ldots, \Box B_n$, we have $m = 2$, $n = 1$ (so we are dealing with $A_1, A_2 \vdash_S B$), and either $A_1 \vdash_S B$ and $B \vdash_S A_2$, or else $A_2 \vdash_S B$ and $B \vdash_S A_1$. Rather, we must say that whenever $\Box A_1, \ldots, \Box A_m \vdash_S \Box B_1, \ldots, \Box B_n$, there are A_i, A_j, and B_k for which $A_i \vdash_S B_k$ and $B_k \vdash_S A_j$.

Here, adapting Lemmon [11, pp. 69–71], we think of a particular n-ary logical relation as given by a set of $\vdash$-statements involving n schematic letters for formulas; for example, the binary relation of *(logical) equivalence* is represented by the set $\{D_1 \vdash D_2; D_2 \vdash D_1\}$, the binary relation of *subcontrariety* by $\{\varnothing \vdash D_1, D_2\}$, the ternary relation of *generalized equivalence* (see McKee [13]) by

$$\{D_1, D_2 \vdash D_3; D_2, D_3 \vdash D_2; D_2, D_3 \vdash D_1\},$$

and so on, where semicolons separate $\vdash$-statements, to avoid confusion with the statement-internal use of commas to separate formulas.[11] When only one such schematic $\vdash$-statement is involved, we speak of a *basic* logical relation. (Thus if we identify the relation with the $\vdash$-statement rather than the latter's unit set, logical relations in general are sets of basic logical relations.)[12] In this terminology we can be more precise about the minimality conditions: the rule-like conditions of congruentiality, monotony, and normality, all say that a certain basic logical relation holds among $\Box$-formulas whenever a set of basic logical relations holds among their immediate subformulas (to get "basic" to apply in the case of congruentiality, see note 8). The associated minimality conditions require that it is *only* when such a set of logical relations holds (among the subformulas) that the basic logical relation in question holds (among the $\Box$-formulas).

These considerations lead, in particular, to the following *minimality conditions* associated with congruentiality, monotony, and normality for a context C, as defined at the end of Section 1 (relative to S):

$C(A_1), \ldots, C(A_m) \vdash_S C(B_1), \ldots, C(B_n)$ implies $A_i \dashv\vdash_S B_j$ for some i, j $(1 \le i \le m, 1 \le j \le n)$ (for congruentiality);

$C(A_1), \ldots, C(A_m) \vdash_S C(B_1), \ldots, C(B_n)$ implies $A_i \vdash_S B_j$ for some i, j $(1 \le i \le m, 1 \le j \le n)$ (for monotony);

$C(A_1), \ldots, C(A_m) \vdash_S C(B_1), \ldots, C(B_n)$ implies $A_1, \ldots, A_m \vdash_S B_j$ for some j $(1 \le j \le n)$ (for normality).

A context C is *minimally congruential* (*minimally monotone*, *minimally normal*) in S if C is congruential (resp., monotone, normal) in S and also satisfies the associated minimality condition listed above. When the context $C(p)$ is $\Box p$, we say that $\Box$ is minimally congruential, and so on, in S, and likewise with other primitive or derived 1-ary connectives. Note that in the case of normal S we may equivalently just impose the $m = 1$ cases of the minimality condition, since we can put $\Box(A_1 \wedge \cdots \wedge A_m)$ ($= \Box\top$ for $m = 0$) in place of $\Box A_1 \wedge \cdots \wedge \Box A_m$. As equivalent characterizations we have the following:

$\Box$ is

(i) minimally congruential in S,
(ii) minimally monotone in S,
(iii) minimally normal in S,

respectively, according as for all $A_1, \ldots, A_m, B_1, \ldots, B_n$, we have

(i) $\Box A_1, \ldots, \Box A_m \vdash_{\mathsf{S}} \Box B_1, \ldots, \Box B_n$ if and only if $A_i \dashv\vdash_{\mathsf{S}} B_j$ for some i, j $(1 \leq i \leq m, 1 \leq j \leq n)$;[13]
(ii) $\Box A_1, \ldots, \Box A_m \vdash_{\mathsf{S}} \Box B_1, \ldots, \Box B_n$ if and only if $A_i \vdash_{\mathsf{S}} B_j$ for some i, j $(1 \leq i \leq m, 1 \leq j \leq n)$;
(iii) $\Box A_1, \ldots, \Box A_m \vdash_{\mathsf{S}} \Box B_1, \ldots, \Box B_n$ if and only if $A_1, \ldots, A_m \vdash_{\mathsf{S}} B_j$ for some j $(1 \leq j \leq n)$.

In all cases, we allow m and n to take on the value zero. Although for strategic purposes our main concern is with minimal congruentiality, minimal normality is of considerable interest in its own right, and we include some remarks on the subject here. Note that the minimality condition associated with normality—the "only if" direction of (c) here—could equivalently be written as follows, using the notation $\Box\Gamma$ for $\{\Box A | A \in \Gamma\}$ (in accordance with which convention, $\Box\Gamma = \varnothing$ when $\Gamma = \varnothing$, and similarly with Δ, these corresponding to the $m, n = 0$ cases above):

$$\text{If } \Box\Gamma \vdash_{\mathsf{S}} \Box\Delta, \text{ then for some } B \in \Delta, \text{ we have } \Gamma \vdash_{\mathsf{S}} B. \tag{3.1}$$

If $\Box$ is assumed normal in S, then, as noted above (in terms of insisting that $m = 1$), we can formulate this with Γ a singleton:

$$\text{If } \Box A \vdash_{\mathsf{S}} \Box\Delta, \text{ then for some } B \in \Delta, \text{ we have } A \vdash_{\mathsf{S}} B. \tag{3.2}$$

A $\Diamond$-formulation is also available for the minimality condition associated with normality (for S presumed normal):

$$\text{If } \Diamond A_1, \ldots, \Diamond A_m \vdash_{\mathsf{S}} \Diamond B, \text{ then for some } A_i, \text{ we have } A_i \vdash_{\mathsf{S}} B. \tag{3.3}$$

Similarly, the minimality condition for congruentiality (with $\Box$ taken as primitive and assumed congruential) can be given an obvious $\Diamond$-formulation:

$$\text{If } \Diamond A_1, \ldots, \Diamond A_m \vdash_{\mathsf{S}} \Diamond B_1, \ldots, \Diamond B_n, \text{ then } A_i \dashv\vdash_{\mathsf{S}} B_j \text{ for some } i, j. \tag{3.4}$$

The minimality condition for normality is described in Humberstone and Williamson [7, p. 40], though not in those terms but rather as a generalized cancellation rule (where cancellation takes us from $\Box A \dashv\vdash \Box B$ to $A \dashv\vdash B$[14]) since it subsumes the "rule of disjunction" (the n-ary such rule, for each $n \in \omega$); we might equally describe it as a *conditional* version of the rule of disjunction in view of the formulations (3.1) and (3.2), the conditional element coming in with what is on the left of the $\vdash$. For bibliographical and other information on the rule of disjunction, as well as for related conditions, consult Williamson [15].

Note that the "for some $B \in \Delta$" part of (3.1) means that for $\Box$ to be minimally normal in S—or indeed to be minimally congruential or minimally monotone in S—we can never have Δ empty when $\Box\Gamma \vdash_{\mathsf{S}} \Box\Delta$: no set of $\Box$-formulas can be

S-inconsistent. (This is a respect in which the minimality condition associated with normality goes beyond the combination of the rule of cancellation and the rule of disjunction.) Thus in no extension of KD is $\Box$ minimally normal (since we have $\Box A, \Box\neg A \vdash_{KD} \varnothing$). Nor can $\Box$ be minimally normal in any consistent extension of what in the nomenclature of Chellas [2] is called KD_c—axiomatically, the normal extension of K by $\Diamond A \to \Box A$—since if S extends KD_c, then $\vdash_S \Box p, \Box\neg p$, so by the "rule of disjunction" ($m = 0$, $n \geq 1$) aspects of minimal normality we should have to have either $\vdash_S p$ or $\vdash_S \neg p$, and S is then inconsistent. Further, from this we see that $\Box$ is not minimally normal in any consistent S-extension of K4, since from the fact that $\Box p \vdash_S \Box\Box p$ the "rule of cancellation" aspect of minimal normality (see note 14), this would give $p \vdash_S \Box p$, and S accordingly extends KD_c.

The following is implicit in the discussion of [7, p. 40]; the proofs of (i) and the "if" half of (ii) are simple, so we show just the "only if" half of (ii). We denote by $R(w)$ the set $\{z \in W \mid wRz\}$ (understood relative to a model $\langle W, R, V\rangle$ with $w \in W$).

Theorem 3.1

(i) $\Box$ *is minimally normal in* K.

(ii) *Let* S *be any consistent normal modal logic with* $\mathcal{M}_S = \langle W_S, R_S, V_S\rangle$ *as its canonical model. Then* $\Box$ *is minimally normal in* S *if and only if* $\mathcal{M}_S$ *satisfies the condition that for all finite* $Y \subseteq W_S$ *there exists* $x \in W_S$ *with* $R_S(x) = Y$.

Proof For the "only if" half of (ii), suppose that $\Box$ is minimally normal in (consistent normal) S. Given finite $Y \subseteq W_S$ we get the promised x as any maximally S-consistent superset of

$$\left\{\Diamond B \,\middle|\, B \in \bigcup Y\right\} \cup \left\{\Box\neg A \,\middle|\, A \notin \bigcup Y\right\}.$$

The first term of this union gives $Y \subseteq R(x)$ and would do so even in the case of Y infinite, while the second term gives $R(x) \subseteq Y$, and here the fact that Y is finite is essential: for $Y = \{y_1, \ldots, y_n\}$ (with $y_i \neq y_j$ when $i \neq j$), we show that $R(x) \subseteq Y$ contrapositively, by showing that if $y \notin Y$, then not Rxy. Suppose accordingly that $y \notin Y$, which means that $y \neq y_1$ and ... and $y \neq y_n$. Thus we may choose formulas $D_1, \ldots, D_n$ such that $D_1 \notin y_1$, $D_1 \in y$, $D_2 \notin y_2$, $D_2 \in y, \ldots, D_n \notin y_n$, $D_n \in y$. So $D_1 \wedge \cdots \wedge D_n \in y$ while $D_1 \wedge \cdots \wedge D_n \notin \bigcup Y$. That puts $\Box\neg(D_1 \wedge \cdots \wedge D_n)$ into x, showing that not Rxy (since $D_1 \wedge \cdots \wedge D_n \in y$).

So it remains only to check that this set (the above union) is itself S-consistent. If it is not, we have

$$\Box\neg A_1, \ldots, \Box\neg A_m \vdash_S \Box\neg B_1, \ldots, \Box\neg B_n$$

for some $A_1, \ldots, A_m, B_1, \ldots, B_n$. By the minimality condition (for normality), for some j ($1 \leq j \leq n$), $\neg A_1, \ldots, \neg A_m \vdash_S \neg B_j$. Since B_j belongs to some $y \in Y$, we are in trouble, as each of $\neg A_1, \ldots, \neg A_m$ is in every element of Y, and therefore in y, placing $\neg B_j$ in y too. $\Box$

Here we see the semantic aspects of some syntactically formulated observations above, such as that to the effect that $\Box$ is not minimally normal in any extension of KD or, alternatively put, that no set of $\Box$-formulas is S-inconsistent when $\Box$ is minimally normal in S. In Theorem 3.1(ii) this emerges in the fact that for such S, as $\varnothing$ is a finite set, we need $x \in W_S$ with $R_S(x) = \varnothing$. Cancellation is another special case, in which the relevant finite sets are the unit sets of the points in the model, this being the predecessor condition of [7, p. 37]. There is an interesting contrast with the case of the (also subsumed) rule of disjunction, touched on at [7, p. 40], where it is remarked that satisfying this rule, that is, the n-ary rule of disjunction for all n, is necessary and sufficient for the canonical frame to satisfy: for all finite $Y \supseteq W_{\mathsf{S}}$ there exists $x \in W_{\mathsf{S}}$ with $R_{\mathsf{S}}(x) \supseteq Y$. (Here $\supseteq$ replaces $=$ in the above condition.) The "finite" can be dropped in the rule of disjunction case, exploiting the finitary nature of the property of being an S-inconsistent set, and any (even uncountable) $Y \subseteq W_{\mathsf{S}}$ has a common R_{S}-predecessor. But in the present case, with an $x \in W_{\mathsf{S}}$ such that $R_{\mathsf{S}}(x) = Y$ for any given finite Y, the reference to finiteness remains essential. Without it we have a condition that, by Cantor's theorem, no frame could satisfy, since it would require there to be at least as many points in the frame as there are subsets, any distinct Y and Y' giving distinct x and x' with $R(x) = Y$, $R(x') = Y'$. For the same reason, no frame $\langle W, R\rangle$ with W finite can satisfy the condition that for every (of necessity, finite) $Y \subseteq W$, we have $x \in W$ with $R(x) = Y$.

The condition that every finite subset (of the universe of the canonical frame for S) has to have an "exact" common predecessor throws some light on the difficulty of finding a normal modal logic properly extending K in which $\Box$ is minimally normal, because it is difficult—and may turn out to be impossible—to force the canonical frame for a proper consistent extension of K to satisfy this condition, which is in tension with many (and perhaps all) similarly canonically "enforceable" conditions. To give just one example: we considered 4 above syntactically, with its making for a failure of cancellation (and hence minimal normality) for 4; from a semantic point of view, as soon as we have $R_{\mathsf{S}}yz$ with $y \neq z$, the existence of x with $R_{\mathsf{S}}(x) = \{y\}$ is inconsistent with transitivity, since we cannot have $R_{\mathsf{S}}xz$. Similarly the observation, made above, that $\Box$ is minimally normal in no extension S of KD reflects the fact that $\varnothing$, being a finite subset, demands an $x \in W_{\mathsf{S}}$ with $R_{\mathsf{S}}(x) = \varnothing$.

The role, specifically, of the *canonical* frame for S is worth emphasizing here. It is not sufficient for $\Box$'s being minimally normal in (a normal modal logic) S that S should be determined by *some* frame satisfying the "exact predecessor for finite sets" condition.

Example 3.2 A simple counterexample arises with the frame $\mathcal{F}_{\mathrm{HF}} = \langle W_{\mathrm{HF}}, R_{\mathrm{HF}}\rangle$ with W_{HF} the set of all hereditarily finite pure sets and $R_{\mathrm{HF}}xy$ just in case $y \in x$. The exact common predecessor condition is satisfied, since for any $y_1, \ldots, y_n \in W_{\mathrm{HF}}$ we have $x \in W_{\mathrm{HF}}$ with x standing in the relation R_{HF} to precisely $y_1, \ldots, y_n$, by taking x as $\{y_1, \ldots, y_n\}$ itself ($R_{\mathrm{HF}}(x) = x$, for all $x \in W_{\mathrm{HF}}$). Since there is only one $x \in W_{\mathrm{HF}}$ with $R_{\mathrm{HF}}(x) = \varnothing$ (namely, $x = \varnothing$), where S_{HF} is the logic determined by $\mathcal{F}_{\mathrm{HF}}$, we have $\vdash_{\mathsf{S}_{\mathrm{HF}}} \Box(\Box\bot \to p), \Box(\Box\bot \to \neg p)$. But since $\nvdash_{\mathsf{S}_{\mathrm{HF}}} \Box\bot \to p$ and $\nvdash_{\mathsf{S}_{\mathrm{HF}}} \Box\bot \to \neg p$, $\Box$ is not minimally normal in S_{HF}.[15]

Example 3.2 concerns the "rule of disjunction" aspect of minimal normality, so presumably the point about the significance of the canonical frame is known for this case. For example, S4.3 is determined by the frame consisting of the rational numbers with their standard $\leq$-ordering, a frame satisfying the condition that for all finite subsets Y there exists x with $R(x) \supseteq Y$, while this logic conspicuously lacks the disjunction property; consider its most common presentation as an axiomatic extension of S4. But in Example 3.2 we wanted to illustrate the point specifically with the "exact" version of the condition ($R(x) = Y$ rather than $R(x) \supseteq Y$) tailored to the full strength of minimal normality. Since the disjunctive axiom just alluded to is a binary disjunction, we could make this point specifically with the n-ary rule of disjunction and the condition that every n points have a common predecessor, specialized to the case of $n = 1$. For $n = 1$, however (the "rule of denecessitation"), the contrast we have been emphasizing between an arbitrary characteristic frame and the canonical frame lapses: it is easy to see that the logic determined by a converse serial frame has the rule of denecessitation. Similarly, in the conditional form of this rule—cancellation à la note 14—if S is determined by a frame in which every point has a predecessor of which it is the unique successor, then S enjoys cancellation.

Since the main issue raised by embedding E in K arises over minimal congruentiality, from this point on minimal normality is mentioned only in connection with that condition (and minimal monotony). Proposition 3.1(i) above gives that part of the following observation pertaining to minimal normality; the other parts follow from the (soundness and) completeness of E and EM with respect to the neighborhood semantics and the "locale" semantics of Jennings and Schotch [8] (or see Chellas [2, Exercises 7.9 (p. 211), 7.24 (p. 219), 9.27 (p. 256)]), where the *locale* terminology is not used.

Proposition 3.3 *In the logics* E, EM, *and* K, $\Box$ *is minimally congruential, minimally monotone, and minimally normal, respectively.*

The author's original plan of attack on the problem of further simplifying French's translation from [3] and, more specifically, replacing it with the translation induced by the Gasquet–Herzig context from Example 2.5, had been to show that the latter context was minimally congruential in K and conclude that for $\Box' A = \Diamond(\Diamond\Box A \wedge \Diamond\Diamond\Box\neg A)$, the (Boolean connectives plus) $\Box'$-fragment of K was precisely E (with $\Box$ written as $\Box'$), giving us the promised faithful embedding. But this also required something that goes beyond what Proposition 3.3 says about E, namely, that what it says about E applies to no proper consistent extension of E: E is the *only* consistent congruential monomodal logic (whether we write the non-Boolean primitive as $\Box$ or as $\Box'$) in which $\Box$ is minimally congruential. Here we leave this as simply part of a conjecture which raises similar questions about the minimality conditions associated with monotony and normality, though in the digression below, which can be skipped without loss of continuity, we establish a weaker result bearing on the congruentiality case.

Conjecture Among consistent (mono)modal logics, the only congruential logic in which $\Box$ is minimally congruential is E; the only monotone logic in which $\Box$ is minimally monotone is EM; and the only normal modal logic in which $\Box$ is minimally normal is K.

Digression A modal logic S will be said to satisfy the *modal separation condition* just in the case for all (finite) sets of formulas Γ, Δ, Θ, Σ, in which the formulas in $\Gamma \cup \Delta$ are $\Box$-free; we have

$$\Gamma, \Box\Theta \vdash_{\mathsf{S}} \Delta, \Box\Sigma \text{ implies that either } \Gamma \vdash_{\mathsf{S}} \Delta \text{ or } \Box\Theta \vdash_{\mathsf{S}} \Box\Sigma.$$

The name for this condition is adapted from talk of the condition of "separation of variables" (e.g., in Maksimova [12]), which requires that $\Gamma \vdash_{\mathsf{S}} \Delta$ or $\Theta \vdash_{\mathsf{S}} \Sigma$ on the hypothesis that, while $\Gamma, \Theta \vdash_{\mathsf{S}} \Delta, \Sigma$, there are no propositional variables common to the formulas in $\Gamma \cup \Delta$ and the formulas in $\Theta \cup \Sigma$. Being modally separated, that is, satisfying the modal separation condition, is equivalent—though we do not show this here—to being what Zolin [18, Section 5], calls a "modalized" logic.

Proposition 3.4 *If* S *is a consistent modally separated modal logic in which* $\Box$ *is minimally congruential, then* $\mathsf{S} = \mathsf{E}$.

Proof Assume the antecedents here. We get $\mathsf{E} \subseteq \mathsf{S}$ immediately from the assumption of congruentiality. For the converse inclusion suppose, for a contradiction, that $\vdash_{\mathsf{S}} A$ while $\nvdash_{\mathsf{E}} A$, and, without loss of generality, that A is a formula for which there is no formula of lower modal degree witnessing the failure of the inclusion $\mathsf{S} \subseteq \mathsf{E}$ (i.e., for any formula A' for which $\vdash_{\mathsf{S}} A'$ and $\nvdash_{\mathsf{E}} A'$, the modal degree of A' is greater than or equal to that of A). Write A in conjunctive normal form (CNF) with $\Box$-formulas that are not proper subformulas of other $\Box$-subformulas of A treated as atoms. A conjunct of this CNF formula looks like this:

$$\neg B_1 \vee \cdots \vee \neg B_k \vee \neg\Box C_1 \vee \cdots \vee \neg\Box C_\ell \vee D_1 \vee \cdots \vee D_m \vee \Box E_1 \vee \cdots \vee \Box E_n,$$

in which the B_i and the D_j can be taken to be propositional variables. One such conjunct must be S-provable without being E-provable, or else A could not have this status. For such a conjunct, A^*, say, we then have

$$B_1, \ldots, B_k, \Box C_1, \ldots, \Box C_\ell \vdash_{\mathsf{S}} D_1, \ldots, D_m, \Box E_1, \ldots, \Box E_n.$$

By the hypothesis that S satisfies the modal separation condition, we have either

(a) $\Gamma \vdash_{\mathsf{S}} \Sigma$ or
(b) $\Box\Delta \vdash_{\mathsf{S}} \Box\Theta$,

where $\Gamma = \{B_1, \ldots, B_k\}$, $\Delta = \{C_1, \ldots, C_\ell\}$, $\Sigma = \{D_1, \ldots, D_m\}$, and $\Theta = \{E_1, \ldots, E_n\}$. Now alternative (a) does not obtain, since this would contradict the consistency of S (given that A^* is not E-provable), so we are left with alternative (b), and we have $\Box\Delta \vdash_{\mathsf{S}} \Box\Theta$, while $\Box\Delta \nvdash_{\mathsf{E}} \Box\Theta$ (since otherwise we should have $\vdash_{\mathsf{E}} A^*$). Accordingly by the minimality condition associated with congruentiality, we have $C_i \dashv\vdash_{\mathsf{S}} E_j$ for some $C_i \in \Delta$, $E_j \in \Theta$, and again this does not hold for $\vdash_{\mathsf{E}}$ in place of $\vdash_{\mathsf{S}}$, as this would imply that $\Box\Delta \vdash_{\mathsf{E}} \Box\Theta$. So either $C_i \nvdash_{\mathsf{E}} E_j$ or $E_j \nvdash_{\mathsf{E}} C_i$ (or both). We work the former case (the latter running identically), having now a formula $C_i \to E_j$ which, like the original A^*, is S-provable but not E-provable. Since $C_i \to E_j$ is S-provable but not E-provable, and also of lower modal degree than A, this contradicts the choice of A as a witness of lowest degree to the noninclusion $\mathsf{S} \not\subseteq \mathsf{E}$. $\Box$

Thus E is the unique consistent modal logic in which $\Box$ is minimally congruential and which is modally separated, since we already know from Proposition 3.3 that $\Box$ is minimally congruential in E, and it is not hard to see that E is modally separated.
End of Digression

Even if the above conjecture had been affirmatively settled insofar as it bears on the case of E (and without the modal separation restriction in Proposition 3.4), another serious obstacle barred the path of the original plan sketched above. The context supplied by Example 2.5 turns out not to be minimally congruential in K after all, as we now illustrate.

Example 3.5 We repeat here the context in question, with gaps marking the place of the context variable (the p of $C(p)$): $\Diamond(\Diamond\Box\_ \wedge \Diamond\Diamond\Box\neg\_)$. Now observe that filling the blanks with $\Box\bot$, $\top$, and $\bot$ reveals the ternary logical relation $\{D_1 \vdash D_2,$

$D_3\}$ to hold in K among the resulting three formulas in the order just given, displayed here with the blank-fillers underlined (as a visual aid), signaling a failure of this context to satisfy the minimality condition associated with congruentiality, because $\Box\bot$ is K-equivalent neither to $\top$ nor to $\bot$:

$$\begin{gathered}\Diamond(\Diamond\Box\underline{\Box\bot} \wedge \Diamond\Diamond\Box\neg\underline{\Box\bot}) \vdash_{\mathsf{K}} \Diamond(\Diamond\Box\underline{\top} \wedge \Diamond\Diamond\Box\neg\underline{\top}),\\ \Diamond(\Diamond\Box\underline{\bot} \wedge \Diamond\Diamond\,\Box\neg\underline{\bot}).\end{gathered} \tag{3.5}$$

We have (3.5) because, removing outer $\Diamond$'s, dropping the underlining, and rewriting $\neg\top$ and $\neg\bot$ as $\bot$ and $\top$, respectively:

$$\Diamond\Box\Box\bot \wedge \Diamond\Diamond\Box\neg\Box\bot \vdash_{\mathsf{K}} \Diamond\Box\top \wedge \Diamond\Diamond\Box\bot, \Diamond\Box\bot \wedge \Diamond\Diamond\Box\top. \tag{3.6}$$

To see that (3.6) is satisfied, note that $\Diamond\Box\Box\bot \vdash_{\mathsf{K}} \Diamond\Box\top$ and (rewriting $\Diamond\Diamond\Box\neg\Box\bot$ as $\Diamond\Diamond\Box\Diamond\top$) $\Diamond\Diamond\Box\Diamond\top \vdash_{\mathsf{K}} \Diamond\Diamond\Box\top$, so it remains to note that $\Diamond\Box\Box\bot \vdash_{\mathsf{K}} \Diamond\Diamond\Box\bot, \Diamond\Box\bot$, or, removing initial $\Diamond$s, that $\Box\Box\bot \vdash_{\mathsf{K}} \Diamond\Box\bot, \Box\bot$ (which follows by substitution of $\Box\bot$ for p and $\bot$ for q from the fact that $\Box p \vdash_{\mathsf{K}} \Diamond p, \Box q$). Thus, with $\Box'$ as above—that is, $\Box' A = \Diamond(\Diamond\Box A \wedge \Diamond\Diamond\Box\neg A)$—we have $\Box'\Box\bot \vdash_{\mathsf{K}} \Box'\top, \Box'\bot$ without $\Box\bot \dashv\vdash_{\mathsf{K}} \top$ or $\Box\bot \dashv\vdash_{\mathsf{K}} \bot$, showing $\Box'$ not to be minimally congruential in K.

The failure of the context from Example 2.5 to be minimally congruential in K does not show that the translation it induces fails to embed E in K: the translation would replace every $\Box$ in a formula with the $\Box'$ of Example 3.5, whereas crucially in that example, we have an unreplaced $\Box$ in one of the formulas involved, namely, $\Box'\Box\bot$. Thus the above counterexample makes use of a formula which lies outside of the image of the translation in question. In fact, French's own context $\Box' p$ (not the current $\Box' p$) from the end of Section 1, which we recall induces a translation embedding E in K, is itself not minimally congruential in K, as we now illustrate.

Example 3.6 Recall that French's $\Box'$ applies to a formula to yield the result of filling the blanks below with that formula:

$$\Diamond[\Diamond(\Box\_ \;\wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond(\Box\neg\_ \;\wedge \Box!\,\Diamond\top) \wedge \Diamond\Diamond\Box\bot)].$$

Thus, filling the blanks with $\top$ and $\Box\Diamond\top$ gives (3.7) and (3.8), respectively, in which $\neg\top$ and $\neg\Box\Diamond\top$ have been rewritten as $\bot$ and $\Diamond\Box\bot$, but other simplifications have not been made (so that, for instance, (3.7) includes a redundant $\Box\top$ conjunct in one subformula, and in (3.8) we have a conjunctive subformula with two identical conjuncts):

$$\Diamond[\Diamond(\Box\top \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond(\Box\bot \wedge \Box\Diamond\top) \wedge \Diamond\Diamond\Box\bot)], \tag{3.7}$$

$$\Diamond[\Diamond(\Box\Box\Diamond\top \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond(\Box\Diamond\Diamond\Box\bot \wedge \Box\Diamond\top) \wedge \Diamond\Diamond\Box\bot)]. \tag{3.8}$$

(Here we refrain from following the lead of Example 3.5 and removing the outer $\Diamond$s, in order to reduce the number of different formulas in play.) Making the simplifications alluded to, and some others, we see that (3.7) and (3.8) are respectively K-equivalent to (3.9) and (3.10):

$$\Diamond[\Diamond\Box\Box\Diamond\top \wedge \Diamond(\Diamond\Box\bot \wedge \Diamond\Diamond\Box\bot], \tag{3.9}$$

$$\Diamond[\Diamond\Box\Box\Diamond\top \wedge \Diamond(\Diamond\Box\Diamond\Diamond\Box\bot \wedge \Diamond\Diamond\Box\bot)]. \tag{3.10}$$

With this processing, we see that (3.9) $\vdash_{\mathsf{K}}$ (3.10), and (3.7) $\vdash_{\mathsf{K}}$ (3.8). That is, $\Box'\top \vdash_{\mathsf{K}} \Box'\Box\Diamond\top$, even though we do not have $\top \dashv\vdash_{\mathsf{K}} \Box\Diamond\top$ (since $\nvdash_{\mathsf{K}} \Box\Diamond\top$), revealing $\Box'$ not to be minimally congruential in K.

At the risk of laboring the obvious, we spell out the point illustrated by Examples 3.5 and 3.6 in general terms. Let τ be the translation induced by French's context ($\Box' p$ for short), and suppose that the formulas $D_1, \ldots, D_m, E_1, \ldots, E_n$ are all in the range of τ (i.e., each is $\tau(A)$ for some formula A). Then if we have (3.11),

$$\Box' D_1, \ldots, \Box' D_m \vdash_{\mathsf{K}} \Box' E_1, \ldots, \Box' E_n, \tag{3.11}$$

then we must have $D_i \dashv\vdash_{\mathsf{K}} E_j$ for some i, j, because (3.11) is (3.12), for some $A_1, \ldots, A_m, B_1, \ldots, B_n$:

$$\Box'\tau(A_1), \ldots, \Box'\tau(A_m) \vdash_{\mathsf{K}} \Box'\tau(B_1), \ldots, \Box'\tau(B_n), \tag{3.12}$$

and so

$$\tau(\Box A_1), \ldots, \tau(\Box A_m) \vdash_{\mathsf{K}} \tau(\Box B_1), \ldots, \tau(\Box B_n), \tag{3.13}$$

which, given that τ embeds E faithfully in K, is equivalent to

$$\Box A_1, \ldots, \Box A_m \vdash_{\mathsf{E}} \Box B_1, \ldots, \Box B_n. \tag{3.14}$$

And so $A_i \dashv\vdash_{\mathsf{E}} B_j$ for some i, j, by the minimal congruentiality of E (see Proposition 3.3), and hence $\tau(A_i) \dashv\vdash_{\mathsf{K}} \tau(B_j)$, that is, $D_i \dashv\vdash_{\mathsf{K}} E_j$, as promised. But this promise does not amount to the claim that $\Box'$ is minimally congruential in K, since that would be a matter of (3.11)'s implying $D_i \dashv\vdash_{\mathsf{K}} E_j$ for some i, j, without the further qualification that the various D_i and E_j were of the form $\tau(A_i)$, $\tau(B_j)$.

We conclude with some open questions.

- What is the status of the conjecture above—and, in particular, are there consistent congruential (resp., normal) proper extensions of E (resp., K) in which $\Box$ is minimally congruential (resp., minimally normal)?
- Does the translation induced by the context $C(p) = \Diamond(\Diamond\Box p \wedge \Diamond\Diamond\Box\neg p)$ (from Example 2.5), embed E faithfully in K, notwithstanding the failure of the original plan to establish this by showing this context to be minimally congruential in K?
- Is there in fact *any context at all* which is minimally congruential in K?

Though not of comparable general significance, an incidental question was also raised by Example 3.2 as to how to axiomatize the logic determined by the frame $\mathcal{F}_{\mathrm{HF}}$—or, for that matter, the logic determined by the variant frame with urelements (mentioned in n. 15), should that logic turn out to be distinct from K. A further incidental question arising from our earlier discussion is whether French's original translation (see note 5) faithfully embeds E in K.

Notes

1. We take these propositional variables as $p_1, \ldots, p_n, \ldots$ and abbreviate p_1 and p_2 to p and q. In addition, we assume that some functionally complete set of Boolean connectives are available, which for convenience (so that we can have formulas constructed without the aid of propositional variables) includes nullary $\top$ and $\bot$. The sole non-Boolean primitive is the 1-ary $\Box$. As usual, a *modal logic* in this language is a set of formulas containing all truth-functional tautologies and closed under modus ponens and uniform substitution (of formulas for propositional variables).

2. In general, for a 1-ary context $C(p)$ we do not require that no variables other than p occur in C, so those contexts at issue in the above definition might more explicitly be called *simple* 1-ary contexts. The insistence on simple contexts in the conditions on

τ means that $C(p)$ is a candidate definiens for an S_1-style $\Box$ operator within S_2, and indeed the translations of current concern are often called definitional translations (see, e.g., Wójcicki [17, p. 70]). More explicitly one might say, $\Box$-definitional translations. Of course, if one does wish to use the chosen context to define a box operator within a language already containing one, some renotation will be called for, as with the example following shortly below, introducing $\Box'$ alongside $\Box$.

3. The proof that the trimodal translation embeds E into K appears as [6, Theorem 21]; the bimodal simplification is from [6, Remark 22, p. 307]. Some of the results of [6] cited here also appeared in Kracht and Wolter [10].

4. The role of these formulas is best grasped by inspection of French [3, Figure 1, p. 426]. As a referee for the present journal observed, French's comment in the closing sentence of [3, p. 428], that his "translation maps formulas of modal degree n to modal formulas of degree $5n$," is not correct.

5. In the originally submitted paper, French had the translation as

$$\tau(\Box A) = \Diamond\big(\Diamond(\Box\tau(A) \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond\Box\neg\tau(A) \wedge \Diamond\Diamond\Box\bot)\big),$$

but there was a problem (noted by a referee for the journal in which [3] appeared) with the proof that this embedded E in K—not that any concrete counterexample had emerged to show that it did not. The modal degree data for this translation are as for the original τ, while for the increase in the modal complexity on prefixing a $\Box$, the "14" in the earlier description is replaced by "12."

6. This formula interchanges the inner $\Box_1$ and $\Box_2$ in the bimodal variant of their trimodal prototype, mentioned above, which is actually cited by Gasquet and Herzig in Remark 22 of [6].

7. Where τ is the translation derived from Thomason's reduction of bimodal to monomodal formulas given on [3, p. 424], $\mathrm{mc}(\tau(\Box A))$, for A with $\mathrm{mc}(A) = n$, is $2n + 16$, while $\mathrm{md}(\tau(A))$ is 0, 6, respectively, for A with $\mathrm{md}(A) = 0, 1$, for A of modal degree $2, 3, 4, \ldots, \mathrm{md}(\tau(A)) = 10, 14, 18, \ldots$, so eventually the modal degree of the translation rises by 4 (per unit increase in the modal degree of the formula translated), just as with French's own translation. Thus the degree of the simplification achieved is overstated in French's remark (see [3, p. 428]) that "our translation maps formulas of modal degree n to formulas of modal degree $5n$, while the translation derived from the Thomason translation maps such a formula to one of modal degree $7n$." (The $n \mapsto 5n$ error was already mentioned in note 4 above.)

8. We could equally well say, for congruentiality, that $A \dashv\vdash_{\mathsf{S}} B$ implies $C(A) \vdash_{\mathsf{S}} C(B)$ (for all A, B).

9. $C(p)$'s being *antitone* in S means that $A \vdash_S B$ always implies $C(B) \vdash_{\mathsf{S}} C(A)$.

10. Indeed, one sees here that, regardless of length, for no successful $\Diamond(Xp \wedge Y\neg p)$ can X or Y consist entirely of occurrences of $\Diamond$.

11. So these semicolons are not needed when the relation is spelled out explicitly: thus S-equivalence is the relation $\{\langle D_1, D_2\rangle \mid D_1 \vdash_{\mathsf{S}} D_2 \text{ and } D_2 \vdash_{\mathsf{S}} D_1\}$, and so on.

12. In a more generous spirit, one might regard these—basic or otherwise—as strong or "positive" logical relations and make room also for "weak" (or "negative," since we use negated $\vdash$-statements) logical relations such as consistency and independence, as well as mixed cases such as—shall we say?—strict subcontrariety: $\{\varnothing \vdash D_1, D_2; D_1, D_2 \nvdash \varnothing\}$; again, a basic/nonbasic distinction could be introduced for these. What matters to the present discussion is only the strong (basic and other) logical relations, however.

13. Compare the condition which differs from this in having $A_i = B_j$ for some i, j after the "if and only if," which is easily seen to be satisfied when S is the smallest modal logic (see note 1). We might call $\Box$ "minimally modal" in this case. A variation on this theme appears in Williamson [16, Proposition 3, p. 32] for the (still not congruential) extension of this choice of S by all instances of the T-schema $\Box A \to A$. (Because of this, Williamson has to give a weakened version of $\Box$'s being modal in the logic concerned, with the added restrictions that $A_1, \ldots, A_m, B_1, \ldots, B_n$ are $\Box$-free and that $\{A_1, \ldots, A_m\}$ is truth-functionally consistent.)

14. Or equivalently, when only normal modal logics are under consideration, from $\Box A \vdash \Box B$ to $A \vdash B$, or indeed again, from $\Diamond A \vdash \Diamond B$ to $A \vdash B$ (see Humberstone and Williamson [7]).

15. The frame $\mathcal{F}_{\mathrm{HF}}$ was mentioned more tentatively in the originally submitted version of this paper, at which stage the author was unsure as to whether it validated any nontheorems of K. A referee pointed out that for any k and ℓ, the frame validated $\Diamond^k(\Box\bot \wedge p) \to \Box^\ell(\Box\bot \to p)$, none of which are K-provable when $k, \ell \geq 1$; here we have made use of the $k = \ell = 1$ case. It would be interesting to know if, instead of $\mathcal{F}_{\mathrm{HF}}$ as described here, we dropped the restriction to pure sets and allowed denumerably many urelements, the universe of the frame comprising them, and all hereditarily finite sets based on them (with y accessible to x, again, iff $y \in x$), would the frame validate any nontheorems of K? And, if so, might the logic determined by it even turn out to be a proper extension of K in which $\Box$ remained minimally normal? (Not according to the conjecture after Proposition 3.3 below, to the effect that in no normal proper extension of K is $\Box$ minimally normal.)

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