

An Integer Construction of Infinitesimals: Toward a Theory of Eudoxus Hyperreals

Alexandre Borovik, Renling Jin, and Mikhail G. Katz

Abstract A construction of the real number system based on almost homomorphisms of the integers \mathbb{Z} was proposed by Schanuel, Arthan, and others. We combine such a construction with the ultrapower or limit ultrapower construction to construct the hyperreals out of integers. In fact, any hyperreal field, whose universe is a set, can be obtained by such a one-step construction directly out of integers. Even the maximal (i.e., *On*-saturated) hyperreal number system described by Kanovei and Reeken (2004) and independently by Ehrlich (2012) can be obtained in this fashion, albeit not in *NBG*. In *NBG*, it can be obtained via a one-step construction by means of a definable ultrapower (modulo a suitable definable class ultrafilter).

1 From Kronecker to Schanuel

Kronecker (see [41]) famously remarked that, once we have the natural numbers in hand, “everything else is the work of man.” Does this apply to infinitesimals, as well?

The exposition in this section follows R. Arthan [7]. A function f from \mathbb{Z} to \mathbb{Z} is said to be an *almost homomorphism* if and only if the function d_f from $\mathbb{Z} \times \mathbb{Z}$ to \mathbb{Z} defined by

$$d_f(p, q) = f(p + q) - f(p) - f(q)$$

has bounded (i.e., finite) range, so that for a suitable integer C , we have $|d_f(p, q)| < C$ for all $p, q \in \mathbb{Z}$. The set denoted

$$\mathbb{Z} \rightarrow \mathbb{Z} \tag{1.1}$$

of all functions from \mathbb{Z} to \mathbb{Z} becomes an abelian group if we add and negate functions pointwise:

$$(f + g)(p) = f(p) + g(p), \quad (-f)(p) = -f(p).$$

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It is easily checked that if f and g are almost homomorphisms, then so are $f + g$ and $-f$. Let S be the set of all almost homomorphisms from \mathbb{Z} to \mathbb{Z} . Then S is a subgroup of $\mathbb{Z} \rightarrow \mathbb{Z}$. Let us write B for the set of all functions from \mathbb{Z} to \mathbb{Z} whose range is bounded. Then B is a subgroup of S . The *Eudoxus reals* are defined as follows.¹

Definition 1.1 The abelian group \mathbb{E} of *Eudoxus reals* is the quotient group S/B .

Elements of \mathbb{E} are equivalence classes, $[f]$ say, where f is an almost homomorphism from \mathbb{Z} to \mathbb{Z} , that is, f is a function from \mathbb{Z} to \mathbb{Z} such that $d_f(p, q) = f(p, q) - f(p) - f(q)$ defines a function from $\mathbb{Z} \times \mathbb{Z}$ to \mathbb{Z} whose range is bounded. We have $[f] = [g]$ if and only if the difference $f - g$ has bounded range, that is, if and only if $|f(p) - g(p)| < C$ for some C and all p in \mathbb{Z} .

The addition and additive inverse in \mathbb{E} are induced by the pointwise addition and inverse of representative almost homomorphisms:

$$[f] + [g] = [f + g], \quad -[f] = [-f],$$

where $f + g$ and $-f$ are defined by

$$(f + g)(p) = f(p) + g(p)$$

and

$$(-f)(p) = -f(p)$$

for all p in \mathbb{Z} .

The group \mathbb{E} of Eudoxus reals becomes an *ordered* abelian group if we take the set $P \subset \mathbb{E}$ of positive elements to be

$$P = \left\{ [f] \in \mathbb{E} : \sup_{m \in \mathbb{N} \subset \mathbb{Z}} f(m) = +\infty \right\}.$$

The multiplication on \mathbb{E} is induced by *composition* of almost homomorphisms. The multiplication turns \mathbb{E} into a commutative ring with unit. Moreover, this ring is a field. Even more surprisingly, \mathbb{E} is an ordered field with respect to the ordering defined by P .

Theorem 1.2 (see Arthan [7]) \mathbb{E} is a complete ordered field and is therefore isomorphic to the field of real numbers \mathbb{R} .

The isomorphism $\mathbb{R} \rightarrow \mathbb{E}$ assigns to every real number $\alpha \in \mathbb{R}$ the class $[f_\alpha]$ of the function

$$\begin{aligned} f_\alpha : \mathbb{Z} &\longrightarrow \mathbb{Z}, \\ n &\mapsto \lfloor \alpha n \rfloor, \end{aligned}$$

where $\lfloor \cdot \rfloor$ is the integer part function.

In the remainder of the paper, we combine the above one-step construction of the reals with the ultrapower or limit ultrapower construction to obtain hyperreal number systems directly out of the integers. We show that any hyperreal field, whose universe is a set, can be so obtained by such a one-step construction. Following this, working in *NBG* (von Neumann–Bernays–Gödel set theory with the axiom of global choice), we further observe that by using a suitable *definable ultrapower*, even the maximal (i.e., the *On-saturated*)² hyperreal number system described by Kanovei and Reeken [21, Theorem 4.1.10(i)], and more recently by Ehrlich [14], can be obtained in a one-step fashion directly from the integers. As Ehrlich [14, Theorem 20]

showed, the ordered field underlying an On -saturated hyperreal field is isomorphic to J. H. Conway’s ordered field No, an ordered field Ehrlich describes as the *absolute arithmetic continuum* (modulo NBG).

2 Passing It through an Ultraproduct

Let now $\mathbb{Z} = \mathbb{Z}^{\mathbb{N}}$ be the ring of integer sequences with operations of componentwise addition and multiplication. We define a *rescaling* to be a sequence $\rho = \langle \rho_n : n \in \mathbb{N} \rangle$ of almost homomorphisms $\rho_n : \mathbb{Z} \rightarrow \mathbb{Z}$. Rescalings are thought of as acting on \mathbb{Z} , hence the name. A rescaling ρ is called *bounded* if each of its components, ρ_n , is bounded.

Rescalings factorized modulo bounded rescalings form a commutative ring \mathcal{E} with respect to addition and composition. Quotients of \mathcal{E} by its maximal ideals are hyperreal fields. Thus, hyperreal fields are factor fields of the ring of rescalings of integer sequences. This description is a tautological translation of the classical construction, due to E. Hewitt [19], but it is interesting for the sheer economy of the language used. We will give further details in the sections below.

3 Cantor, Dedekind, and Schanuel

The strategy of Cantor’s construction of the real numbers³ can be represented schematically by the diagram

$$\mathbb{R} := (\mathbb{N} \rightarrow (\mathbb{Z} \times \mathbb{Z})_{\alpha})_{\beta}, \tag{3.1}$$

where the subscript α evokes the passage from a pair of integers to a rational number; the arrow \rightarrow alludes to forming sequences; and subscript β reminds us to select Cauchy sequences modulo equivalence. Meanwhile, Dedekind proceeds according to the scheme

$$\mathbb{R} := (\mathcal{P}(\mathbb{Z} \times \mathbb{Z})_{\alpha})_{\gamma}, \tag{3.2}$$

where α is as above, \mathcal{P} alludes to the set-theoretic power operation, and γ selects his cuts. For a history of the problem, see P. Ehrlich [13].

An alternative approach was proposed by Schanuel, and developed by N. A’Campo [1], R. Arthan [6], [7],⁴ T. Grundhöfer [18], R. Street [40], O. Deiser [12, pp. 112–27], and others, who follow the formally simpler blueprint

$$\mathbb{R} := (\mathbb{Z} \rightarrow \mathbb{Z})_{\sigma}, \tag{3.3}$$

where σ selects certain almost homomorphisms from \mathbb{Z} to itself, such as the map

$$a \mapsto \lfloor ra \rfloor \tag{3.4}$$

for real r , modulo equivalence. (Think of r as the “large-scale slope” of the map.)⁵ Such a construction has been referred to as the *Eudoxus reals*.⁶ The construction of \mathbb{R} from \mathbb{Z} by means of almost homomorphisms has been described as “skipping the rationals \mathbb{Q} .”

We will refer to the arrow in (3.3) as the *space dimension*, so as to distinguish it from the *time dimension* occurring in the following construction of an extension of \mathbb{N} :

$$(\mathbb{N} \rightarrow \mathbb{N})_{\tau_{\text{cof}}}, \tag{3.5}$$

where τ_{cof} identifies sequences $f, g : \mathbb{N} \rightarrow \mathbb{N}$ which *differ* on a finite set of indices:

$$\{n \in \mathbb{N} : f(n) = g(n)\} \text{ is cofinite.}^7 \tag{3.6}$$

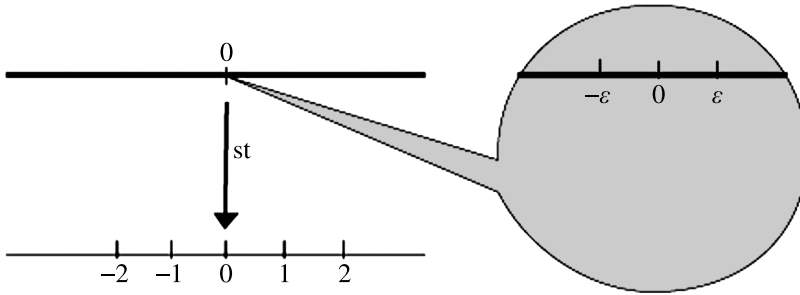


Figure 1 Zooming in on infinitesimal ϵ

Here the *constant* sequences induce an inclusion

$$\mathbb{N} \rightarrow (\mathbb{N} \rightarrow \mathbb{N})_{\tau_{\text{cof}}}.$$

Such a construction is closely related to (a version of) the Ω -calculus of Schmieden and Laugwitz [37]. The resulting semiring has zero divisors. To obtain a model which satisfies the first-order Peano axioms, we need to quotient it further. Note that up to this point the construction has not used any nonconstructive foundational material such as the axiom of choice or the weaker axiom of the existence of nonprincipal ultrafilters.

4 Constructing an Infinitesimal-Enriched Continuum

The traditional ultrapower construction of the hyperreals proceeds according to the blueprint

$$(\mathbb{N} \rightarrow \mathbb{R})_{\mathcal{U}},$$

where \mathcal{U} is a fixed ultrafilter on \mathbb{N} . Replacing \mathbb{R} by any of the possible constructions of \mathbb{R} from \mathbb{Z} , one in principle obtains what can be viewed as a direct construction of the hyperreals out of the integers \mathbb{Z} . Formally, the most economical construction of this sort passes via the Eudoxus reals.

An infinitesimal-enriched continuum can be visualized by means of an infinite-magnification microscope as in Figure 1.

To construct such an infinitesimal-enriched field, we have to deal with the problem that the semiring $(\mathbb{N} \rightarrow \mathbb{N})_{\tau_{\text{cof}}}$ constructed in the previous section contains zero divisors.

To eliminate the zero divisors, we need to quotient the ring further. This is done by extending the equivalence relation by means of a maximal ideal defined in terms of an ultrafilter. Thus, we extend the relation defined by (3.6) to the relation declaring f and g equivalent if

$$\{n \in \mathbb{N} : f(n) = g(n)\} \in \mathcal{U}, \tag{4.1}$$

where \mathcal{U} is a fixed ultrafilter on \mathbb{N} , and we add negatives. The resulting modification of (3.5), called an *ultrapower*, will be denoted

$$\mathbb{I}\mathbb{N} := (\mathbb{N} \rightarrow \mathbb{N})_{\tau} \tag{4.2}$$

and is related to Skolem’s [39] construction in 1934 of a nonstandard model of arithmetic. We refer to the arrow in (4.2) as *time* to allude to the fact that a sequence that

increases without bound for large time will generate an infinite “natural” number in $\mathbb{I}\mathbb{N}$. A “continuous” version of the ultrapower construction was exploited by Hewitt [19] in constructing his hyperreal fields in 1948.

The traditional ultrapower approach to constructing the hyperreals is to start with the field of real numbers \mathbb{R} and build the ultrapower

$$(\mathbb{N} \rightarrow \mathbb{R})_\tau, \tag{4.3}$$

where the subscript τ is equivalent to that of (4.2) (see, e.g., Goldblatt [17]). For instance, relative to Cantor’s procedure (3.1), this construction can be represented by the scheme

$$(\mathbb{N} \rightarrow (\mathbb{N} \rightarrow (\mathbb{Z} \times \mathbb{Z})_\alpha)_\beta)_\tau;$$

however, this construction employs needless intermediate procedures as described above. Our approach is to follow instead the “skip the rationals” blueprint

$$\mathbb{I}\mathbb{R} := (\mathbb{N} \rightarrow \mathbb{Z}^\mathbb{N})_{\sigma\tau}, \tag{4.4}$$

where the image of each $a \in \mathbb{N}$ is the sequence $u^a \in \mathbb{Z}^\mathbb{N}$ with general term u_n^a , so that $u^a = \langle u_n^a : n \in \mathbb{N} \rangle$. Thus a general element of $\mathbb{I}\mathbb{R}$ is generated (represented) by the sequence

$$\langle a \mapsto (n \mapsto u_n^a) : a \in \mathbb{N} \rangle. \tag{4.5}$$

Here one requires that for each fixed element $n_0 \in \mathbb{N}$ of the exponent copy of \mathbb{N} , the map

$$a \mapsto u_{n_0}^a$$

is an almost homomorphism (space direction), while τ in (4.4) alludes to the ultrapower quotient in the time direction n . For instance, we can use almost homomorphisms of type (3.4) with $r = \frac{1}{n}$. Then the sequence

$$\left\langle a \mapsto \left(n \mapsto \left\lfloor \frac{a}{n} \right\rfloor \right) : a \in \mathbb{N} \right\rangle \tag{4.6}$$

generates an infinitesimal in $\mathbb{I}\mathbb{R}$ since the almost homomorphisms are “getting flatter and flatter” for large time n .

Theorem 4.1 *Relative to the construction (4.4), we have a natural inclusion $\mathbb{R} \subset \mathbb{I}\mathbb{R}$. Furthermore, $\mathbb{I}\mathbb{R}$ is isomorphic to the model ${}^*\mathbb{R}$ of the hyperreals obtained by quotienting $\mathbb{R}^\mathbb{N}$ by the chosen ultrafilter, as in (4.3).*

Proof Given a real number $r \in \mathbb{R}$, we choose the constant sequence given by $u_n^a = \lfloor ra \rfloor$. (The sequence is constant in time n .) Sending r to the element of $\mathbb{I}\mathbb{R}$ defined by the sequence

$$\langle a \mapsto (n \mapsto \lfloor ra \rfloor) : a \in \mathbb{N} \rangle$$

yields the required inclusion $\mathbb{R} \hookrightarrow \mathbb{I}\mathbb{R}$. The isomorphism $\mathbb{I}\mathbb{R} \rightarrow {}^*\mathbb{R}$ is obtained by letting

$$U_n = \lim \frac{u_n^a}{a}$$

for each $n \in \mathbb{N}$, and sending the element of $\mathbb{I}\mathbb{R}$ represented by (4.5) to the hyperreal represented by the sequence $\langle U_n : n \in \mathbb{N} \rangle$. \square

Denoting by Δx the infinitesimal generated by the integer object (4.6), we can then define the derivative of $y = f(x)$ at x following Robinson as the real number $f'(x)$ infinitely close (or, in Fermat’s terminology, *adequal*)⁸ to the infinitesimal ratio $\frac{\Delta y}{\Delta x} \in \mathbb{IR}$.

Applications of infinitesimal-enriched continua range from aid in teaching calculus (see Ely [15], Katz and Katz [22], [23], Katz and Tall [31]) to the Boltzmann equation (see L. Arkeryd [4], [5]); modeling of timed systems in computer science (see H. Rust [36]); mathematical economics (see R. Anderson [3]); and mathematical physics (see Albeverio et al. [2]). A comprehensive reappraisal of the historical antecedents of modern infinitesimals has been undertaken in recent work by Błaszczyk, Katz, and Sherry [8], Borovik and Katz [9], Bråting [10], Kanovei [20], Katz and Katz [24]–[27], Katz and Leichtnam [28], Katz and Sherry [29], [30], and others. A construction of infinitesimals by “splitting” Cantor’s construction of the reals is presented in Giordano and Katz [16].

5 Formalization

In this and the next sections we formalize and generalize the arguments in the previous sections. We show that by a one-step construction from \mathbb{Z} -valued functions we can obtain any given (set) hyperreal field. We can even obtain a universal hyperreal field which contains an isomorphic copy of every hyperreal field, by a one-step construction from \mathbb{Z} -valued functions.

We assume that the reader is familiar with some basic concepts of model theory. Consult Chang and Keisler [11] or Keisler [32] for concepts and notation undefined here.

Let \mathfrak{A} and \mathfrak{B} be two models in a language \mathcal{L} with base sets A and B , respectively. The model \mathfrak{B} is called an \mathcal{L} -elementary extension of the model \mathfrak{A} , or \mathfrak{A} is an \mathcal{L} -elementary submodel of \mathfrak{B} , if there is an embedding $e : A \rightarrow B$ called an *\mathcal{L} -elementary embedding*, such that for any first-order \mathcal{L} -sentence $\varphi(a_1, a_2, \dots, a_n)$ with parameters $a_1, a_2, \dots, a_n \in A$, $\varphi(a_1, a_2, \dots, a_n)$ is true in \mathfrak{A} if and only if $\varphi(e(a_1), e(a_2), \dots, e(a_n))$ is true in \mathfrak{B} .

Let \mathfrak{A} and \mathfrak{B} be two models in language \mathcal{L} with base sets A and B , respectively. Let

$$\mathcal{L}' = \mathcal{L} \cup \{P_R : \exists m \in \mathbb{N}, R \subseteq A^m\}.$$

That is, \mathcal{L}' is formed by adding to \mathcal{L} an m -dimensional relational symbol P_R for each m -dimensional relation R on A for any positive integer m . Let \mathfrak{A}' be the natural \mathcal{L}' -model with base set A ; that is, the interpretation of P_R in \mathfrak{A}' for each $R \subseteq A^m$ is R . The model \mathfrak{B} is called a *complete elementary extension* of \mathfrak{A} if \mathfrak{B} can be expanded to an \mathcal{L}' -model \mathfrak{B}' with base set B such that \mathfrak{B}' is an \mathcal{L}' -elementary extension of \mathfrak{A}' .

It is a well-known fact that if \mathfrak{B} is an ultrapower of \mathfrak{A} or a limit ultrapower of \mathfrak{A} , then \mathfrak{B} is a complete elementary extension of \mathfrak{A} .

In this section we always view the set \mathbb{R} as the set of all Eudoxus reals.

An ordered field is called a hyperreal field if it is a proper complete elementary extension of \mathbb{R} . Let

$$\mathcal{L}' = \{+, \cdot, \leq, 0, 1, P_R\}_{R \in \mathcal{R}},$$

where \mathcal{R} is the collection of all finite-dimensional relations on \mathbb{R} . We do not distinguish between \mathbb{R} and the \mathcal{R}' -model $\mathfrak{R} = (\mathbb{R}; +, \cdot, \leq, 0, 1, R)_{R \in \mathcal{R}}$. By saying that

${}^*\mathbb{R}$ is a hyperreal field we will sometimes mean that ${}^*\mathbb{R}$ is the base set of the hyperreal field, but at other times we mean that ${}^*\mathbb{R}$ is the hyperreal field viewed as an \mathcal{L}' -model. We will spell out the distinction when it becomes necessary.

Recall that S is the set of all bounded functions from \mathbb{Z} to \mathbb{Z} . For a pair of almost homomorphisms $f, g : \mathbb{Z} \rightarrow \mathbb{Z}$, we will write $f \sim_\sigma g$ if and only if $f - g \in S$. Let I be an infinite set. If $F(x, y)$ is a two-variable function from $\mathbb{Z} \times I$ to \mathbb{Z} and i is a fixed element in I , we write $F(x, i)$ for the one-variable function $F_i(x) = F(x, i)$ from \mathbb{Z} to \mathbb{Z} .

Definition 5.1 Let I be any infinite set. We set

$$\begin{aligned} \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) \\ = \{F \in \mathbb{Z}^{\mathbb{Z} \times I} : \forall i \in I, F(x, i) \text{ is an almost homomorphism}\}. \end{aligned}$$

Let \mathcal{U} be a fixed nonprincipal ultrafilter on I . For a pair of functions $f, g : I \rightarrow J$ for some set J , we set

$$f \sim_\tau g \quad \text{if and only if} \quad \{i \in I : f(i) = g(i)\} \in \mathcal{U}.$$

Let $[f]_\tau = \{g \in I^J : g \sim_\tau f\}$.

Definition 5.2 For any $F, G \in \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})$ we will write

$$F \sim_{\sigma\tau} G \quad \text{if and only if} \quad \{i \in I : F(x, i) \sim_\sigma G(x, i)\} \in \mathcal{U}.$$

It is easy to check that $\sim_{\sigma\tau}$ is an equivalence relation on $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})$. For each $F \in \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})$ let

$$[F]_{\sigma\tau} = \{G \in \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) : G \sim_{\sigma\tau} F\}.$$

For each $F(x, y) \in \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})$ we can consider $[F(\cdot, y)]_\sigma$ as a function of y from I to \mathbb{R} . Thus the map

$$\Phi : \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) / \sim_{\sigma\tau} \rightarrow \mathbb{R}^I / \mathcal{U}$$

such that $\Phi([F]_{\sigma\tau}) = [[F]_\sigma]_\tau$ is an isomorphism from $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) / \sim_{\sigma\tau}$ to $\mathbb{R}^I / \mathcal{U}$. Hence $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) / \sim_{\sigma\tau}$ can be viewed as an ultrapower of \mathbb{R} . Therefore, the quotient

$$\mathbb{R}_I = \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) / \sim_{\sigma\tau}$$

is a hyperreal field constructed in one step from the set of \mathbb{Z} -valued functions $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})$.

If the set I is \mathbb{N} , then $\mathcal{A}(\mathbb{Z} \times \mathbb{N}, \mathbb{Z}) / \sim_{\sigma\tau}$ is exactly the hyperreal field \mathbb{R} mentioned in the previous sections. Since I can be any infinite set, we can construct a hyperreal field of arbitrarily large cardinality in one step from a set of \mathbb{Z} -valued functions $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})$.

6 Limit Ultrapowers and Definable Ultrapowers

If we consider a limit ultrapower instead of an ultrapower, we can obtain any (set) hyperreal field by a one-step construction from a set of \mathbb{Z} -valued functions $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) \mid \mathcal{G}$. The reader could consult Keisler [32] for the notations, definitions, and basic facts about limit ultrapowers. The main fact that we need here is the following theorem (see Keisler [32, Theorem 3.7]).

Theorem 6.1 *If \mathfrak{A} and \mathfrak{B} are two models of the same language, then \mathfrak{B} is a complete elementary extension of \mathfrak{A} if and only if \mathfrak{B} is a limit ultrapower of \mathfrak{A} .*

Given any (set) hyperreal field ${}^*\mathbb{R}$, let \mathcal{U} be the ultrafilter on an infinite set I , and let \mathcal{G} be the filter on $I \times I$ such that ${}^*\mathbb{R}$ is isomorphic to the limit ultrapower $(\mathbb{R}^I/\mathcal{U}) \mid \mathcal{G}$. We can describe the limit ultrapower $(\mathbb{R}^I/\mathcal{U}) \mid \mathcal{G}$ in one step from the set of \mathbb{Z} -valued functions $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) \mid \mathcal{G}$. For each $F \in \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})$, let

$$\text{eq}(F) = \{(i, j) \in I \times I : [F(x, i)]_\sigma = [F(x, j)]_\sigma\}.$$

Let

$$\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) \mid \mathcal{G} = \{F \in \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) : \text{eq}(F) \in \mathcal{G}\}.$$

Notice that $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) \mid \mathcal{G}$ is a subset of $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})$. Hence $(\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) \mid \mathcal{G})/\sim_{\sigma\tau}$ can be viewed as a subset of $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})/\sim_{\sigma\tau}$. Again, for each

$$F \in \mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})/\sim_{\sigma\tau}$$

let

$$\Phi([F]_{\sigma\tau}) = [[F]_\sigma]_\tau.$$

Then Φ is an isomorphism from

$$(\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) \mid \mathcal{G})/\sim_{\sigma\tau}$$

to $(\mathbb{R}^I/\mathcal{U}) \mid \mathcal{G}$. Therefore, $(\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) \mid \mathcal{G})/\sim_{\sigma\tau}$ as an elementary subfield of $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z})/\sim_{\sigma\tau}$ is isomorphic to the hyperreal field ${}^*\mathbb{R}$.

Theorem 6.2 *An isomorphic copy of any (set) hyperreal field ${}^*\mathbb{R}$ can be obtained by a one-step construction from a set of \mathbb{Z} -valued functions $\mathcal{A}(\mathbb{Z} \times I, \mathbb{Z}) \mid \mathcal{G}$ for some filter \mathcal{G} on $I \times I$.*

7 Universal and On-Saturated Hyperreal Number Systems

We call a hyperreal field \mathbb{R} *universal* if any hyperreal field, which is a set or a proper class of *NBG*, can be elementarily embedded in \mathbb{R} . Obviously a universal hyperreal field is necessarily a proper class. We now want to construct a definable hyperreal field with the property that any definable hyperreal field that can be obtained in *NBG* by a one-step construction from a collection of \mathbb{Z} -valued functions can be elementarily embedded in it. In a subsequent remark we point out that in *NBG* we can actually construct a definable hyperreal field so that every hyperreal field (definable or nondefinable) can be elementarily embedded in it. Moreover, the universal hyperreal field so constructed is isomorphic to the *On*-saturated hyperreal field described in [14].

Notice that *NBG* implies that there is a well-ordering \leq_V on V where V is the class of all sets. A class $X \subseteq V$ is called Δ_0 -definable if there is a first-order formula $\varphi(x)$ with set parameters in the language $\{\in, \leq\}$ such that for any set $a \in V$, $a \in X$ if and only if $\varphi(a)$ is true in V . Trivially, every set is Δ_0 -definable. We work within a model of *NBG* with set universe V plus all Δ_0 -definable proper subclasses of V . By saying that a class A is definable we mean that A is Δ_0 -definable.

Let Σ be the class of all finite sets of ordinals; that is, let, $\Sigma = On^{<\omega}$. Notice that Σ is a definable proper class. Let \mathcal{P} be the collection of all definable subclasses of Σ . Notice that we can code \mathcal{P} by a definable class.⁹ Using the global choice, we can form a nonprincipal definable ultrafilter $\mathcal{F} \subseteq \mathcal{P}$ such that for each $\alpha \in On$, the definable class

$$\hat{\alpha} = \{s \in \Sigma : \alpha \in s\}$$

is in \mathcal{F} . Again \mathcal{F} can be coded by a definable class. Let $\mathcal{A}_0(\mathbb{Z} \times \Sigma, \mathbb{Z})$ be the collection of all definable class functions F from $\mathbb{Z} \times \Sigma$ to \mathbb{Z} such that for each

$s \in \Sigma$, $F(x, s)$ is an almost homomorphism from \mathbb{Z} to \mathbb{Z} . For any two functions F and G in $\mathcal{A}_0(\mathbb{Z} \times \Sigma, \mathbb{Z})$, we write $F \sim_{\sigma\tau} G$ if and only if the definable class

$$\{s \in \Sigma : F(x, s) - G(x, s) \in S\}$$

is in \mathcal{F} . Let $[F]_{\sigma\tau}$ be the collection of all definable classes G in $\mathcal{A}_0(\mathbb{Z} \times \Sigma, \mathbb{Z})$ such that $F \sim_{\sigma\tau} G$. Then $\sim_{\sigma\tau}$ is an equivalence relation on $\mathcal{A}_0(\mathbb{Z} \times \Sigma, \mathbb{Z})$. Let

$$\mathbb{IR}_0 = \mathcal{A}_0(\mathbb{Z} \times \Sigma, \mathbb{Z}) / \sim_{\sigma\tau}.$$

By the arguments employed before, we can show that \mathbb{IR}_0 is isomorphic to the definable ultrapower of \mathbb{R} modulo \mathcal{F} . Hence \mathbb{IR}_0 is a complete elementary extension of \mathbb{R} . Therefore, \mathbb{IR}_0 is a hyperreal field. By slightly modifying the proof of [11, Theorem 4.3.12, p. 255] we can prove the following theorem.

Theorem 7.1 *\mathbb{IR}_0 is a class hyperreal field, and any definable hyperreal field ${}^*\mathbb{R}$ admits an elementary imbedding into \mathbb{IR}_0 .*

Proof We only need to prove the second part of the theorem. For notational convenience we view \mathbb{IR}_0 as $\mathbb{R}^\Sigma / \mathcal{F}$ instead of $\mathcal{A}_0(\mathbb{Z} \times \Sigma, \mathbb{Z}) / \sim_{\sigma\tau}$ in this proof.

Given a definable hyperreal field ${}^*\mathbb{R}$, recall that \mathcal{L} is the language of ordered fields, and

$$\mathcal{L}' = \mathcal{L} \cup \{P_R : R \text{ is a finite-dimensional relation on } \mathbb{R}\}.$$

Let $\Lambda^*_{\mathbb{R}}$ be all quantifier-free \mathcal{L}' -sentences $\varphi(r_1, r_2, \dots, r_m)$ with parameters $r_i \in {}^*\mathbb{R}$ such that $\varphi(r_1, r_2, \dots, r_m)$ is true in ${}^*\mathbb{R}$. Since ${}^*\mathbb{R}$ is a definable class, $\Lambda^*_{\mathbb{R}}$ is a definable class (under a proper coding). Let κ be the size of $\Lambda^*_{\mathbb{R}}$; that is, κ is the cardinality of ${}^*\mathbb{R}$ if ${}^*\mathbb{R}$ is a set and $\kappa = On$ if ${}^*\mathbb{R}$ is a definable proper class. Let j be a definable bijection from κ to $\Lambda^*_{\mathbb{R}}$.

For each $r \in {}^*\mathbb{R}$ we need to find a definable function $F_r : \Sigma \rightarrow \mathbb{R}$ such that the map $r \mapsto [F_r]_\tau$ is an \mathcal{L}' -elementary embedding. We define these F_r simultaneously.

Let $s \in \Sigma$. If $s \cap \kappa = \emptyset$, let $F_r(s) = 0$. Suppose $s \cap \kappa \neq \emptyset$, and let $s' = s \cap \kappa$. Notice that if $\kappa = On$, then $s = s'$. Let $\varphi_s(r_1, r_2, \dots, r_m) = \bigwedge_{\alpha \in s'} j(\alpha)$. Since

$$\exists x_1, x_2, \dots, x_m \varphi_s(x_1, x_2, \dots, x_m)$$

is true in ${}^*\mathbb{R}$, it is also true in \mathbb{R} . Let $(a_1, a_2, \dots, a_m) \in \mathbb{R}^m$ be the \leq_V -least m -tuple such that $\varphi_s(a_1, a_2, \dots, a_m)$ is true in \mathbb{R} . If $r \notin \{r_1, r_2, \dots, r_m\}$, let $F_r(s) = 0$. If $r = r_i$ for $i = 1, 2, \dots, m$, then let $F_r(s) = a_i$. Since φ_s is quantifier-free, the functions F_r are definable classes in *NBG*.

We now verify that $\Phi : {}^*\mathbb{R} \rightarrow \mathbb{IR}_0$ such that $\Phi(r) = [F_r]_\tau$ is an \mathcal{L}' -elementary embedding.

Let $\varphi(r_1, r_2, \dots, r_m)$ be an arbitrary \mathcal{L}' -sentence with parameters $r_1, r_2, \dots, r_m \in {}^*\mathbb{R}$.

Suppose that $\varphi(r_1, r_2, \dots, r_m)$ is true in ${}^*\mathbb{R}$. Since $\varphi(x_1, x_2, \dots, x_m)$ defines an m -ary relation R_φ on \mathbb{R} , we have that the \mathcal{L}' -sentence

$$\eta =: \forall x_1, x_2, \dots, x_m (\varphi(x_1, x_2, \dots, x_m) \leftrightarrow R_\varphi(x_1, x_2, \dots, x_m))$$

is true in \mathbb{R} . Hence η is true in ${}^*\mathbb{R}$ and in \mathbb{IR}_0 . One of the consequences of this is that $R_\varphi(r_1, r_2, \dots, r_m)$ is true in ${}^*\mathbb{R}$, and hence it is in $\Lambda^*_{\mathbb{R}}$. Let $\alpha \in \kappa$ be such that $j(\alpha) = R_\varphi(r_1, r_2, \dots, r_m)$. If $\alpha \in s$, then

$$\varphi_s = R_\varphi(r_1, r_2, \dots, r_m) \wedge \bigwedge_{\beta \in s', \beta \neq \alpha} j(\beta).$$

Hence $R_\varphi(F_{r_1}(s), F_{r_2}(s), \dots, F_{r_m}(s))$ is true in \mathbb{R} by the definition of the F_r 's. Since η is true in \mathbb{R} , we have that $\varphi(F_{r_1}(s), F_{r_2}(s), \dots, F_{r_m}(s))$ is true in \mathbb{R} . Thus

$$\{s \in \Sigma : \varphi(F_{r_1}(s), F_{r_2}(s), \dots, F_{r_m}(s)) \text{ is true in } \mathbb{R}\} \supseteq \hat{\alpha}.$$

Since $\hat{\alpha}$ is a member of \mathcal{F} , we have that $\varphi([F_{r_1}]_\tau, [F_{r_2}]_\tau, \dots, [F_{r_m}]_\tau)$ is true in $\mathbb{R}^\Sigma/\mathcal{F}$.

Suppose that $\varphi(r_1, r_2, \dots, r_m)$ is false in ${}^*\mathbb{R}$. Then $\neg\varphi(r_1, r_2, \dots, r_m)$ is true in ${}^*\mathbb{R}$. Hence by the same argument we have that

$$\neg\varphi([F_{r_1}]_\tau, [F_{r_2}]_\tau, \dots, [F_{r_m}]_\tau) \text{ is true in } \mathbb{R}^\Sigma/\mathcal{F},$$

which implies that

$$\varphi([F_{r_1}]_\tau, [F_{r_2}]_\tau, \dots, [F_{r_m}]_\tau) \text{ is false in } \mathbb{R}^\Sigma/\mathcal{F}.$$

Hence $\Phi(r) = [F_r]_\tau$ is an \mathcal{L}' -elementary embedding from ${}^*\mathbb{R}$ to \mathbb{IR}_0 . □

Remark 7.2 We have shown that every definable hyperreal field can be elementarily embedded into \mathbb{IR}_0 . If we want to show that \mathbb{IR}_0 is universal, we need to elementarily embed every (definable or nondefinable) hyperreal field ${}^*\mathbb{R}$ into the definable hyperreal field \mathbb{IR}_0 . Notice that there are models of *NBG* with nondefinable classes. The proof of Theorem 7.1 may not work when ${}^*\mathbb{R}$ is a nondefinable class because the bijection j may not be definable and F_r may not be definable. If F_r is not definable, $[F_r]_\tau$ may not be an element in \mathbb{IR}_0 .

The idea of making every hyperreal field embeddable into \mathbb{IR}_0 is that we can make \mathbb{IR}_0 *On*-saturated by selecting a definable ultrafilter \mathcal{F} more carefully. Notice that *NBG* implies that every proper class has the same size *On*. Hence ${}^*\mathbb{R}$ can be expressed as the union of *On*-many sets. If we can make sure that \mathbb{IR}_0 is *On*-saturated, that is, α -saturated for any set cardinality α , then ${}^*\mathbb{R}$ can be elementarily embedded into \mathbb{IR}_0 although such an embedding may be nondefinable.

The ultrafilter \mathcal{F} used in the construction of \mathbb{IR}_0 in the proof of Theorem 7.1 is a definable version of a *regular* ultrafilter. In order to make sure that \mathbb{IR}_0 is *On*-saturated, we need to require that \mathcal{F} be a special kind of definable regular ultrafilter called a definable *good* ultrafilter. The definition of a (set) good ultrafilter can be found in [11, p. 386]. The construction of an α^+ -good ultrafilter can be found in either [11, Theorem 6.1.4] or Kunen [33]. By the same idea of constructing \mathcal{F} above we can follow the steps in [33] or [11] to construct a definable class good ultrafilter \mathcal{F} on *On*. Now the ultrapower \mathbb{IR}_0 of \mathbb{R} modulo the definable class good ultrafilter \mathcal{F} is *On*-saturated. The proof of this fact is similar to that in [11]. However, since the definition of a definable class good ultrafilter and the proof of the saturation property of the ultrapower modulo a definable class good ultrafilter are long and technical, and since the ideas are similar to what we have already presented above, we will not include them in this paper.

Another way of constructing a definable *On*-saturated hyperreal field \mathbb{IR}_0 is by taking the union of an *On*-long definable elementary chain of set hyperreal fields $\{{}^*\mathbb{R}_\alpha : \alpha \in \text{On}\}$ with the property that ${}^*\mathbb{R}_\alpha$ is $|\alpha|$ -saturated. However, this construction cannot be easily translated into a “one-step” construction. Moreover, if we allow higher-order classes, we can express \mathbb{IR}_0 as a one-step limit ultrapower following the same idea as in the proof of [11, Theorem 6.4.10]. However, this is not possible in *NBG* since all classes allowed in a model of *NBG* are subclasses of V . On the

other hand, as we indicated above, the process of constructing \mathbb{R}_0 as a definable ultrapower can be carried out in *NBG*, and done so in a “one-step” fashion.

Notes

1. The term *Eudoxus reals* has gained some currency in the literature (see, e.g., Arthan [7]). Shenitzer [38, p. 45] argues that Eudoxus anticipated nineteenth century constructions of the real numbers. The attribution of such ideas to Eudoxus, based on an interpretation involving Eudoxus, Euclid, and Book 5 of *The Elements*, may be historically questionable.
2. Recall that a model M is *On-saturated* if M is κ -saturated for any cardinal κ in On . Here On (or ON) is the class of all ordinals (cf. Kunen [34, p. 17]). A hyperreal number system $\langle \mathbb{R}, {}^*\mathbb{R}, S \in \mathfrak{F} \rangle$ is *On-saturated* if it satisfies the following condition: If X is a set of equations and inequalities involving real functions, hyperreal constants, and variables, then X has a hyperreal solution whenever every finite subset of X has a hyperreal solution (see Ehrlich [14, Section 9, p. 34]).
3. The construction of the real numbers as equivalence classes of Cauchy sequences of rationals, usually attributed to Cantor, is actually due to H. Méray [35] who published three years earlier than E. Heine.
4. Arthan’s “Irrational construction of \mathbb{R} from \mathbb{Z} ” (see [6]) describes a different way of skipping the rationals, based on the observation that the Dedekind construction can take as its starting point any Archimedean densely ordered commutative group. The construction delivers a completion of the group, and one can define multiplication by analyzing its order-preserving endomorphisms. Arthan uses the additive group of the ring $\mathbb{Z}[\sqrt{2}]$, which can be viewed as $\mathbb{Z} \times \mathbb{Z}$ with an ordering defined using a certain recurrence relation.
5. One could also represent a real by a string based on its decimal expansion, but the addition in such a presentation is highly nontrivial due to carryover, which can be arbitrarily long. In contrast, the addition of almost homomorphisms is term by term. Multiplication on the reals is induced by composition in $\mathbb{Z} \rightarrow \mathbb{Z}$ (see formula (1.1)).
6. See n. 1 for a discussion of the term.
7. Note that addition is term by term in the time direction as well.
8. See A. Weil [42, p. 1146].
9. This is true because each definable subclass of Σ can be effectively coded by the Gödel number of a first-order formula in the language of $\{\in, \leq_V\}$ and a set in V . By the well-ordering of V we can determine a unique code for each definable class in \mathcal{P} .

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Borovik
School of Mathematics
University of Manchester
Oxford Street, Manchester, M13 9PL
UK
alexandre.borovik@manchester.ac.uk

Jin
Department of Mathematics
College of Charleston
Charleston, South Carolina 29424
USA
JinR@cofc.edu

Katz
Department of Mathematics
Bar Ilan University
Ramat Gan 52900
Israel
katzmik@macs.biu.ac.il