

Denumerably Many Post-Complete Normal Modal Logics with Propositional Constants

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Abstract We show that there are denumerably many Post-complete normal modal logics in the language which includes an additional propositional constant. This contrasts with the case when there is no such constant present, for which it is well known that there are only two such logics.

1 Introduction

Say that a logic is *Post-complete* if it is consistent and has no consistent proper extension. It is well known that there are only two Post-complete normal monomodal logics—**KT!** and the smallest normal modal logic to contain $\Box\perp$ —alias **KVer**.¹ This contrasts starkly with the case where we are simply considering Post-complete quasi-normal monomodal logics (i.e., Post-complete modal logics which extend **K**) of which Segerberg has shown there are nondenumerably many.²

As the above examples show, the phenomenon of Post-completeness is quite sensitive to the lattice of logics under investigation. When we fix upon a particular lattice of modal logics, the Post-complete logics in that lattice will be its co-atoms. Nothing about this point requires us to be concerned with modal logics, of course, as can be seen in the discussion of the lattice relativity of Post-completeness in Humberstone [5]—the discussion in footnote 10 therein providing a number of further references on the issue.³

If we shift to the case where we are considering bimodal normal modal logics—modal logics containing two modal operators, both of which are normal—the situation changes again. Here we have an increase in strength from the monomodal case, but, rather than there being fewer Post-complete modal logics, we instead find that there are now nondenumerably many such logics, as shown in [12]. This is a case where we have changed the lattice of modal logics under investigation, not by changing the closure conditions we place upon logics in the lattice, but by changing the

Received November 10, 2011; accepted November 20, 2011

2010 Mathematics Subject Classification: Primary 03B45

Keywords: modal logic, post-completeness, propositional constants

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language under investigation. Here we will investigate the number of Post-complete normal modal logics when we change the language by adding a propositional constant. In Section 2 we will go over some formal preliminaries before proving our main result in Section 3. In Section 4 we will end by discussing some of the ramifications of this result.

2 Formal Preliminaries

Let L be the propositional language constructed in the usual way from denumerably many propositional variables p_0, p_1, p_2, \dots , using the connectives \rightarrow, \neg , and \Box —the other connectives being defined in terms of these as usual. Let L_κ be a proposition language just like L except that it also contains an additional propositional constant κ . When wanting to make comparative comments about normal modal logics formulated in the two languages, we will denote logics in the language L_κ by subscripting their name with κ . So, for example, \mathbf{K}_κ is the smallest normal modal logic in the language L_κ .

A *frame* in this setting will be an ordered triple $\langle W, R, C \rangle$, where W is a non-empty set, R is a binary relation on W , and $C \subseteq W$. Throughout we will adopt the convention of identifying isomorphic frames. A *model* is a frame along with a valuation function V which maps every propositional variable p_i to a subset of W (the set of members of W at which p_i is true). We will define truth at a point w in a model $\mathcal{M} = \langle W, R, C, V \rangle$ (“ $\mathcal{M} \models_w A$ ”) inductively as follows:

$$\begin{aligned} \mathcal{M} \models_w p_i &\iff w \in V(p_i); \\ \mathcal{M} \models_w \neg A &\iff \mathcal{M} \not\models_w A; \\ \mathcal{M} \models_w A \rightarrow B &\iff \mathcal{M} \not\models_w A \text{ or } \mathcal{M} \models_w B; \\ \mathcal{M} \models_w \Box A &\iff \forall u (Rwu \Rightarrow \mathcal{M} \models_u A); \\ \mathcal{M} \models_w \kappa &\iff w \in C. \end{aligned}$$

A formula A is *true throughout a model* $\mathcal{M} = \langle W, R, C, V \rangle$ (“ $\mathcal{M} \models A$ ”) whenever A is true at all points $w \in W$ in \mathcal{M} , *valid on a frame* $\mathfrak{F} = \langle W, R, C \rangle$ (“ $\mathfrak{F} \models A$ ”) whenever A is true throughout all models on that frame, and *valid at a point w in a frame* \mathfrak{F} (“ $\mathfrak{F} \models_w A$ ”) whenever it is true at that point in all models on that frame. Given a class of frames \mathcal{C} , let $\text{Log}(\mathcal{C})$ be the set of all formulas A such that A is valid on all the frames in \mathcal{C} . As usual we will write $\text{Log}(\mathfrak{F})$ for $\text{Log}(\{\mathfrak{F}\})$.

3 Main Result

In this section we will exhibit a denumerable collection of distinct normal modal logics in the language L_κ , each one of which is Post-complete. Consider the frames $\mathfrak{L}_n = \langle W_n, R_n, C_n \rangle$, for $n \geq 1$:

- $W_n := \{0, 1, \dots, n, n+1\}$;
- $R_n := \{(n+1, 0)\} \cup \{(i, j) \mid j = i+1\}$;
- $C_n := \{0\}$.

We make the following observation without proof.

Proposition 3.1 \mathfrak{L}_n is the only (point-generated) frame for $\text{Log}(\mathfrak{L}_n)$.

Consider, now, the formula L_n^κ :

$$L_n^\kappa : \kappa \rightarrow \Diamond(\Box^n \neg \kappa \wedge \Diamond^{n+1} \kappa).$$

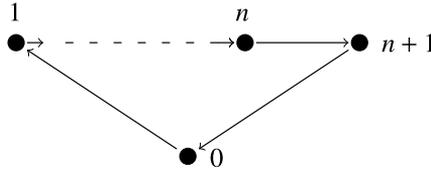


Figure 1 The Frame \mathfrak{L}_n

Here $\Box A$ is an abbreviation for $\Box A \wedge A$. Note also that, despite the fact that the logics $\text{Log}(\mathfrak{L}_n)$ are extensions of $\mathbf{KD!}_\kappa$, the use of \Diamond and \Box (rather than simply one or the other) is intended to be suggestive. It is easy to see that, for all n , L_n^κ is valid on the frame \mathfrak{L}_n . We now show that this formula is valid on none of the frames \mathfrak{L}_m for which $m \neq n$.

Lemma 3.2 For all $n, m \in \text{Nat}$ we have the following:

$$\mathfrak{L}_n \not\models L_m^\kappa \text{ whenever } m < n.$$

Proof Suppose that $\mathfrak{L}_n \models L_m^\kappa$. Then, in particular, it follows that $\mathfrak{L}_n \models (\star)$, where

$$(\star) : \kappa \rightarrow \Diamond^{m+2}\kappa.$$

Suppose, then, that $\mathfrak{L}_n \models_0 (\star)$, from which it follows that $\mathfrak{L}_n \models_0 \Diamond^{m+2}\kappa$. Thus $\mathfrak{L}_n \models_1 \Diamond^{m+1}\kappa$, and so $\mathfrak{L}_n \models_m \Diamond^2\kappa$. Now as $m + 2 < n + 2$, and $R^{n+2}(0) = \{0\}$, and the only point in C_n is 0, we thus have that $\mathfrak{L}_n \models_{R^2(m)} \kappa$ by hypothesis and that $\mathfrak{L}_n \models_{R^2(m)} \neg\kappa$ by construction, giving us a contradiction, and so the result follows. \square

Lemma 3.3 For all $n, m \in \text{Nat}$ we have the following:

$$\mathfrak{L}_n \not\models L_m^\kappa \text{ whenever } m > n.$$

Proof Suppose that $m = n + k$ for some $k > 0$. Then in \mathbf{K} we have $L_m^\kappa \rightarrow (\star')$, where

$$(\star') : \kappa \rightarrow \Diamond(\Box^n\neg\kappa \wedge \Box^n(\Box^k\neg\kappa)).$$

Suppose, then, that $\mathfrak{L}_n \models L_m^\kappa$, so in particular $\mathfrak{L}_n \models (\star')$, and hence as $\mathfrak{L}_n \models_0 \kappa \wedge \Diamond\neg\kappa$ we have that $\mathfrak{L}_n \models_0 \Diamond(\Box^n\neg\kappa \wedge \Box^n(\Box^k\neg\kappa))$. In particular it follows that $\mathfrak{L}_n \models_1 \Box^n\neg\kappa \wedge \Box^n(\Box^k\neg\kappa)$, and so $\mathfrak{L}_n \models_n \neg\kappa \wedge \Box \Box^k\neg\kappa$. Consequently, $\mathfrak{L}_n \models_{n+1} \Box^k\neg\kappa$, and in particular $\mathfrak{L}_n \models_{n+1} \Box\neg\kappa$, from which it follows that $\mathfrak{L}_n \models_0 \kappa$ and $\mathfrak{L}_n \models_0 \neg\kappa$, giving the result. \square

Theorem 3.4 The logics $\text{Log}(\mathfrak{L}_n)$ are all distinct normal modal logics.

Proof The proof follows by Lemmas 3.2 and 3.3. \square

To see that each logic $\text{Log}(\mathfrak{L}_n)$ is Post-complete begin by noting that the frames \mathfrak{L}_n are *distinguishing* in the sense that for each point x there is a formula D_x such that, for all models on \mathfrak{L}_n , D_x is true at a point y in a model iff $x = y$. The relevant formulas are given in the following table:

$x \in W_n$	D_x
1	$\diamond^{n+1}\kappa$
2	$\diamond^n\kappa$
\vdots	\vdots
i	$\diamond^{(n+2)-i}\kappa$
\vdots	\vdots
n	$\diamond^2\kappa$
$n + 1$	$\diamond\kappa$
0	κ

Say that a formula is *variable-free* if it is constructed using the Boolean connectives, κ and \square . Further, say that a substitution σ is variable-free if for all propositional variables p_i , $\sigma(p_i) = A_i$ for some variable-free formula A_i . In what follows we will make use of the fact that the distinguishing formulas D_x above are all variable-free formulas.

Consider now the following function f from $\wp(W_n)$ to variable-free formulas:

$$f(X) = \bigvee \{D_x \mid x \in X\}.$$

Lemma 3.5 *For all valuations V we have the following for all formulas A and points $x \in W_n$:*

$$\langle \mathfrak{L}_n, V \rangle \models_x A \text{ if and only if } \langle \mathfrak{L}_n, V \rangle \models_x \sigma_f(A),$$

where $\sigma_f(p_i) = f(V(p_i))$.

Proof By induction upon the complexity of A , the only case of interest being the basis case, where $A = p_i$. We show the following:

$$\langle \mathfrak{L}_n, V \rangle \models_x p_i \iff \langle \mathfrak{L}_n, V \rangle \models_x \sigma_f(p_i).$$

For the “ \Rightarrow ” direction, suppose that $\langle \mathfrak{L}_n, V \rangle \models_x p_i$. By the construction of σ_f this means that $\sigma_f(p_i) = \sigma_f(V(p_i)) = f(V(p_i) \setminus \{x\}) \vee D_x$. Since, for all y , $\langle \mathfrak{L}_n, V \rangle \models_x D_y$ iff $x = y$, it follows that $\langle \mathfrak{L}_n, V \rangle \models_x D_x$ and hence that $\langle \mathfrak{L}_n, V \rangle \models_x \sigma_f(p_i)$.

For the “ \Leftarrow ” direction, suppose that $\langle \mathfrak{L}_n, V \rangle \models_x \sigma_f(p_i)$. By the definition of σ_f this means that $\langle \mathfrak{L}_n, V \rangle \models_x f(V(p_i))$. It is easy to see, though, that $\langle \mathfrak{L}_n, V \rangle \models_x f(X)$ iff $x \in X$, for all $X \subseteq W$. Consequently, it follows that $x \in V(p_i)$ and thus that $\langle \mathfrak{L}_n, V \rangle \models_x p_i$ as desired. \square

In particular, as $\sigma_f(A)$ is a variable-free formula, for all formulas A , its truth or falsity at a point in a model does not depend upon V , giving us the following corollary.

Corollary 3.6 *For all valuations V we have the following for all formulas A and points $x \in W_n$:*

$$\langle \mathfrak{L}_n, V \rangle \models_x A \text{ if and only if } \mathfrak{L}_n \models_x \sigma_f(A),$$

where $\sigma_f(p_i) = f(V(p_i))$.

Corollary 3.7 *If A is a nontheorem of $\text{Log}(\mathfrak{L}_n)$, then for some variable-free substitution σ we have that $\sigma(A)$ is a nontheorem of $\text{Log}(\mathfrak{L}_n)$.*

Proof The proof follows from Corollary 3.6 and Proposition 3.1 \square

Let the formula A_n be defined as follows:

$$\bigwedge_{0 \leq j < n+2} \Box^j A.$$

Lemma 3.8 ⁽⁴⁾ *If A is a variable-free formula that is not valid in \mathfrak{L}_n , then $\neg A_n$ is valid on \mathfrak{L}_n .*

Proof If $\mathfrak{L}_n \not\models A$, then there is some point $y \in W_n$ such that $\mathfrak{L}_n \not\models_y A$. Then for any $x \in W_n$, let $j < n + 2$ be such that $R_n^j(x) = \{y\}$. (The existence of such a j is guaranteed by the fact that R_n is functional.) Then A is false at $R_n^j(x)$, and so $\mathfrak{L}_n \not\models_x \Box^j A$, and thus $\mathfrak{L}_n \not\models A_n$. As this is so for all $x \in W_n$, it follows that $\mathfrak{L}_n \models \neg A_n$. \square

Theorem 3.9 *For all $n \in \text{Nat}$, $\text{Log}(\mathfrak{L}_n)$ is a Post-complete normal modal logic in the language with a single propositional constant.*

Proof Suppose, for a reductio, that there is a logic S which is a consistent proper normal extension of $\text{Log}(\mathfrak{L}_n)$. Then there must be some formula A such that $\vdash_S A$ and $\not\models_{\text{Log}(\mathfrak{L}_n)} A$. By Corollary 3.7 it follows that there is some variable-free substitution σ such that $\not\models_{\text{Log}(\mathfrak{L}_n)} \sigma(A)$. Thus there is a point-generated model on a frame for $\text{Log}(\mathfrak{L}_n)$ which invalidates $\sigma(A)$. By Proposition 3.1 it follows that $\mathfrak{L}_n \not\models \sigma(A)$. By Lemma 3.8 it follows that $\mathfrak{L}_n \models \neg(\sigma(A))_n$, and thus $\vdash_{\text{Log}(\mathfrak{L}_n)} \neg(\sigma(A))_n$.

As $\vdash_S A$ and S is closed under uniform substitution it follows that $\vdash_S \sigma(A)$, and hence as S is normal we have the following as a theorem of S :

$$\bigwedge_{0 \leq j < n+2} \Box^j \sigma(A).$$

As $S \supseteq \text{Log}(\mathfrak{L}_n)$ it follows, though, that $\vdash_S \neg(\sigma(A))_n$, contradicting the supposition that S was consistent, and the result follows. \square

Corollary 3.10 *There are denumerably many Post-complete normal modal logics in the language with propositional constants.*

Proof The proof follows directly from Theorems 3.4 and 3.9. \square

As mentioned above, all the logics $\text{Log}(\mathfrak{L}_n)$ are extensions of $\mathbf{KD!}_\kappa$. This provides an interesting contrast with $\mathbf{KD!}$, which has $\mathbf{KT!}$ as its sole Post-complete normal extension, while, as shown above, $\mathbf{KD!}_\kappa$ has denumerably many Post-complete normal extensions.

4 Conclusion

What we have shown above is that there are at least denumerably many Post-complete normal modal logics in the language with propositional constants. Rather than telling us something interesting about monomodal logics (properly speaking), though, this is better thought of as telling us something interesting about a rather odd lattice of *bimodal* logics. Typically when people talk about modal operators we implicitly restrict attention to unary operators \Box , but this does not have to be so. For example, in Goguadze [3]—drawing inspiration from Jónsson and Tarski’s work on Boolean algebras with operators—polyadic modal operators are defined where the semantic

interpretation of an n -ary modal operator $\blacksquare(p_1, \dots, p_n)$ is given in terms of an $(n + 1)$ -ary relation R_\blacksquare as follows:

$$\begin{aligned} \mathcal{M} \models_x \blacksquare(A_1, \dots, A_n) &\iff \text{for all } v_1, \dots, v_n \\ &\text{if } R_{\blacksquare}xv_1, \dots, v_n \text{ then } \mathcal{M} \models_{v_i} A_i \text{ for } 1 \leq i \leq n. \end{aligned}$$

In this setting a propositional constant corresponds to the limit case of a polyadic modal operator, interpreted in terms of the world of evaluation being in a given set. Now, of course, this new 0-place modal operator is not going to be *normal*, as it is not going to satisfy the 0-place version of necessitation—from the provability of A infer κ —as κ in the logics under consideration here is not a theorem. That simply means that we have a 0-place non-normal modal operator on our hands, not that we do not have a modal operator at all.⁵

To more easily connect with the point made above that the language with a single unary modal operator and a propositional constant used here is a bimodal language in disguise, we can think of our propositional constant as a unary modal operator which does not depend upon its argument. This would mean that, for example, the constant-masquerading-as-unary-modal-operator \Box_κ would validate the schema $\Box_\kappa A \leftrightarrow \Box_\kappa B$ for all formulas A and B . Again, as mentioned above, this new operator \Box_κ is not going to be normal, as we can have $\vdash A$ without having $\vdash \Box_\kappa A$ —this is really just saying that κ is a theorem.

We end by presenting the following open problem.

Open Problem 4.1 Are there nondenumerably many Post-complete normal modal logics in the language L_κ ?

Update. Since the submission of this paper several people acquainted with its contents notified the author that there are indeed nondenumerably many Post-complete normal modal logics in the present language. Model-theoretic proofs of the stronger result were supplied by Robert Goldblatt and by a referee for this journal, and an algebraic proof was sketched by Tomasz Kowalski. Rather than reproducing any of their arguments here, the author has opted for simply notifying the reader of the stronger result and leaving those mentioned (and perhaps others) free to publish their proofs.

Notes

1. We will assume that the reader is familiar with modal logic, any unexplained terminology being taken from Chellas [2]. Logics here will be thought of as sets of formulas, but rather than writing $A \in S$ to denote that A is a theorem of S , we will instead use the more perspicuous notation $\vdash_S A$.
2. Segerberg has written extensively on Post-completeness in modal logics. The interested reader is referred to Segerberg [9], [10], and [11], as well as to Makinson [7] and Kohn [6].
3. This distinction is recorded in Williamson [12] by referring to the logics which we will call *Post-complete normal modal logics*, as maximal consistent normal modal logics. This seems like a mistake to the present author, making it seem as if we are applying different concepts rather than applying the same concept in different settings.

4. The following lemma was suggested to me by Robert Goldblatt to patch an error in a previous proof of Theorem 3.9, as well as being independently suggested to me by Lloyd Humberstone for reasons of elegance.
5. This observation lends some credence to the complaints sometimes raised against Anderson [1] that we are smuggling in all the relevant deontic content with our constant for the sanction. Nowell-Smith and Lemmon [8, p. 291] make the point that the Andersonian reduction will only work if our constant has deontic content or represents a “deontic concept.” As it happens, the sanction constant \mathcal{S} is just a strange deontic modal operator, and so what the Anderson result shows is that we can define a unary deontic modal operator in terms of a unary alethic modal operator and a 0-place deontic modal operator. An interesting result, to be sure, but hardly a reduction of the deontic modalities to the alethic. Of course, if this kind of reduction is executed so as to avoid other problems, this objection is not so clear cut (cf. Humberstone [4]).

References

- [1] Anderson, A., “A reduction of deontic logic to alethic modal logic,” *Mind*, vol. 67 (1958), pp. 100–103. [555](#)
- [2] Chellas, B. F., *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, 1980. [Zbl 0431.03009](#). [MR 0556867](#). [554](#)
- [3] Goguadze, G., C. Piazza, and Y. Venema, “Simulating polyadic modal logics by monadic ones,” *Journal of Symbolic Logic*, vol. 68 (2003), pp. 419–62. [Zbl 1059.03008](#). [MR 1976585](#). [553](#)
- [4] Humberstone, L., “Relative necessity revisited,” *Reports on Mathematical Logic*, vol. 13 (1981), pp. 33–42. [Zbl 0519.03009](#). [MR 0657330](#). [555](#)
- [5] Humberstone, L., “Contra-classical logics,” *Australasian Journal of Philosophy*, vol. 78 (2000), pp. 438–74. [549](#)
- [6] Kohn, R. V., “Some post-complete extensions of S2 and S3,” *Notre Dame Journal of Formal Logic*, vol. 18 (1977), pp. 467–70. [Zbl 0348.02026](#). [MR 0497940](#). [554](#)
- [7] Makinson, D., and K. Segerberg, “Post completeness and ultrafilters,” *Mathematical Logic Quarterly*, vol. 20 (1974), pp. 385–88. [Zbl 0298.02014](#). [MR 0381936](#). [554](#)
- [8] Nowell-Smith, P., and E. Lemmon, “Escapism: The logical basis of ethics,” *Mind*, vol. 69 (1960), pp. 289–300. [555](#)
- [9] Segerberg, K., “Post completeness in modal logic,” *Journal of Symbolic Logic*, vol. 37 (1972), pp. 711–15. [Zbl 0268.02012](#). [MR 0316217](#). [554](#)
- [10] Segerberg, K., “Halldén’s theorem on Post completeness,” pp. 206–9 in *Modality, Morality and Other Problems of Sense and Nonsense: Essays Dedicated to Sören Halldén*, CWK Gleerup Bokforlag, Lund, 1973. [MR 0453497](#). [554](#)
- [11] Segerberg, K., “The truth about some Post numbers,” *Journal of Symbolic Logic*, vol. 41 (1976), pp. 239–44. [Zbl 0331.02008](#). [MR 0432405](#). [554](#)
- [12] Williamson, T., “Continuum many maximal consistent normal bimodal logics with inverses,” *Notre Dame Journal of Formal Logic*, vol. 39 (1998), pp. 128–34. [Zbl 0968.03027](#). [MR 1671746](#). [549](#), [554](#)

Acknowledgments

The author would like to thank Lloyd Humberstone for his insightful comments, Robert Goldblatt for spotting an error in a previous version of this paper and suggesting a solution to Open Problem 4.2, Tomasz Kowalski for (also) suggesting a solution to Open Problem 4.2, the audience at the Melbourne University Logic Seminar, and a referee for this journal.

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