Notre Dame Journal of Formal Logic
Volume 53, Number 4, 2012

On Existence in Set Theory

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Abstract The aim of the present paper is to provide a robust classification of valid sentences in set theory by means of existence and related notions and, in this way, to capture similarities and dissimilarities among the axioms of set theory. In order to achieve this, precise definitions for the notions of productive and nonproductive assertions, constructive and nonconstructive productive assertions, and conditional and unconditional productive assertions, among others, will be presented. These definitions constitute the result of a semantical analysis of the notions involved. The conceptual clarification developed here results in a classification of valid sentences of set theory that goes against the standard view that extensionality is not an existence assertion.

1 Introduction

In this introduction, some of the problems concerning the notion of existence in set theory that motivated this investigation are stated. The aim of this paper is to classify the valid sentences of set theory in terms of existence and related notions, and, in order to do so, some precise concepts and distinctions must first be introduced. This will be done gradually in the forthcoming sections.

The work on the axiomatization of set theory provides a list of principles about sets that constitutes the basis of the contemporary conception of mathematical existence. Some principles of set theory, such as the axiom of pairs, are understood as existence assertions: given sets x and y, the axiom of pairs asserts the existence of the pair $\{x, y\}$. Other principles are usually understood in a different way, as axioms about the nature of sets: the extensionality axiom, for example, is understood as defining the equality of sets. The axioms are therefore usually separated in two groups, the axioms of the nature of sets and the existence axioms. The extensionality and regularity axioms constitute the first group, and the remaining axioms constitute the second group. This separation is not made on the basis of a syntactic property

Received September 10, 2011; accepted November 12, 2011
2010 Mathematics Subject Classification: Primary 03E30
Keywords: existence axioms, foundations of set theory
 10.1215/00294527-1722737

such as the occurrence of an existential quantifier in the prefix of a prenex normal form of a formula expressing the axiom. Rather, it is presented as a conceptual distinction.

There is a further division of the existence axioms in two groups: the constructive existence axioms and the nonconstructive ones. The second group is constituted by the axiom of choice alone, and the first group by the other existence axioms. In this case, the separation is presented, at least in part, as a property of the system: the capacity of introducing, or not, abstraction terms that instantiate the existence axiom. Joseph Shoenfield [9, p. 259] formulates this distinction as follows:

> We shall also prove a related result: if ZF is consistent, then neither the axiom of choice nor its negation can be proved in ZF. One might ask why this is of special interest, since the axiom of choice is certainly true for sets. One answer is that the axiom of choice is of a special nature. The sets asserted to exist by the existence axioms of ZF (such as the power set of a set) can be explicitly described in ZF; in fact they are of the form $\{x \mid Q(x, v_1, \ldots, v_n)\}$. On the other hand, there is no reason to suppose that for every set v, there is a choice function on v which can be described in this way. Thus it is conceivable that for some notion of set which involves using only collections which can be described, the axioms of ZF are true while the axiom of choice is false.

Together with the previous separation of the axioms into the axioms of nature and the so-called *existence axioms*, this further splitting of the existence axioms results in a trichotomy in the axioms of Zermelo–Fraenkel set theory with choice (ZFC). Up until now, this trichotomy has not been founded on solid grounds. Although the constructive/nonconstructive separation, as formulated by Shoenfield, is not established "by decree" as is the separation between existence/nonexistence, it relies on the previous one (nature versus existence), and the above extract is still far from giving a precise definition of a constructive existence axiom or theorem. The analysis of some examples will show that a first attempt at giving precision to these distinctions encounters some difficulties. Consider the axiom of extensionality:

$$\forall x \forall y \big(\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y\big).$$

This formula is equivalent, by the definition of one quantifier in terms of the other and tautological consequence, to

$$\forall x \forall y \big(x \neq y \rightarrow \exists z \big((z \in x \wedge z \notin y) \vee (z \notin x \wedge z \in y)\big)\big),$$

which says that if x is different from y, then there exists a witness z of their difference. The above formula is equivalent, by prenex operations, to the formula

$$\forall x \forall y \exists z \big(x \neq y \rightarrow \big((z \in x \wedge z \notin y) \vee (z \notin x \wedge z \in y)\big)\big).$$

Why is this not an existence axiom? One could say that the axiom is not used as such, to prove the existence of sets, but only to prove the equality or the difference of sets. One could, however, in principle, prove that two sets are different by showing that they are distinguished by a property and then conclude, by extensionality, that there is a witness of their difference. This would be an existential use of this axiom, similar to Cantor's proof of the existence of a transcendental real number. Of course, Cantor's diagonal method constructs a transcendental number, making the use of extensionality superfluous. Nevertheless, this does not exclude the possibility of an existential use of this axiom. Furthermore, an adequate existence/nonexistence distinction should be based on a priori grounds, not on the actual use of the principle.

The separation of existence axioms into the constructive or instantiated ones on one side, and the axiom of choice on the other, is not immune to this undefined situation concerning the axiom of extensionality. If extensionality is to be considered an existence axiom, then there is also no reason to suppose that there is a witness of the difference of any two sets that can be described by an abstraction term. In fact, all such witnesses could be choice functions for complex sets. Therefore, it is not sufficient to simply stipulate the axiom of extensionality as a nonexistence axiom. One must provide a satisfactory definition for the notion of an existence axiom and prove that this axiom is on the nonexistence side. But this is not the only difficulty with the constructive/nonconstructive division usually found in discussions of the axiom of choice. Consider the following tautology:

$$\exists x P(x) \to \exists x P(x), \quad P(x) \text{ stands for "} x \text{ is a well-ordering of } \mathbb{R}.\text{"}$$

The formula above is equivalent, by prenex operations, to

$$\exists x \big(\exists x P(x) \to P(x)\big),$$

and this is an existential theorem of standard first-order logic such that even the system ZFC cannot provide an abstraction term that instantiates it (see Levy [4], p. 130). Azriel Levy gives a similar example and comments on this in [6, p. 175]. Levy ends his commentary in a pessimistic tone:

> If the reader will claim that the counterexamples we mentioned here are unnatural he will not be totally wrong, but he will probably be at a loss suggesting a clear-cut distinction between natural and unnatural statements $\Gamma(x)$.

Therefore, "constructive subtheories of ZFC" have nonconstructive consequences, in Shoenfield's sense, and the axiom of choice cannot be considered at fault for the nonconstructive consequences of set theory. The separation of the axiom of choice from the other axioms presented by Shoenfield is not robust enough: it is confused by logical deduction. Satisfactory distinctions must not have this lack of stability. For a proper analysis of the existence requirements of the axioms, it is necessary that adequate definitions be presented. The exposition by Shoenfield lacks such definitions.

The forthcoming analysis refers, above all, to the system of set theory ZFC by which twentieth-century mathematics is canonically organized.[1] The consistency of the system ZFC is assumed throughout. A precise formulation of ZFC that is adequate for the purposes of the present investigation of existence will be postponed until Section 6 on conditional and unconditional degrees of existence requirements. The reason for a later formulation of the system is twofold: first, the specific features of the definition given here are not really necessary before then, and, second, this avoids introducing an overwhelming apparatus all at once. The last section, dedicated to concluding remarks, will provide additional explanations of the main notions of the present paper and a short discussion on the foundational relevance of the results.

2 A Gradation of Existence Requirements for ZFC

The existential character of a statement is not adequately captured by the syntactic features of a formula expressing it. In fact, any formula A is equivalent to an existential formula, for example, $\exists x(x = x \wedge A)$, in which x is assumed not to occur free in A. Even if one allows only prenex operations, the tautology $\exists x A \to \exists x A$ and the formula

$$\exists z(z \in x) \to \exists z(z \in x \wedge z \cap x = \emptyset)$$

expressing the regularity axiom are equivalent to existential formulas by prenex operations. These theorems of set theory should not be considered as existence assertions: they are implications such that the entire existence requirement of the consequent is already present in the antecedent.

The forthcoming analysis of the existential character of a statement will not be formulated in terms of syntactic features of a formula expressing it. Instead, the semantic notion of the existence requirement of a statement will play a conspicuous role. The existence requirement of a statement is evaluated in terms of degrees of closure of the domains of existence in which the validity of the statement is tested. If a statement is valid in every domain presenting a certain closure property, then that closure property is sufficient to fulfill the existential demands of the statement. Now, there are natural closure properties for a domain in set theory: transitivity, supertransitivity, the property of being a level V_α, the property of being the universe V, among others. These constitute a hierarchy of closure properties. Therefore, a natural gradation of existence requirements can be introduced.

Although any set or class is a domain of existence in set theory, only nonempty domains are considered in the evaluation of the existence requirement of a statement. The contribution of the empty domain to the classification of valid sentences of set theory is not a quantitative one. Instead, the contribution of the empty domain for the present analysis is qualitative. The reason for this is that the degree of closure of the empty domain is that of a level. Thus, the consideration of this domain in the gradation of existence requirements would only artificially raise the existence requirement of a sentence like $\exists x(x = x)$. Indeed, if the empty domain is considered in the definition of degrees, then the associated existence requirement of this sentence is higher than that of separation axioms, among others. The identification of the existence requirement of a statement with the least degree of closure that is sufficient to fulfill the existential demands of the statement is plausible only if the domains are restricted to the nonempty ones. The consideration of the empty domain gives an entirely qualitative distinction that will be expounded in Section 6.

Definition 1, and the results immediately following it, introduce a gradation of existence requirements in terms of the degree of closure of the domains in which the validity of the statement is tested. For example, a statement A admits a lower degree of existence requirement than B if the existential demands of A are already supplied by the transitivity of the domain, while the demands of B are supplied by the supertransitivity of the domain (closure under taking elements and subsets) but not by the transitivity only. Any set or class can be seen as a domain of existence. The universe presents the maximum degree of closure. If a domain presents a higher closure degree than another one, then it more closely resembles the universe. The analysis that follows is restricted to *sentences*.

Definition 1 The sentence A in $L(ZF)$ is said to admit the following degrees of existence requirements:

- degree zero of existence requirement, if A holds in every (nonempty) $\in$-interpretation I of $L(ZF)$ in an extension by definitions or by introduction of constants T of ZFC;[2] that is, for each $\in$-interpretation I,

$$T \vdash A^I;$$

- degree 1 of existence requirement, if A holds in every transitive $\in$-interpretation I of $L(ZF)$ in an extension by definitions or by introduction of constants T of ZFC; that is,

$$T \vdash \forall x \forall y \big(U_I(x) \to (y \in x \to U_I(y))\big) \to A^I;$$

- degree 2 of existence requirement, if A holds in every supertransitive $\in$-interpretation I of $L(ZF)$ in an extension by definitions or by introduction of constants T of ZFC; that is,

$$T \vdash \forall x \forall y \big(U_I(x) \to ((y \in x \vee y \subset x) \to U_I(y))\big) \to A^I;$$

- degree 3 of existence requirement, if

$$ZFC \vdash \mathrm{Ord}(\alpha) \to A^{V_\alpha},$$

in which $\mathrm{Ord}(\alpha)$ stands for "α is an ordinal";[3]
- degree 4 of existence requirement, if

$$ZFC \vdash \mathrm{Lim\,Ord}(\alpha) \to A^{V_\alpha},$$

in which $\mathrm{Lim\,Ord}(\alpha)$ stands for "α is a limit ordinal";
- degree 4ω of existence requirement, if

$$ZFC \vdash \mathrm{Lim\,Ord}(\alpha) \wedge \omega < \alpha \to A^{V_\alpha};$$

- degree 5 of existence requirement, if for the identity interpretation $\mathbf{V}$ of $L(ZF)$ in ZFC it holds that $ZFC \vdash A^{\mathbf{V}}$; that is,

$$ZFC \vdash A.$$

Remark 2 An $\in$-interpretation of $L(ZF)$ in an extension of ZFC is just a domain and need not satisfy the axioms.

Notation 3 Denote by $E(d)$ the set of sentences in $L(ZF)$ that admit a degree d of existence requirement, $0 \le d \le 5$ or $d = 4\omega$. If d and d' are degrees, denote by $d \prec d'$ the ordering obtained from the usual lexicographic ordering on pairs by the correspondence of the degrees $0, 1, 2, 3, 4, 5$ with the pairs $(0,0), (1,0), (2,0), (3,0), (4,0), (5,0)$, respectively, and the degree 4ω with the pair $(4, \omega)$.

Lemma 4 *The degrees of existence requirement are linearly ordered by strength in the sense that $E(d) \subset E(d')$ if $d \prec d'$. All inclusions are proper.*

Proof All the inclusions are trivial, except that $E(4\omega) \subset E(5)$, and this follows from the reflection principle. In Section 3, the evaluation of the axioms will show that the inclusions are proper. □

Proposition 5 is an immediate corollary of Lemma 4. It is important mainly because it identifies the scope of the notion of existence requirement and defines the existence requirements associated to a sentence within this scope.

Proposition 5 (The existence requirement of a sentence) *A sentence in $L(ZF)$ is a theorem of ZFC if and only if it admits a degree of existence requirement. If this is the case, then the existence requirement associated to the sentence A is defined to be the least degree admitted by A and is denoted by $r(A)$.*

The gradation of existence requirements considered here is restricted to statements that hold in the universe; it makes no sense to speak of the existence requirement associated to a sentence that is not valid. If A is a valid sentence then one can ask how much closure of the universe is required by the validity of A. Is the transitivity of the domain sufficient for the existential demands of A? Is the supertransitivity sufficient for the existential demands of A? The validity of A may require the following: no closure property (degree zero); closure under taking elements (degree 1); closure under taking elements and subsets (degree 2); closure under any operation not increasing rank (degree 3); closure under any operation not increasing rank, or increasing rank by a finite ordinal (degree 4); closure under any operation not increasing rank, or increasing rank by a finite ordinal, or producing sets of rank at most ω (degree 4ω); closure under everything (degree 5). Intuitively, if $0 \prec r(A)$, then the sentence A demands some closure on a nonempty domain. If some closure is required, then A is a *productive assertion*; it has the power of producing sets on nonempty domains. If $r(A) = 0$, then A is a *nonproductive assertion*. This new productive/nonproductive assertion distinction is more restrictive than the syntactical distinction that relies on the mere occurrence of an existential quantifier in the prefix of a prenex normal form of a sentence, as Proposition 11 will show, and it can be formulated as follows.

Definition 6 (Productive and nonproductive assertions) The sentence A is a *nonproductive assertion* (in ZFC) if A admits degree zero of existence requirement. On the other hand, A is a *productive assertion* (in ZFC) if A admits a degree of existence requirement and $0 \prec r(A)$.

Therefore, the productive assertions in ZFC are defined to be those sentences that admit only nonzero degrees of existence requirements. Proposition 7 below expresses a very important property of the classification of valid sentences in terms of existence requirements: stability under logical deduction. If a sentence B is proved from another sentence A that admits a degree of existence requirement, then B also admits a degree of existence requirement and $r(B)$ is at most equal to $r(A)$. Furthermore, it may be seen that the value $r(A)$ is stable under logical variations of A.

Proposition 7 *If the sentence A admits a degree of existence requirement (equivalently, $ZFC \vdash A$) and B is a sentence such that $\vdash A \rightarrow B$, then B also admits a degree of existence requirement and $r(B) \prec r(A)$ or $r(B) = r(A)$. In particular, if $\vdash A \leftrightarrow B$ (and $ZFC \vdash A$), then $r(B) = r(A)$.*

Proof If $\vdash A \rightarrow B$ and if A is valid in a nonempty domain, then so is B. □

Corollary 8 *If $ZFC \vdash A$, then any proof of A makes use of axioms with existence requirement at least $r(A)$. Furthermore, if Γ is a subtheory of ZFC such that $\Gamma \vdash B$ and if A is a nonlogical axiom of Γ, then $r(A) \prec d$, for $0 \prec d$, then $r(B) \prec d$.*

Definition 9 (d-equivalence) Two sentences A and B are *d-equivalent* if the sentence $A \leftrightarrow B$ admits a degree d of existence requirement.

Remark 10 The relation of d-equivalence is clearly an equivalence relation. Furthermore, if A and B are d-equivalent and A admits a degree d of existence requirement, then B also admits a degree d of existence requirement. This follows from the fact that relativization of sentences preserves the propositional connectives.

It is worth noting that in any proof of the sentence A, with associated existence requirement $r(A)$, there occurs an axiom with associated existence requirement at least $r(A)$. This is a consequence of Proposition 7. Proposition 11 shows that the semantical definition of productive assertion is stronger than the mere syntactical property of occurrence of an existential quantifier in the prefix of the sentence.

Proposition 11 *If the sentence A admits a degree of existence requirement and $0 \prec r(A)$, then the symbol $\exists$ occurs in the prefix of any sentence in prenex normal form zero-equivalent to A.*

Proof If the symbol $\exists$ does not occur in the prefix of a sentence B in prenex normal form zero-equivalent to A, then this sentence is universal. Since it is a theorem, its relativization to any nonempty domain holds. Indeed, the effect of the relativization on B is only the restriction of the universal quantifiers. Since B is zero-equivalent to A, A admits degree zero. This is a contradiction. □

In the following section, the axioms of ZFC are evaluated in terms of the classification introduced above. Before doing so, however, the pathological example given in the introduction may be easily evaluated:

$$\exists x P(x) \rightarrow \exists x P(x), \quad P(x) \text{ stands for "} x \text{ is a well-ordering of } \mathbb{R}.\text{"}$$

This sentence admits a degree zero of existence requirement, and although the symbol $\exists$ occurs in the prefixes of all its prenex normal forms, it is in fact a nonproductive assertion.

3 The Existence Requirements of the Axioms

If $ZFC \vdash A$ and $r(A) = 5$, then by Corollary 8 the theory ZFC must have axioms with existence requirement 5, and this is indeed the case. In fact, the principles of set theory have a uniform distribution in terms of existence requirements in the sense that all levels are populated by some statements. This uniform distribution of the axioms indicates further that they are not accidental or arbitrarily chosen: there are axioms for all levels of existence requirements. Propositions 12 and 13 evaluate the existence requirements of the axioms. The main results needed for this evaluation can be found in standard expositions of set theory (see Kunen [3, pp. 112–17], Drake [2, pp. 107–8]).

Proposition 12 (Evaluation of axioms) *If the sentence A is*

- *the regularity axiom or the axiom of the empty set, then $r(A) = 0$;*
- *the extensionality axiom, then $r(A) = 1$;*
- *the union axiom or the axiom of the choice set, then $r(A) = 3$;*
- *the power set axiom or the axiom of the choice function or the axiom of pairs, then $r(A) = 4$;*
- *the axiom of infinity, then $r(A) = 4\omega$.*

Proof If I is any $\in$-interpretation of $L(ZF)$ in (an extension by definitions or by introduction of constants T of) ZFC and x is a set in I, then, by the regularity axiom, there is a minimal element y in the set $\{z \in x \mid U_I(z)\}$, U_I standing for the universe of the interpretation. The element y is a minimal element of x in I, and the regularity axiom holds in I. For the axiom of the empty set, any set of minimal rank in I is an empty set in I.

It is fairly well known that the extensionality axiom holds in any transitive $\in$-interpretation of $L(ZF)$ in (an extension by definitions or by introduction of constants T of) ZFC. However, it is not the case that this axiom holds in any $\in$-interpretation: it suffices to consider an interpretation given by the domain $\{x, y\}$ such that $x \neq y$ and $x \notin y$ and $y \notin x$. (In this interpretation there are two "empty sets.")

The supertransitive domain $\big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\big\}$ shows that both the axiom of union and the axiom of the choice set do not hold in any supertransitive $\in$-interpretation. However, both axioms hold in all levels V_α because taking a choice set or the union are rank-decreasing operations.

The power set axiom, the axiom of pairs, and the axiom of the choice function do not hold in all successor levels because the corresponding operations increase rank. For instance, all three axioms fail in $V_2 = \{\emptyset, \{\emptyset\}\}$. Nevertheless, all these axioms hold in limit levels since the corresponding operations can increase rank by a finite ordinal only.

The axiom of infinity does hold in any transitive $\in$-interpretation that contains ω, and in any V_α for any $\alpha < \omega$ in particular, and does not hold in V_ω, for example. □

Before treating the schemas, it is already a fact that the productive assertions among the axioms do not coincide with the existence axioms in Shoenfield's sense and as presented in the introduction. Indeed, the extensionality axiom *is* a productive assertion, although a weak one. It admits degree 1, but not degree zero. In the case of the regularity axiom, it is indeed a nonproductive assertion. The axiom of the empty set admits a degree zero of existence requirement. A finer analysis is needed to distinguish the axiom of the empty set from the regularity axiom. This will be done in Section 6 on conditional and unconditional degrees.

Proposition 13 (Evaluation of schemas) *The existence requirements of the schemas are evaluated differently and defined to be the least upper bound of $r(A)$ for A an instance of the axiom:*

- *the existence requirement of the separation schema is* 2*;*
- *the existence requirement of the replacement schema is* 5.

Proof The transitive domain $\big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\big\}$ shows that not all the instances of the separation schema hold in transitive domains. Nevertheless, all instances of this schema hold in supertransitive domains since supertransitive domains contain all subsets of a given set.

It is well known that not all instances of the replacement schema hold in $V_{\omega+\omega}$. Of course, all instances of this axiom hold in $\mathbf{V}$. □

The analysis presented so far already shows differences with some of the usual conceptions regarding the axioms, and Remark 14 explains some of these. Nevertheless, it is important to note that this section contains only the first part of the analysis, which at this point is still far from being complete.

Remark 14 (The logic of $\in$) The evaluation of the axioms shows that the only nonproductive assertions among them are the regularity axiom and the axiom of the empty set. If one is to speak of a logic of $\in$, analogous to the logic of $=$, and if such a logic must hold in every domain adapted to it, that is, in every interpretation in which $\in$ and $=$ are interpreted as the true membership and equality relations, then its principles should be only these nonproductive axioms.

The extensionality axiom is commonly considered as the definition of equality among sets. Dana Scott writes in [8, p. xiii]:

> An axiom like the Extensionality Axiom, which says that sets are uniquely determined by their elements, is also sufficiently logical in character, because of its most definitional nature. So we can throw it to the side of logic.

George Boolos goes one step further in [1, p. 501]:

> The axiom of extensionality enjoys a special epistemological status shared by none of the other axioms of ZF. Were someone to deny another of the axioms of ZF, we would be rather more inclined to suppose, on the basis of his denial alone, that he believed that axiom false than we would if he denied the axiom of extensionality. Although "there are unmarried bachelors" and "there are no bachelors" are equally preposterous things to say, if someone were to say the former, he would far more invite the suspicion that he did not mean what he said than someone who said the latter. Similarly, if someone were to say, "there are distinct sets with the same members," he would thereby justify us in thinking his usage nonstandard far more than someone who asserted the denial of some other axiom. Because of this difference, one might be tempted to call the axiom of extensionality "analytic," true by virtue of the meanings of the words contained in it, but not to consider the other axioms analytic.

If logic is to be understood in the way explained above, then the extensionality axiom should not figure among the principles of the logic of $\in$. On the contrary, aside from the usual logical axioms, the principles of $\in$-logic should be just the regularity axiom and the axiom of the empty set. Since these axioms admit a simple Skolem normal form, the theorems of such theory can be characterized by an application of Herbrand's theorem (see [9, p. 54]) in a simple way. In fact, consider the Skolem normal form of these axioms:

$$\forall x \forall y \forall z \big(z \in x \rightarrow (f(x) \in x \wedge \neg(y \in x \wedge y \in f(x)))\big) \quad \text{(regularity axiom)},$$
$$\forall w \neg(w \in c) \quad \text{(axiom of the empty set)}.$$

The term $f(x)$ stands for a $\in$-minimal element in x, where f is a unary function symbol. The constant symbol c stands for an empty set. A sentence in prenex normal form is a theorem of the $\in$-logic if and only if there is a disjunction of instances of the matrix of its Herbrand normal form that is a tautological consequence of instances of the identity and equality axioms, and the axioms above (in Skolem normal form). All those instances will be in the expansion of $L(ZF)$ obtained by the addition of c and f.

4 Constructive and Nonconstructive Productive Assertions

The purpose of this section is to introduce another important definition, intended to capture the constructive/nonconstructive distinction. As in the case of the productive/nonproductive distinction, the approach to the problem is purely semantical: the constructivity of a productive/nonproductive assertion cannot be read from the symbols. The constructive character of a theorem, on the contrary, is evaluated when the validity of the sentence is tested against all domains of constructive productivity: an assertion is constructive if it holds in every $\mathbf{L}(x)$, the constructible sets relative to x.

An exposition of some syntactical features related to the definition will be given in Remark 17, where a relation to the Levy hierarchy will be explored and will further motivate the definition.

Regarding the domains of constructive productivity, it is important to consider not only the absolute constructible sets but also the relative constructible sets. The reason for this will be made clear by the consideration of the definition of conditional/unconditional existence degrees that will be introduced in Section 6. The point is that productive assertions may have a conditional existence degree. Consider the power set axiom: it implies the existence of a power set of a set, under the condition that the set is given. Therefore, if this assertion is constructive, then it must constructibly assign a power set for any given set, not only for any constructible set.

The constructive/nonconstructive distinction is entirely qualitative: there will be no contribution to the existence requirements. It results in a further division: it introduces the constructive productive assertions, the nonconstructive productive assertions, and the constructive nonproductive assertions, without changing the existence requirements associated to the productive assertions, or the degree zero admitted by the nonproductive ones.

Definition 15 Suppose that A is a productive assertion according to Definition 6. In this case, A is a *constructive productive assertion in* ZFC if for the interpretation $\mathbf{L}(x)$ of $L(ZF)$ in ZFC, given that x is the first variable in the alphabetical ordering that does not occur in A, it holds that $ZFC \vdash \forall x A^{\mathbf{L}(x)}$. If $ZFC \nvdash \forall x A^{\mathbf{L}(x)}$, then A is a *nonconstructive productive assertion*. If A admits degree zero of existence requirement, then $ZFC \vdash \forall x A^{\mathbf{L}(x)}$ and A is said to be a *constructive nonproductive assertion*.

In Definition 15, the variable x can be instantiated by any set. For the empty set, $\mathbf{L}(\emptyset) = \mathbf{L}$. The important property of this definition is stability under deduction: A set theory with only constructive axioms cannot have nonconstructive consequences. The constructive/nonconstructive distinction is stable under logical deduction. Proposition 16 identifies the nonconstructive character of the axiom of choice and the constructive character of the other axioms of ZFC.

Proposition 16 *If the sentence A is such that $ZF \vdash A$ and A is a productive assertion, then A is a constructive productive assertion in ZFC. If A admits degree zero, then A is a constructive nonproductive assertion. The axiom of choice is a nonconstructive productive assertion in ZFC.*

Proof $\mathbf{L}(x)$ is an interpretation of ZF, and the first part of the proposition follows from the soundness theorem for interpretations. It is known that if M is a countable transitive interpretation of ZFC, then, for an appropriate partial order, the generic extensions $M[G]$ are such that $ZFC \vdash \neg AC^{\mathbf{L}(\mathbb{R})^{M[G]}}$ (see [3, p. 245]). This shows that $ZFC \nvdash AC^{\mathbf{L}(\mathbb{R})}$ and proves the second part. □

The somewhat extended Remark 17 provides some connections with syntactical properties of sentences. In doing so, it also motivates the definition of constructive productive assertion, and this is the most important reason for including it. The scenario will be further clarified by the development of the conditional and unconditional productive assertions in Section 6.

Remark 17 (Relation with the Levy hierarchy)

- If the sentence A is a theorem of ZFC and is in the class Π_1 in the Levy hierarchy, then A admits degree 1 and is either a constructive productive or a constructive nonproductive assertion in ZFC.
- Suppose that A is Π_2 (and not Π_1) and is a theorem of ZFC. In particular, A is not a theorem of ZF, but A is equivalent (in ZF) to a sentence of the form $\forall z \exists y B$, for B a bounded formula. In this case, $ZFC \vdash \forall x A^{\mathbf{L}(x)}$ if and only if, omitting the quantifier $\forall x$,
$$ZFC \vdash (\forall z \exists y B)^{\mathbf{L}(x)},$$
that is, if and only if
$$ZFC \vdash \big(\forall z \in \mathbf{L}(x)\big)\big(\exists y \in \mathbf{L}(x)\big) B,$$
because B is absolute. If this holds, then
$$ZFC \vdash \big(x \in \mathbf{L}(x)\big) \to \big(\exists y \in \mathbf{L}(x)\big) B'$$
for B', the formula obtained by replacing the free occurrences of z in B by x. But then
$$ZFC \vdash \big(\exists y \in \mathbf{L}(x)\big) B'$$
and, replacing back x by z,
$$ZFC \vdash \big(\exists y \in \mathbf{L}(z)\big) B.$$
This means that if A is a Π_2-constructive productive assertion in ZFC equivalent to $\forall z \exists y B$, then ZFC proves that given a set z there exists a set y in $\mathbf{L}(z)$ such that B is valid.
- Similarly, if A is Π_3, equivalent (in ZF) to $\forall z \exists y \forall w B$, and is a constructive productive assertion in ZFC, then
$$ZFC \vdash \big(\exists y \in \mathbf{L}(z)\big)\big(\forall w \in \mathbf{L}(z)\big) B.$$
This analysis holds in general for theorems of ZFC of complexity Π_n.
- Analogously, if A is a constructive productive assertion in ZFC of complexity Σ_1 and is equivalent to $\exists y C$, then
$$ZFC \vdash (\exists y \in \mathbf{L}) C.$$
A trivial generalization applies to Σ_n-theorems of ZFC.

5 A Unified Treatment for Stronger Set Theories

The distinctions introduced so far have a somewhat limited scope because the notion of existence requirement applies only to theorems of ZFC. It is desirable to include further principles of set theory in the analysis. In order to do so, it is sufficient to consider stronger theories in the place of ZFC and to generalize the notions and results introduced into this new setting. This will be done in what follows. Avoiding tedious repetitions, the main notions and results will be given in outline.

Consider T a simple extension[4] of ZFC as, for example, a theory obtained from ZFC by adding to it some generalized axiom of infinity. In this section the consistency of each theory T under consideration is assumed.

Definition 18 The sentence A in $L(T)$ is said to admit, in T, the following degrees of existence requirement:

- degree zero of existence requirement, if A holds in every (nonempty) $\in$-interpretation I of $L(T)$ in an extension by definitions or by introduction of constants T' of T;[5] that is, for each $\in$-interpretation I,

$$T' \vdash A^I;$$

- degree 1 of existence requirement, if A holds in every transitive $\in$-interpretation I of $L(T)$ in an extension by definitions or by introduction of constants T' of T; that is,

$$T' \vdash \forall x \forall y \big(U_I(x) \to (y \in x \to U_I(y))\big) \to A^I;$$

- degree 2 of existence requirement, if A holds in every supertransitive $\in$-interpretation I of $L(T)$ in an extension by definitions or by introduction of constants T' of T; that is,

$$T' \vdash \forall x \forall y \big(U_I(x) \to ((y \in x \vee y \subset x) \to U_I(y))\big) \to A^I;$$

- degree 3 of existence requirement, if

$$T \vdash \mathrm{Ord}(\alpha) \to A^{V_\alpha},$$

  in which $\mathrm{Ord}(\alpha)$ stands for "α is an ordinal";[6]
- degree 4 of existence requirement, if

$$T \vdash \mathrm{Lim\,Ord}(\alpha) \to A^{V_\alpha},$$

  in which $\mathrm{Lim\,Ord}(\alpha)$ stands for "α is a limit ordinal";
- degree 4β of existence requirement, if

$$T \vdash \mathrm{Lim\,Ord}(\alpha) \wedge \beta < \alpha \to A^{V_\alpha},$$

  where β is a constant for a limit regular ordinal introduced by definition in T;
- degree 5 of existence requirement, if for the identity interpretation $\mathbf{V}$ of $L(T)$ in T it holds that $T \vdash A^{\mathbf{V}}$; that is,

$$T \vdash A.$$

Notation 19 Denote by $E(d)$ the set of sentences in $L(T)$ that admit, in T, degree d of existence requirement, $0 \le d \le 5$ or $d = 4\beta$, for some β. If d and d' are degrees, denote by $d \prec d'$ the ordering obtained from the usual lexicographic ordering on pairs of ordinals by the correspondence of the degrees $0, 1, 2, 3, 4, 5$ with the pairs $(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0)$, respectively, and the degrees 4β with the pairs $(4, \beta)$.

Remark 20 (On the metatheory) Definitions 15 and 19 cannot, in general, be formulated in a finitary metatheory like the corresponding Definitions 1 and 3, but for some particular theories they can be recast in a finitary setting.

The following results are straightforward generalizations of the corresponding statements, Lemma 4 and Propositions 5, 7, and 11. The relevance of these results has already been explained above. Only the most important features associated with them will be emphasized here.

Lemma 21 *The degrees of existence requirement are linearly ordered by strength in the sense that $E(d) \subset E(d')$ if $d \prec d'$. All inclusions are proper.*

Proof The nontrivial inclusions are again consequences of the reflection principle. □

The next proposition/definition establishes the existence requirement associated with a sentence within the scope of this notion.

Proposition 22 *A sentence is a theorem of T if and only if it admits, in T, a degree of existence requirement. If this is the case, then the existence requirement associated to A is defined to be the least degree admitted by A and is denoted by $r(A)$.*

Definition 23 (Productive and nonproductive assertions) The sentence A is a *nonproductive assertion* (in T) if A admits a degree zero of existence requirement. On the other hand, A is a *productive assertion* (in T) if A admits a degree of existence requirement and $0 \prec r(A)$.

The stability of the existence requirements under deduction is established by Proposition 24.

Proposition 24 *If the sentence A admits a degree of existence requirement (equivalently, $T \vdash A$) and B is a sentence such that $\vdash A \to B$, then B also admits a degree of existence requirement and $r(B) \prec r(A)$ or $r(B) = r(A)$. In particular, if $\vdash A \leftrightarrow B$ (and $T \vdash A$), then $r(B) = r(A)$.*

Definition 25 (d-equivalence) Two sentences A and B are *d-equivalent* if the sentence $A \leftrightarrow B$ admits a degree d of existence requirement.

The semantical definition of productive assertion is stronger than the mere syntactical characterization of existentials in this unified context as well.

Proposition 26 *If the sentence A admits a degree of existence requirement in T and $0 \prec r(A)$, then the symbol $\exists$ occurs in the prefix of any sentence in prenex normal form zero-equivalent to A.*

The following simple result will be useful in the subsequent exposition. It is also interesting since it states precisely that bounded existential quantifiers cannot produce strong productive assertions.

Proposition 27 *If the sentence A admits a degree of existence requirement in T and is of the form $\forall x_1 \cdots \forall x_n B$, where B in $L(ZF)$ is such that all its quantifiers are bounded, then A admits a degree 1 of existence requirement.*

Proof Bounded formulas are absolute for any transitive $\in$-interpretation of $L(ZF)$. Therefore, the relativization of A to the transitive $\in$-interpretation I of $L(ZF)$ consists of a restriction of the universal quantifiers $\forall x_1, \ldots, \forall x_n$ to the domain of I. Since A is a theorem, A^I is also a theorem. □

If a sentence A of the form $\forall x_1 \cdots \forall x_n B$, where B in $L(ZF)$ is such that all its quantifiers are bounded, is a productive assertion, then Proposition 27 shows that it is a productive assertion of the weakest kind, that is, $r(A) = 1$.

Proposition 28 below asserts that if a sentence A in $L(ZF)$ is provably consistent with ZFC by a method of extending transitive $\in$-interpretations and admits degree 1 in the theory $ZFC + A$, then $ZFC \vdash A$. In particular, one cannot change the set $E(1)$ for ZFC, as defined in Notation 3, using forcing without further assumptions. If A admits degree 1 in the theory $ZFC + A$, holds in $M[G]$, and $M[G]$ is a generic extension of M constructed within the resources of ZFC, then A also holds in M. After the statement and proof of Proposition 28, it will be made clear that the same argument does not apply to $E(2)$.

In order to formulate this precisely, consider the theory ZFC_M, obtained from ZFC by adding a constant M, an axiom saying that M is transitive, and, for each *sentence* A in $L(ZF)$, an axiom saying that M reflects A. By the reflection principle, ZFC_M is conservative over ZFC (see [4, pp. 132–33]).

Proposition 28 *Suppose that A is a sentence in $L(ZF)$, and suppose that A admits degree 1 in the theory $ZFC + A$. Suppose further that, in the theory ZFC_M, M' is a transitive $\in$-interpretation of $ZFC + A$ and that $M \subset M'$. In these conditions it results that $ZFC \vdash A$.*

Proof If x is a new variable, and Trans(x) is the predicate expressing that x is transitive, then $ZFC + A \vdash \text{Trans}(x) \rightarrow A^x$. By hypothesis, M' is a transitive $\in$-interpretation of $ZFC + A$ in ZFC_M. Since Trans(x) is absolute for transitive $\in$-interpretations, $\text{Trans}(x) \rightarrow A^x$ relativized to M' holds:

$$ZFC_M \vdash x \in M' \rightarrow \big(\text{Trans}(x) \rightarrow A^x\big).$$

Now, M is a transitive interpretation of ZFC in ZFC_M. The reflection principle for A, relativized to M, gives

$$ZFC_M \vdash \exists x\big(x \in M \wedge \text{Trans}(x) \wedge (A^x \leftrightarrow A^M)\big).$$

Since $M \subset M'$, it holds that $ZFC_M \vdash A^M$. Therefore, $ZFC_M \vdash A$, because $A \leftrightarrow A^M$ is an axiom of ZFC_M. Conservativity of ZFC_M over ZFC gives $ZFC \vdash A$. □

Notice that the proof of Proposition 28 does not apply to higher degrees. The argument above fails for supertransitive domains since supertransitivity is not absolute for M and M'.

If A is a sentence in $L(ZF)$ that is consistent with ZFC and admits degree 1 in $ZFC + A$, then, in general, it does not hold that $ZFC \vdash A$. In fact, following Gödel and Rosser, consider A a Π_1-sentence of the form $\forall x_1 \cdots \forall x_n B$, where B in $L(ZF)$ is such that all its quantifiers are bounded, and such that if ZFC is consistent, then neither A nor $\neg A$ are theorems.[7] Proposition 27 states that sentences of the form $\forall x_1 \cdots \forall x_n B$ are preserved under restriction, and that in $ZFC + A$ the sentence A admits degree 1. Since A is equivalent in ZF to the Rosser sentence for ZFC, it is not a theorem of ZFC.

In this general setting, further principles of set theory can be evaluated. Most notable are the so-called large cardinal notions. Proposition 29 provides an evaluation of some simple examples.

Proposition 29 (Evaluation of further axioms)

- *If the standard sentence A expressing "there exist weakly inaccessible cardinals" is an axiom of T and κ is a constant for the least weakly inaccessible cardinal, then $r(A) = 4\kappa$.*
- *If the standard sentence A expressing "there exist strongly inaccessible cardinals" is an axiom of T and λ is a constant for the least strongly inaccessible cardinal, then $r(A) = 4\lambda$.*
- *If the standard sentence A expressing "there exist measurable cardinals" is an axiom of T and μ is a constant for the least measurable cardinal, then $r(A) = 4\mu$.*

- *If the sentence A expressing the continuum hypothesis is a theorem of T, then $r(A) = 4\omega$. If it is $\neg A$ that is a theorem of T, then $r(\neg A) = 4\omega$.*

Proof Only the last item requires a proof. It follows from the absoluteness of ω and of the power set operation for supertransitive domains that these sentences admit a degree 4ω of existence requirement. Furthermore, the sentence expressing the continuum hypothesis asserts the existence of ω, and hence it admits no degree lower than 4ω. □

The qualitative constructive/nonconstructive distinction can also be introduced in this setting, and it gives some interesting results. Proposition 31 below provides some examples.

Definition 30 Suppose that A is a sentence that admits a degree of existence requirement in T. In this case, if A is productive, then A is a *constructive productive assertion in* T if for the interpretation $\mathbf{L}(x)$ of $L(T)$ in T, given that x is the first variable in the alphabetical ordering that does not occur in A, it holds that $T \vdash \forall x A^{\mathbf{L}(x)}$. If $T \nvdash \forall x A^{\mathbf{L}(x)}$, then A is a *nonconstructive productive assertion*. If A admits a degree zero of existence requirement, then $T \nvdash \forall x A^{\mathbf{L}(x)}$ and A is said to be a *constructive nonproductive assertion*.

Proposition 31 *If the sentence A expressing "there exist weakly inaccessible cardinals" is a theorem of T, then A is a constructive productive assertion in T. If the sentence A expressing "there exist measurable cardinals" is a theorem of T, then A is a nonconstructive productive assertion in T.*

Proof The first part of the result follows from the absoluteness of weakly inaccessible cardinals for transitive interpretations. The second part is an immediate corollary of a well-known theorem due to Dana Scott: If there exists a measurable cardinal, then $\mathbf{V} \neq \mathbf{L}$ (see [9, p. 314]). □

Remark 32 (The axiom of constructibility) Every theorem of ZFL (ZF + the axiom of constructibility) is either a constructive productive assertion or a constructive nonproductive assertion in the theory ZFL, because $\mathbf{L}$ is an interpretation of ZFL in ZF and $\mathbf{L}(x) = \mathbf{L}$ under the axiom of constructibility.

6 Conditional and Unconditional Degrees of Existence Requirement

Aside from the productive/nonproductive distinction and the subordinated notion of constructive productive assertion, there is a further division of the axioms of set theory. It can be said of an axiom that it expresses an unconditional existence, that is, that it says that some set exists regardless of whether other sets are given or not. The axioms of the empty set and of infinity are unconditional. On the other hand, an axiom may express a conditional existence, stating that given some sets there is another set satisfying some property. The axioms of power set, union, and pairing are examples of conditional existence axioms. This distinction can also be made precise with the notion of validity in the empty domain: roughly, conditional existence holds in the empty domain, and unconditional existence does not hold in the empty domain.

In order to provide a precise formulation of this distinction, a more careful analysis is required. The inclusive logic, that is, the quantification theory allowing empty domains, must be considered: first-order logic, in its standard formulations, is not existentially neutral because it is not valid in the empty domain. In making this restriction, part of the standard logical apparatus is lost: prenex operations are no longer available on purely logical grounds, for example. Nevertheless, it will be shown that the whole first-order logic can be recovered with an unconditional existential.

In this paper, inclusive logic deals only with sentences. Validity in the empty interpretation of $L(ZF)$ in ZFC is defined for all sentences and is truth-functional: a quantified subsentence of a sentence is assigned the value true if it is of the form $\forall x A$, or false if it is of the form $\exists x A$. The truth value of any other sentence in $L(ZF)$ is obtained from these by the truth tables for classical connectives. A sentence is valid in inclusive logic if and only if it is a theorem of standard first-order logic that is also valid in the empty interpretation. Therefore, proofs in inclusive logic can be defined precisely as proofs in standard first-order logic followed by the test of validity in the empty interpretation. Since validity in the empty interpretation is truth-functional, the set of valid sentences in inclusive logic is recursively enumerable. Furthermore, a system of axioms for inclusive logic, with modus ponens as the only inference rule, can be obtained from the system ETH (see Mendelson [7, p. 149]) by adding the following axioms of identity and equality (for $L(ZF)$):

- $\forall x(x = x)$ (identity axiom);
- $\forall x \forall y \forall z \forall w\big(x = z \to (y = w \to (x = y \to z = w))\big)$ (equality axiom);
- $\forall x \forall y \forall z \forall w\big(x = z \to (y = w \to (x \in y \to z \in w))\big)$ (equality axiom).

Remark 33 (Relation between inclusive and standard first-order validity) Let $\top$ denote some fixed sentence $C \vee \neg C$. We have

$$\vdash A \qquad \text{iff} \qquad \vdash_I \exists x \top \to A,$$

where $\vdash_I$ denotes validity in inclusive logic and $\vdash$ denotes validity in standard first-order logic. In fact, if A is a theorem in standard first-order logic, then $\exists x \top \to A$ is also a theorem in standard logic and is valid in the empty domain. Conversely, if $\vdash_I \exists x \top \to A$, then $\vdash \exists x \top \to A$. Since $\vdash \exists x \top$, it follows that $\vdash A$.

The theory ZFC (or the stronger systems considered in the above section) has a relational language and an axiom that implies $\exists x \top$ in the inclusive logic[8] and hence can be formulated directly, taking the inclusive logic with the usual equality axioms as the background logic. Furthermore, the logic free of existential presuppositions is the proper logic for identifying with accuracy the degrees of existence requirements of the sentences in set theory.

For the sake of definiteness, consider the axiom system for ZFC with the system for inclusive logic as the background logic and having the following axioms:

- $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$ (extensionality axiom);
- $\forall x\big(\exists y(y \in x) \to \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y))\big)$ (regularity axiom);
- $\forall x \exists y \forall z(z \in y \leftrightarrow z \in x \wedge A)$, all the variables are distinct, and x and y do not occur in A (separation axioms);
- $\forall x\big(\forall y \exists z \forall w(A \leftrightarrow w \in z) \to \exists v \forall w(\exists y(y \in x \wedge A) \to w \in v)\big)$, all the variables are distinct, and x, z, and v do not occur in A (replacement axioms);
- $\forall x \exists y \forall z(\forall w(w \in z \to w \in x) \to z \in y)$ (power set axiom);
- $\exists x\big(\exists y(y \in x \wedge \forall z(z \notin y)) \wedge \forall y(y \in x \to \exists z(z \in x \wedge \forall w(w \in z \leftrightarrow w \in y \vee w = y)))\big)$ (axiom of infinity);
- $\exists x \forall y(y \notin x)$ (axiom of the empty set);

- $\forall x \exists y \big(\forall z (z \in x \to (\exists w (w \in z) \wedge \forall u (u \in x \to \forall t (t \notin u \vee t \notin z)))) \to \forall z \exists w \forall v (z \in x \to (v \in z \wedge v \in y \leftrightarrow v = w))\big)$ (axiom of the choice set).

From Remark 33 it follows that this is indeed an axiomatization of ZFC. Furthermore, *the above formulation of ZFC (or ZF) is to be considered, except for some inessential variations, the proper formulation for the present analysis*. In particular, all subtheories and stronger theories of the previous sections are to be considered as built over this formulation. The whole analysis presented so far applies without any change. In a word, throughout this paper the labels ZFC or ZF should be understood as referring to the above formulation of the corresponding theories.

Remark 34 All the sentences of the above system, with the exception of the axiom of infinity and the axiom of the empty set, are true in the empty interpretation, and the rule of inference modus ponens preserves validity of sentences. The axiom of infinity and the axiom of the empty set are false in the empty interpretation.

The main definition of this section is the following.

Definition 35 (Conditional and unconditional degrees) The sentence A in $L(ZF)$ admits a *conditional* degree d of existence requirement if A is valid in the empty interpretation and admits a degree d of existence requirement. On the other hand, A admits *unconditional* degree d of existence requirement if A is not valid in the empty interpretation and admits degree d of existence requirement.

Among the principles of ZFC set theory, only the axioms of infinity and of the empty set admit unconditional degrees of existence requirements. The generalized axioms of infinity also admit unconditional existence degrees in the stronger theories in which they are theorems.

Proposition 36 *If A admits a conditional degree of existence requirement, then it does not admit an unconditional degree. Conversely, if A admits an unconditional degree of existence requirement, then it does not admit a conditional degree. In particular, if $ZFC \vdash A$, then $r(A)$ is either a conditional degree or an unconditional degree.*

Proof The proof follows immediately from Definition 30: A cannot be both valid and invalid in the empty interpretation. □

Proposition 37 *If A admits a conditional degree of existence requirement and $\vdash_I A \to B$, then B also admits a conditional degree of existence requirement and $r(B) \prec r(A)$ or $r(B) = r(A)$. Specifically, at least one of either the axiom of infinity or the axiom of the empty set occurs in any proof of a sentence that admits an unconditional degree of existence requirement in the above axiomatization of ZFC.*

Proof This is a corollary of Remark 34 above and Proposition 7. □

Definition 38 introduces the last distinction of the present analysis. It is a further division in the previous dichotomy of the productive/nonproductive assertion. This section has been restricted to existence degrees in ZFC, but straightforward generalizations apply to some stronger theories.

Definition 38 (Conditional and unconditional productive assertions) The sentence A is a *conditional productive assertion (in ZFC)* if it is a productive assertion ($0 \prec r(A)$) and admits a conditional degree of existence requirement. Analogously, the sentence A is an *unconditional productive assertion (in ZFC)* if it is a productive assertion and admits an unconditional degree of existence requirement. The sentence A can also be a *conditional nonproductive assertion* if it admits conditional degree zero or an *unconditional nonproductive assertion* if it admits unconditional degree zero.

Remark 39 Although it may sound strange to speak of an unconditional nonproductive assertion, the examples show that this is reasonable terminology. For example, a logical theorem of the form $\exists x(C \vee \neg C)$, for a sentence C, is an unconditional nonproductive assertion. It simply excludes the empty domain, but it does not make any existence requirement in nonempty domains. On the other hand, strong productive assertions like the power set axiom or the replacement axioms do hold in the empty interpretation. The analysis of the empty interpretation provides a qualitative conditional/unconditional distinction, and it plays no role in the evaluation of the degrees of existence requirements. In this respect, it is similar to the analysis of the constructible domains $\mathbf{L}(x)$.

The prototype syntactical form of an unconditional productive assertion is $\exists x A$, and the prototype of a conditional productive assertion is $\forall x \exists y A$. The connection to the Levy hierarchy exploited in extended Remark 17 makes even more sense now.

The results above show that the distinction introduced in Definition 38 is exhaustive and stable under inclusive logical consequence. It defines four mutually exclusive classes: the conditional productive/nonproductive assertions and the unconditional productive/nonproductive assertions. The subordinate distinction of constructive/nonconstructive productive assertion is not disturbed by this novel component, and it applies in the same way, providing six mutually exclusive classes.

Putting all these distinctions together gives a partition of the sentences that admit a degree of existence requirement in six classes: the conditional constructive productive assertions, the conditional nonconstructive productive assertions, the unconditional constructive productive assertions, the unconditional nonconstructive productive assertions, the conditional constructive nonproductive assertions, and the unconditional constructive nonproductive assertions. The corresponding existence requirements provide a quantitative scale within each class. Now the notion of existence assertion can be defined.

Definition 40 The sentence A is a *nonexistence assertion* if and only if A is a conditional constructive nonproductive assertion. If A lies in any of the remaining five classes mentioned above, then A is an *existence assertion*.

A conditional constructive nonproductive assertion holds in any domain, including the empty one. The sentences lying in any of the other classes are not valid in at least one domain of existence, and this is the motivation for the above definition.

Proposition 41 classifies the axioms present in the above formulation of ZFC and is the final outcome of this section. It does not require a proof as it only aggregates results already established.

Proposition 41 (Classification of the axioms of ZFC)

- *The axioms of identity and equality are all conditional constructive nonproductive assertions.*
- *The extensionality axiom is a conditional constructive productive assertion.*

- *The regularity axiom is a conditional constructive nonproductive assertion.*
- *Each separation axiom is either a conditional constructive productive or a conditional constructive nonproductive assertion.*
- *Each replacement axiom is either a conditional constructive productive or a conditional constructive nonproductive assertion.*
- *The power set axiom is a conditional constructive productive assertion.*
- *The axiom of infinity is an unconditional constructive productive assertion.*
- *The axiom of the empty set is an unconditional constructive nonproductive assertion.*
- *The axiom of the choice set is a conditional nonconstructive productive assertion.*

The existence requirements of those axioms that are existence assertions are as in Proposition 12. The schemas are evaluated as in Proposition 13.

Proposition 41 completes the analysis of the axioms of ZFC. There are not only two or three classes, but six qualitative classes with quantitative distinctions within them. For instance, the distinction between the extensionality axiom and the power set axiom is a quantitative one, that is, a difference of degree: both are conditional constructive productive axioms. The extensionality axiom admits degree 1, but the power set axiom does not. Furthermore, the qualitative difference between the regularity axiom and the axiom of the empty set has been made clear.

This section closes with a brief remark on the inclusive logic of $\in$. Remark 14 provides some context and motivation for considering this topic.

Remark 42 (The inclusive logic of $\in$) According to the present analysis, the inclusive logic of $\in$ should be constituted by ETH or any other formal system for inclusive logic, the axioms of identity and equality, and the regularity axiom (as this last is the only principle of ZFC that admits a conditional degree zero of existence requirement).

7 Final Remarks

The existence requirement expressed by a sentence of set theory is a semantical property; it is not written in the symbols. This property can be evaluated when the validity of the sentence is tested against domains of existence. A gradation of existence requirements was introduced according to the closure degrees of the domains: the minimum degree of closure required for the validity of a sentence is a measure of its existence requirement. If the existence requirement of a sentence is nonzero, then this sentence is productive; it has the power of producing new sets under the condition that other sets are given.

Regarding a productive assertion, its constructivity is also a semantical property, and in order to evaluate it, the validity of the sentence expressing it should also be tested against domains of constructive productivity. These domains are identified in this paper with the relative constructible sets, and this leads to Definition 15 of a constructive productive assertion. A nonproductive assertion is always constructive according to the definition mentioned above.

Finally, the conditional or unconditional character of a degree of existence requirement is also a semantical property, and it is evaluated by testing the validity in the domain of conditional existence only, that is, in the empty domain.

These notions are relative to a background theory: a sentence can only be said to make an existence requirement if it is valid in the background theory. If a sentence is not valid in the theory under consideration, then there is no sense in speaking of its existence requirement. In order to avoid truth definitions for strong set theories, the degrees of existence requirements have been restricted to the theorems of the theory in question, and in this sense a proof-theoretic semantics was adopted. Nevertheless, it is also possible to work with a notion of validity more general than that of being a theorem.

The perspective adopted in the present paper is to be compared and contrasted with more syntactically oriented approaches to the notions of existence and constructive existence, such as the analysis set forth in [5] by Levy. It is enough to examine [5, Table I, p. 129] to confirm that Levy's analysis follows a syntactical path. With the exception of a few particular cases, the results exposed in Levy's paper are negative. This is an indication of the inadequacy of a syntactical perspective on existence and related notions.

An important consequence of the semantical formulation of the distinctions above is the fact that they are stable under the proper notion of consequence—a sentence on one side of the distinction has no consequences, in the inclusive logic, on the other side of the distinction. If the existence requirements associated to the axioms of a set theory T are bounded by d, then the existence requirements associated to the theorems of T are also bounded by d. In a constructive set theory, that is, a theory without nonconstructive productive assertions among its axioms, there are no nonconstructive productive theorems. The consequences of conditional existence axioms admit only conditional degrees of existence requirements. Here it is important to consider inclusive logic. Standard formulations of first-order logic have theorems that admit *unconditional* degree zero.

The final result is achieved in the section on conditional and unconditional degrees (Section 6). This is a classification of the theorems of ZFC (and simple extensions or subtheories) in six mutually exclusive classes: the conditional constructive productive assertions, the conditional nonconstructive productive assertions, the unconditional constructive productive assertions, the unconditional nonconstructive productive assertions, the conditional constructive nonproductive assertions, and the unconditional constructive nonproductive assertions. Theorems in different classes are qualitatively different from the point of view of existence in set theory. Within a given class, the theorems may have quantitative differences given by the degrees of existence requirement.

All these qualitative classes are populated by principles of set theory with different existence requirements. For example, the sentence expressing the existence of a measurable cardinal is an unconditional nonconstructive productive assertion in the theory with this generalized axiom of infinity. The axiom of the empty set is an unconditional constructive nonproductive assertion because, although it simply excludes the empty domain, it holds in every nonempty domain. The axiom of choice (set or function) is a conditional nonconstructive productive assertion. These six classes and the degrees of existence requirement are the most important notions introduced here, and the uniform distribution of the axioms in the classes and in degrees is an important property indicating that they are not accidental.

According to Definition 40, the extensionality axiom is an existence assertion, and this is in disagreement with the standard view. The criticism of the standard view presented in the introduction already pointed to the fact that the received view is based on insufficient grounds. In fact, this paper disputes the standard view, and although there is some agreement between this view and the present analysis, the results achieved here have followed a completely different path. Furthermore, as presented in the introduction, the usual view on existence axioms completely lacks clarification, and the usual explanations for the constructive character of the axioms are seriously flawed. Indeed, the usual characterization of constructivity in terms of set-theoretic definability (or instantiation by abstraction terms) gives the absurd result that there are existential theorems from logic that are nonconstructive. Also, the extensionality axiom is not constructive under this characterization: In general, there is no abstraction term for a witness of the difference of two sets in ZFC.[9] The problem with this characterization becomes more evident if one remembers that the extensionality axiom is the axiom that guarantees the uniqueness condition for the instantiated axioms. The constructivity of an axiom in the standard view is always based on a nonconstructive axiom (under the standard characterization).

The present analysis provided semantical definitions of existence and related notions. The motivation for the definitions of productive, constructive, and conditional character of a statement is simple: These are semantical notions and are evaluated when the validity of the statement is tested against the appropriate domains. Thus, the productive character of a statement, that is, the power of producing new sets from given sets, is evaluated when the validity of the statement is tested against arbitrary nonempty domains. This introduces a gradation of productivity corresponding to a natural hierarchy of closure properties for domains of existence in set theory. Similarly, the constructive character of the productivity of a statement, that is, whether the production of new sets from given sets is constructive, is evaluated when the validity of the statement is tested against the domains of constructive production of sets from arbitrary sets: the relative constructible sets. Finally, the conditional character of a statement is evaluated when the validity of the statement is tested against the domain of conditional existence, that is, the empty domain. The result of these definitions is an exhaustive classification of the valid sentences in set theory.

The motivation for Definition 40 of existence and nonexistence assertions is also clear. A sentence is an existence assertion if it is productive in nonempty domains or if it is unconditional and excludes the empty domain. The corresponding classification of the axioms in existence and nonexistence ones agrees with the usual view, with the exception of the extensionality axiom. This is an existence axiom in the present sense, although it is of the weakest kind: it admits conditional degree 1 of existence requirement.

It was mentioned in the introduction that the extensionality axiom may indeed have an existential use in mathematics. Suppose that two mathematical structures are given, one of them being a substructure of the other. One could prove that these structures are different by showing, for example, a topological property that holds in one structure and not in the other. Then one would conclude, by extensionality, that there is an element in the structure that is not in the substructure. This would be an existential use of extensionality in a mathematical proof.

A last word on the foundational relevance of the present results. The principles of set theory constitute the basis for the contemporary notion of mathematical existence. If the analysis presented here is indeed on the right track, and the usual view on these principles is not entirely correct, then it would constitute a contribution to the understanding of the notion of mathematical existence. Furthermore, the application of the notions introduced here to stronger set theories may have some bearing on the problem of new axioms for set theory that could also change the understanding of the notion of mathematical existence. At the very least, the present paper has shown that the usual view of the extensionality axiom is disputable. Since the extensionality axiom plays an essential role in set theory, this cannot be ignored.

Notes

1. Other systems related to ZFC will also be considered.

2. The exact definition of $\in$-interpretation of $L(ZF)$ in T can be found in Shoenfield's book [9, p. 261]. Shoenfield requires that T proves that the interpretation is nonempty. For the present purposes, it is preferable to include $\exists x U_I(x)$ as a hypothesis in each clause. Thus, for example, the first clause in Definition 1 should read $T \vdash \exists x U_I(x) \rightarrow A^I$. For the sake of legibility, the hypothesis that I is nonempty will be omitted. In what follows, whenever an $\in$-interpretation is specified, such as V_α, it is not necessary to indicate what extensions by definitions or by introduction of constants of ZFC are concerned.

3. In this item, and in the following ones, some $\in$-interpretations of $L(ZF)$ are specified, and, although it is implicit that the function symbol V_α is introduced, it is not necessary to indicate an extension of ZFC. That is the reason for writing $ZFC \vdash$ instead of $T \vdash$, for some specified T.

4. T is a simple extension of T' if they have the same language.

5. In what follows, whenever an $\in$-interpretation is specified, such as V_α, it is not necessary to indicate what extensions by definitions or by introduction of constants of T are concerned.

6. In this item, and in the following ones, some $\in$-interpretations of $L(T)$ are specified, and, although it is implicit that the function symbol V_α is introduced, it is not necessary to indicate an extension of T. That is the reason for writing $T \vdash$ instead of $T' \vdash$, for some specified T'.

7. In fact, the Rosser sentence for ZFC translates into a Δ_1-sentence in the Levy hierarchy (see [2, p. 160]).

8. For example, consider the existential sentence $\exists x A$ expressing the axiom of infinity. In the inclusive logic it holds that $\exists x A \rightarrow \exists x \top$.

9. This was already explained in the introduction: It suffices to take two sets X and Y constituted by choice functions for other complicated sets, and such that ZFC can prove $X \neq Y$. This is an important remark since, according to the present analysis, the extensionality axiom is an existence assertion.

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Acknowledgments

I am very indebted to Roderick Batchelor for several discussions on this subject. I am also indebted to an anonymous reviewer, to my postdoctoral supervisor Walter Carnielli, and to Hugo Mariano, João Schwarz, and Daniel Tausk. Author's work supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (Brazil), process 2009/10751-8.

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