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The Logic of Finite Order

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Abstract This paper develops a formal system, consisting of a language and semantics, called *serial logic* (**SL**). In rough outline, **SL** permits quantification over, and reference to, some finite number of things *in an order*, in an ordinary everyday sense of the word "order," and superplural quantification over things thus ordered. Before we discuss **SL** itself, some mention should be made of an issue in philosophical logic which provides the background to the development of **SL**, and with respect to which I wish to contend that the system permits progress.

1 Plural Interpretations of Second-Order Logic

It is by now familiar that there is a plural interpretation of monadic second-order logic (**MSOL**). This is first developed by George Boolos in [7] and [8]. In Boolos's initial presentation, English plurals are used to interpret **MSOL**. More recently, however, significant work has been done on formal plural logics,[1] and these may be used in preference to natural language for the interpretation of **MSOL**. For example, Rayo [24], [26] renders the key clauses of the Boolos interpretation in terms of the system[2] **PFO**:

$$\mathrm{Tr}'(X_j x_i) = x_i \prec xx_j, \tag{1}$$

$$\mathrm{Tr}'(\exists X_j.\varphi) = \exists xx_j.\,\mathrm{Tr}'(\varphi) \vee \mathrm{Tr}'(\varphi*), \tag{2}$$

where[3] $\varphi*$ is the result of substituting $x_i \neq x_i$ everywhere for $X_j x_i$ in φ.

Plural interpretations of **MSOL** are of interest for at least two reasons. First, they permit a second-order formulation of set theory to be undertaken while maintaining that the first-order variables of set theory range with unrestricted generality over all sets. If, as standardly thought, second-order variables range over sets of entities from the first-order domain, then this instance of the comprehension schema for **MSOL**,

$$\exists X \forall x X x \leftrightarrow x \notin x, \tag{3}$$

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commits us, given unrestrictedly general quantification, to the existence of the Russell set. Plural interpretations avoid this unwelcome commitment. By the lights of the Boolosian interpretation, (3) is committed to *the nonself-membered sets*, but not to a set which contains all and only those sets.

Another claim which is sometimes made on behalf of plural interpretations is that they safeguard the logicality of **MSOL** (see Linnebo [19]). What precisely is meant by this is a delicate question, and judgments as to what logicality consists in vary considerably from author to author. However, Linnebo supplies three criteria which he deems necessary (although not necessarily sufficient) for logicality, and these allow us to form an idea of what is being claimed when it is claimed that plural interpretations safeguard logicality. Note that even those authors who deny that there is a fact of the matter whether some system is logical[4] might agree that the features Linnebo identifies are of value and philosophical interest, such that plural interpretations may be worthy of investigation in spite of the fruitlessness of claims to secure logicality. Linnebo's criteria are the following:

- *Ontological innocence* The acceptance of a logic ought not to involve us in ontological commitments prior to the investigation of the nature of reality by extra-logical sciences.
- *Universal applicability* Logic is topic neutral, in that a logic can be "applied to any realm of discourse." Linnebo notes that second-order logic with the second-order variables interpreted as ranging over all subsets of the domain fails this test, since it cannot be applied to domains which are not set-sized.
- *Cognitive primacy* Logic can be immediately understood without necessary reference to ideas which are not themselves logical.[5]

The standard suggestion is that plural logic possesses these features, and that they are preserved across the interpretation, so that **MSOL** is shown to possess them by its interpretability in a plural logic.

It is not our purpose here to adjudicate the claims made on behalf of plural interpretations of **MSOL**. A limitation should be noted, however: plural interpretations are restricted to the monadic case,[6] and there is no immediately obvious way to extend them to encompass full second-order logic, with second-order variables of arbitrarily large finite adicity.[7] Boolos himself does not consider this a serious limitation, since most mathematically interesting theories have a pair function on their domain, and in this context we can simulate n-adic quantification as monadic quantification over n-tuples. But if our interests are more general than Boolos's purely mathematical ones, so that we want to be able to second-order quantify in contexts where our first-order variables can range over anything of any sort, we might be nervous about recourse to pairing—the assumption that the set-theoretic membership relation is absolutely general, which is required in order to allow that anything whatsoever can be a coordinate of a tuple, is not unproblematic (see Uzquiano [34]). In any case, if the proponent of a plural interpretation wishes to claim not simply that plural logic *can be used* to interpret second-order logic, but rather that the plural interpretation has a privileged status, shedding philosophical light on, and perhaps disclosing the meaning of, second-order locutions, there is an additional problem recognizably similar to that identified by Benacerraf [4] for the ontology of number. We are being invited to extend **MSOL** through the use of pair functions. But *which* pair functions? There are a multiplicity of implementations of pairing, all of them adequate. For example,

$f(x, y) = 2^x \cdot 3^y$ is a pair function for arithmetic, but so are $f'(x, y) = 3^x \cdot 5^y$ and $f''(x, y) = 859^x \cdot 1009^y$. For set theory, $g(x, y) = \{\{x\}, \{x, y\}\}$ is a pair function; but so is $g'(x, y) = \{\{\{x\}, \emptyset\}, \{\{y\}\}\}$. This abundance might be thought to be of not much importance. As long as we have some pairs to quantify over, does it matter which ones? It does if the appeal to pairs within **MSOL** is intended, not solely as a logico-mathematical device, but as offering some kind of insight into what we are *really* doing when—as we imagine—we quantify into polyadic predicate position. This point is crucial. Whatever we conclude about quantification into polyadic predicate position will tell us something about that position itself. Polyadic predicates are commonplace in our theories. Are we content to rest with the suggestion that they are indeterminate: "<" denotes either being among these Kuratowski pairs, or being among those Wiener pairs, and so on? If we are not, then we will have to look elsewhere than **MSOL** plus pair functions for full second-order resources.[8]

Recourse to sui generis tuples might be suggested here as a principled resolution. The thought here is that for any x and y, not necessarily distinct, $\langle x, y\rangle$ exists; but the pair does not have a birthday at any stage in the cumulative hierarchy, nor is it any other kind of mathematical object,[9] if these are not identified with sets. It is not altogether clear that this navigates successfully around the Benacerraf-style difficulty; even if there are sui generis pairs, these are no *more* adequate to the task than, say, implementations of pairing using cumulatively generated sets, so what is the basis for declaring the sui generis pairs to be the ones picked out by our second-order discourse?[10] Even if this worry is abated, the invocation of sui generis pairs sins against one of Linnebo's criteria for logicality. If (polyadic) second-order quantification is over tuples, then it is not ontologically innocent. Both the initial proposal that polyadic quantification be read as monadic quantification over pairs, and the modification of this by allowing that pairs be sui generis, fall at this hurdle.

There is reason, then, to explore ways of extending the plural interpretation to full second-order logic which do not make use of pairing.[11] The main proposal of the present paper is that **SL** provides us with such a way.

2 Motivational Preliminaries

I will develop a recognizably plural reading of **SOL** using (**SL**). Serial logic may be thought of as the logic of *finite order* in the following sense: **SL** permits quantification over some (finite number of) things *in a certain order*; first this one, then that one, and so on. Call some things in an order[12] a *seriality*, where this usage should not be understood as reifying in import. A seriality is just some things in an order. It is not, for example, a set or any other kind of collective or plural entity. In fact, **SL** permits plural quantification over serialities, this being a special form of superplural quantification.[13] Before developing **SL** formally, a few points of business are in order. Presently, I shall supply motivation for the thought that **SL** is properly considered logical, with an eye to the desiderata outlined in the previous section. Before that, it is necessary to get more precise about the plural notion of *some things*, which we have just invoked to introduce the idea of a seriality.

2.1 Liberalizing extensionality Call *some one or more things together* a *plurality*.[14] In standard treatments of plural logic, pluralities are individuated by a simple principle of extensionality:

$$\forall xx \forall yy(xx \approx yy) \leftrightarrow \forall x(x \prec xx \leftrightarrow x \prec yy), \qquad \text{(PL-EXT)}$$

where "$\approx$" reads "are the same things as" and "$\prec$" reads "is one of."

I do not deny for one moment that (PL-EXT) describes *a* plural notion which is properly thought of as logical. I do deny that it describes the only such notion. There is another one, which validates only the left–right direction of (PL-EXT), giving

$$\forall xx \forall yy(xx \approx yy) \rightarrow \forall x(x \prec xx \leftrightarrow x \prec yy). \qquad \text{(LIB-EXT)}$$

On the understanding of pluralities expressed by (LIB-EXT), there may be *non-trivial repetition* of objects in pluralities. A full statement of the principle of sameness governing pluralities thus understood would express that *xx* and *yy* are the same iff every object is included n times in *xx* iff it is included n times in *yy*, for every $n \in \mathbb{Z}^+$. It is convenient to enrich the vocabulary of a plural logic founded on this understanding with some dyadic predicates $\ulcorner \prec_n \urcorner$ which read "is included n times in." Motivation for this liberalization of extensionality may be had through a consideration of the natural language phenomena of *unordered lists*. Examples of lists simpliciter are easy to come by in everyday language and thought. Caution is required, though, since many of these appear to impose some kind of ordering on the items denoted by the nouns occurring in them ("the first five members of the Fibonacci sequence are 0, 1, 1, 2, and 3"). In the next section, we will develop a logic which formalizes this form of plural denotation. For the time being, we do not want to imply ordering. Rather the form of plural denotation which motivates our relaxation of extensionality is that found in "the first five members of the Fibonacci sequence, but not in order, are 1, 3, 2, 0, and 1." I take it that this makes perfect sense, and that quantification over things in plurality, given such relaxation, does not stray from logicality.[15]

It is an interesting question whether a logic of pluralities with this liberalized principle of extensionality has any applications (beyond the regimentation of certain natural language forms). Investigation would take us too far afield, though; our noting that "some things" need not be understood in the way suggested by (PL-EXT) is merely a stepping stone on our way to **SL**. Having clarified the sense of "some things" in which a seriality is some things in an order, let us now consider the basis for a logic which permits quantification over, and reference to, some things qua serialities.

2.2 Serialities First, I wish to suggest that talk about serialities is correctly understood as a form of plural talk. This will ultimately secure the continuity of our interpretation of **SOL** with the Boolosian treatment of **MSOL**. To remove any ambiguity about what is being claimed for seriality talk when it is said that it is plural, I mean here to impute both syntactic and semantic plural features to the natural language analogues of my subsequently developed formal talk of serialities, so as to justify claiming the same features of it. By syntactic plural features, I mean belonging grammatically with plural (rather than singular) forms. By semantic plural features, I mean that the salient terms each denote some entities together rather than a single entity (a set-theoretic tuple, or a plural property).[16] It is not hard to support the syntactic part of this position with respect to some *serial terms*—noun phrases which refer to some things in an order. Consider here the already used example of "the first five Fibonacci numbers are 0, 1, 1, 2, and 3"; similarly, "the top three teams in the Scottish premiership on April 5th 2011 were Celtic, Rangers, and Hearts," and "we came here via (in order) Peterborough, York, and Newcastle." In each of these cases the list is a syntactically plural noun phrase (with the commas and *and*'s functioning as term-forming operators on singular terms), as, in the first two cases, is the description to the left of the relevant part of the verb "to be" (although, for Russellian reasons, I would want to deny that this is a *term*). Our ultimate interest will be not just (or primarily) in serial terms in general, but in serial variables, both free and bound, in particular. Are there any examples of natural language analogues to these? Sometimes the plural demonstratives "those" and "these" give the appearance of being serial: "these are the three prizewinners in the largest pumpkin competition"; "those are the stages of the surgical procedure." A likely note of dissent to be sounded at this point is that, while these demonstrative usages are certainly syntactically plural, they are simply not serial. According to the dissenter, "these" in the pumpkin competition case refers plurally to the prizewinning pumpkins, but not in any order. Any imputation of order is generated by context—if I say "these are the three prizewinners in the largest pumpkin competition" and gesticulate toward the vegetables, which are arranged in the order of the prizes, then it may well be that I have communicated to you that *these* pumpkins won the prizes in *that* order, but this is not in virtue of the logical function of the demonstrative. So it would be argued. By way of response, while one might concede (for the sake of argument, at least) that matters are marginal in the case as precisified by the dissenter, uses in other contexts favor the view that demonstratives may be serial. Think about an anaphoric use: "Ms. Smith's, Professor McTaggart's, and Mr. Singh's; *those* are the three prizewinners in the largest pumpkin competition."[17] Here, "those" surely inherits the ordering of the serial noun phrase it is anaphoric for. In the logic we will develop (enhanced with a serial description operator "$\overrightarrow{\iota\iota}$," which can be understood as an abbreviation) we will be able to express the logical form of the claim made here (this should be clear enough in broad terms, even to those not yet familiar with **SL**):

$$\overrightarrow{aa} \bowtie \overrightarrow{\iota\iota}\,\overrightarrow{xx}\varphi(\overrightarrow{xx}). \tag{4}$$

Perhaps the serial nature of some anaphoric plural demonstratives is even more apparent in cases of dispute:

> **Plato**: *OK Computer*, *The Bends*, and *Kid A*; those are the top three Radiohead albums.
>
> **Socrates**: Nonsense. The guitar melodies in *The Bends* make that album better than *OK Computer*.

Plato's utterance here looks not only serial, but serially quantificational (as a first step toward regimentation: "There are some things ([in an order]; they are *OK Computer*, *The Bends*, and *Kid A*, and they are the top three Radiohead albums").

It is one thing to allow that there are such things as serial denotation and quantification. It is another thing altogether to admit that they are not dependent on mathematical or other resources, either ontologically or cognitively, in such a way that their use to interpret **SOL** would represent anything more than a recreational curiosity. In particular, it might be thought a tall order to defend the position that serial forms are part of logic, rather than an outpost of significant mathematics in relatively everyday thought and language. The response one anticipates to the contrary stance, and certainly the one I have encountered when trying to articulate it in the course of the research giving rise to the present paper, is that—whatever one might say about unordered plurals, or even higher-level plurals—serial discourse is either concerned with set-theoretic, or other mathematical, objects, or depends cognitively somehow on mathematics (the theory of the ordinals, perhaps, or combinatorics), or both. Upon encountering this response, the advocate of logicality should question who the burden of proof belongs to here.[18] After all, if I talk (as we are prone to say pre-theoretically) about three prizewinning pumpkins in an order, it is surely plausible that I am talking about the pumpkins themselves, and nothing but the pumpkins. I am not talking about a tuple of pumpkins; nor am I talking about any other kind of mathematical object, or about some kind of plural property or state of affairs. Exactly the same intuitions that so many philosophers find compelling in Boolos's Cheerios case[19] with respect to simple plurals transfer to the serial case. As for cognitive dependence on extralogical resources, are we really to believe that in simply listing some finite number of things in an order, or in quantifying over things thus ordered, I am engaged in anything mathematically significant? Is it not more likely that a conceptually basic, and logical, grasp of order precedes and motivates the mathematical development of advanced combinatorial and ordinal theories? It is important to make a distinction here: there are notions of order which are uncontroversially dependent on substantial advanced mathematics. On the one hand, once the nature of the ordering under consideration ceases to be the simple one present in serialities (which is, apart from the possibility of nontrivial repetition, a well-order[20]), then we enter the province of the various mathematical theories of order, typically explored set-theoretically, and importantly instanced in the likes of preorders and lattices. On the other hand, the theory of even a well-order in cases where the number of things being ordered is transfinite could well be thought to require resources from outside of logic. That such things may be well-ordered is not trivial. That this is possible requires that they be (under some relation) isomorphic to some ordinal under its standard ordering, and that there will always be such an isomorphism is equivalent to the set-theoretic axiom of choice. While some sort of choice principle might properly be regarded as logical (there are choice axioms in standard deductive systems for **SOL** and in some plural logics), there may be a suspicion that a theory of transfinite order is unavoidably cognitively dependent upon the set-theoretic theory of ordinals, or some theory of equivalent power, and so cannot be part of logic. I make no attempt to resolve here whether or not this is the case.[21] I do propose, however, the following diagnosis of why there is a seeming reluctance to admit serial talk as logical: recognizing that there *is* thought and language concerning order which transgresses the boundaries of what may properly be called "logical," philosophers overreact by admitting *no* thought or language of this kind as logical. That this is an overreaction is apparent from the answerability of concerns about ontological commitment and cognitive primacy, the two likely bases for attack on the logicality of serial talk.

2.3 Plurally serial quantification **SL** makes use not only of serial resources, but also of *plurally serial* quantification (and reference). This is a special case of superplural quantification, which is the subject of some discussion in the literature on plural logic (see [20], [19], [25]). The present setting is not apposite for supplying a detailed defense of superplural quantification, and its status as a bona fide part of logic. Some brief remarks are in order, though; there have been attempts to identify superplural locutions in natural language, notably [20] and [21]. Although suggestive, too much should not be invested in the decisiveness of these researches. The occurrence of some form in a suitable fashion in natural language might very well serve as evidence for cognitive primacy, and so for logicality. But if no such occurrence is forthcoming upon examination of natural languages, the supporter of superplurals need not despair. Linnebo observes that the legitimacy of superplurals can be argued for in isolation from any natural language evidence:

> What really matters is presumably whether we can iterate the principles and considerations on which our understanding of ordinary first-level plural quantification is based: if we can, then higher-level plural quantification will be justified in much the same way as ordinary first-level plural quantification; and if not, then not. Thus, even if there were no higher-level plural locutions in natural languages, this would at best provide some weak prima facie evidence that no such iteration is possible. And this prima facie evidence could be defeated by pointing to independent reasons why higher-level plural locutions are scarce in natural languages. One such independent reason may simply be that ordinary speakers aren't very concerned about their ontological commitments and thus find it more convenient to express facts involving second-level pluralities by positing objects to represent the first-level pluralities (for instance by talking about two pairs of shoes) rather than by keeping track of additional grammatical device for second-level plurals. [20]

Allowing superplurals in general, can we admit plurally serial forms, which stand to serial forms as ordinary superplurals do to plurals? Enough has been said already to justify their admission since, as with superplurals *simpliciter*, they result from the iteration of principles which we have accepted independently as logical. Nevertheless, it is not overly difficult to imagine examples of plurally serial quantification in natural language. Return in your thoughts to the village show in which the earlier-mentioned largest pumpkin competition occurs. Now suppose that I wish to talk about the prizewinners in a number of competitions, not only the largest pumpkin competition, but also the most elegantly decorated Victoria sponge competition, the oddest-looking carrot competition, the floral arrangements competition, and so forth. Specifically, I wish to talk about the prizewinners in several of those competitions at once, and I wish to do so *while preserving the orderings of the prizewinners*, for reasons that will be obvious when one considers the utterance "some lots of winners of various competitions stood on some podiums." Podiums are, after all, ordered—the first prizewinner standing higher than the second, and the second higher than the third—and I wish to communicate that the prizewinners stood on the podiums *in the correct order*. I submit that the concept "prizewinners," and the phrase "some lots of winners," have the desired serial content, at least as used in many contexts, including the one under present consideration. And so the utterance, for all its undeniable inelegance, should be understood as involving plurally serial quantification.[22]

3 The Logic SL

These motivational preliminaries completed, let us go on to examine the logic **SL**. For immediate purposes, "a / the logic" (as in "the logic **SL**") will mean simply a formal language together with a semantics. It would be interesting to develop a proof system for **SL**, but this will have to be a topic for future research.

The logic we develop is similar in some respects to the logic of vectors that Hazen and Taylor present in [33]. There are important differences, though. (Readers may wish to defer considering these until after the formalism of **SL** has been laid out.) Serial variables, unlike vector variables, are typed according to the size of the serialities they take as values. The absence of such typing serves well certain philosophical purposes of Hazen's and Taylor's (see here also Hossack's invocation of vector logic in developing his metaphysics of facts [14, pp. 47–53])[23]. It does, however, make the resulting logic less tractable, and serves our present purposes less well. Hazen and Taylor also do not spell out the intended ontology of their system. I am clear that serialities are to be understood as a case of pluralities, consisting of nothing over and above the things that are in them. The distinction (in the finite case) between some things *in an order* and those things *simpliciter* is one in thought and manner of reference, not in underlying ontology.

We begin our exposition by laying out the language of **SL**.

3.1 The language $\mathscr{L}_{\mathbf{SL}}$

3.1.1 Lexicon The lexicon of the language $\mathscr{L}_{\mathbf{SL}}$ contains the following items:

(1) All of the items in the lexicon of first-order logic with identity (**FOL**$^=$) other than nonlogical predicates, including constants.
(2) For all $n \in \mathbb{Z}^+$, denumerably many n-place *serial variables*, $\ulcorner \overrightarrow{xx}^n_1 \urcorner$, $\ulcorner \overrightarrow{xx}^n_2 \urcorner$, . . . , we frequently omit the superscript, and permit ourselves the use of "$\overrightarrow{yy}$," "$\overrightarrow{zz}$," and so on.
(3) For all $n \in \mathbb{Z}^+$, denumerably many n-place *serial constants*: $\ulcorner \overrightarrow{aa}^n_1 \urcorner$, $\ulcorner \overrightarrow{aa}^n_2 \urcorner$, . . . , we frequently omit the superscript, and permit ourselves the use of "$\overrightarrow{bb}$," "$\overrightarrow{cc}$," and so on.
(4) For all $n \in \mathbb{Z}^+$, a special dyadic predicate $\ulcorner \ltimes_n \urcorner$. This will be read "is the nth member of."
(5) The special dyadic predicate "$\bowtie$." This will be read "is the same seriality as."
(6) For all $n \in \mathbb{Z}^+$, denumerably many plurally serial variables over n-place serialities, $\ulcorner \overbrace{xx^n_1} \urcorner$, $\ulcorner \overbrace{xx^n_2} \urcorner$, . . . , We frequently omit the superscript, and permit ourselves the use of "$\overbrace{yy}$," "$\overbrace{zz}$," and so on.
(7) For all $n \in \mathbb{Z}^+$, denumerably many plurally serial constants over n-place serialities, $\ulcorner \overbrace{aa^n_1} \urcorner$, $\ulcorner \overbrace{aa^n_2} \urcorner$, . . . , we frequently omit the superscript, and permit ourselves the use of "$\overbrace{bb}$," "$\overbrace{cc}$," and so on.
(8) The special dyadic predicate "$\overbrace{\prec}$," which will signify the inclusion of a seriality in a plural seriality.
(9) The special dyadic predicate "$\overbrace{\approx}$," which will signify sameness of plural seriality.

3.1.2 Formation rules Next we go on to specify the well-formed formulas (wffs) of $\mathscr{L}_{SL}$.

We do this recursively. First we state the following.

[(1P)] Every wff of the fragment of **FOL**$^=$ which does not include nonlogical predicates is a wff.

Then we go on to describe those atomic wffs which are not such in virtue of (1P). Here, and throughout the specification of the formation rules, t is a metavariable ranging over object language terms (constants or variables), $\overrightarrow{tt}$ (and $\overrightarrow{uu}$ etc.) over serial terms, and $\overbrace{tt}$ over plurally serial variables. Only variables may be bound by quantifiers, so in the rules involving quantifiers the range of the term metavariables should be understood as restricted accordingly:

(1A) For all $n \in \mathbb{Z}^+$, and all $m \leq n$, $\ulcorner t \ltimes_m \overrightarrow{tt}^n \urcorner$ is a wff.
(2A) $\ulcorner \overrightarrow{tt} \bowtie \overrightarrow{uu} \urcorner$ is a wff.
(3A) $\ulcorner \overrightarrow{tt} \overbrace{\prec} \overbrace{tt} \urcorner$ is a wff.
(4A) $\ulcorner \overbrace{tt} \overbrace{\approx} \overbrace{uu} \urcorner$ is a wff.

Next, we move on to the molecular wffs:

(1M) If φ is a wff with no bound occurrence of $\ulcorner \overrightarrow{tt} \urcorner$, then $\ulcorner \exists \overrightarrow{tt} \varphi \urcorner$ is a wff.
(2M) If φ is a wff with no bound occurrence of $\ulcorner \overbrace{tt} \urcorner$, then $\ulcorner \exists \overbrace{tt} \varphi \urcorner$ is a wff.
(3M) If φ is a wff with no bound occurrence of $\ulcorner t \urcorner$, then $\ulcorner \exists t \varphi \urcorner$ is a wff.
(4M) If φ and ψ are wffs, then $\ulcorner \varphi \rightarrow \psi \urcorner$ is a wff.
(5M) If φ is a wff, then $\ulcorner \neg \varphi \urcorner$ is a wff.

The universal quantifier, and the connectives other than negation and the conditional, are defined as abbreviations in the usual fashion. It remains to state the closure of the wffs of $\mathcal{L}_{\mathbf{SL}}$ under these rules:

[1C] No string formed other than in accordance with (1P), (1A) to (4A), or (1M) to (5M) by a finite number of applications is a wff.

Having characterized the wffs of $\mathcal{L}_{\mathbf{SL}}$, we may proceed to specify an informal semantics for **SL**.

3.2 Informal semantics for SL Given a domain $\mathcal{D}$, not necessarily set-sized,[24] we can sketch out the intended semantics for **SL** as follows. Serial variables range over serialities on $\mathcal{D}$, where a seriality is some finite number of objects in an order. So, for example, where $\mathcal{D}$ consists of a, b, and c, and where we denote serialities by enclosing a list of their members between two occurrences of "‖", we have that $\|a, b, c\|$ is a seriality on $\mathcal{D}$, as are $\|c, a, b\|$ and $\|a, a, b\|$. Note that there will never be fewer than denumerably many serialities on any given domain. Even the domain whose sole member is a has on it the serialities $\|a\|$, $\|a, a\|$, $\|a, a, a\|$, and so on. In **SL**, serial variables range over every seriality on the domain. Importantly, no more than "plurality," should "seriality" be understood as denoting a special kind of object (a set-theoretic tuple, for example); instead, a seriality is just some things in an order.

Plurally serial variables take as each of their values *some serialities* on $\mathcal{D}$. So, continuing to focus on our example domain of three objects, one plurally serial variable might be assigned $\|a, a\|$ and $\|b, a, c\|$ on some valuation; another might be assigned $\|b, b, c, a\|$, $\|a, b, c\|$, and $\|b\|$, and so on. For any given domain, there will never be fewer than continuum-many values available for the plurally serial variables.[25] In **SL**, plurally serial variables range over every collection[26] formed from any seriality on $\mathcal{D}$. We can now give truth clauses for the distinctive formula types of $\mathcal{L}_{\mathbf{SL}}$. Define an **SL**-valuation v to be an assignment to every singular variable an object from $\mathcal{D}$, to every n-place serial variable an n-place seriality on $\mathcal{D}$, and to every plurally serial variable over n-place serialities, some n-place serialities on $\mathcal{D}$. In the background is an **SL**-interpretation, which assigns to every individual constant an object from $\mathcal{D}$, to every n-place serial constant an n-place seriality on $\mathcal{D}$, and to every plurally serial constant for n-place serialities, some n-place serialities on $\mathcal{D}$.

We use $\ulcorner v(\tau)\urcorner$ to denote the object, seriality, or serialities assigned by v, or its background interpretation, to the lexicon item τ. The truth clauses follow (bivalence is assumed)[27]:

- $\ulcorner t \ltimes_m \overrightarrow{tt}^n \urcorner$ is true iff $v(t)$ is the mth among $v(\overrightarrow{tt}^n)$;
- $\ulcorner \overrightarrow{tt} \bowtie \overrightarrow{uu} \urcorner$ is true iff $v(\overrightarrow{tt})$ is the same seriality[28] as $v(\overrightarrow{uu})$;
- $\ulcorner \overrightarrow{tt} \overbrace{\prec} \overbrace{tt} \urcorner$ is true iff $v(\overrightarrow{tt})$ is among $v(\overbrace{tt})$;
- $\ulcorner \overbrace{tt} \overbrace{\approx} \overbrace{uu} \urcorner$ is true iff the serialities among $v(\overbrace{tt})$ are all and only the serialities among $v(\overbrace{uu})$;
- $\ulcorner \exists \overrightarrow{xx} \varphi \urcorner$ is true iff φ is true in some **SL**-valuation which differs from v at most with respect to $\ulcorner \overrightarrow{xx} \urcorner$;
- $\ulcorner \exists \overbrace{xx} \varphi \urcorner$ is true iff φ is true in some **SL**-valuation which differs from v at most with respect to $\overbrace{xx}$.

The truth clauses for the connectives and for the formulae types shared with **FOL**$^=$ are the obvious ones, and we omit them.

3.3 Model-theoretic semantics for SL In passing we should note the availability of a model-theoretic semantics of the usual sort for **SL**. Here one must be careful to distinguish between the two senses of "semantics" in order not to be misled with respect to the philosophical motivation for the development of **SL**.[29] Providing a model-theoretic semantics for **SL** might be a convenient means, for certain mathematical and metalogical purposes, to model the conditions under which the wffs of the logic are true. It is badly misleading, however, to suppose that the existence of a set-based model theory for **SL** licenses any conclusions about the commitments of serial quantification, or about the meaning of formulae involving such quantification. It does not follow from the mere fact that I can produce an **SL**-model which assigns, say, ordered tuples to serial variables, that when I quantify serially (as I naively put it) over some domain, I am *really* quantifying not over the items in that domain, but over set-theoretic tuples of those items.

This cautionary note having been sounded, it is entirely straightforward to lay out the model theory. Let an **SL**-model be $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$, where $\mathcal{D}$ is a nonempty set and $\mathcal{I}$ is a function from the individual constants of $\mathcal{L}_{\mathbf{SL}}$ to $\mathcal{D}$, the n-place serial constant of $\mathcal{L}_{\mathbf{SL}}$ to $\mathcal{D}^n$, and plurally serial variables to $\wp(\mathcal{D}^n)$. In terms of valuations, n-place serial variables are assigned elements of $\mathcal{D}^n$, and plurally serial variables are assigned elements of $\wp(\mathcal{D}^n)$. The details of valuations, as well as the definitions of truth, consequence, and validity in **SL**-models are unproblematic, and do not require detailed coverage here.

4 The Interpretation of Full Second-Order Logic

We can interpret **SOL** using **SL**. By analogy with the Rayo formalization of monadic case,[30] we have the following clauses:

$$\mathrm{Tr}'(X^n_j x_{i1} \ldots x_{in}) = \exists \overrightarrow{xx^n_i}(x_{i1} \ltimes_1 \overrightarrow{xx^n_i} \ldots \wedge x_{in} \ltimes_n \overrightarrow{xx^n_i}) \wedge \overrightarrow{xx^n_i} \overbrace{\prec} \overbrace{xx_j}, \tag{5}$$

$$\mathrm{Tr}'(\exists X^n_j . \varphi) = \exists \overbrace{xx_j} . \mathrm{Tr}'(\varphi) \vee \mathrm{Tr}'(\varphi *), \tag{6}$$

where $\varphi *$ is the result of substituting $x_{i1} \neq x_{i1}$ everywhere for $X^n_j x_{i1} \ldots x_{in}$ in φ.

As before, we deal with atomic predications involving a (nonlogical) *predicate* in the obvious way, this time invoking plurally serial constants. Two things should be noted immediately. First, we can tidy up the interpretation by treating the monadic case as a limiting case of the polyadic, as in the following instance of (5):

$$\mathrm{Tr}'(X^1_j x_i) = \exists \overrightarrow{xx^1_i}(x_i \ltimes_1 \overrightarrow{xx^1_i}) \wedge \overrightarrow{xx^1_i} \overbrace{\ \prec\ } \overbrace{xx_j}\ . \tag{7}$$

Here we are using a one-membered seriality to stand proxy for its member, relative to (1), and similarly using a serial superplurality, consisting of some one-membered serialities, to stand proxy for a simple plurality. I will defend in a moment the philosophy of predication that falls out of this as consonant with a plural account of predication. Before that, something should be said about (5). We have here a schema, with instances for every natural number ≥ 2. There is no serious issue about comprehensibility here—strictly speaking, we have denumerably many clauses in our translation scheme, but there is an effective procedure for generating the $(n + 1)$st instance, given the nth.

4.0.1 Logical consequence A more ambitious project is to provide a thoroughgoing metatheory for **SOL** using the resources of **SL**, including, importantly, an analysis of logical consequence. Second-order consequence is fraught with philosophical difficulties. Two issues are especially pressing: the *overgeneration problem* and the question of whether an account of consequence can be forthcoming for a *non-set-sized domain*. These are dealt with extensively in the literature. On overgeneration see Shapiro [29, pp. 105–6], Jané [15], and Etchemendy [10], and on domain size see Rayo and Uzquiano [27]. Boolos provides a plural-based account of validity for the second-order set theory in [7]. This lacks an analysis of logical consequence. Rayo and Uzquiano repair the defect in [27]. Their approach, which is impressive, is nonetheless confined to the case of second-order set theory. It turns out that a full metatheory for **SOL**, including accounts of truth and consequence, *can* be developed on the basis of the **SL**-interpretation. I postpone presenting this until a subsequent paper.

4.0.2 Inter-translatability **SOL** and **SL** are inter-translatable in the following sense: there is an effective procedure for transforming each wff of the language of each logic into a unique wff of the language of the other.[31] The proof of this is deferred until an appendix.

5 Philosophical Issues Arising

5.1 A general analysis of predication Returning for the purposes of comparison to plural interpretations of **MSOL**, one criticism which has been directed against these is that the account of predication implicit in Rayo's clause (1), and Boolos's English equivalent, is unattractive. We are being invited to understand (monadic) atomic predication in terms of plural inclusion, the relation *is one of*. But, as Timothy Williamson articulates the worry,

> plurals are not predicative: the expressions 'red things' and 'is red' belong to grammatically distinct categories, plural N and VP respectively. To read atomic '*Px*' as 'it is one of them' is to impose more structure than appears to be present in the object language. [36, p. 67]

Other worries suggest themselves thick and fast. Many writers on plural logic accept the following principle,

$$x \prec xx \rightarrow \Box x \prec xx, \tag{NecInc}$$

whereas the following is clearly false:

$$Xx \rightarrow \Box Xx. \tag{NecInc}$$

But the Boolosian clause plus (NecInc) force (NecPred) upon us.

My current concern is not to undertake a point-by-point defense of a plural understanding of predication; I go some of the way toward that in a paper in preparation. For now, I will raise the possibility of a minimalist response to these objections to plural interpretations of **MSOL**, before going on the offensive and outlining positive reasons to accept a plural account of predication and, crucially, discussing whether these reasons extend naturally to the account of (n-adic) predication implicit in our **SL**-interpretation of **SOL**.

5.1.1 Minimalism about the plural interpretation The path of least resistance against the opponent of a plural analysis of predication denies that the plural interpretation of **MSOL** supplies any such analysis. For the *minimalist*, the plural interpretation is an interpretation only in a very specific sense; nothing deep is disclosed about the nature of predication, but rather the translatability of monadic second-order logic into a plural system tells us about equivalence of expressive resources and the availability of plural logics in some context where second-order systems have been used standardly. An important case in point is the foundations of mathematics. For example, there is a plural statement of the principle of mathematical induction:

$$\forall xx(0 \prec xx \wedge \forall y[z \prec xx \rightarrow y' \prec xx]) \rightarrow \forall yy \prec xx. \tag{PL-IND}$$

The result of adding this to the first-order axioms of PA is, given the assumption that plural variables range over every plurality on the domain (called by Rayo "fullness"), a categorical characterization of the natural numbers.

A clear statement of the minimalist stance is provided by MacKay: "For some purposes, a plural first-order language will have some of the advantages that are ordinarily associated with second-order logic" [22, p. 139]. Minimalism provides a refuge if objections to a plural analysis of predication prove insurmountable, both in the case of **MSOL** and of **SOL**. In other words, there is a minimalist reading of the present paper which takes our achievement to consist in showing that **SL** is available for the same foundational purposes as **SL**. Suppose that we are not content to rest with minimalism, though. What can be said in favor of a plural analysis of predication? We continue discussing the case of monadic second-order logic, plurally interpreted.

5.1.2 The maximalist case for *the plural interpretation* Why might one be attracted to the claim that predication should be analyzed in terms of plural inclusion? A major advantage of the view over all accounts other than the Quine–Davidson deflationary view, which understands predicational in a purely disquotational view in the context of a truth theory for a language, the plural account accrues the benefit of lacking distinctive metaphysical commitments. Consider some alternatives: a rival theorist of predication might claim the following: "Fa" is true iff$_{df}$ the particular a instantiates the universal F. A second rival might retort that "Fa" is true iff$_{df}$ the saturated object a falls under the unsaturated concept F. Yet another theorist might propose that "Fa"

is true iff$_{df}$ the object a has an F trope among its mereological constituents. And another would venture that "Fa" is true iff$_{df}$ a is an element of the set of Fs. Each of these competing theories involves our philosophy of logic in significant metaphysical commitments.[32] The plural understanding avoids such a commitment, and this is a virtue.

The motivation for this attribution of virtue is not nominalism—although a plural account of predication would be of value to a nominalist—but a recognition that we do well to afford ontologists more freedom in their assays, instead of requiring that they postulate a universal (or a concept, or a type of trope, etc.[33]) for every predicate. It is vital to be aware in this context that predicates are abundant. According to standard semantics, there are 2^{κ} available predicate values, where κ is the cardinality of the universe—although, given that the predicate letters are countable, there will not be *that* many predicates in the mathematically interesting transfinite case—but it strains credibility to claim that the universals[34] are that many in number. Universals, at least for most of their contemporary proponents, are sparse. They correspond to the perfectly natural properties, real respects of resemblance in nature which feature in laws of nature. The canonical expression of this view is by Armstrong (see [1], [2], [3]), who explicitly denies that the universals can be assayed by surveying the predicates. From a perspective neutral between competing accounts of the nature of perfectly natural properties, Lewis also marks carefully the distinction between sparse and abundant properties—sparse properties being natural respects of resemblance; abundant properties being the values of predicates (see [18]).

If we understand predication plurally, we set the ontologist free to assay the universals according to her own proper methods—attention to the data of the natural sciences, to a theory of causation, or whatever—instead of demanding that she read the structure of the world off the structure of a language. Metaphysics gains a flexibility: for some xx, perhaps it is true that "$a \prec xx$" iff a instantiates a particular universal. For other xx there will be no such equivalence. To acknowledge this is to say of pluralities what Stout said of classes: some of them have a *distributive unity*, others do not (see [32]). The things of which I am prepared to say "has electric charge $+\frac{2}{3}$" could well have a distributive unity; the things of which I am prepared to say "is grue" are less promising candidates. Reading predication as the attribution of plural inclusion allows us to remain neutral about the metaphysics of any particular case, while acknowledging that there are cases where a plurality has distributive unity, and yet retaining an account of predication which is uniform between the joint-carving and gruesome cases. This is exactly what we want from an account of predication, and that the plural account delivers it is a major point in its favor.

5.1.3 Maximalism and the ***SL****-interpretation* The foregoing justification of a plural account of predication attended exclusively to standard plural-based accounts of *monadic* second-order logic; "predication" in the previous subsection means monadic predication. But we have an interpretation of *full* **SOL** on the table, implicit in which is an account of polyadic predication, encompassing monadic predication as a limiting case. Two questions arise. First, is the account of predication implicit in the **SL**-interpretation defensible philosophically? Second, if the defense of a certain plural account of monadic predication above was compelling, how can we be justified in replacing this with a *new* account, implicit in (5)? Tackling this last question first, the argument of above supports a plural account of predication. We are still proposing a plural account of monadic predication. It is simply that when only **MSOL** was being considered, there was only one obvious such account. Once we generalize to the polyadic case, decisions have to be made—is the account already discussed above the correct one for the monadic case, and therefore should we develop a separate philosophical case for the implicit account of n-adic predication for $n \geq 2$? Or should we hold it correct to apply (5) to the monadic case as well, and seek to justify the account of predication implicit therein as applicable generally? I advocate the second course of action. It is arbitrary to exclude the monadic case from a general theory, however much the temptation to do so might be fueled by an undermotivated privileging of monadic predication present at times in the Western philosophical tradition.[35] A theory which at once adequately analyzes *all* predication has the virtues of elegance and simplicity, and so commands our assent (in preference to the disjunctive theory) by the usual criteria of theory choice. Immediately the next issue arises: is the general theory on offer, that implicit in (5), deserving of our assent?

What does the theory at issue say? In logicians' English, to say that $X^n_j x_1 \dots x_n$ is to say that there is some n-adic seriality consisting of $x_1 \dots x_n$, and that this seriality is among the X^n_j serialities. An example is in order;[36] when I say that Edinburgh is north of London, I say (a) that there are some *things in an order* which are Edinburgh and London, *in that order*, and (b) that those things in that order are among the lot of *things in orders* which are the *things in orders* which we designate with "is north of." In this case, these *things in orders* are of clear geographical interest to us; we might naturally say that plurally serial talk of the "to the north of" serialities is relatively joint-carving. As in the simple monadic case of the previous chapter, however, it is easy to construct gruesome polyadic predicates; on the present account these correspond to some serialities which together are not joint-carving. And, as in the monadic case, it is to the account's advantage that it admits such predicates (and, more generally, values of polyadic second-order variables) without insisting that they come with ontological commitments to abundant universals or suchlike. Independent of this benefit, the account has an inherent plausibility. Once one has unshackled oneself from the dogma that predicates pick out, and second-order variables range over, some special kind of one-over-many object (whether relations, or concepts, or sets), the obvious recourse is to analyze predication in terms solely of the objects satisfying the predicate, or (for a given valuation) variable, under consideration. This is the controlling thought behind plural interpretations, understood maximally. **SL** provides us with the simplest way of doing this for predication in general that secures the logicality of relevant parts of **SOL**, since it treats order as primitive, instead of simulating it. In fact, it is very difficult to see how any analysis which involves a *simulation* of order could make any tenable claim to be a sound basis for a maximally conceived interpretation. Armed with **SL**, we are able to overcome this difficulty, and the resulting interpretation of **SOL** is both formally adequate and philosophically appealing.

5.2 Logics of higher order Can the account developed in this paper be generalized for nth level logic,[37] for any $n \in \mathbb{Z}^+$? Apart from its intrinsic interest, the question may be of importance for those who wish to use some higher-order logic for a particular philosophical purpose. An example of this would be the use of third-order logic in some neo-Fregean attempts to detail the foundations of analysis. Assuming that the assumption of a plural hierarchy incorporating serial quantification is in good logical standing, there is no obstacle to generalization, although any rigorous metatheory will quickly become prohibitively unwieldy. For example, third-order quantification would be understood in terms of fourth-level plural quantification over plural serialities themselves arranged serially. We will not develop any higher-order system here.

6 Other Applications of SL

SL may prove to be of use beyond the important, but localized, project of supplying a plural interpretation for **SOL**. Here is an example of a potential use. We are taught as undergraduates to understand logical consequence as a binary relationship between a set (possibly empty) of *premises* and a singleton containing a *conclusion*. In the case of multiple conclusion logics, the requirement that the latter set be a singleton is removed (see [31]). Upon philosophical reflection we might conclude, however, that the use of set-theoretic machinery here is artificial, and that logical consequence should really be understood as a (plural or multigrade) relation obtaining between some premises and a conclusion (or some conclusions). There is no problem here, as long as consequence is understood *classically*. The situation is complicated when *substructural logics* are considered. These lack one or more of the classical structural rules.[38] The absence of some rules can be accommodated within a plural understanding of premises readily, at least given the liberalization of extensionality described in Section 2.1 above and the availability in the metalogic of the family of relations of the form $\ulcorner \prec_n \urcorner$. Consider the rule of *contraction*, a key feature of the metalogic of idempotent logics:

$$X; X \vdash A \Rightarrow X \vdash A. \tag{CON}$$

Absent (CON), it matters how many tokens of the same formula type occur in the premise of an argument.[39] But we can be confident that liberalized plural resources allow us to keep track of this.[40] But now consider the rule characteristic of commutative logics:

$$X; Y \vdash A \Rightarrow Y; X \vdash A. \tag{COM}$$

Without (COM) the *ordering* of the premises matters. An important instance of a system lacking (COM) is the Lambek calculus, used to model the syntactic structure of language. No longer will a mere weakening of extensionality allow us to understand the premises together as a plurality. Proof-theoretically important information would be lost in so doing. We can, however, think about them as a seriality; and the recognition of this opens the door for the use of **SL** as a tool for metalogical study.

More usual approaches invoke multisets and tuples (possibly themselves multisets) to accommodate these substructural logics within a set-theoretic account of proofs. An interesting project would be to examine whether the theory of serialities can be developed in a conceptually natural fashion so as to permit serial talk to function as an across-the-board alternative to multisets. As **SL** stands, it lacks the resources for this; minimally, an equivalent of the extended plural $\ulcorner \prec_n \urcorner$ would be required.

7 An Option for the Cautious, and Some for the Adventurous

Suppose that you are prepared to follow the argument of Section 2 above only so far. You are prepared to entertain, say, both the liberalization of extensionality and quantification over serialities. Plurally serial quantification, however, is a step too far for you. This position is far from unimaginable; it is an analogue of a common stance with regard to philosophical debate about plural quantification *simpliciter*: plural quantification is comprehensible and useful, but there is something suspicious about superplural quantification. If you adopt this viewpoint, it is open to you to adopt the system **SL**-, obtained by removing plurally serial vocabulary from the lexicon of $\mathscr{L}_{\mathbf{SL}}$ and all plurally serial features from the semantics of **SL**. This system will still be of some use, both in regimenting natural language and for some metalogical purposes. Useful in the same areas, although less so than **SL**-, are those systems obtained by substituting (LIB-EXT) for (PL-EXT) in the proof systems of standard plural logics and by making appropriate modifications to the semantics.

These cautious approaches have duals of a sort. Once the availability of a number of modifications and extensions of standard plural logics—along the lines of **SL**, **SL**-, and logics with liberalized extensionality—is noted, the person less worried about philosophical motivation might advocate further extensions along the same lines, or developing new systems which mix-and-match various features of distinct extensions of plural logic. Why not, this more adventurous logician may ask, have a system which admits serialities but does not allow nontrivial repetition? Why not allow serial quantification over ordinary pluralities, or over serialities? Why not admit so-called superduperplural quantification (plural quantification over superpluralities) within a system which also allows serial quantification?

Why not indeed? Many of the systems resulting from this kind of exercise will not prove very useful, but there is no harm in marking the available territory. Let a thousand flowers bloom!

Appendix A Inter-translatability

Without loss of generality, we consider those fragments of the languages of **SOL** and **SL** containing only logical vocabulary. Call these $\mathscr{L}_{\mathbf{SOL}^-}$ and $\mathscr{L}_{\mathbf{SL}^-}$. It is entirely routine to verify that our proofs can be generalized to the case of the full languages, containing logical vocabulary. The lexicons of $\mathscr{L}_{\mathbf{SOL}^-}$ and $\mathscr{L}_{\mathbf{SL}^-}$ are denumerable sets

$$\begin{aligned}\mathrm{Lex}(\mathscr{L}_{\mathbf{SOL}^-}) &= \{=, x_0, x_1, x_2 \ldots X^1_0, X^1_1 \ldots X^2_0 \ldots \exists, \wedge, \neg, (,), \},\\ \mathrm{Lex}(\mathscr{L}_{\mathbf{SL}^-}) &= \{=, x_0, x_1 \ldots \overrightarrow{xx}^1_0, \overrightarrow{xx}^1_1 \ldots \overbrace{xx^1_0}, \overbrace{xx^1_1} \ldots \ltimes_1 \ldots \bowtie,\\ &\qquad \overbrace{\prec}, \overbrace{\approx}, \exists, \wedge, \neg, (,), \}.\end{aligned}$$

The other standard connectives, functional apparatus, and the universal quantifier are definable in the usual way. We specify the sets of wffs of each language in Backus–Naur form. First the wffs of $\mathscr{L}_{\mathbf{SOL}^-}$:

$$\varphi ::= X^n x_{i0} \ldots x_{in} | x_i = x_j | \neg\varphi | (\varphi \wedge \varphi) | \exists x_i \varphi | \exists X_i \varphi.$$

And the wffs of $\mathscr{L}_{\mathbf{SL}^-}$:

$$\begin{aligned}\varphi ::= x_i = x_j | x_i \ltimes_{m \leq n} \overrightarrow{xx}^n | xx_i \bowtie xx_j | \overrightarrow{xx}_i \overbrace{\prec} \overbrace{xx_j} \,|\, \overbrace{xx_i} \overbrace{\approx} \overbrace{xx_j}\\ | \neg\varphi | (\varphi \wedge \varphi) | \exists x_i \varphi | \exists \overrightarrow{xx}^n_i \varphi | \exists \overbrace{xx^n_i} \varphi.\end{aligned}$$

Now to prove inter-translatability.

Theorem 1 *Every wff of $\mathscr{L}_{\mathbf{SOL}^-}$ can be translated into a wff of $\mathscr{L}_{\mathbf{SL}^-}$.*

Proof We proceed by induction on the complexity of $\mathscr{L}_{\mathbf{SOL}^-}$.

There are two atomic cases to consider. Statements of identity may be dealt with trivially, since they are already wffs of $\mathscr{L}_{\mathbf{SL}^-}$. Other predications are translated in accordance with (5) above.

For the induction hypothesis, assume that φ and ψ are wffs of $\mathscr{L}_{\mathbf{SOL}^-}$, and that $\mathrm{Tr}'(\varphi)$ and $\mathrm{Tr}'(\psi)$ are their translations into $\mathscr{L}_{\mathbf{SL}^-}$. We translate as follows: $\ulcorner\neg\varphi\urcorner$ as $\ulcorner\neg\,\mathrm{Tr}'(\varphi)\urcorner$; $\ulcorner(\varphi\wedge\psi)\urcorner$ as $\ulcorner(\mathrm{Tr}'(\varphi)\wedge\mathrm{Tr}'(\psi))\urcorner$; and $\ulcorner\exists x_i\varphi\urcorner$ as $\ulcorner\exists x_i\,\mathrm{Tr}'(\varphi)\urcorner$. We translate $\ulcorner\exists X_i\varphi$ in accordance with (6), according to the rule that, if the second-order bound variable occurs in any atomic predications in the open formula following the quantifier, the serial variable bound by the quantifier in the translation is the same as that already used to translate the atomic predication; otherwise a new serial variable is introduced. This exhausts the wffs of $\mathscr{L}_{\mathbf{SOL}^-}$. □

Theorem 2 *Every wff of $\mathscr{L}_{\mathbf{SL}^-}$ can be translated into a wff of $\mathscr{L}_{\mathbf{SOL}^-}$.*

Proof Again, we proceed by induction on complexity.

This time we have five atomic cases to consider. As before, statements of singular identity are trivial. The others are translated as follows: variables are new unless otherwise stated, in order to avoid clash, modulo the proviso that tokens of the same variable are dealt with consistently. We translate $\ulcorner x_i \ltimes_n \overrightarrow{xx}_j^m\urcorner$ as $\ulcorner\exists x_{k0}\ldots x_{kn-1},[x_{kn+1}\ldots x_{km}]X_j^m x_{k0}\ldots x_{kn-1}x_i[x_{kn+1}\ldots x_{km}]\urcorner$; $\ulcorner xx_i^n \bowtie xx_j^m$ as "$x_1 \neq x_1$" when $n \neq m$, otherwise as $\ulcorner\forall x(\mathrm{Tr}'(x\ltimes_1\overrightarrow{xx}_i^n \leftrightarrow x\ltimes_1\overrightarrow{xx}_j^m)\wedge\ldots\wedge \mathrm{Tr}'(x\ltimes_n\overrightarrow{xx}_i^n \leftrightarrow x\ltimes_n\overrightarrow{xx}_j^n))\urcorner$; $\ulcorner\overrightarrow{xx}_i^n\overbrace{\prec}\overbrace{xx_j^m}\urcorner$ as "$x_1\neq x_1$" when $n\neq m$, otherwise as $\ulcorner\exists x_1,\ldots,x_n X_j^m x_1\ldots x_n\urcorner$; and $\ulcorner\overbrace{xx_i^n}\overbrace{\approx}\overbrace{xx_j^m}\urcorner$ as "$x_1\neq x_1$" when $n\neq m$, otherwise as $\ulcorner\forall x_1,\ldots,x_n X_i x_1\ldots x_n \leftrightarrow X_j x_1\ldots x_n\urcorner$. □

Corollary 1 *$\mathscr{L}_{\mathbf{SL}^-}$ and $\mathscr{L}_{\mathbf{SOL}^-}$ are inter-translatable.*

Proof The proof is immediate from Theorems 1 and 2. □

Notes

1. See, for example, [20] and [26].

2. Strictly speaking, only the fragment of **PFO** which lacks singular predicates is required here. Call this **PFO**-. It is straightforward to prove a bi-interpretability result for **MSOL** and **PFO**- by induction on complexity of the formulae of both systems.

3. The second disjunct of (2) ensures that "empty" instances of comprehension, such as $\exists X\forall x Xx \leftrightarrow x\neq x$, come out true. In the standard set-theoretic semantics, this is secured by the existence of the "empty" set. But it seems odd to suggest that there is an empty *plurality*.

4. A prominent proponent of this position is Shapiro [30].

5. Wagner calls this criterion *fundamentality* and emphasizes helpfully that the issue is not one about the order of (actual) belief acquisition, but rather about the necessary order of justification and definition (see [35], [9]). I may in fact believe some proposition which is entirely logical in content because I believe some other empirically grounded proposition. This is not, of itself, injurious to logicality.

6. On **MSOL**, see [12].

7. For our purposes, *full* second-order logic encompasses the n-adic case for all $n \in \mathbb{Z}^+$, where $0 \notin \mathbb{Z}^+$. Zero-adic second-order quantification, which is very naturally understood as propositional quantification, would be very difficult to incorporate into the approach developed in this paper, and probably requires separate treatment.

8. For some interesting thoughts on the set-theoretic implementation of pairs see [11].

9. Other than, trivially, a pair.

10. Analogously, Benacerraf's argument is not simply that numbers cannot be sets, but that they cannot be *objects* of any kind, including (presumably) numbers, where numbers are a species of object!

11. Similar difficulties to those identified for the pairing-based approach can be generated for any attempt to use the Lewis–Hazen–Burgess system developed in [16], [17], and [13], to interpret full second-order logic.

12. In an everyday sense of "order," to be distinguished from other, mathematical, uses below.

13. *Superplural quantification* is iterated plural quantification, plural quantification *over* pluralities. See [20] and [25].

14. It cannot be emphasized enough that a plurality is *some things*—the word "plurality" being a grammatical convenience permitting reference to things collectively. Pluralities are not collective objects, ones created by the coming together of many. In this they differ importantly from both sets and fusions.

15. I accept that it may be difficult to detect in natural language when this type of plural quantification, rather than the more hard-line extensional variant present in standard plural logics, is being used. Linguistic context may, of course, serve to disambiguate—"some things are the first five members of the Fibonacci sequence." A major advantage to formalization is that ambiguity of this sort is eliminated straightforwardly.

16. These distinctions are owing to Salvatore Florio.

17. "Those" strikes me as marginally more grammatical here than "these."

18. There may be a concern that this sort of burden-of-proof shifting makes for a weak argument. This point was pressed by an anonymous referee. To an extent, the point may be conceded, but the unavailability of decisive arguments here just seems to be a feature of disputes about logicality. That said, it is not arbitrary to insist that the burden of proof

rests with an interlocutor who wishes to argue that some utterance incurs additional commitments.

19. "Entities are not to be multiplied beyond necessity. One might doubt, for example, that there is such a thing as the set of Cheerios in the . . . bowl on the table. There are, of course, quite a lot of Cheerios in the bowl, well over two hundred of them. But is there, in addition to the Cheerios, also a set of them all? And what about the $>10^{60}$ subsets of that set? (And don't forget all the sets of sets of Cheerios in the bowl.) It is haywire to think that when you have some Cheerios you are eating a *set*—what you're doing is: eating THE CHEERIOS. Maybe there are some reasons for thinking there is such a set—there are, after all, $>10^{60}$ ways to divide the Cheerios into two portions—but it doesn't follow just from the fact that there are some Cheerios in the bowl that, as some who theorize about the semantics of plurals would have it, there is also a set of them all." [8, p. 72].

20. More exactly, the ordering is a total, nonreflexive, asymmetric, transitive relation R on some things xx such that xx have an R-minimal member.

21. A referee comments on the drawing of a distinction between finite and transfinite serialities: "It is somewhat surprising that there should be such a divide, since it doesn't exist when we deal only with plural quantification: plural quantification, whether over a finite or infinite number of things, is deemed logical (and cognitively primary) by its partisans." There is some force to this. However, I do think that the addition of an ordering does make a difference, for the reasons given in the body of the text. In the final analysis, no decision of the question is required for present purposes—finite serialities suffice for the task of interpreting full **SOL**. In spite of this, the topic is clearly deserving of further discussion.

22. I accept that intuitions about meaning in this case vary. For this reason I emphasize the importance of the considerations drawn from Linnebo earlier in this section, to the effect that the logicality of a form can be motivated in the absence of natural language examples.

23. Hossack's system is not identical to Hazen's and Taylor's. Significantly, Hossack does not allow vector quantification.

24. A domain should be understood as *some objects*, rather than as a set of objects. So there are domains consisting of the sets (all of them), of the ordinals (all of them), and so on.

25. To prove this, consider the smallest possible domain, where $\mathcal{D}$ consists of arbitrary a: there are $\aleph_0$ serialities on $\mathcal{D}$. Every possible collection of serialities determines a value for a plurally serial variable. By elementary combinatorics, there are $2^{\aleph_0} - 1$ of these, which by cardinal arithmetic is shown to equal $2^{\aleph_0}$, as required.

26. In a nonreifying sense of "collection."

27. In these clauses $\overrightarrow{xx}$ and $\overbrace{xx}$ are metavariables, ranging over object language serial variables and plurally serial variables, respectively.

28. Two serialities are the same iff they have the same number n of members, and for all $m \leq n$ the mth among one is the same as the mth among the other. Note that this gives

us the R-L, but not the L-R, direction of the standard identity condition for *multisets*, $\forall x \forall y (\forall z \forall n (z \in^n x \leftrightarrow z \in^n y) \leftrightarrow x = y)$ (see [5]), reading "$\in^n$" as "is included n times in" (although **SL** as it stands lacks the expressive resources to express this). There is clearly a fairly close conceptual relationship between serialities and finite multisets. However, a multiset is a distinct entity over and above its elements, and is not ordered. It would be an interesting exercise to explore which, if any, of the customary uses of multisets could make use of **SL** instead.

29. On this see Burgess [9].

30. See (1) above.

31. The same relationship obtains between **MSOL** and **PFO**-.

32. Within **SOL**, $\models \exists X \exists x Xx$. So, supposing that I am understanding predication in terms of the instantiation of a universal by a particular, it is a truth of logic that there are universals and particulars.

33. The proponent of a set-theoretic account of predication is exempt from this criticism. Sets are abundant. She has difficulties of her own, though; quite apart from the question of ontological innocence, a moment's reflection on the predicate "is non-self-membered" should convince us that not every predicate can correspond to a set, at least if we are capable of picking out sets with singular terms capable of combining with predicates to form declarative units of meaning (as we surely are).

34. Hereafter "or concepts, or tropes, etc." should be taken as implicit.

35. On this, see Mertz [23].

36. Of course, the example consists of something which would be said formally using a nonlogical predicate and individual constants, rather than variables. But exactly the same principles are in play in both cases.

37. There is also an interesting question about logics of infinite orders, but this would require more attention than is possible here.

38. For details, see [28].

39. Examples include linear logic and Łukasiewicz's many-valued logic.

40. Slightly more carefully, for any logic which allows a given formula to occur only finitely many times among some premises.

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