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On the Indecomposability of ω^n

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Abstract We study the reverse mathematics of pigeonhole principles for finite powers of the ordinal ω. Four natural formulations are presented, and their relative strengths are compared. In the analysis of the pigeonhole principle for ω^2, we uncover two weak variants of Ramsey's theorem for pairs.

1 Introduction

In the set-theoretic literature, one finds two formulations of the indecomposability of an ordinal α.

Additive indecomposability: If $\beta_0 + \cdots + \beta_{k-1} = \alpha$, then $\beta_i = \alpha$ for some $i < k$.

Combinatorial indecomposability: If $B_0 \cup \cdots \cup B_{k-1} = \alpha$, then B_i has order-type α for some $i < k$.

Prima facie, combinatorial indecomposability is stronger since additive indecomposability corresponds to the special case where the parts $B_0, \ldots, B_{k-1}$ are required to be nonoverlapping (possibly empty) intervals. However, the additively indecomposable ordinals and the combinatorially indecomposable ordinals are precisely the ordinal powers of ω, so the two properties are actually equivalent.

The fact that ω is combinatorially indecomposable is also known as the infinite pigeonhole principle. In reverse mathematics, the infinite pigeonhole principle was first studied by Hirst [11], who showed that it was equivalent to the Π^0_1-bounding principle ($\mathsf{B}\Pi^0_1$). The additive indecomposability of the ordinal powers ω^α was also studied by Hirst [10], but the formally stronger combinatorial indecomposability of ω^α was not directly explored.

In this paper, we analyze the combinatorial indecomposability of ω^n for $2 \leq n < \omega$. One difficulty with the analysis is that "B_i has order-type ω^n" has several different interpretations in second-order arithmetic. In Section 2, we analyze

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the reverse mathematics of four natural interpretations which are all equivalent assuming arithmetic comprehension (ACA_0) but diverge assuming only recursive comprehension (RCA_0).

The analysis of the case $n = 2$ has led us to two combinatorial principles related to Ramsey's theorem for pairs (RT^2_k), which has been intensely studied in reverse mathematics (see Seetapun and Slaman [14], Cholak, Jockusch, and Slaman [1], Hirschfeldt and Shore [9], Dzhafarov and Hirst [5], Dzhafarov and Jockusch [6]).

RT^2_k *For every finite coloring $c : \mathbb{N}^2 \to \{0, \dots, k-1\}$, there are a color $d < k$ and an infinite set H such that $c(x, y) = d$ for all $x, y \in H$ with $x < y$.*

These two weaker principles are the weak Ramsey theorem for pairs,

WRT^2_k *For every finite coloring $c : \mathbb{N}^2 \to \{0, \dots, k-1\}$, there are a color $d < k$ and an infinite set H such that $\{y \in \mathbb{N} : c(x, y) = d\}$ is infinite for every $x \in H$;*

and the hyperweak Ramsey theorem for pairs,

HWRT^2_k *For every finite coloring $c : \mathbb{N}^2 \to \{0, \dots, k-1\}$, there are a color $d < k$ and an increasing function $h : \mathbb{N} \to \mathbb{N}$ such that, for all $0 < i_1 < i_2$, the rectangle*

$$[h(i_1 - 1), h(i_1) - 1] \times [h(i_2 - 1), h(i_2) - 1]$$

contains a pair with color d.

In Section 3 we compare HWRT^2_2 to other known combinatorial principles. In particular, we show that HWRT^2_2 is strictly weaker than WRT^2_2. In addition, we give a direct proof that $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \mathsf{HWRT}^2_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.

Conventions. A standard reference for subsystems of second-order arithmetic and their use in reverse mathematics is Simpson [15]. Formal definitions of the basic systems RCA_0 and ACA_0 can be found there. Another standard reference for induction principles used in this paper is Hájek and Pudlák [8]. While this last reference focuses on first-order arithmetic, it is generally straightforward to relativize their definitions and results to the second-order setting.

Our general approach is model-theoretic rather than proof-theoretic. Throughout the paper $\mathbb{N}$ will denote the first-order part of the model currently under consideration; we will use ω to denote the set of standard natural numbers. Every result in this paper indicates in parentheses the base system over which the result is formulated. Some of the results are parameterized by a standard natural number, which is also indicated in parentheses.

In Section 3, for the purpose of forcing, we will find it convenient to use a functional interpretation of the basic system RCA_0. Such an interpretation was described by Kohlenbach [13], but we use the equivalent system described by Dorais [3]. Basic structures are of the form $\mathfrak{N} = (\mathbb{N}, \mathcal{N}_1, \mathcal{N}_2, \dots)$, where each $\mathcal{N}_k$ is a set of functions $\mathbb{N}^k \to \mathbb{N}$ which together form an algebraic clone: each $\mathcal{N}_k$ contains all the constant functions, the projections $\pi_i(x_1, \dots, x_k) = x_i$, and if $f \in \mathcal{N}_\ell$ and $g_1, \dots, g_\ell \in \mathcal{N}_k$, then the superposition $f(g_1(x_1, \dots, x_k), \dots, g_\ell(x_1, \dots, x_k))$ belongs to $\mathcal{N}_k$.

On top of this basic structure, we require closure under *primitive recursion*: there are distinguished $0 \in \mathbb{N}$ (zero) and $\sigma \in \mathcal{N}_1$ (successor) such that for any $f \in \mathcal{N}_{k-1}$ and $g \in \mathcal{N}_{k+1}$ there is a unique $h \in \mathcal{N}_k$ such that

$$h(0, \bar{w}) = f(\bar{w}) \qquad \text{and} \qquad h\big(\sigma(x), \bar{w}\big) = g\big(h(x, \bar{w}), x, \bar{w}\big)$$

for all $x, \bar{w} \in \mathbb{N}$. Note that the uniqueness requirement on h is crucial since this is the only form of induction in this system.

Using primitive recursion, we can define the usual arithmetic operations such as addition, multiplication, truncated subtraction ($x \dot{-} y = \max(x - y, 0)$) together with the usual identities between them. We will also assume the *dichotomy axiom* $x \dot{-} y = 0 \vee y \dot{-} x = 0$, which is necessary to show that the relation $x \leq y$ defined by $x \dot{-} y = 0$ is a linear ordering of $\mathbb{N}$.

Finally, in addition to the basic axioms described above, we will consider the second-order *uniformization axiom*: For every $f \in \mathcal{N}_{k+1}$ such that $\forall \bar{w}\ \exists x\ f(x, \bar{w}) = 0$, there is a $g \in \mathcal{N}_k$ such that $\forall \bar{w}[f(g(\bar{w}), \bar{w}) = 0]$. This axiom ensures closure under general recursion, which is essentially equivalent to recursive comprehension.

Every functional structure $\mathfrak{N}$ corresponds to a set-based structure $(\mathbb{N}; \mathcal{S}; 0, 1, +, \cdot)$ for second-order arithmetic as described in [15], where $\mathcal{S}$ consists of all subsets of $\mathbb{N}$ whose characteristic function is in $\mathcal{N}_1$. The latter structure is a model of RCA_0 if and only if the uniformization axiom holds in $\mathfrak{N}$. Conversely, given a traditional model $(\mathbb{N}; \mathcal{S}; 0, 1, +, \cdot)$ of RCA_0, we can define $\mathcal{N}_k$ to be the class of all functions $\mathbb{N}^k \to \mathbb{N}$ whose coded graph belongs to $\mathcal{S}$, and the resulting structure is a functional model which satisfies uniformization. Since our choice to adopt functional models is a matter of convenience, we will freely use this translation between functional models and traditional models.

2 Combinatorial Indecomposability

In this section, we describe four different interpretations of the statement that "ω^n is combinatorially indecomposable" and examine their strength over RCA_0. We will state the indecomposability principles in terms of a canonical representation of the ordinal ω^n, namely the lexicographic ordering of $\mathbb{N}^n$, which is defined by letting $(x_0, \ldots, x_{n-1}) < (y_0, \ldots, y_{n-1})$ when

$$x_0 = y_0 \wedge \cdots \wedge x_{i-1} = y_{i-1} \wedge x_i < y_i$$

holds for some $i < n$. We also use the term lexicographic to describe functions from $f : \mathbb{N}^n \to \mathbb{N}^n$ which preserve the lexicographic ordering.

Usually, "X has order-type ω^n" is interpreted as saying that the given ordering X is order-isomorphic to ω^n. Using this interpretation, combinatorial indecomposability corresponds to the following principle.

$\mathsf{Iso\text{-}Indec}^n_k$ *For every finite coloring $c : \mathbb{N}^n \to \{0, \ldots, k-1\}$ there is a color $d < k$ such that the set*

$$A_d = \{(x_1, \ldots, x_n) \in \mathbb{N}^n : c(x_1, \ldots, x_n) = d\}$$

is lexicographically isomorphic to $\mathbb{N}^n$.

We use $\mathsf{Iso\text{-}Indec}^n$ to denote $(\forall k)\mathsf{Iso\text{-}Indec}^n_k$. Since a set $A \subseteq \mathbb{N}$ is order-isomorphic to $\mathbb{N}$ if and only if it is infinite, the statement $\mathsf{Iso\text{-}Indec}^1$ is precisely equivalent to $\mathsf{B}\Pi^0_1$ by Hirst's result. However, the very next case $\mathsf{Iso\text{-}Indec}^2_2$ already implies arithmetic comprehension.

Proposition 2.1 (RCA_0) $\mathsf{Iso\text{-}Indec}^2_2$ *implies arithmetic comprehension.*

Proof We show that $\mathsf{Iso\text{-}Indec}^2_2$ implies that the range of an arbitrary injection $f : \mathbb{N} \to \mathbb{N}$ exists. Consider the coloring $c : \mathbb{N}^2 \to 2$ defined by letting $c(x, y) = 1$ if and only if $x \in \{f(0), \ldots, f(y-1)\}$. Note that $c(x, 0) = 0$ for every x.

Let $A_0 = c^{-1}(0)$ and $A_1 = c^{-1}(1)$. On the one hand, if $h : \mathbb{N}^2 \to A_1$ is an isomorphism, then $h_1(n, 0)$ must be the $(n+1)$th element of the range of f. On the other hand, if $h : \mathbb{N}^2 \to A_0$ is an isomorphism, then $h_1(n+1, 0) - 1$ must be the $(n+1)$th element in the complement of the range of f. □

Of course, it is easy to see that ACA_0 proves $\mathsf{Iso\text{-}Indec}^n$ for all $n < \omega$.

2.1 Indecomposability and induction The weakest statements of indecomposability for ω^n that we will consider are the following Π^1_1 statements.

$\mathsf{Elem\text{-}Indec}^n_k$ *For every finite coloring $c : \mathbb{N}^n \to \{0, \ldots, k-1\}$ there is a color $d < k$ such that*

$$(\exists^\infty x_1)(\exists^\infty x_2) \cdots (\exists^\infty x_n)[c(x_1, x_2, \ldots, x_n) = d].$$

We will use $\mathsf{Elem\text{-}Indec}^n$ to denote $(\forall k)\mathsf{Elem\text{-}Indec}^n_k$. Note that $\mathsf{Elem\text{-}Indec}^n_k$ is provable in RCA_0 for every $k < \omega$, but the principle $\mathsf{Elem\text{-}Indec}^n$ is nontrivial.

The statement $\mathsf{Elem\text{-}Indec}^1$ says that for every finite coloring $c : \mathbb{N} \to \{0, \ldots, k-1\}$ there is a color $d < k$ such that the set $A_d = \{x : c(x) = d\}$ is infinite—this statement is equivalent to $\mathsf{B}\Pi^0_1$. We can generalize this as follows.

Theorem 2.2 (RCA_0; $1 \le n < \omega$)

(a) $\mathsf{Elem\text{-}Indec}^n$ *implies* $\mathsf{B}\Pi^0_n$.
(b) $\mathsf{I}\Sigma^0_{n+1}$ *implies* $\mathsf{Elem\text{-}Indec}^n$.

Part (a) of Theorem 2.2 will follow from Proposition 2.4. Part (b) is proved in Proposition 2.6.

A principle equivalent to bounding will be used in the proof of Theorem 2.2. In [8, Section I.2(b)], Hájek and Pudlák introduced the regularity principle $\mathsf{R}\Gamma$ which says that if $\varphi(x, y)$ is a Γ formula, then

$$(\exists^\infty x)(\exists y < k)\varphi(x, y) \leftrightarrow (\exists y < k)(\exists^\infty x)\varphi(x, y)$$

holds for all $k \in \mathbb{N}$. They further show that $\mathsf{R}\Sigma^0_{n+1}$, $\mathsf{R}\Pi^0_n$, and $\mathsf{B}\Pi^0_{n+1}$ are equivalent for every $n < \omega$ (see [8, Section I.2.23(4)]).

The regularity principle is useful in handling a certain class of colorings. A function $c : \mathbb{N}^{m+n} \to \mathbb{N}$ is *weakly n-stable* if for all $x_1, \ldots, x_m \in \mathbb{N}$ there is a $y \in \mathbb{N}$ such that

$$(\forall^\infty z_1) \cdots (\forall^\infty z_n)[y = c(x_1, \ldots, x_m, z_1, \ldots, z_n)].$$

This is very similar to saying that the iterated limit

$$\lim_{z_1 \to \infty} \cdots \lim_{z_n \to \infty} c(x_1, \ldots, x_m, z_1, \ldots, z_n)$$

exists for all $x_1, \ldots, x_m \in \mathbb{N}$. However, the usual definition of such limits requires that intermediate limits all exist too, which is not required by weak n-stability. We say that c is *strongly n-stable* if it is weakly i-stable for each $1 \le i \le n$; this guarantees the existence of all intermediate limits and corresponds to the usual meaning of iterated limit. The two notions agree when $n = 1$, and they agree with the definition of *stable* introduced by Cholak, Jockusch, and Slaman [1].

If $c : \mathbb{N}^{m+n} \to \mathbb{N}$ is strongly n-stable, then the iterated limit

$$f(x_1, \dots, x_m) = \lim_{z_1 \to \infty} \cdots \lim_{z_n \to \infty} c(x_1, \dots, x_m, z_1, \dots, z_n)$$

is a total Σ^0_{n+1}-definable map $f : \mathbb{N}^m \to \mathbb{N}$. (More precisely, the graph of f is Σ^0_{n+1}-definable. Note that the map f need not be a function of the current model.) The converse of this fact is due to Švejdar [16].

Lemma 2.3 **($\mathsf{RCA}_0 + \mathsf{B}\Pi^0_{n-1}$; $1 \leq n < \omega$)** *Every total Σ^0_{n+1}-definable map $f : \mathbb{N} \to \mathbb{N}$ is representable in the form*

$$f(x) = \lim_{z_1 \to \infty} \cdots \lim_{z_n \to \infty} c(x, z_1, \dots, z_n),$$

where $c : \mathbb{N}^{n+1} \to \mathbb{N}$ is a strongly n-stable function.

Proof Švejdar [16, Theorem 1] shows under $\mathsf{B}\Pi^0_{n-1}$ that for every total Σ^0_{n+1}-definable map $f : \mathbb{N} \to \mathbb{N}$ there is a 1-stable Σ^0_n-definable (indeed, $\Sigma^0_0(\Sigma^0_{n-1})$-definable) map $f' : \mathbb{N}^2 \to \mathbb{N}$ such that

$$f(x) = \lim_{z_1 \to \infty} f'(x, z_1)$$

for all $x \in \mathbb{N}$. Iterating this result, we find Σ^0_{n+1-i}-definable strongly i-stable maps $f^{(i)} : \mathbb{N}^{i+1} \to \mathbb{N}$ such that

$$f(x) = \lim_{z_1 \to \infty} \cdots \lim_{z_i \to \infty} f^{(i)}(x, z_1, \dots, z_i)$$

for all $x \in \mathbb{N}$. The nth such map is Σ^0_1-definable and hence corresponds to an actual function $c : \mathbb{N}^{n+1} \to \mathbb{N}$ in our model, which acts as claimed. □

Proposition 2.4 **(RCA_0; $1 \leq n < \omega$)** *$\mathsf{B}\Pi^0_n$ is equivalent to the following statement. For any strongly $(n-1)$-stable $c : \mathbb{N}^n \to \{0, 1, \dots, k-1\}$, there is a $d < k$ such that*

$$(\exists^\infty x_1)(\exists^\infty x_2) \cdots (\exists^\infty x_n)[c(x_1, x_2, \dots, x_n) = d].$$

Proof We will prove equivalence with $\mathsf{R}\Pi^0_{n-1}$ instead of equivalence with $\mathsf{B}\Pi^0_n$.

Let $g : \mathbb{N} \to \mathbb{N}$ be the total Σ^0_n-definable function given by

$$g(x_1) = \lim_{x_2 \to \infty} \cdots \lim_{x_n \to \infty} c(x_1, x_2, \dots, x_n).$$

Therefore, by $\mathsf{R}\Sigma^0_n$, we have that there is a $d < k$ such that $(\exists^\infty x_1)[g(x) = d]$. In particular,

$$(\exists^\infty x_1)(\forall^\infty x_2) \cdots (\forall^\infty x_n)[c(x_1, x_2, \dots, x_n) = d],$$

and the conclusion follows immediately.

Let $\varphi(x, y)$ be Π^0_{n-1}, and suppose that $(\exists^\infty x)(\exists y < k)\varphi(x, y)$. Consider the total Σ^0_n-definable function g such that $g(x_0) = y_0$ if and only if there is an x such that $x_0 \leq x$, $y_0 < k$, and $\varphi(x, y_0) \wedge (\forall y < y_0)\neg\varphi(x, y)$, but $\neg\varphi(x', y')$ for all x', y' such that $x_0 \leq x' < x$ and $y' < k$. Since g is a total Σ^0_n-definable function, Lemma 2.3 ensures that there is a strongly $(n-1)$-stable $c : \mathbb{N}^n \to \{0, \dots, k-1\}$ such that

$$g(x_1) = \lim_{x_2 \to \infty} \cdots \lim_{x_n \to \infty} c(x_1, x_2, \dots, x_n)$$

for all x_1. By hypothesis, there is a $d < k$ such that

$$(\exists^\infty x_1)(\exists^\infty x_2) \cdots (\exists^\infty x_n)[c(x_1, x_2, \dots, x_n) = d].$$

It follows that $(\exists^\infty x)[g(x) = d]$ and hence that $(\exists^\infty x)\varphi(x, d)$. □

Note that part (a) of Theorem 2.2 will follow immediately from Proposition 2.4. Now we prove part (b). We will need the following result, which is essentially due to Jockusch and Stephan [12].

Lemma 2.5 (RCA_0) *Given a sequence of sets $A = \langle A_n \rangle_{n=0}^\infty$ such that A'' exists, there is an infinite set X such that $(X \oplus A)'' \equiv_T A''$ and, for all n, either $X \subseteq^* A_n$ or $X \subseteq^* \mathbb{N} \setminus A_n$.*

Here and elsewhere, the notation $X \subseteq^* Y$ means that $(\forall^\infty x)(x \in X \to x \in Y)$. A close inspection of the proof of [12, Theorem 2.1] shows that the above is provable in RCA_0.

Proposition 2.6 (RCA_0; $1 \leq n < \omega$) $\mathsf{I}\Sigma^0_{n+1}$ *implies* $\mathsf{Elem\text{-}Indec}^n$.

Proof Let $\mathfrak{N}$ be a model of $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n+1}$, and let $c_0 : \mathbb{N}^n \to \{0, 1, \ldots, k-1\}$ be a coloring in $\mathfrak{N}$. Let $\mathfrak{M}$ be the model of RCA_0 whose second-order part consists of all Δ^0_{n+1}-definable sets with parameters from $\mathfrak{N}$.

Given $\bar{x} \in \mathbb{N}^n$ and $i < k$, let $A_{\bar{x},i} = \{y \in \mathbb{N} : c_0(\bar{x}, y) = i\}$, and let $A = \langle A_n \rangle_{n=0}^\infty$ effectively enumerate all such $A_{\bar{x},i}$. Since $A'' \equiv_T c_0'' \in \mathfrak{M}$, by Lemma 2.5 there is an infinite set X_1 such that $(c_0 \oplus X_1)'' \equiv_T c_0''$ and, for all $\bar{x}$ and i, either $X_1 \subseteq^* A_{\bar{x},i}$ or $X_1 \subseteq^* \mathbb{N} \setminus A_{\bar{x},i}$. We now define a new coloring $c_1 : \mathbb{N}^{n-1} \to \{0, 1, \ldots, k-1\}$ by

$$c_1(z_1, z_2, \ldots, z_{n-1}) = \lim_{x \in X_1} c_0(z_1, z_2, \ldots, z_{n-1}, x),$$

which is computable from $(c_0 \oplus X_1)'$. Note also that $c_1' \leq_T c_0''$.

If $n \geq 3$, we now repeat this process for the coloring c_1. For this construction to work, use the fact that $c_1'' \leq_T (c_0 \oplus X_1)''' \equiv_T c_0''' \in \mathfrak{M}$ in order to apply Lemma 2.5 as above. We are left with an infinite set X_2 such that $(c_1 \oplus X_2)'' \equiv_T c_1'' \leq_T c_0'''$ and which defines a coloring

$$c_2(z_1, \ldots, z_{n-2}) = \lim_{x \in X_2} c_1(z_1, \ldots, z_{n-2}, x),$$

which is computable in $(c_1 \oplus X_2)'$.

Continuing this process as necessary we end with a set X_{n-1} such that $(c_{n-2} \oplus X_{n-1})'' \equiv_T c_{n-2}'' \in \mathfrak{M}$ and

$$c_{n-1}(z_1) = \lim_{x \in X_{n-1}} c_{n-2}(z_1, x)$$

exists for all z_1. Since $c_{n-1}' \leq_T c_{n-2}'' \leq_T c_0^{(n)} \in \mathfrak{M}$, there is a d for which there are infinitely many z such that $c_1(z) = d$. Unraveling the definition of all the colorings we see that

$$(\exists^\infty x_1) \cdots (\exists^\infty x_n)[c_0(x_1, \ldots, x_n) = d]$$

holds in $\mathfrak{M}$. Therefore the same holds in $\mathfrak{N}$ since this is an arithmetical statement with parameters in $\mathfrak{N}$. □

2.2 Indecomposability and embeddings We now consider an indecomposability principle between $\mathsf{Elem\text{-}Indec}^n$ and $\mathsf{Iso\text{-}Indec}^n$. Much of the strength of $\mathsf{Iso\text{-}Indec}^n$ comes from the isomorphism requirement. This can be relaxed by asking instead that one of the pieces of the partition *contain* a lexicographically isomorphic copy of $\mathbb{N}^n$. Indeed, this is generally how combinatorial indecomposability is understood for nonordinal order types (see Fraïssé [7]). This leads us to our next formulation of combinatorial indecomposability:

$\mathsf{Lex\text{-}Indec}^n_k$ *For every finite coloring $c : \mathbb{N}^n \to \{0, \dots, k-1\}$, there is a lexicographic embedding $h : \mathbb{N}^n \to \mathbb{N}^n$ such that $c \circ h$ is constant.*

We will use $\mathsf{Lex\text{-}Indec}^n$ to denote $(\forall k)\mathsf{Lex\text{-}Indec}^n_k$. Again, we see that $\mathsf{Lex\text{-}Indec}^1$ is equivalent to $\mathsf{B}\Pi^0_1$. The main result of this section is that $\mathsf{RCA}_0 + \mathsf{Lex\text{-}Indec}^3_2$ is equivalent to ACA_0. Additionally, we show that $\mathsf{Lex\text{-}Indec}^n_k$ implies $\mathsf{Elem\text{-}Indec}^n_k$. Note that $\mathsf{Lex\text{-}Indec}^2_k$ is weaker than $\mathsf{Iso\text{-}Indec}^2_k$, since it follows from Ramsey's theorem for pairs, which is known to be weaker than ACA_0 (see Seetapun and Slaman [14]; see also [1]).

To begin our analysis of $\mathsf{Lex\text{-}Indec}^n$, we will first establish three facts about the behavior of lexicographic embeddings in RCA_0. Except when explicitly stated otherwise, we will write h_i for the ith coordinate of a lexicographic embedding $h : \mathbb{N}^n \to \mathbb{N}^n$.

Lemma 2.7 (RCA_0; $1 \le n < \omega$) *If $h : \mathbb{N}^n \to \mathbb{N}^n$ is a lexicographic embedding, then*

$$x_1 \le h_1(x_1, x_2, \dots, x_n) < h_1(x_1 + 1, 0, \dots, 0)$$

for all $x_1, \dots, x_n \in \mathbb{N}$.

Proof We prove the lemma by (external) induction on $1 \le n < \omega$. The case $n = 1$ is trivial.

Suppose that the result is true for some n. Work in RCA_0. Let $h : \mathbb{N}^{n+1} \to \mathbb{N}^{n+1}$ be a lexicographic embedding. For convenience, we will index our coordinates for $\mathbb{N}^{n+1}$ from zero to n instead of 1 to $n+1$. Thus $h_0 : \mathbb{N}^{n+1} \to \mathbb{N}$ is the first coordinate of h.

We show that

$$h_0(x_0, x_1, \dots, x_n) < h_0(x_0 + 1, 0, \dots, 0)$$

for all $x_0, x_1, \dots, x_n \in \mathbb{N}$; the fact that $x_0 \le h_0(x_0, x_1, \dots, x_n)$ then follows by induction. Suppose, for the sake of contradiction, that $h_0(x_0, x_1, \dots, x_n) = h_0(x_0+1, 0, \dots, 0) = y_0$, say. Then the function $\tilde{h} : \mathbb{N}^n \to \mathbb{N}^n$ such that

$$\tilde{h}_i(z_1, \dots, z_n) = h_i(x_0, x_1 + 1 + z_1, z_2, \dots, z_n)$$

is a lexicographic embedding. By the induction hypothesis,

$$z_1 \le \tilde{h}_1(z_1, 0, \dots, 0) = h_1(x_0, x_1 + 1 + z_1, 0, \dots, 0) \le h_1(x_0 + 1, 0, \dots, 0)$$

for all $z_1 \in \mathbb{N}$, which is clearly impossible. □

Lemma 2.8 (RCA_0; $1 \le n < \omega$) *If $h : \mathbb{N}^n \to \mathbb{N}^n$ is a lexicographic embedding and $1 \le j < i \le n$, then*

$$\lim_{x_i \to \infty} h_j(x_1, \dots, x_{i-1}, x_i, 0, \dots, 0)$$

exists and is bounded above by $h_j(x_1, \dots, x_{i-1} + 1, 0, \dots, 0)$.

Proof We proceed by induction on $j < i$. By the induction hypothesis, find $\tilde{x}_i$ such that

$$h_k(x_1, \dots, x_{i-1}, x_i, 0, \dots, 0) = h_k(x_1, \dots, x_{i-1}, \tilde{x}_i, 0, \dots, 0)$$

for all $x_i \geq \tilde{x}_i$ and $1 \leq k < j$. Note that we must then have

$$\begin{aligned} h_j(x_1, \dots, x_{i-1}, x_i, 0, \dots, 0) &\leq h_j(x_1, \dots, x_{i-1}, x_i', 0, \dots, 0) \\ &\leq h_j(x_1, \dots, x_{i-1} + 1, 0, 0, \dots, 0) \end{aligned}$$

for all $x_i' \geq x_i \geq \tilde{x}_i$. It follows immediately that

$$\lim_{x_i \to \infty} h_j(x_1, \dots, x_{i-1}, x_i, 0, \dots, 0)$$

exists and is bounded above by $h_j(x_1, \dots, x_{i-1} + 1, 0, 0, \dots, 0)$. □

Lemma 2.9 (RCA$_0$; $1 \leq n < \omega$) *If $h : \mathbb{N}^n \to \mathbb{N}^n$ is a lexicographic embedding and $1 \leq i \leq n$, then*

$$\lim_{x_i \to \infty} h_i(x_1, \dots, x_{i-1}, x_i, 0, \dots, 0) = \infty$$

for all $x_1, \dots, x_{i-1} \in \mathbb{N}$.

Proof By Lemma 2.8, we can find $\tilde{x}_i$ such that

$$h_j(x_1, \dots, x_{i-1}, x_i, 0, \dots, 0) = h_j(x_1, \dots, x_{i-1}, \tilde{x}_i, 0, \dots, 0)$$

for all $x_i \geq \tilde{x}_i$ and all $1 \leq j < i$. Note that the function $\tilde{h} : \mathbb{N}^{n-i+1} \to \mathbb{N}^{n-i+1}$ defined by

$$\tilde{h}_k(y_1, \dots, y_{n-i+1}) = h_{i+k-1}(x_1, \dots, x_{i-1}, \tilde{x}_i + y_1, y_2, \dots, y_{n-i+1})$$

is then a lexicographic embedding and the result follows immediately by applying Lemma 2.7 to $\tilde{h}$. □

Theorem 2.10 (RCA$_0$) Lex-Indec$^3_2$ *implies arithmetic comprehension.*

Proof We show how to compute the range of a function $f : \mathbb{N} \to \mathbb{N}$ using Lex-Indec$^3_2$. For each z, let $f[z] = \{f(0), \dots, f(z)\}$. Consider the coloring $c : \mathbb{N}^3 \to \{0, 1\}$ defined by

$$c(x, y, z) = \begin{cases} 0 & \text{when } (\forall w \leq x)(w \in f[y] \leftrightarrow w \in f[z]), \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that $h : \mathbb{N}^3 \to \mathbb{N}^3$ is a lexicographic embedding such that $c \circ h$ is constant. First, note that $c \circ h$ must have constant value zero.

To determine whether x is in the range of f, use the following procedure:

> First find y such that $h_1(x, y, 0) = h_1(x, y + 1, 0)$. Answer yes if $x \in f[h_2(x, y + 1, 0)]$; otherwise answer no.

This procedure will never return false positive answers, so suppose that $x = f(s)$ and we check that the algorithm answers yes on input x. The existence of a y such that $h_1(x, y, 0) = h_1(x, y + 1, 0)$ is guaranteed by Lemma 2.8. Given such a y we can then use Lemma 2.9 to find z such that $s \leq h_3(x, y, z)$. Since

$$h_1(x, y, 0) = h_1(x, y, z) = h_1(x, y + 1, 0),$$

we then have

$$h_2(x, y, 0) \leq h_2(x, y, z) \leq h_2(x, y + 1, 0).$$

Since $c(h(x, y, z)) = 0$ and $x \leq h_1(x, y, z)$ by Lemma 2.7, we know that $x \in f[h_2(x, y, z)] \leftrightarrow x \in f[h_3(x, y, z)]$. Since $s \leq h_3(x, y, z)$ we know that $x \in f[h_3(x, y, z)]$, and since $h_2(x, y, z) \leq h_2(x, y+1, 0)$ we conclude that $x \in f[h_2(x, y+1, 0)]$. □

We end this section by proving that $\mathsf{Lex\text{-}Indec}^n_k$ implies $\mathsf{Elem\text{-}Indec}^n_k$. In light of Theorems 2.10 and 2.2, this is really only interesting in the case $n = 2$.

Proposition 2.11 (RCA_0; $1 \leq n < \omega$) *For every positive integer k, $\mathsf{Lex\text{-}Indec}^n_k$ implies $\mathsf{Elem\text{-}Indec}^n_k$.*

Proof By (external) induction on n, we show that *for any coloring $c : \mathbb{N}^n \to \{0, \ldots, k-1\}$, if there is a lexicographic embedding $h : \mathbb{N}^n \to \mathbb{N}^n$ such that $c \circ h$ is constant with value $d < k$, then*

$$(\exists^\infty x_1) \cdots (\exists^\infty x_n)[c(x_1, \ldots, x_n) = d].$$

The result is trivial for $n = 1$. Suppose that the result is true for some n. Work in RCA_0. Let $c : \mathbb{N}^{n+1} \to \{0, \ldots, k-1\}$ be a coloring, and let $h : \mathbb{N}^{n+1} \to \mathbb{N}^{n+1}$ be a lexicographic embedding $h : \mathbb{N}^{n+1} \to \mathbb{N}^{n+1}$ such that $c \circ h$ is constant with value $d < k$. For convenience, we will index our coordinates for $\mathbb{N}^{n+1}$ from zero to n instead of 1 to $n+1$. Thus $h_0 : \mathbb{N}^{n+1} \to \mathbb{N}$ is the first coordinate of h.

Let $w_0 \in \mathbb{N}$ be given; we want to show that

$$(\exists x_0 \geq w_0)(\exists^\infty x_1) \cdots (\exists^\infty x_n)[c(x_0, x_1, \ldots, x_n) = d]. \tag{†}$$

By Lemma 2.7, we have $w_0 \leq h_0(w_0, w_1, \ldots, w_n) < h_0(w_0 + 1, 0, \ldots, 0)$ for all $w_1, \ldots, w_n \in \mathbb{N}$. By $\mathsf{I}\Sigma^0_1$, let

$$\begin{aligned} x_0 &= \max\{h_0(w_0, w_1, \ldots, w_n) : w_1, \ldots, w_n \in \mathbb{N}\} \\ &= \max\{h_0(w_0, w_1, 0, \ldots, 0) : w_1 \in \mathbb{N}\}, \end{aligned}$$

and pick w_1 such that $x_0 = h_0(w_0, w_1, 0, \ldots, 0)$.

Define the coloring $c' : \mathbb{N}^n \to \{0, \ldots, k-1\}$ by

$$c'(x_1, \ldots, x_n) = c(x_0, x_1, \ldots, x_n),$$

and define the function $h' : \mathbb{N}^n \to \mathbb{N}^n$ by

$$h'_i(z_1, \ldots, z_n) = h_i(w_0, w_1 + z_1, z_2, \ldots, z_n).$$

Then h' is a lexicographic embedding such that $c' \circ h'$ is constant with value d. By the induction hypothesis applied to h' and c',

$$(\exists^\infty x_1) \cdots (\exists^\infty x_n)[c'(x_1, \ldots, x_n) = d].$$

Since $x_0 \geq w_0$, this implies (†). □

Corollary 2.12 (RCA_0) $\mathsf{Lex\text{-}Indec}^2$ *implies* $\mathsf{B}\Pi^0_2$.

Proof $\mathsf{Elem\text{-}Indec}^2$ implies $\mathsf{B}\Pi^0_2$ by Theorem 2.2. □

2.3 Indecomposability and games Another formulation of combinatorial indecomposability is obtained by interpreting the conclusion of $\mathsf{Elem\text{-}Indec}^n_k$ in Hintikka's game-theoretic semantics. This process leads to the following game.

Definition 2.13 Given a finite coloring $c : \mathbb{N}^n \to \{0, \dots, k-1\}$, the game $\mathsf{G}_n(c)$ between player $\oplus$ and player $\ominus$ is played as follows.

- To start the game, player $\oplus$ chooses a color $d < k$.
- Then, player $\ominus$ and player $\oplus$ alternately play

$$\begin{array}{c|cccccccc} \text{player } \ominus & a_1 & & a_2 & & \cdots & & a_n & \\ \hline \text{player } \oplus & & b_1 & & b_2 & & \cdots & & b_n \end{array}$$

  such that $a_i \leq b_i$ for $i = 1, \dots, n$.

Player $\oplus$ wins this play if $c(b_1, b_2, \dots, b_n) = d$, otherwise player $\ominus$ wins.

Of course, player $\ominus$ can never have a winning strategy for this game.

Proposition 2.14 (RCA_0; $1 \leq n < \omega$) *For every finite coloring $c : \mathbb{N}^n \to \{0, \dots, k-1\}$, player $\ominus$ does not have a winning strategy in the game $\mathsf{G}_n(c)$.*

Proof Suppose that $(\sigma_d)_{d<k}$ is such that $\sigma_d : \mathbb{N}^{<n} \to \mathbb{N}$ is a winning strategy for player $\ominus$ in $\mathsf{G}_n(c)$ when player $\oplus$'s first move is d. Define $b_1, \dots, b_n$ by

$$b_m = \max_{d<k} \sigma_d(b_1, \dots, b_{m-1})$$

for $m = 1, \dots, n$. Then, for every $d \in \{0, \dots, k-1\}$, $b_1, b_2, \dots, b_n$ is a valid sequence of play for player $\oplus$ against player $\ominus$'s strategy σ_d, which means that $c(b_1, b_2, \dots, b_n) \neq d$. Therefore, $c(b_1, \dots, b_n) \notin \{0, \dots, k-1\}$—a contradiction. □

If the game $\mathsf{G}_n(c)$ is determined, then player $\oplus$ must have a winning strategy, which leads to the following principle.

$\mathsf{Game\text{-}Indec}^n_k$ *For every finite coloring $c : \mathbb{N}^n \to \{0, \dots, k-1\}$, player $\oplus$ has a winning strategy in the game $\mathsf{G}_n(c)$.*

As usual, we use $\mathsf{Game\text{-}Indec}^n$ to denote $(\forall k)\mathsf{Game\text{-}Indec}^n_k$. Again, it is easy to see that $\mathsf{Game\text{-}Indec}^1$ is equivalent to $\mathsf{B}\Pi^0_1$.

It turns out that $\mathsf{Game\text{-}Indec}^n_k$ is equivalent to a strong version of $\mathsf{Lex\text{-}Indec}^n_k$. A *strong lexicographic embedding* $h : \mathbb{N}^n \to \mathbb{N}^n$ is a lexicographic embedding with the additional property that

$$x_1 = y_1, \dots, x_i = y_i \Rightarrow h_i(x_1, \dots, x_n) = h_i(y_1, \dots, y_n)$$

holds for $i = 1, \dots, n$. Characterizing $\mathsf{Game\text{-}Indec}^n$ as the existence of such strong lexicographic embedding relates $\mathsf{Game\text{-}Indec}^n$ to WRT^2_k and RT^2_k.

Proposition 2.15 (RCA_0; $1 \leq n < \omega$) *Given a finite coloring $c : \mathbb{N}^n \to \{0, \dots, k-1\}$, player $\oplus$ has a winning strategy in $\mathsf{G}_n(c)$ if and only if there is a strong lexicographic embedding $h : \mathbb{N}^n \to \mathbb{N}^n$ such that $c \circ h$ is constant.*

Proof Suppose that $\sigma : \mathbb{N}^{\leq n} \to \mathbb{N}$ is a winning strategy for player $\oplus$. Let $d < k$ be player $\oplus$'s color choice. For $i = 1, \dots, n$, define the function $h_i : \mathbb{N}^i \to \mathbb{N}$ by primitive recursion as follows:

$$h_i(a_1,\dots,a_{i-1},0) = \sigma\big(h_1(a_1),\dots,h_{i-1}(a_1,\dots,a_{i-1}),0\big),$$
$$h_i(a_1,\dots,a_{i-1},a+1) = \sigma\big(h_1(a_1),\dots,h_{i-1}(a_1,\dots,a_{i-1}), h_i(a_1,\dots,a_{i-1},a)+1\big).$$

(The definition of h_i depends on the prior definition of $h_1,\dots,h_{i-1}$, but since n is standard this is not problematic.) Then the function

$$h(a_1,\dots,a_n) = \big(h_1(a_1),\dots,h_n(a_1,\dots,a_n)\big)$$

is a strong lexicographic embedding such that $c \circ h$ is constant with value d.

Conversely, suppose that $h : \mathbb{N}^n \to \mathbb{N}^n$ is a strong lexicographic embedding such that $c \circ h$ is constant with value d. Define the strategy $\sigma : \mathbb{N}^{\leq n} \to \mathbb{N}$ as follows. Let d be the initial color choice for σ, and then define $\sigma(a_1,\dots,a_i) = h_i(a_1,\dots,a_i,0,\dots,0)$. Then, by definition of strong lexicographic embedding, we always have

$$h(a_1,\dots,a_n) = \big(\sigma(a_1),\dots,\sigma(a_1,\dots,a_n)\big),$$

which ensures that σ is a winning strategy for player $\oplus$. $\square$

Corollary 2.16 ($\mathbf{RCA}_0$) $\mathsf{Game\text{-}Indec}^2_k$ *is equivalent to* WRT^2_k.

Corollary 2.17 ($\mathbf{RCA}_0$) RT^2_k *implies* $\mathsf{Game\text{-}Indec}^2_k$.

Proposition 2.18 ($\mathbf{RCA}_0$; $1 \leq n < \omega$) *Suppose that $f : \mathbb{N}^{1+n} \to \mathbb{N}$ is a weakly n-stable function. Then there is a coloring $c : \mathbb{N}^{1+2n} \to \{0,1\}$ such that if player $\oplus$ has a winning strategy in the game $\mathsf{G}_{1+2n}(c)$, then there are an infinite set H and a function $f_\infty : H \to \mathbb{N}$ such that*

$$(\forall^\infty z_1)\cdots(\forall^\infty z_n)[f_\infty(x) = f(x,z_1,\dots,z_n)]$$

holds for every $x \in H$.

Proof Let $c : \mathbb{N}^{1+2n} \to \{0,1\}$ be defined by

$$c(x,y_1,\dots,y_n,z_1,\dots,z_n) = \begin{cases} 1 & \text{when } f(x,\bar{y}) = f(x,\bar{z}), \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\sigma : \mathbb{N}^{\leq 1+2n} \to \mathbb{N}$ is a winning strategy for player $\oplus$ in $\mathsf{G}_{1+2n}(c)$. First note that since f is weakly n-stable, the color 1 must be player $\oplus$'s first move.

Now, knowing that player $\oplus$'s first move is 1, let

$$H = \big\{\sigma(w) : w \in \mathbb{N}\big\} = \big\{x \in \mathbb{N} : (\exists w \leq x)[x = \sigma(w)]\big\}.$$

This is clearly an infinite set. For $x \in H$, define $f_\infty(x)$ as follows: let $w \leq x$ be least such that $x = \sigma(w)$; then let

$$y_1 = \sigma(w,0), \qquad y_2 = \sigma(w,0,0),\dots, \qquad y_n = \sigma(w,0,\dots,0);$$

finally, set $f_\infty(x) = f(x,y_1,\dots,y_n)$. The remainder of player $\oplus$'s strategy σ witnesses that

$$(\exists^\infty z_1)\cdots(\exists^\infty z_n)[f_\infty(x) = f(x,z_1,\dots,z_n)].$$

Since f is n-stable, it follows that

$$(\forall^\infty z_1)\cdots(\forall^\infty z_n)[f_\infty(x) = f(x,z_1,\dots,z_n)],$$

as required. $\square$

Corollary 2.19 ($\mathsf{RCA}_0 + \mathsf{B}\Pi^0_{2n-1}$; $1 \leq n < \omega$) *Suppose that $f : \mathbb{N}^{1+n} \to \mathbb{N}$ is a weakly n-stable function. Then there is a coloring $c : \mathbb{N}^{1+2n} \to \{0,1\}$ such that if player $\oplus$ has a winning strategy in the game $\mathsf{G}_{1+2n}(c)$, then there is a function $f_\infty : \mathbb{N} \to \mathbb{N}$ such that*

$$(\forall^\infty z_1) \cdots (\forall^\infty z_n)[f_\infty(x) = f(x, z_1, \ldots, z_n)]$$

for every x.

Proof The function $\bar{f} : \mathbb{N}^{1+n} \to \mathbb{N}$ defined by

$$\bar{f}(x, z_1, \ldots, z_n) = \langle f(0, z_1, \ldots, z_n), \ldots, f(x, z_1, \ldots, z_n)\rangle$$

is also weakly n-stable by $\mathsf{B}\Pi^0_{2n-1}$; apply Proposition 2.18 to $\bar{f}$. □

Here is a partial converse of Proposition 2.18 for strongly n-stable functions.

Proposition 2.20 (RCA_0; $1 \leq n < \omega$) *If $f^{(i)} : \mathbb{N}^{1+i} \to \mathbb{N}$ are such that*

$$f^{(i-1)}(x, y_1, \ldots, y_{i-1}) = \lim_{y_i \to \infty} f^{(i)}(x, y_1, \ldots, y_{i-1}, y_i)$$

for $i = 1, \ldots, n$, then player $\oplus$ has a winning strategy in the game $\mathsf{G}_{1+2n}(c)$, where $c : \mathbb{N}^{1+2n} \to \{0,1\}$ is the coloring associated to $f^{(0)}$ as in Proposition 2.18.

Player $\oplus$'s strategy is to simply pick sufficiently large natural numbers with the value prescribed by the functions $f^{(i)}$.

When $n = 1$, Propositions 2.18 and 2.20 are exact converses for stable f. In general, these two propositions show that every particular instance of Δ_{1+n}-comprehension corresponds to player $\oplus$ having a winning strategy in a particular instance of the game G_{1+2n}.

3 The Hyperweak Ramsey Theorem

In the last section, we left open some of the questions regarding the various statements of indecomposability for ω^2. Not too surprisingly, these principles are closely related to Ramsey's theorem for pairs. We have shown in Corollary 2.16 that $\mathsf{Game\text{-}Indec}^2_k$ is equivalent to the principle WRT^2_k from the introduction. The other principle from the introduction, namely, HWRT^2_k, turns out to be a close relative of $\mathsf{Lex\text{-}Indec}^2_k$. In its general form, the hyperweak Ramsey theorem is as follows.

HWRT^n_k *For every finite coloring $c : \mathbb{N}^n \to \{0, \ldots, k-1\}$, there are a color $d < k$ and an increasing function $h : \mathbb{N} \to \mathbb{N}$ such that, for all $0 < i_1 < \cdots < i_n$, the box*

$$[h(i_1 - 1), h(i_1) - 1] \times \cdots \times [h(i_n - 1), h(i_n) - 1]$$

contains an n-tuple with color d.

The main result of this section is the following.

Theorem 3.1 *Every countable model of $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ has an ω-extension that satisfies $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \mathsf{HWRT}^2_2$.*

It follows immediately that $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \mathsf{HWRT}^2_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.

The principle HWRT^2_k can be reformulated as follows.

Proposition 3.2 (RCA_0) *The principle* HWRT^2_k *is equivalent to the following statement.*

— *For every finite coloring* $c : \mathbb{N}^2 \to \{0, \dots, k-1\}$, *there are a color* $d < k$ *and an increasing function* $g : \mathbb{N} \to \mathbb{N}$ *such that*

$$\bigcup_{x=g(i-1)}^{g(i)-1} \{y \in \mathbb{N} : c(x, y) = d\} \tag{1}$$

is infinite for every $i \geq 1$.

Proof The fact that HWRT^2_k implies this statement is clear. For the converse, let $d < k$ and $g : \mathbb{N} \to \mathbb{N}$ be as in the statement. For each $i \geq 1$ let $f_i(z)$ be the first $y \geq z$ such that $c(x, y) = d$ for some $g(i-1) \leq x \leq g(i)-1$. Define the sequence $0 < i_0 < i_1 < i_2 < \cdots$ so that $g(i_{\ell+1}) > f_{i_0}(g(i_\ell)), \dots, f_{i_\ell}(g(i_\ell))$ for each ℓ. Then $h(\ell) = g(i_\ell)$ is as required. □

If $c : \mathbb{N}^2 \to \{0, \dots, k-1\}$ is a coloring and $h : \mathbb{N}^2 \to \mathbb{N}^2$ is a lexicographic embedding such that $c \circ h$ is constant with value $d < k$, then the first coordinate function $g(i) = h_1(i, 0)$ is such that each of the sets (1) is infinite.

Corollary 3.3 (RCA_0) $\mathsf{Lex\text{-}Indec}^2_k$ *implies* HWRT^2_k.

The principle HWRT^2_2 is also related to the principle ADS of Hirschfeldt and Shore [9]. A coloring $c : [\mathbb{N}]^2 \to \{0, \dots, k-1\}$ is *transitive* if, for all $x < y < z$, if $c(x, y) = c(y, z)$, then $c(x, z) = c(x, y) = c(y, z)$.

ADS *Every transitive coloring* $c : [\mathbb{N}]^2 \to \{0, 1\}$ *has an infinite homogeneous set.*

For every transitive coloring $c : [\mathbb{N}]^2 \to \{0, 1\}$ there is a unique linear ordering $\prec$ such that, for all $x < y$,

$$c(x, y) = \begin{cases} 0 & \text{when } x \succ y, \\ 1 & \text{when } x \prec y. \end{cases}$$

Thus ADS is equivalent to the statement that every linear ordering of $\mathbb{N}$ has an infinite ascending or descending sequence, hence the name. The principle SADS is the restriction of ADS to stable transitive colorings.

Proposition 3.4 (RCA_0) HWRT^2_2 *implies* SADS.

Proof Let $c : [\mathbb{N}]^2 \to \{0, 1\}$ be a stable transitive coloring. By HWRT^2_2, there are a color and an increasing function $h : \mathbb{N} \to \mathbb{N}$ such that, for all $0 < i < j$, the rectangle

$$[h(i-1), h(i)-1] \times [h(j-1), h(j)-1]$$

contains a pair with color d. Let $\prec$ be the linear ordering of $\mathbb{N}$ which agrees with color d, and for each i let $m(i)$ be the $\prec$-minimal element of the interval $[h(i-1), h(i)-1]$. Note that $m(i)$ is necessarily $\prec$-below all but finitely many elements of $\mathbb{N}$. Therefore, we can define the sequence $i_0 = 0 < i_1 < i_2 < \cdots$ so that each i_{k+1} is the least $i > i_k$ such that $c(m(i_k), m(i)) = d$. By transitivity, the sequence $\{m(i_k)\}_{k=0}^\infty$ is an infinite c-homogeneous set. □

It was shown by Chong, Lempp, and Yang [2] that SADS implies $\mathsf{B}\Pi^0_1$.

Corollary 3.5 (RCA_0) HWRT^2_2 *implies* $\mathsf{B}\Pi^0_1$.

3.1 Forcing preliminaries The forcings we will be interested in are forcings with finite conditions. That is, our poset $\mathbb{Q}$ of forcing conditions is a Σ^0_1-definable set, and so are the order relation $\leq$ and the incompatibility relation $\perp$. We will first develop the general theory of such forcings before we deal with actual examples to prove Theorem 3.1.

Our approach to forcing follows that of Dorais [3]. In particular, we work within the functional interpretation of RCA_0 presented in the introduction. For the rest of this section we show how to adapt the forcing machinery of [3] to forcings with finite conditions.

We first develop the basic machinery necessary to define the internal forcing language. The base level of this are the forcing names, which are the terms of the forcing language. A *partial k-ary name* is a Σ^0_1-definable set $F \subseteq \mathbb{Q} \times \mathbb{N}^{k+1}$ (with ground model parameters) such that

- if $(p, \bar{x}, y) \in F$ and $q \leq p$, then $(q, \bar{x}, y) \in F$;
- if $(p, \bar{x}, y), (p, \bar{x}, y') \in F$, then $y = y'$.

We say that F is a *p-local* if for every $q \leq p$ and every $\bar{x} \in \mathbb{N}$ there are $y \in \mathbb{N}$ and $r \leq q$ such that $(r, \bar{x}, y) \in F$.

Before we discuss the syntax of the forcing language, we will discuss the semantics of these names. A filter $G \subseteq \mathbb{Q}$ is *Π^0_n-generic* (over $\mathfrak{N}$) if for every set $D \subseteq \mathbb{Q}$ which is Π^0_n definable over $\mathfrak{N}$, there is a $p \in G$ such that either $p \in D$ or else p has no extension in D at all.

If G is Π^0_1-generic and F is a p-local k-ary name for some $p \in G$, then the *evaluation* F^G is the total k-ary function defined by

$$F^G(\bar{x}) = y \Leftrightarrow (\exists q \in G)\big((q, \bar{x}, y) \in F\big).$$

The basic projections, constants, and indeed all ground model functions F have *canonical names* $\check{F}$ defined by

$$(p, \bar{x}, y) \in \check{F} \Leftrightarrow y = F(\bar{x}),$$

which invariably evaluate to F. The *generic extension* $\mathfrak{N}[G]$ is the ω-extension of $\mathfrak{N}$ whose functions consist of the evaluations of all names that are p-local for some $p \in G$.

In a typical language, the basic terms are composed to form the class of all terms. This is not so for the forcing language since composition and other operations can be done directly at the semantic level. If F is a partial ℓ-ary name and $F_1, \ldots, F_\ell$ and are partial k-ary names, then the superposition $H = F \circ (F_1, \ldots, F_\ell)$ is defined by

$$(p, \bar{x}, z) \in H \Leftrightarrow \exists \bar{y}\big((p, \bar{x}, y_1) \in F_1 \wedge \cdots \wedge (p, \bar{x}, y_\ell) \in F_\ell \wedge (p, \bar{y}, z) \in F\big).$$

This is a partial k-ary name, and if each of $F, F_1, \ldots, F_\ell$ is p-local, then so is H. Primitive recursion can be handled in a similar way. Given a partial $(k-1)$-ary name F_0 and a $(k+1)$-ary name F, the k-ary name H is defined by $(p, \bar{x}, y, z) \in H$ iff there is a finite sequence $\langle z_0, \ldots, z_y \rangle$ with $z = z_y$ such that $(p, \bar{x}, z_0) \in F_0$ and $(p, \bar{x}, i, z_i, z_{i+1}) \in F$ for every $i < y$. This is again a partial k-ary name, and if F_0, F are p-local, then so is H. Other recursive operations can be handled via Proposition 3.8.

The *formulas of the forcing language* are defined in the usual manner as the smallest family which is closed under the following formation rules.

- If F is a partial k-ary name, F' is a partial k'-ary name, and $\bar{v} = v_1, \dots, v_k$, $\bar{v}' = v'_1, \dots, v'_{k'}$ are variable symbols, then $(F(\bar{v}) = F'(\bar{v}'))$ is a formula.
- If φ is a formula, then so is $\neg\varphi$.
- If φ and ψ are formulas, then so is $(\varphi \wedge \psi)$.
- If φ is a formula and v is a variable symbol, then $(\forall v)\psi$ is also a formula.

Free and bound variables are defined in the usual manner. The *sentences* of the forcing language are formulas without free variables. Although not present in the formal language, we will freely use $\vee$, $\to$, $\leftrightarrow$, and $\exists$ as abbreviations:

$$(\varphi \vee \psi) \equiv \neg(\neg\varphi \wedge \neg\psi), \qquad (\varphi \to \psi) \equiv \neg(\varphi \wedge \neg\psi),$$
$$(\varphi \leftrightarrow \psi) \equiv (\varphi \to \psi) \wedge (\psi \to \varphi), \qquad (\exists v)\varphi \equiv \neg(\forall v)\neg\varphi.$$

Because our language lacks $\leq$, bounded quantifiers are defined by

$$(\forall v \leq F)\varphi \equiv (\forall v)(v \dot{-} F = 0 \to \varphi), \qquad (\exists v \leq F)\varphi \equiv (\exists v)(v \dot{-} F = 0 \wedge \varphi).$$

Bounded formulas are those whose quantifiers are all of this form, where the name F does not depend on the quantified variable v. The usual arithmetic hierarchy is then built from these in the usual manner by alternation of quantifiers.

We are now ready to define the forcing relation. The definition for atomic sentences is motivated by the above definition of the forcing extension. The remaining cases follow the classical definition of forcing. The definition of $p \Vdash \theta$ is by induction on the complexity of the sentence θ. Assume that all names occurring in the sentences below are p-local:

- $p \Vdash (F = F')$ iff, for all $q \leq p$ and $y, y' \in \mathbb{N}$, if $(q, y) \in F$ and $(q, y') \in F'$, then $y = y'$;
- $p \Vdash (\varphi \wedge \psi)$ iff $p \Vdash \varphi$ and $p \Vdash \psi$;
- $p \Vdash (\forall v)\varphi(v)$ iff $p \Vdash \varphi(\check{x})$, for all $x \in \mathbb{N}$;
- $p \Vdash \neg\varphi$ iff there is no $q \leq p$ such that $q \Vdash \varphi$.

The meaning of the forcing relation for the abbreviations defined above can be computed as usual:

- $p \Vdash (F \neq F')$ iff, for all $q \leq p$ and $y, y' \in \mathbb{N}$, if $(q, y) \in F$ and $(q, y') \in F'$, then $y \neq y'$;
- $p \Vdash (\varphi \vee \psi)$ iff for every $q \leq p$ there is an $r \leq q$ such that either $r \Vdash \varphi$ or $r \Vdash \psi$;
- $p \Vdash (\varphi \to \psi)$ iff for every $q \leq p$ such that $q \Vdash \varphi$, there is an $r \leq q$ such that $r \Vdash \psi$;
- $p \Vdash (\exists v)\varphi(v)$ iff for every $q \leq p$ there are an $r \leq q$ and an $x \in \mathbb{N}$ such that $r \Vdash \varphi(\check{x})$.

Moreover, this is a classical forcing:

- $p \Vdash \neg\neg\varphi$ iff $p \Vdash \varphi$.

Lemma 3.6 *For every bounded formula $\varphi(\bar{v})$ of the forcing language, there is a partial name $T_\varphi(\bar{v})$ such that $\varphi(\bar{v})$ is p-local if and only if $T_\varphi(\bar{v})$ is p-local, and then $p \Vdash (\forall \bar{v})[\varphi(\bar{v}) \leftrightarrow T_\varphi(\bar{v}) = 0]$.*

Proof We define $T_\varphi(\bar{v})$ by induction on the complexity of $\varphi(\bar{v})$:

- $T_{F=F'}(\bar{v}) = |F(\bar{v}) - F'(\bar{v})|$;
- $T_{\neg\varphi}(\bar{v}) = 1 \dot{-} T_\varphi(\bar{v})$;
- $T_{\varphi\wedge\psi}(\bar{v}) = T_\varphi(\bar{v}) + T_\psi(\bar{v})$;

- $T_{(\forall w \leq F)\varphi}(\bar{v}) = \sum_{w \leq F(\bar{v})} T_\varphi(\bar{v}, w)$.

The fact that $T_\varphi(\bar{v})$ is p-local when φ is p-local follows from the fact that p-local names are closed under superposition and primitive recursion. $\square$

The following fact is then easy to check by induction on n.

Proposition 3.7 (RCA$_0$; $1 \leq n < \omega$) *If $\theta(v_1, \ldots, v_k)$ is a p-local Π^0_n-formula of the forcing language, then the relation $p \Vdash \theta(\check{x}_1, \ldots, \check{x}_k)$ is Π^0_n, uniformly in the parameter p.*

Note, however, that the Σ^0_n forcing relation is not generally Σ^0_n.

It follows that if G is a Π^0_n-generic filter over $\mathfrak{N}$, then for every Π^0_n-sentence θ which is p-local for some $p \in G$, there is a condition $q \in G$ such that either $q \Vdash \theta$ or $q \Vdash \neg\theta$. Working through the inductive definitions, we see that in this scenario

$$\mathfrak{N}[G] \models \theta^G \Leftrightarrow q \Vdash \theta \text{ for some } q \in G,$$

where θ^G is the standard formula obtained by replacing all names of θ by their evaluations.

Proposition 3.8 (RCA$_0$) *If $\theta(\bar{v}, w)$ is a p-local Σ^0_1-formula of the forcing language such that $p \Vdash (\forall \bar{v})(\exists w)\theta(\bar{v}, w)$, then there is a p-local name F such that $p \Vdash (\forall \bar{v})\theta(\bar{v}, F(\bar{v}))$.*

Proof We may assume that $\theta(\bar{v}, w)$ is actually bounded and moreover of the form $T(\bar{v}, w) = 0$ for some p-local name T as in Lemma 3.6. Let $(q_n, \bar{x}_n, y_n)$ enumerate the Σ^0_1-definable set

$$\left\{(q, \bar{x}, y) \in \mathbb{Q} \times \mathbb{N}^{k+1} : q \leq p \wedge (q, \bar{x}, y, 0) \in T\right\}.$$

Then define the name F by $(q_n, \bar{x}_n, y_n) \in F$ iff for every $m < n$, we have $q_m \perp q_n$, $\bar{x}_m \neq \bar{x}_n$, or $y_m = y_n$. $\square$

If G is a filter over $\mathbb{Q}$, then let $\mathfrak{N}[G]$ be the model obtained by evaluating all G-local names at G. Since elements of $\mathfrak{N}$ all have canonical names, we see that $\mathfrak{N}[G]$ is an ω-extension of $\mathfrak{N}$.

Theorem 3.9 (RCA$_0$) *If G is Π^0_2-generic for $\mathbb{Q}$ over $\mathfrak{N}$, then $\mathfrak{N}[G] \models$ RCA$_0$.*

This is a consequence of Proposition 3.8 and the fact that G-local names are closed under superposition and primitive recursion.

Note that Theorem 3.9 can fail if the assumption on G is weakened to Π^0_1-genericity. This is because we use only names which are p-local for some $p \in G$, which is not always sufficient to guarantee closure under recursive comprehension. However, if $\mathfrak{N}[G] \models \mathsf{I}\Sigma^0_1$, then $\mathfrak{N}[G]$ can be closed under recursive comprehension to form a model of RCA$_0$.

3.2 Forcing construction Let $\mathbb{P} = (P, \leq)$ be the poset of all (codes for) finite increasing functions $p : \{0, 1, \ldots, |p| - 1\} \to \mathbb{N}$, ordered by end extension. (This is a variant of Cohen forcing.) Let $c : \mathbb{N}^2 \to \{0, 1\}$ be a coloring in $\mathfrak{N}$ for which there is no increasing $h : \mathbb{N} \to \mathbb{N}$ such that the set

$$\bigcup_{x=h(n)}^{h(n+1)-1} \left\{y \in \mathbb{N} : c(x, y) = 1\right\} \tag{2}$$

is infinite for every n. Let $\mathbb{P}' = (P', \leq')$ be the subposet consisting of all $p \in \mathbb{P}$ such that

$$\bigcup_{x=p(n)}^{p(n+1)-1} \{y \in \mathbb{N} : c(x, y) = 0\} \tag{3}$$

is *cofinite* for every $n < |p| - 1$. The poset $\mathbb{P}'$ is Σ^0_2-definable over $\mathfrak{N}$, so our methods do not necessarily apply for forcing with $\mathbb{P}'$ over $\mathfrak{N}$. Instead, we force over a Σ^0_2-*envelope of* $\mathfrak{N}$: an ω-extension $\mathfrak{N}'$ of $\mathfrak{N}$ such that $\mathfrak{N}' \models \mathsf{RCA}_0$ and every total Σ^0_2-definable function over $\mathfrak{N}$ belongs to $\mathfrak{N}'$. (Since $\mathfrak{N} \models \mathsf{I}\Sigma^0_2$, the model $\mathfrak{N}'$ consisting of all total functions which are Σ^0_2-definable over $\mathfrak{N}$ is as required, but we will need the more general notion later.)

The hypothesis that there is no increasing function $h : \mathbb{N} \to \mathbb{N}$ such that the sets (2) are all infinite clearly implies that each one of the sets $D'_n = \{p \in \mathbb{P}' : |p| \geq n\}$ is open dense. Since a generic filter G for $\mathbb{P}'$ must meet each one of these open dense sets, we see that $g = \bigcup G$ is a well-defined increasing function $g : \mathbb{N} \to \mathbb{N}$ such that (3) is cofinite for each n. This function g is the *generic real* associated to G. The generic filter G is in fact completely determined by g since $G = \{p \in \mathbb{P} : p \subseteq g\}$. Since g will be of greater interest, we will systematically work with g instead of G throughout the following.

The following fact is the keystone to showing that forcing with $\mathbb{P}'$ over $\mathfrak{N}'$ leads to a generic function g which is well behaved over the ω-submodel $\mathfrak{N}$.

Lemma 3.10 *Suppose that $U \subseteq \mathbb{P}$ is Σ^0_1-definable over $\mathfrak{N}$. If every $p \in \mathbb{P}'$ is such that for every $q \leq' p$ there is an $r \leq q$ such that $r \in U$, then for every $q \leq' p$ there is an $r \leq' q$ such that $r \in U$.*

Proof Suppose that every $q \leq' p$ has an extension in U but there is some $q \leq' p$ which has no extension in $U \cap \mathbb{P}'$. We will use such a q to construct an increasing function $h : \mathbb{N} \to \mathbb{N}$ in $\mathfrak{N}$ such that the set (2) is infinite for every $i \geq 1$, thereby contradicting our hypothesis that there are no such functions.

First, find $h(0) \in \mathbb{N}$ such that $q_0 \in \mathbb{P}'$ where $q_0 = q \frown h(0)$. By hypothesis, we can find an extension $r_0 \geq q_0$ in U. It follows that $r_0 \notin \mathbb{P}'$, which means that

$$\bigcup_{x=r_0(i-1)}^{r_0(i)-1} \{y \in \mathbb{N} : c(x, y) = 0\}$$

is coinfinite for some $0 < i < |r_0|$. Since $q_0 \in \mathbb{P}'$, this i must be greater than or equal to $|q_0|$. So if we set $h(1) = \max(r_0) = r_0(|r_0| - 1)$, we necessarily have that

$$\bigcup_{x=h(0)}^{h(1)-1} \{y \in \mathbb{N} : c(x, y) = 1\}$$

is infinite.

Once $h(n)$ has been defined, set $q_n = q \frown h(n)$, and note that q_n is necessarily in $\mathbb{P}'$ since $h(n) \geq h(0)$. As above, we can then find an extension $r_n \leq q_n$ in U and set $h(n+1) = \max(r_n) = r_n(|r_n| - 1)$. As before, we then have that

$$\bigcup_{x=h(n)}^{h(n+1)-1} \{y \in \mathbb{N} : c(x, y) = 1\}$$

is infinite.

This construction can be carried out completely inside $\mathfrak{N}$. Indeed, all we need to do at each stage is to search for an extension $r_n \le q_n$ in U, which can be done by enumerating U until such an r_n is found. $\square$

Proposition 3.11 *If $p \in \mathbb{P}'$ and F is a p-local name for $\mathbb{P}$ over $\mathfrak{N}$, then F is also p-local for $\mathbb{P}'$ over $\mathfrak{N}'$.*

Proof Apply Lemma 3.10 to the sets $U_{\bar{x}} = \{r \in \mathbb{P} : (\exists y)[(r, \bar{x}, y) \in F]\}$. $\square$

Note that there are names for $\mathbb{P}$ over $\mathfrak{N}$ which are p-local for $\mathbb{P}'$ over $\mathfrak{N}'$, but not p-local for $\mathbb{P}$ over $\mathfrak{N}$. One such name is the 2-ary name f such that $f(n, m)$ is the $(m+1)$th element of

$$\bigcup_{x=g(n)}^{g(n+1)-1} \{y \in \mathbb{N} : c(x, y) = 0\}.$$

In particular, the generic function g is not Π^0_2-generic for $\mathbb{P}$ over $\mathfrak{N}$. It is, however, weakly Σ^0_2-generic for $\mathbb{P}$ over $\mathfrak{N}$, as we will now show.

Proposition 3.12 *Let θ be a p-local Σ^0_2-sentence of the forcing language for $\mathbb{P}$ over $\mathfrak{N}$. If $p \in \mathbb{P}'$ and $p \Vdash' \theta$, then there is a $q \le' p$ such that $q \Vdash \theta$.*

Proof By Lemma 3.6, we may suppose that θ is of the form $(\exists u)(\forall v)[F(u, v) = 0]$, where F is a 2-ary p-local name for $\mathbb{P}$ over $\mathfrak{N}$. Without loss of generality, we may further assume that there is some $x \in \mathbb{N}$ such that, for all $y \in \mathbb{N}$, $p \Vdash' F(\check{x}, \check{y}) = 0$. Applying Lemma 3.10 to the set $U = \{r \in \mathbb{P} : (\exists y, z)[(r, x, y, z) \in F \wedge z \neq 0]\}$, we see that there must be a $q \le' p$ with no extension in U at all. This is equivalent to saying that $q \Vdash F(\check{x}, \check{y}) = 0$ for all $y \in \mathbb{N}$. Hence, $q \Vdash \theta$. $\square$

Together with the generic extension $\mathfrak{N}'[g]$ of $\mathfrak{N}'$, we obtain an ω-extension $\mathfrak{N}[g]$ of $\mathfrak{N}$ by evaluating all partial names in $\mathfrak{N}$ which are g-local for $\mathbb{P}'$ over $\mathfrak{N}'$. This is not a generic extension, but it does satisfy RCA_0. In order to iterate the forcing construction, we will need to make sure that the generic extension $\mathfrak{N}'[g]$ is a Σ^0_2-envelope for $\mathfrak{N}[g]$. The key to proving this is the following fact.

Proposition 3.13 *Let $\theta(u, v)$ be a p-local Σ^0_2-formula of the forcing language for $\mathbb{P}$ over $\mathfrak{N}$. If $p \Vdash' (\forall u)(\exists w)\theta(u, w)$, then there is a p-local name F for $\mathbb{P}'$ over $\mathfrak{N}'$ such that $p \Vdash' (\forall u)\theta(u, F(u))$.*

Proof Suppose that $\theta(u, v) \equiv (\exists w)\varphi(u, v, w)$, where $\varphi(u, v, w)$ is a p-local Π^0_1-formula of the forcing language for $\mathbb{P}$ over $\mathfrak{N}$. By Proposition 3.7, the relation

$$R = \{(q, x, y, z) \in \mathbb{P} \times \mathbb{N}^3 : q \Vdash \varphi(\check{x}, \check{y}, \check{z})\}$$

is Π^0_1-definable over $\mathfrak{N}$. Therefore $R' = R \cap \mathbb{P}' \times \mathbb{N}^3 \in \mathfrak{N}'$. Fix an enumeration $\langle r_n, x_n, y_n, z_n \rangle$ of R', and define the partial name F by $(q, x, y) \in F$ iff there is an n such that $x = x_n$, $y = y_n$, and $q \le r_n$ but $q \not\le r_m$ for $m < n$. Proposition 3.12 shows that F is a p-local name and that $p \Vdash' (\forall u)\theta(u, F(u))$, as required. $\square$

Thus, if the Σ^0_2-formula $\theta(u, v)$ defines a map over $\mathfrak{N}[g]$, then this function actually belongs to the generic envelope $\mathfrak{N}'[g]$. In other words, $\mathfrak{N}'[g]$ is a Σ^0_2-envelope for $\mathfrak{N}[g]$.

Proof of Theorem 3.1 We start with a model $\mathfrak{N}_0$ of $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$. Then we find a Σ^0_2-envelope $\mathfrak{N}'_0$ for $\mathfrak{N}_0$ as explained above. Let $c : \mathbb{N}^2 \to \{0, 1\}$ be a coloring in $\mathfrak{N}_0$ for which there is no increasing $h : \mathbb{N} \to \mathbb{N}$ such that

$$\bigcup_{x=h(n)}^{h(n+1)-1} \{y : c(x, y) = 0\}$$

is infinite for every n. Then we force with $\mathbb{P}'$ over $\mathfrak{N}'_0$ to obtain a generic extension $\mathfrak{N}'_1 = \mathfrak{N}'_0[g]$ and at the same time an ω-extension $\mathfrak{N}_1 = \mathfrak{N}_0[g]$, where $g : \mathbb{N} \to \mathbb{N}$ is an increasing function such that

$$\bigcup_{x=g(n)}^{g(n+1)-1} \{y : c(x, y) = 0\}$$

is cofinite for every n. By Proposition 3.13, we then have that $\mathfrak{N}'_1$ is a Σ^0_2-envelope for $\mathfrak{N}_1$, and hence $\mathfrak{N}_1 \vDash \mathsf{I}\Sigma^0_2$.

We can iterate this process to obtain two parallel sequences of ω-extensions

$$\begin{array}{ccccccccccc} \mathfrak{N}_0 & \subseteq & \mathfrak{N}_1 & \subseteq & \mathfrak{N}_2 & \subseteq & \cdots & \subseteq & \mathfrak{N}_\omega & = & \bigcup_{i<\omega} \mathfrak{N}_i \\ \cap & & \cap & & \cap & & & & \cap & & \\ \mathfrak{N}'_0 & \subseteq & \mathfrak{N}'_1 & \subseteq & \mathfrak{N}'_2 & \subseteq & \cdots & \subseteq & \mathfrak{N}'_\omega & = & \bigcup_{i<\omega} \mathfrak{N}'_i \end{array}$$

At each stage, we have that $\mathfrak{N}_i$, $\mathfrak{N}'_i$ are both models of RCA_0 and $\mathfrak{N}'_i$ is a Σ^0_2-envelope of $\mathfrak{N}_i$. It follows that these facts are also true for $\mathfrak{N}_\omega$ and $\mathfrak{N}'_\omega$. Therefore $\mathfrak{N}_\omega \vDash \mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$. Moreover, with careful bookkeeping to deal with every potential counterexample $c : \mathbb{N}^2 \to \{0, 1\}$ of HWRT^2_2, we can make sure that $\mathfrak{N}_\omega \vDash \mathsf{HWRT}^2_2$. In the end, $\mathfrak{N}_\omega$ is the required ω-extension. □

3.3 Forcing over ω-models When $\mathfrak{N}$ is an ω-model of RCA_0, the forcing methods of the last section are slight overkill. Indeed, there is no risk of breaking induction by adjoining more second-order elements to $\mathfrak{N}$. Nevertheless, the forcing posets used in Section 3.2 can be used to shed some light on the situation for ω-models.

Proposition 3.14 (ACA_0) *For every computable coloring $c : \mathbb{N}^2 \to \{0, 1\}$ one of the following is true.*

- *There is a computable increasing function $h : \mathbb{N} \to \mathbb{N}$ such that*

$$\bigcup_{x=h(n)}^{h(n+1)-1} \{y \in \mathbb{N} : c(x, y) = 1\}$$

  is infinite for every n.
- *There is a $0'$-computable 1-generic increasing function $g : \mathbb{N} \to \mathbb{N}$ such that*

$$\bigcup_{x=g(n)}^{g(n+1)-1} \{y \in \mathbb{N} : c(x, y) = 0\}$$

  is cofinite for every n.

Proof Suppose that there is no computable increasing function $h : \mathbb{N} \to \mathbb{N}$ such that

$$\bigcup_{x=h(n)}^{h(n+1)-1} \{y \in \mathbb{N} : c(x, y) = 1\}$$

is infinite for every n. Let $\mathbb{P}$ and $\mathbb{P}'$ be defined as in Section 3.2. Note that Lemma 3.10 applies to the countable coded ω-model REC whose second-order part consists of all computable sets, as computed in our ambient model of ACA_0. In our ambient model of ACA_0, we have an effective listing $\langle U_n \rangle_{n=0}^{\infty}$ of all computably enumerable (Σ_1^0 over REC) subsets of $\mathbb{P}$.

Since $\mathbb{P}'$ is Σ_2^0-definable (without parameters) we have a $0'$-computable enumeration $\langle p_i \rangle_{i=0}^{\infty}$ of $\mathbb{P}'$. Define the sequence $\langle q_n \rangle_{n=0}^{\infty}$ of elements of $\mathbb{P}'$ as follows.

- Let q_0 be an arbitrary element of $\mathbb{P}'$.
- Once q_n has been defined, let q_{n+1} be the first p_i in our enumeration such that $p_i \leq' q_n$ and either $p_i \in U_n$ or else there is no extension $r \leq p_i$ such that $r \in U_n$.

Lemma 3.10 ensures that there always is a $q_{n+1} \leq' q_n$ as required by the second condition. Furthermore, since each U_n is computably enumerable, the requirements for the second condition can be checked using $0'$ as an oracle.

It follows that the sequence $\langle q_n \rangle_{n=0}^{\infty}$ is $0'$-computable, and hence so is $g = \bigcup_{n=0}^{\infty} q_n$. Note that g is a well-defined increasing function $\mathbb{N} \to \mathbb{N}$ since our listing $\langle U_n \rangle_{n=0}^{\infty}$ includes all of the open dense sets $\{p \in \mathbb{P} : |p| \geq i\}$. Moreover, g is clearly 1-generic, and since every initial segment of g is in $\mathbb{P}'$, we see that

$$\bigcup_{x=g(n)}^{g(n+1)-1} \{y \in \mathbb{N} : c(x, y) = 0\}$$

is cofinite for every n. □

Since 1-generic degrees below $0'$ are low, by iterating the relativized form of Proposition 3.14, we see that the following holds.

Corollary 3.15 (ACA_0) *There is a countable coded ω-model of* $\mathsf{RCA}_0 + \mathsf{HWRT}_2^2$ *whose second-order part consists entirely of low sets.*

It was shown by Downey, Hirschfeldt, Lempp, and Solomon [4] that SRT_2^2 has no ω-model whose second-order part consists entirely of low sets. A close inspection of their argument shows that this can be formalized in ACA_0.

Corollary 3.16 (ACA_0) HWRT_2^2 *does not imply* SRT_2^2 *over* RCA_0.

It follows that HWRT_2^2 also does not imply WRT_2^2 since the latter implies SRT_2^2 over RCA_0.

4 Conclusions and Questions

In the first part of this paper, we investigated various formulations of the pigeonhole principle for finite ordinal powers of the ordinal ω. The weakest such principle $\mathsf{Elem\text{-}Indec}^n$ was found to be sandwiched between two standard induction principles. Namely, Theorem 2.2 showed that

$$\mathsf{I}\Sigma_{n+1}^0 \longrightarrow \mathsf{Elem\text{-}Indec}^n \longrightarrow \mathsf{B}\Pi_n^0.$$

It is known that $\mathsf{I}\Sigma^0_{n+1}$ is strictly stronger than $\mathsf{B}\Pi^0_n$ (see [8]), but we do not know how Elem-Indec^n sits in between the two.

Question 4.1 Does Elem-Indec^n lie strictly between $\mathsf{I}\Sigma^0_{n+1}$ and $\mathsf{B}\Pi^0_n$ in the hierarchy of induction principles?

Hirst's result [11] that Elem-Indec^1 is equivalent to $\mathsf{B}\Pi^0_1$ suggests that Elem-Indec^n might be equivalent to $\mathsf{B}\Pi^0_n$. Indeed, there is hope that some induction could be shaved off from our proof of Proposition 2.6.

The two stronger principles Lex-Indec^n_k and Game-Indec^n_k turned out to be equivalent to ACA_0 when $n \geq 3$ and $k \geq 2$. However, for $n = 2$, both Lex-Indec^2_k and Game-Indec^2_k follow from RT^2_k (indeed WRT^2_k), which is known to be strictly weaker than ACA_0 (see [14], [1]). This led us to consider two weak forms of Ramsey's theorem for pairs, namely, WRT^2_k and HWRT^2_k. Another interesting possible weakening of RT^2_k was considered by Dzhafarov and Hirst [5], namely the increasing polarized theorem for pairs (IPT^2_k), which is sandwiched between RT^2_k and WRT^2_k. The known implications between these principles in the case $k = 2$ are summarized in the following diagram:

$$\begin{tikzcd}
\mathsf{RT}^2_2 \arrow[r] & \mathsf{IPT}^2_2 \arrow[r] & \text{Game-Indec}^2_2 \arrow[r, leftrightarrow] \arrow[d] & \mathsf{WRT}^2_2 \arrow[r] \arrow[d] & \mathsf{SRT}^2_2 \arrow[d] & \\
 & & \text{Lex-Indec}^2_2 \arrow[r] & \mathsf{HWRT}^2_2 \arrow[r] \arrow[ur, "/" marking] & \mathsf{SADS} \arrow[r] & \mathsf{B}\Pi^0_1
\end{tikzcd}$$

Besides the nonimplications $\mathsf{HWRT}^2_2 \not\rightarrow \mathsf{SRT}^2_2$, $\mathsf{B}\Pi^0_1 \not\rightarrow \mathsf{SADS}$, and their consequences, we do not know whether any of the remaining implications are strict. Many of the resulting questions are special cases or refinements of the open questions from [1], [9], and [5]. For example, it is still an open question whether SRT^2_2 implies RT^2_2 (see [1, Question 13.6]). Of the remaining questions, we wonder the following.

Question 4.2 Is HWRT^2_2 strictly weaker than Lex-Indec^2_2?

Question 4.3 Is WRT^2_2 strictly stronger than Lex-Indec^2_2?

Of course, a negative answer to Question 4.2 would provide a positive answer to Question 4.3. Similarly, a negative answer to Question 4.3 would provide a positive answer to Question 4.2. However, it is plausible that both questions have a positive answer.

In another line of thought, we wonder how HWRT^2_2 is related to other combinatorial consequences of Ramsey's theorem for pairs. Of particular interest is the following.

Question 4.4 How is HWRT^2_2 related to the principle SCAC of Hirschfeldt and Shore [9]?

Indeed, the similarity between the forcing construction from Section 3 and those used by Hirschfeldt and Shore suggests that there might be some nontrivial ties between these two principles.

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