

A General Characterization of the Variable-Sharing Property by Means of Logical Matrices

Gemma Robles and José M. Méndez

Abstract As is well known, the variable-sharing property (vsp) is, according to Anderson and Belnap, a necessary property of any relevant logic. In this paper, we shall consider two versions of the vsp, what we label the “weak vsp” (wvsp) and the “strong vsp” (svsp). In addition, the “no loose pieces property,” a property related to the wvsp and the svsp, will be defined. Each one of these properties shall generally be characterized by means of a class of logical matrices. In this way, any logic verified by an actual matrix in one of these classes has the property the class generally represents. Particular matrices (and so, logics) in each class are provided.

1 Introduction

As is well known, according to Anderson and Belnap, the following is a necessary property of any relevant logic S (see [1]).

Definition 1.1 (Variable-sharing property—vsp) If $A \rightarrow B$ is a theorem of S , then A and B share at least one propositional variable.

The main results on the vsp are the following [cf. [1]):

1. Any logic equivalent to (or included in) the logic of relevant implication, R , has the vsp.
2. The logic R -Mingle, RM , and so, any of its extensions does not have the vsp.

The logic R -Mingle is the result of adding the axiom “mingle”

$$M. A \rightarrow (A \rightarrow A)$$

to R . And despite the fact recorded in (2), the vsp and M are not, in general, incompatible. In [7] (see also [8]), for example, it is proved that the vsp and M are

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compatible if the De Morgan negation characteristic of relevant logics is restricted. And in [2], it is shown that the vsp is preserved when M is added, together with an intensional conjunction, to the implication-negation fragment of R. Therefore, we have a third result on the vsp of some interest:

3. The vsp and the axiom mingle are compatible if either the extensional conjunction and disjunction of standard relevant logics or else the De Morgan negation characteristic of these logics are dropped.

But let us now return to the fact stated in (1). Anderson and Belnap actually prove a much stronger result in [1]. They prove that any logic equivalent to (or included in) R has what we shall here name the “strong variable-sharing property” and the “no loose pieces property.” In order to discuss these properties, the following definition of antecedent parts and consequent parts of wff is needed (cf. [1], p. 240).

Definition 1.2 (Antecedent parts and consequent parts) Let A be a wff. Then

1. A is a cp of A ,
2. If $B \wedge C$ is a cp (ap) of A , then both B and C are cps (aps) of A ,
3. If $B \vee C$ is a cp (ap) of A , then both B and C are cps (aps) of A ,
4. If $B \rightarrow C$ is a cp (ap) of A , then B is an ap (cp) of A and C is a cp (ap) of A ,
5. If $\neg B$ is a cp (ap) of A , then B is an ap (cp) of A .

The properties referred to above are the following.

Definition 1.3 (Strong variable-sharing property—svsp) If $A \rightarrow B$ is provable, then some variable occurs as an ap of both A and B , or else as a cp of both A and B .

Definition 1.4 (No loose pieces property—nlpp) If A is provable and A contains no conjunction as aps and no disjunctions as cps, every variable in A occurs once as ap and once as cp.

The nlpp is named after an observation by Anderson and Belnap (cf. [1], p. 255) remarking that any logic equivalent to (or included in R) lacks wff such as

$$\begin{aligned} (q \rightarrow p) &\rightarrow (p \rightarrow p) \\ p &\rightarrow [(q \rightarrow p) \rightarrow p] \\ (p \rightarrow q) &\rightarrow (p \rightarrow p) \\ p &\rightarrow [(p \rightarrow q) \rightarrow p] \\ [(p \rightarrow q) \rightarrow p] &\rightarrow p \end{aligned}$$

where q is in them, so to speak, a “loose piece.” On the other hand, the property stated in Definition 1.3 is labeled the “strong” variable-sharing property because, as it is proved in Theorem 3.10 below, it is a sufficient but not necessary condition for the vsp. Now, as pointed out above, Anderson and Belnap actually prove that any logic equivalent to (or included in) R has both the svsp and nlpp.

The aim of this paper is to define a general class of logical matrices for each one of the properties in Definitions 1.1, 1.3, and 1.4, especially for those in Definition 1.3 and Definition 1.4. That is to say, the purpose of this paper is to define vsp-matrices, svsp-matrices, and nlpp-matrices in a general way. Each one of these classes has the following property. Let S be a logic verified (cf. §2) by a, say, svsp-matrix. Then S has the svsp. And it is similarly the case with vsp-matrices and nlpp-matrices. These facts are proved following Anderson and Belnap’s strategy in [1], §22.1.3.

In this way, we have a simple method for checking immediately if there is some logic with any of the referred properties in, say, a sequence of matrices generated by a computer. Or, indeed, a simple method for building up (with or without a computer) matrices such that any logic verified by them has one of the aforementioned properties. In addition, these properties are, generally, algebraically characterized, and so, in some respect, the logics possessing them are also characterized in the same way.

Nevertheless, the usefulness of the method can be questioned on the basis of facts (1) and (2) recalled at the beginning of this introduction. That is to say, no method is needed to know that any logic included in R has the svsp and the nlpp; neither is it needed to know that no logic including RM has these properties. But, as was also recalled in (3), there are strong interesting logics with the said properties in the vicinity of RM . And, moreover, as it will be shown in the sequel, there are strong, and maybe interesting, logics not included in RM with one or more of the properties considered in this paper.

The structure of the paper is as follows. In Section 2, a series of preliminary definitions (“logical matrix,” etc.) is recalled. The aim of this short section is essentially to set some terminology used throughout the paper. In Section 3, *weak relevant matrices* (wr-matrices) are defined (wr-matrices are vsp-matrices). And it is proved that the svsp is not a necessary condition for the vsp. In Section 4, Section 5, svsp-matrices and nlpp-matrices are, respectively, characterized. In Section 6, we define *strong relevant matrices* (sr-matrices). Sr-matrices are svsp-matrices as well as nlpp-matrices. Finally, in Section 7, a note is included on relevance, the vsp and two alternatives proposed to this last property: depth relevance and the relevant equivalence property. Examples of each class of matrices featuring logics with the respective property (properties) are provided.

2 Logical Matrices. Preliminary Definitions

We shall consider propositional languages with a set of denumerable propositional variables and the following connectives: \rightarrow (conditional), \wedge (conjunction), \vee (disjunction), \neg (negation), the biconditional (\leftrightarrow) being defined in the customary way. The set of wff is also defined in the usual way. Then the notion of logical matrix is defined as follows.

Definition 2.1 A logical matrix M is a structure $(K, T, F, f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg})$ where

1. K is a set,
2. T and F are nonempty subsets of K such that $T \cup F = K$ and $T \cap F = \emptyset$,
3. $f_{\rightarrow}, f_{\wedge}, f_{\vee}$ are binary functions (distinct of each other) on K , and f_{\neg} is a unary function on K .

It is said that K is the *set of elements* of M ; T is the *set of designated elements*, and F is the *set of nondesignated elements*. The functions $f_{\rightarrow}, f_{\wedge}, f_{\vee}$, and f_{\neg} interpret in M the conditional, conjunction, disjunction, and negation, respectively. In some cases one or more of these functions may not be defined.

Now, let L be a propositional language, A_1, \dots, A_n, B be any wff of L , and S be a logic defined on L . On the other hand, let M be a logical matrix and v_m an assignment of elements of M to the propositional variables of B . That B is assigned the element j of K is expressed as follows: $v_m(B) = j$.

Then we set the following definition.

Definition 2.2 Let M be a logical matrix. M *verifies* B if and only if for any assignment, v_m , of elements of K to the propositional variables of B , $v_m(B) \in T$.

Definition 2.3 Let M be a logical matrix. M *falsifies* B if and only if for some assignment, v_m , of elements of K to the propositional variables of B , $v_m(B) \in F$.

Definition 2.4 Let $A_1, \dots, A_n \vdash_S B$ be a rule of derivation of S , and M be a logical matrix. Then, M *verifies* $A_1, \dots, A_n \vdash_S B$ if and only if for any assignment, v_m , of elements of K to the variables of A_1, \dots, A_n and B , if $v_m(A_1) \in T, \dots, v_m(A_n) \in T$, then, $v_m(B) \in T$.

Finally, we have the following.

Definition 2.5 Let M be a logical matrix. M *verifies* S if and only if M verifies all axioms and rules of derivation of S .

3 The SVSP Is Not a Necessary Condition for the VSP

First, it is clear that we have the following.

Proposition 3.1 *Let S be a logic with the svsp. Then S has the vsp.*

Proof It is obvious. □

Therefore, the svsp is a sufficient condition for the vsp, but it is not a necessary condition: as it will be shown, there are logics with the vsp lacking the svsp. In order to prove this, we set the following definition.

Definition 3.2 (wr-matrices) Let M be a logical matrix in which a_i, a_r , and a_F are elements of K distinct of each other such that $a_F \in F$ and the following conditions are fulfilled:

1. $F_\wedge(a_i, a_i) = F_\vee(a_i, a_i) = F_\rightarrow(a_i, a_i) = F_\neg(a_i) = a_i$,
2. $F_\wedge(a_r, a_r) = F_\vee(a_r, a_r) = F_\rightarrow(a_r, a_r) = F_\neg(a_r) = a_r$,
3. $F_\rightarrow(a_i, a_r) = a_F$.

It is said that M is a weak relevant matrix (wr-matrix, for short).

Remark 3.3 The label weak relevant matrix is intended to distinguish wr-matrices from strong relevant matrices to be defined with stricter conditions in Section 6.

Next, the following is proved.

Proposition 3.4 *Let M be a wr-matrix and S a logic verified by it. Then S has the vsp.*

Proof Assume the hypothesis of Proposition 3.4, and let $A \rightarrow B$ be a wff in which A and B do not share propositional variables. Then, let v_m be an assignment of elements of M to the variables of A and B such that $v_m(p_n) = a_i$ for each variable p_n in A , and $v_m(p_n) = a_r$ for each variable p_n in B . By conditions (1) and (2) in Definition 3.2, $v_m(A) = a_i$ and $v_m(B) = a_r$. So $v_m(A \rightarrow B) = a_F$ by condition (3) in Definition 3.2. Therefore, if $A \rightarrow B$ is a formula in which A and B do not share a propositional variable, $A \rightarrow B$ is not a theorem of S . Consequently, if $A \rightarrow B$ is a theorem of S , A and B do share at least one propositional variable. In other words, S has the vsp. □

Now we shall provide a class of logics with the vsp but lacking the svsp.

Definition 3.5 Consider the matrix $M_{DF3.5} = (K, T, F, f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg})$ where

1. $K = \{0, 1, 2, 3\}$
2. $T = \{1, 2, 3\}$
3. $F = \{0\}$
4. the functions $f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}$ are defined as shown in the tables below:

\rightarrow	0	1	2	3	\neg	\wedge	0	1	2	3	\vee	0	1	2	3
0	3	3	3	3	3	0	0	0	0	0	0	0	1	2	3
1	0	1	2	3	1	1	0	1	1	1	1	1	1	2	3
2	0	0	2	3	2	2	0	1	2	2	2	2	2	2	3
3	0	0	0	3	2	3	0	1	2	3	3	3	3	3	3

5. $a_i = 2$
6. $a_r = 1$
7. $a_F = 0$

Remark 3.6 The following is a Hasse diagram of $M_{DF3.5}$.



Proposition 3.7 Matrix $M_{DF3.5}$ is a wr-matrix.

Proof By checking that conditions (1)–(3) in Definition 3.2 are fulfilled by $M_{DF3.5}$. □

Let us now describe some of the logics verified by $M_{DF3.5}$.

Proposition 3.8 The schema

$$(1) A \rightarrow A$$

and the rules

$$(2) \text{ If } A, \text{ then } (\neg A \wedge B) \rightarrow \neg B$$

and

$$\text{Modus Ponens (MP) If } A \text{ and } A \rightarrow B, \text{ then } B$$

are verified by $M_{DF3.5}$.

Proof It is left to the reader. □

Proposition 3.9 Let S be any logic verified by $M_{DF3.5}$ in which (1), (2), and MP are derivable. Then S has the vsp but not the svsp.

Proof (a) S has the vsp. By Propositions 3.4 and 3.7. (b) S does not have the svsp. It is clear that

$$(3) [\neg(p \rightarrow p) \wedge q] \rightarrow \neg q$$

is a theorem of S . But there is no propositional variable as ap or as cp of both antecedent and consequent of (3). Therefore, S does not have the svsp. □

Finally, the section is ended with the following theorem.

Theorem 3.10 *The svsp is a sufficient but not a necessary condition for the vsp.*

Proof Propositions 3.1 and 3.9. □

4 SVSP-Matrices

In this section svsp-matrices shall be characterized generally. Then, it will be proved that any logic S verified by an svsp-matrix has the svsp. That is, in any theorem of S of the form $A \rightarrow B$ some variable occurs as an ap or as a cp of both A and B .

Definition 4.1 (svsp-matrices) Let M be a logical matrix in which $a_T \in T$, $a_F \in F$, and a_i, a_r, a_l , and a_s are elements of K . And let us designate by K_1 , K_2 , and K_3 the subsets of K $\{a_i, a_r\}$, $\{a_l, a_s\}$, and $\{a_T, a_F\}$, respectively. The sets K_1 , K_2 , and K_3 are disjoint, and the members in K_1 as well as those in K_2 are possibly (but not necessarily) distinct from each other. Finally, the following conditions are fulfilled:

1.
 - (a) $f_\wedge(a_T, a_T) = f_\vee(a_T, a_T) = f_\rightarrow(a_F, a_T) = f_\rightarrow(a_T, a_F) = f_-(a_F) = a_T$.
 - (b) $f_\wedge(a_F, a_F) = f_\vee(a_F, a_F) = f_\rightarrow(a_T, a_F) = f_-(a_T) = a_F$.
2.
 - (a) $\forall x \forall y \in K_1 f_\wedge(x, y) \ \& \ f_\vee(x, y) \ \& \ f_\rightarrow(x, y) \ \& \ f_-(x) \in K_1$.
 - (b) $\forall x \forall y \in K_2 f_\wedge(x, y) \ \& \ f_\vee(x, y) \ \& \ f_\rightarrow(x, y) \ \& \ f_-(x) \in K_2$.
3. $\forall x \in K_1$
 - (a) $f_\wedge(a_T, x) \ \& \ f_\wedge(x, a_T) \ \& \ f_\vee(a_T, x) \ \& \ f_\vee(x, a_T) \ \& \ f_\rightarrow(x, a_T) \ \& \ f_\rightarrow(a_F, x) \in K_1 \cup \{a_T\}$.
 - (b) $f_\wedge(a_F, x) \ \& \ f_\wedge(x, a_F) \ \& \ f_\vee(a_F, x) \ \& \ f_\vee(x, a_F) \ \& \ f_\rightarrow(a_T, x) \in K_1 \cup \{a_F\}$.
4. $\forall x \in K_2$
 - (a) $f_\wedge(a_T, x) \ \& \ f_\wedge(x, a_T) \ \& \ f_\vee(a_T, x) \ \& \ f_\vee(x, a_T) \ \& \ f_\rightarrow(x, a_T) \ \& \ f_\rightarrow(a_F, x) \in K_2 \cup \{a_T\}$.
 - (b) $f_\wedge(a_F, x) \ \& \ f_\wedge(x, a_F) \ \& \ f_\vee(a_F, x) \ \& \ f_\vee(x, a_F) \ \& \ f_\rightarrow(x, a_F) \in K_2 \cup \{a_F\}$.
5.
 - (a) $\forall x \in K_1 f_\rightarrow(x, a_F) = a_F$
 - (b) $\forall x \in K_2 f_\rightarrow(a_T, x) = a_F$
 - (c) $\forall x \in K_1 \forall y \in K_2 f_\rightarrow(x, y) \in F$.

Then it is said that M is a matrix for the *strong variable sharing property* (svsp-matrix, for short).

Remark 4.2 The following diagram is intended to help the reader to visualize the general structure of svsp-matrices. Only elements of K referred to in Definition 4.1 are included. The order in which the elements are displayed in the matrix schema is of course purely conventional. That is, a_T, a_F, a_i, a_r, a_l , and a_s can be displayed differently in actual svsp-matrices. Similar diagrams for other classes of matrices to be defined in this paper shall be included in the following sections.

\rightarrow	F	i	r	l	s	T	\neg
F		Tir	Tir	Tls	Tls	T	T
i	F	ir	ir	FF	FF	Tir	ir
r	F	ir	ir	FF	FF	Tir	ir
l	Fls			ls	ls	Tls	ls
s	Fls			ls	ls	Tls	ls
T	F	Fir	Fir	F	F		F

\wedge	F	i	r	l	s	T
F	F	Fir	Fir	Fls	Fls	
i	Fir	ir	ir			Tir
r	Fir	ir	ir			Tir
l	Fls			ls	ls	Tls
s	Fls			ls	ls	Tls
T		Tir	Tir	Tls	Tls	T

\vee	F	i	r	l	s	T
F	F	Fir	Fir	Fls	Fls	
i	Fir	ir	ir			Tir
r	Fir	ir	ir			Tir
l	Fls			ls	ls	Tls
s	Fls			ls	ls	Tls
T		Tir	Tir	Tls	Tls	T

Remark 4.3 $T, F, i, r, l,$ and s designate $a_T, a_F, a_i, a_r, a_l,$ and $a_s,$ respectively. F designates an arbitrary element of F . Certain blocks contain more than one element. It means, of course, that any one of them can appear in svsp-matrices. Empty blocks can contain any element of M .

Next, we prove that any logic S verified by a svsp-matrix has the svsp. In order to do this, the following lemma is useful.

Lemma 4.4 (Cf. [1], §22.1.3) *Let M be a svsp-matrix and $A \rightarrow B$ be a wff in which no variable occurs as an ap of both A and B or as a cp of both A and B . Then there is an assignment v_m of elements of M to the variables of $A \rightarrow B$ such that*

1. *for every ap C of A , $v_m(C) \in \{a_i, a_r, a_F\}$, and for every cp C of A , $v_m(C) \in \{a_i, a_r, a_T\}$;*
2. *for every ap C of B , $v_m(C) \in \{a_l, a_s, a_T\}$, and for every cp C of B , $v_m(C) \in \{a_l, a_s, a_F\}$.*

Proof Assume the hypothesis of Lemma 4.4. Then each variable p occurring in $A \rightarrow B$ has to appear in A and/or in B in one of the six situations tabulated below:

	A	B
$p :$	cp	-
	ap	-
	-	ap
	-	cp
	cp	ap
	ap	cp

The first row is read “ p occurs as a cp in A , but does not occur in B ,” and the rest of the rows are read similarly.

According to these possibilities, the following assignment v_m of elements of M is defined for each variable p in $A \rightarrow B$:

	A	B	
$p :$	cp	-	a_i
	ap	-	a_r
	-	ap	a_l
	-	cp	a_s
	cp	ap	a_T
	ap	cp	a_F

We remark that if p appears both as a cp and an ap of A (respectively, B) and not in B (respectively, A), then p can equivalently be assigned a_i or a_r (a_l or a_s). Let us, for definiteness, assign a_i (a_l) to p if such is the case.

Then the proof of Lemma 4.4 is by induction on the length of C . Let us prove by way of an example a couple of cases. By (3a), (4b), (5c) and so on we refer to the clauses in Definition 4.1.

First the conditional case in Lemma 4.4(1): Suppose that C is of the form $D \rightarrow E$.

1. C is an ap. Then D is a cp and E is an ap by Definition 1.2. By hypothesis of induction (H.I), $v_m(D) \in \{a_i, a_r, a_T\}$ and $v_m(E) \in \{a_i, a_r, a_F\}$, whence by (1b), (2a), (3b), and (5a), $v_m(D \rightarrow E) \in \{a_i, a_r, a_F\}$ as it was to be proved.
2. C is a cp. Then D is an ap and E is a cp by Definition 1.2. By H.I, $v_m(D) \in \{a_i, a_r, a_F\}$ and $v_m(E) \in \{a_i, a_r, a_T\}$, whence by (1a), (2a), and (3a), $v_m(D \rightarrow E) \in \{a_i, a_r, a_T\}$, as it was to be proved.

Next the negation case in Lemma 4.4(2): Suppose that C is of the form $\neg D$:

1. C is an ap. Then D is a cp by Definition 1.2. By H.I, $v_m(D) \in \{a_l, a_s, a_F\}$. Then $v_m(\neg D) \in \{a_l, a_s, a_T\}$ by (1a) and (2b).
2. C is a cp. The proof is similar to the previous one by (1b) and (2b). \square

By Lemma 4.4, the following is proved.

Theorem 4.5 *Let M be a svsp-matrix and S be a logic verified by it. Then, if $A \rightarrow B$ is a theorem of S , some variable occurs as an ap or else as a cp of both A and B . That is, S has the svsp.*

Proof Let $A \rightarrow B$ be a wff in which no variable occurs as an ap of both A and B or as a cp of both A and B . And let M be an svsp-matrix verifying S . By Lemma 4.4 there is an assignment v_m of elements of M to the variables of $A \rightarrow B$ such that for every cp C of A , $v_m(C) \in \{a_i, a_r, a_T\}$ and for every cp C of B , $v_m(C) \in \{a_l, a_s, a_F\}$. As A and B are cps of themselves, $v_m(A) \in \{a_i, a_r, a_T\}$ and $v_m(B) \in \{a_l, a_s, a_F\}$, whence by conditions (1b), (5a), (5b), and (5c) in Definition 4.1, $v_m(A \rightarrow B) \in F$. That is, $A \rightarrow B$ is not a theorem of S . Consequently, if $A \rightarrow B$ is a theorem of S , some variable occurs as an ap of both A and B , or else as a cp of both A and B . \square

Next, as a way of an example, a strong logic with the svsp not included in R-Mingle shall be provided.

Definition 4.6 Consider the matrix $M_{DF4.6} = (K, T, F, f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg})$ where

1. $K = \{0, 1, 2, 3, 4, 5\}$,
2. $T = \{1, 2, 3, 4, 5\}$,
3. $F = \{0\}$,
4. the functions $f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}$ are defined as shown in the tables below:

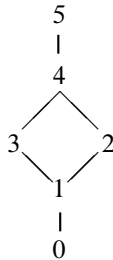
\rightarrow	0	1	2	3	4	5	\neg
0	1	1	2	3	4	5	5
1	0	1	2	3	4	5	4
2	0	0	2	0	4	5	2
3	0	0	0	3	4	5	3
4	0	0	0	0	4	5	1
5	0	0	0	0	0	5	0

\wedge	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	1	2	2
3	0	1	1	3	3	3
4	0	1	2	3	4	4
5	0	1	2	3	4	5

\vee	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	4	5
2	2	2	2	4	4	5
3	3	3	4	3	4	5
4	4	4	4	4	4	5
5	5	5	5	5	5	5

5. $K_1 = \{2\}$
6. $K_2 = \{3\}$
7. $a_T = 5$
8. $a_F = 0$.

Remark 4.7 The following is a Hasse diagram of $MD_{DF4.6}$.



Definition 4.8 Consider the following logic $S_{DF4.6}$. The logic $S_{DF4.6}$ is axiomatized by adding to Routley and Meyer’s basic positive logic B_+ (cf. [10]) the following axioms and rules:

- a1. $[(A \rightarrow A) \rightarrow B] \rightarrow B$
- a2. $[(A \rightarrow (A \rightarrow B))] \leftrightarrow (A \rightarrow B)$
- a3. $(B \rightarrow A) \rightarrow (A \rightarrow A)$
- a4. $A \rightarrow [(B \rightarrow A) \rightarrow A]$
- a5. $[(A \rightarrow B) \rightarrow A] \rightarrow A$
- a6. $\neg\neg A \leftrightarrow A$

a7. If $A \rightarrow B$, then $\neg B \rightarrow \neg A$

a8. If $A \rightarrow B$, then $(A \rightarrow \neg B) \rightarrow \neg A$.

Remark 4.9 We shall try here to axiomatize neither $M_{DF4.6}$ nor other matrices to be provided in the sequel.

Then the following is proved.

Proposition 4.10 *Matrix $M_{DF4.6}$ is an svsp-matrix.*

Proof By checking that conditions (1)–(5) in Definition 4.1 are fulfilled by $M_{DF4.6}$. \square

Proposition 4.11 *The logic $S_{DF4.6}$ is verified by $M_{DF4.6}$.*

Proof It is left to the reader. \square

Proposition 4.12 *The logic $S_{DF4.6}$ has the svsp.*

Proof By Proposition 4.10, Proposition 4.11, and Theorem 4.5. \square

Now, notice that $S_{DF4.6}$ does not include and neither is it included in R-Mingle (in RM3, actually): (a3), (a4), and (a5) are not theorems of RM3 (on the logic RM, see [10]; on RM3, an extension of RM, cf. [4] and Definition 5.4 below).

The section is ended with the following remark.

Remark 4.13 Note that a series of logics with the svsp not included in RM3 is, in fact, provided in $M_{DF4.6}$. One of these logics is used in [9] to define deep relevant logics not included in RM.

5 NIPP-Matrices

In this section nlpp-matrices are generally characterized. Then it is proved that any logic S verified by an nlpp-matrix has the nlpp. That is, in any theorem A of S with no conjunctions as aps and no disjunctions as cps, every variable in A occurs at least once as ap and at least once as cp.

Definition 5.1 Let M be a logical matrix in which $a_T \in T$, $a_F \in F$, and a_i, a_r are elements of K . And let us designate by K_1 and K_2 the subsets of K $\{a_i, a_r\}, \{a_T, a_F\}$, respectively. The sets K_1 and K_2 are disjoint, and the members in K_1 are possibly (but not necessarily) distinct from each other. Finally, the following conditions are fulfilled:

1.
 - (a) $f_{\vee}(a_T, a_T) = f_{\rightarrow}(a_F, a_T) = f_{\neg}(a_F) = a_T$.
 - (b) $f_{\wedge}(a_F, a_F) = f_{\rightarrow}(a_T, a_F) = f_{\neg}(a_T) = a_F$.
2. $\forall x \forall y \in K_1 f_{\wedge}(x, y) \ \& \ f_{\vee}(x, y) \ \& \ f_{\rightarrow}(x, y) \ \& \ f_{\neg}(x) \in K_1$.
3. $\forall x \in K_1$
 - (a) $f_{\vee}(a_T, x) = f_{\vee}(x, a_T) = f_{\rightarrow}(x, a_T) = f_{\rightarrow}(a_F, x) = a_T$.
 - (b) $f_{\wedge}(a_F, x) = f_{\wedge}(x, a_F) = f_{\rightarrow}(a_T, x) = f_{\rightarrow}(x, a_F) = a_F$.

It is said that M is a *matrix for the no loose pieces property* (nlpp-matrix, for short).

Nlpp-matrices generally look as in the following diagram (cf. Remark 4.2 and Remark 4.3).

\rightarrow	F	i	r	T	\neg
F		T	T	T	T
i	F	ir	ir	T	ir
r	F	ir	ir	T	ir
T	F	F	F		F

\wedge	F	i	r	T
F	F	F	F	
i	F	ir	ir	
r	F	ir	ir	
T				

\vee	F	i	r	T
F				
i		ir	ir	T
r		ir	ir	T
T		T	T	T

Lemma 5.2 (Cf. [1], §22.1.3) *Let M be an nlpp-matrix and A be any wff with no conjunctions as aps and no disjunctions as cps. Moreover, let p be a propositional variable which appears only as a cp, or only as an ap, in A . Then there is an assignment v_m of elements of M to the variables of A such that*

1. *if B is a subformula of A in which p does not occur, then $v_m(B) \in K_1$;*
2. *if B is a subformula of A in which p does occur, then (a) if B is an ap of A , then $v_m(B) = a_T$, and (b) if B is a cp of A , then $v_m(B) = a_F$.*

Proof Assume the hypothesis of Lemma 5.2 with p appearing only as a cp. Then we set the following assignment under M : $v_m(p) = a_F$ and $v_m(q) = a_i$ for each variable q distinct from p .

1. Induction on the length of B . The proof is trivial as K_1 is closed under \rightarrow , \wedge , \vee , and \neg . (By condition (2) in Definition 5.1).
2. Induction on the length of B . If B is a propositional variable, then B is p and $v_m(B) = a_F$. (Recall that p occurs only as a cp). Regarding complex formulas, we prove the conditional case and leave the rest of the cases to the reader.

B is of the form $C \rightarrow D$:

- (a) B is an ap. Then C is a cp and D is an ap (cf. Definition 1.2):
 - (i) p occurs in C and D : by H.I, $v_m(C) = a_F$ and $v_m(D) = a_T$. So $v_m(C \rightarrow D) = a_T$ by (1a).
 - (ii) p occurs in C but not in D : by H.I, $v_m(C) = a_F$. By Lemma 5.2(1), $v_m(D) \in \{a_i, a_r\}$. So $v_m(C \rightarrow D) = a_T$ by (3a).
 - (iii) p occurs in D but not in C : by Lemma 5.2(1), $v_m(C) \in \{a_i, a_r\}$; by H.I, $v_m(D) = a_T$. Then $v_m(C \rightarrow D) = a_T$ by (3a).
- (b) B is a cp. Then C is an ap and D is a cp (cf. Definition 1.2).
- (c) p occurs in C and D : by H.I, $v_m(C) = a_T$ and $v_m(D) = a_F$. Then $v_m(C \rightarrow D) = a_F$ by (1b).
- (d) p occurs in C but not in D : by H.I, $v_m(C) = a_T$. By Lemma 5.2(1), $v_m(D) \in \{a_i, a_r\}$. Then $v_m(C \rightarrow D) = a_F$ by (3b).
- (e) p occurs in D but not in C : by Lemma 5.2(1), $v_m(C) \in \{a_i, a_r\}$; by H.I, $v_m(D) = a_F$. Then $v_m(C \rightarrow D) = a_F$ by (3b).

The proof of the conditional case is now finished. The proof of the conjunction, disjunction, and negation cases is similar (recall that conjunctions can only appear as cps and disjunctions as aps). If, on the other hand, p occurs only as an ap, assign p the value a_T (the proof is similar). \square

Leaning on Lemma 5.2 we prove the following.

Theorem 5.3 *Let M be an nlpp-matrix and S a logic verified by it. Then, if A is a theorem of S with no conjunctions as aps and no disjunctions as cps, every variable in A occurs at least once as ap and at least once as cp. That is, S has the nlpp.*

Proof Let A be a wff with no conjunctions as aps and no disjunctions as cps in which p appears only as cp or only as an ap. And let M be an nlpp-matrix and S a logic verified by it. As A is a cp of itself, there is, by Lemma 5.2, an assignment under M v_m to the variables of A such that $v_m(A) = a_F$. That is, A is not a theorem of S . Consequently, if A is a theorem of S with no conjunctions as aps and no disjunctions as cps, each variable in A appears at least once as cp and at least once as ap. That is, S has the nlpp. \square

Next, it shall be shown that RM has the nlpp and, on the other hand, that there are logics not included in RM having also this property.

Definition 5.4 Consider the matrix $M_{DF5.4} = (K, T, F, f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg})$ where

1. $K = \{0, 1, 2\}$
2. $T = \{1, 2\}$
3. $F = \{0\}$
4. the functions $f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}$ are defined as shown in the tables below:

\rightarrow	0	1	2	\neg	\wedge	0	1	2	\vee	0	1	2
0	2	2	2	2	0	0	0	0	0	0	1	2
1	0	1	2	1	1	0	1	1	1	1	1	2
2	0	0	2	0	2	0	1	2	2	2	2	2

5. $K_1 = \{1\}$
6. $a_T = 2$
7. $a_F = 0$.

We recall that $M_{DF5.4}$ is the characteristic matrix for the logic RM3, an extension of RM (cf. [4]). Then, similarly as in the case of the logic $S_{DF4.6}$, we prove the following.

Proposition 5.5 *The logic RM3 (and so, the logic RM) has the nlpp.*

Proof Given that $M_{DF5.4}$ is characteristic for RM3, by checking that $M_{DF5.4}$ is an nlpp-matrix. \square

Next, a logic (actually, a series of them, cf. Remark 4.13) with the nlpp and not included in RM3 is defined.

Definition 5.6 Consider the matrix $M_{DF5.6} = (K, T, F, f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg})$ where $K, T, F, K_1, a_T, a_F, f_{\wedge}, f_{\vee}, f_{\neg}$ are as in Definition 5.4 and f_{\rightarrow} is as shown in the table below:

\rightarrow	0	1	2
0	1	2	2
1	0	1	2
2	0	0	1

Definition 5.7 Consider the following logic $S_{DF5.6}$. The logic $S_{DF5.6}$ is axiomatized by adding to Routley and Meyer's logic B (cf. [10]) the following axioms:

- a1. $[(A \rightarrow A) \rightarrow B] \rightarrow B$
- a2. $A \rightarrow [(B \rightarrow B) \rightarrow A]$
- a3. $[[[(B \rightarrow B) \rightarrow A] \rightarrow A] \rightarrow (A \rightarrow A)]$
- a4. $A \rightarrow [[[[[(B \rightarrow B) \rightarrow A] \rightarrow A] \rightarrow A] \rightarrow A]]$

- a5. $\neg(A \rightarrow B) \rightarrow (B \rightarrow A)$
a6. $(A \rightarrow \neg A) \rightarrow \neg A$.

Similarly as in the case of RM3 above, the following proposition is proved.

Proposition 5.8 *The logic $S_{DF5.6}$ has the nlpp.*

Proof By checking that $M_{DF5.6}$ is an nlpp-matrix and that the logic $S_{DF5.6}$ is verified by it. \square

Now we notice that $S_{DF5.6}$ does not include and neither is it included in R-Mingle (in RM3, actually): (a3) and (a4) are not theorems of RM3.

6 Strong Relevant Matrices

Strong relevant matrices (sr-matrices, for short) are labeled by contrast to weak relevant matrices characterized in Section 3. Sr-matrices are svsp-matrices as well as nlpp-matrices. So they are defined after Definition 4.1 and Definition 5.1.

Definition 6.1 Let M be a logical matrix in which $a_T \in T$, $a_t \in T$, $a_F \in F$, $a_f \in F$ and a_i, a_r, a_l, a_s, a_k , and a_m are elements of K . And let us designate by K_1, K_2, K_3, K_4 , and K_5 the subsets of K $\{a_i, a_r\}, \{a_l, a_s\}, \{a_k, a_m\}, \{a_T, a_t\}$, and $\{a_F, a_f\}$, respectively. The sets K_1, K_2, K_4 , and K_5 are disjoint as well as K_3, K_4 , and K_5 . On the other hand, the members belonging to each one of these subsets of K are possibly (but not necessarily) distinct from each other. Finally, the following conditions are fulfilled:

1.
 - (a) $f_\wedge(a_T, a_T) = f_\vee(a_T, a_T) = f_\rightarrow(a_F, a_T) = f_\neg(a_F) = a_T$.
 - (b) $f_\vee(a_t, a_t) = f_\rightarrow(a_f, a_t) = f_\neg(a_f) = a_t$.
 - (c) $f_\wedge(a_F, a_F) = f_\vee(a_F, a_F) = f_\rightarrow(a_T, a_F) = f_\neg(a_T) = a_F$.
 - (d) $f_\wedge(a_f, a_f) = f_\rightarrow(a_t, a_f) = f_\neg(a_t) = a_f$.
2.
 - (a) $\forall x \forall y \in K_1 f_\wedge(x, y) \ \& \ f_\vee(x, y) \ \& \ f_\rightarrow(x, y) \ \& \ f_\neg(x) \in K_1$.
 - (b) $\forall x \forall y \in K_2 f_\wedge(x, y) \ \& \ f_\vee(x, y) \ \& \ f_\rightarrow(x, y) \ \& \ f_\neg(x) \in K_2$.
 - (c) $\forall x \forall y \in K_3 f_\wedge(x, y) \ \& \ f_\vee(x, y) \ \& \ f_\rightarrow(x, y) \ \& \ f_\neg(x) \in K_3$.
3. $\forall x \in K_1$
 - (a) $f_\wedge(a_T, x) \ \& \ f_\wedge(x, a_T) \ \& \ f_\vee(a_T, x) \ \& \ f_\vee(x, a_T) \ \& \ f_\rightarrow(x, a_T) \ \& \ f_\rightarrow(a_F, x) \in K_1 \cup \{a_T\}$.
 - (b) $f_\wedge(a_F, x) \ \& \ f_\wedge(x, a_F) \ \& \ f_\vee(a_F, x) \ \& \ f_\vee(x, a_F) \ \& \ f_\rightarrow(a_T, x) \in K_1 \cup \{a_F\}$.
4. $\forall x \in K_2$
 - (a) $f_\wedge(a_T, x) \ \& \ f_\wedge(x, a_T) \ \& \ f_\vee(a_T, x) \ \& \ f_\vee(x, a_T) \ \& \ f_\rightarrow(x, a_T) \ \& \ f_\rightarrow(a_F, x) \in K_2 \cup \{a_T\}$.
 - (b) $f_\wedge(a_F, x) \ \& \ f_\wedge(x, a_F) \ \& \ f_\vee(a_F, x) \ \& \ f_\vee(x, a_F) \ \& \ f_\rightarrow(x, a_F) \ \& \ f_\rightarrow(a_T, x) \in K_2 \cup \{a_F\}$.
5. $\forall x \in K_3$
 - (a) $f_\vee(a_t, x) = f_\vee(x, a_t) = f_\rightarrow(x, a_t) = f_\rightarrow(a_f, x) = a_t$.
 - (b) $f_\wedge(a_f, x) = f_\wedge(x, a_f) = f_\rightarrow(a_t, x) = f_\rightarrow(x, a_f) = a_f$.
6.
 - (a) $\forall x \in K_1 f_\rightarrow(x, a_F) = a_F$
 - (b) $\forall x \in K_2 f_\rightarrow(a_T, x) = a_F$

$$(c) \forall x \in K_1 \forall y \in K_2 f_{\rightarrow}(x, y) \in F.$$

It is said that M is a *strong relevant matrix* (sr-matrix, for short).

Sr-matrices generally look as in the following diagram (cf. Remark 4.2).

\rightarrow	F	f	i	r	l	s	k	m	t	T	\neg
F			Tir	Tir	Tls	Tls				T	T
f							t	t	t		t
i	F		ir	ir	FF	FF				Tir	ir
r	F		ir	ir	FF	FF				Tir	ir
l	Fls				ls	ls				Tls	ls
s	Fls				ls	ls				Tls	ls
k	f						km	km	t		km
m	f						km	km	t		km
t	f						f	f			f
T	F		Fir	Fir	F	F					F

\wedge	F	f	i	r	l	s	k	m	t	T
F	F		Fir	Fir	Fls	Fls				
f		f					f	f		
i	Fir		ir	ir						Tir
r	Fir		ir	ir						Tir
l	Fls				ls	ls				Tls
s	Fls				ls	ls				Tls
k		f					km	km		
m		f					km	km		
t										
T			Tir	Tir	Tls	Tls				T

\vee	F	f	i	r	l	s	k	m	t	T
F	F		Fir	Fir	Fls	Fls				
f										
i	Fir		ir	ir						Tir
r	Fir		ir	ir						Tir
l	Fls				ls	ls				Tls
s	Fls				ls	ls				Tls
k							km	km	t	
m							km	km	t	
t							t	t	t	
T			Tir	Tir	Tls	Tls				T

Remark 6.2 $T, F, f, i, r, l, s, k, m, t$ designate $a_T, a_F, a_f, a_i, a_r, a_l, a_s, a_k, a_m,$ and $a_t,$ respectively. F designates an arbitrary element of F . Certain blocks contain more than one element. It means, of course, that any one of them can appear in sr-matrices. Empty blocks can contain any element of M .

Next the following is proved.

Proposition 6.3 *Let M be an sr-matrix. Then M is an svsp-matrix as well as an nlpp-matrix.*

Proof M is svsp-matrix: by conditions (1a), (1c), (2a), (2b), (3), (4), and (6) in Definition 6.1. M is an nlpp-matrix by conditions (1b), (1d), (2c), and (5) in Definition 6.1. \square

And consequently, we have the following theorem.

Theorem 6.4 *Let M be an sr-matrix and S be a logic verified by it. Then S has the svsp and the nlpp.*

Proof By Proposition 6.3, Theorem 4.5, and Theorem 5.3. \square

Matrices in Definition 6.1 can be simplified if we let T , F , and K_1 or K_2 play the role of t , f , and K_3 , respectively. That is, if $T = t$, $F = f$, and $K_3 = K_1$ (or $K_3 = K_2$). Then sr-matrices can be defined as follows.

Definition 6.5 (sr-matrices—simplified definition) Let M be a logical matrix in which $a_T \in T$, $a_F \in F$, and a_i, a_r, a_l , and a_s are elements of K . And let us designate by K_1 , K_2 , and K_3 the subsets of K $\{a_i, a_r\}$, $\{a_l, a_s\}$, and $\{a_T, a_F\}$, respectively. The sets K_1 , K_2 , and K_3 are disjoint, and the members in K_1 as well as those in K_2 are possibly (but not necessarily) distinct from each other. Finally, the following conditions are fulfilled:

1.
 - (a) $f_\wedge(a_T, a_T) = f_\vee(a_T, a_T) = f_\rightarrow(a_F, a_T) = f_-(a_F) = a_T$.
 - (b) $f_\wedge(a_F, a_F) = f_\vee(a_F, a_F) = f_\rightarrow(a_T, a_F) = f_-(a_T) = a_F$.
2.
 - (a) $\forall x \forall y \in K_1 f_\wedge(x, y) \ \& \ f_\vee(x, y) \ \& \ f_\rightarrow(x, y) \ \& \ f_-(x) \in K_1$.
 - (b) $\forall x \forall y \in K_2 f_\wedge(x, y) \ \& \ f_\vee(x, y) \ \& \ f_\rightarrow(x, y) \ \& \ f_-(x) \in K_2$.
3. $\forall x \in K_1$
 - (a) $f_\vee(a_T, x) = f_\vee(x, a_T) = f_\rightarrow(x, a_T) = f_\rightarrow(a_F, x) = a_T$.
 - (b) $f_\wedge(a_F, x) = f_\wedge(x, a_F) = f_\rightarrow(a_T, x) = f_\rightarrow(x, a_F) = a_F$.
 - (c) $f_\wedge(a_T, x) \ \& \ f_\wedge(x, a_T) \in K_1 \cup \{a_T\}$.
 - (d) $f_\vee(a_F, x) \ \& \ f_\vee(x, a_F) \in K_1 \cup \{a_F\}$.
4. $\forall x \in K_2$
 - (a) $f_\wedge(a_T, x) \ \& \ f_\wedge(x, a_T) \ \& \ f_\vee(a_T, x) \ \& \ f_\vee(x, a_T) \ \& \ f_\rightarrow(x, a_T) \ \& \ f_\rightarrow(a_F, x) \in K_2 \cup \{a_T\}$.
 - (b) $f_\wedge(a_F, x) \ \& \ f_\wedge(x, a_F) \ \& \ f_\vee(a_F, x) \ \& \ f_\vee(x, a_F) \ \& \ f_\rightarrow(x, a_F) \in K_2 \cup \{a_F\}$.
5.
 - (a) $\forall x \in K_2 f_\rightarrow(a_T, x) = a_F$
 - (b) $\forall x \in K_1 \forall y \in K_2 f_\rightarrow(x, y) \in F$.

Then it is said that M is a simplified strong relevant matrix (simplified sr-matrix, for short).

Simplified sr-matrices look as follows (cf. Remark 4.2, Remark 4.3).

\rightarrow	F	i	r	l	s	T	\neg
F		T	T	Tls	Tls	T	T
i	F	ir	ir	FF	FF	T	ir
r	F	ir	ir	FF	FF	T	ir
l	Fls			ls	ls	Tls	ls
s	Fls			ls	ls	Tls	ls
T	F	F	F	F	F		F

\wedge	F	i	r	l	s	T
F	F	F	F	Fls	Fls	
i	F	ir	ir			Tir
r	F	ir	ir			Tir
l	Fls			ls	ls	Tls
s	Fls			ls	ls	Tls
T		Tir	Tir	Tls	Tls	T

\vee	F	i	r	l	s	T
F	F	Fir	Fir	Fls	Fls	
i	Fir	ir	ir			T
r	Fir	ir	ir			T
l	Fls			ls	ls	Tls
s	Fls			ls	ls	Tls
T		T	T	Tls	Tls	T

Actually, the following is proved.

Proposition 6.6 *Let M be a simplified sr-matrix. Then M is an sr-matrix.*

Proof M is an svsp-matrix: by conditions (1)–(5) in Definition 6.5. M is an nlpp-matrix: by conditions (1), (2a), (3a), and (3b) in Definition 6.5. \square

Remark 6.7 It is clear that K_1 in Definition 6.5 plays the role K_3 does in Definition 6.1. But Definition 6.5 could of course be rewritten to let K_2 play the role of K_3 .

Next we shall provide some examples, first, a nonsimplified matrix.

Definition 6.8 Consider the matrix $M_{DF6.8} = (K, T, F, f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg})$ where

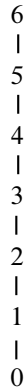
1. $K = \{0, 1, 2, 3, 4, 5, 6\}$
2. $T = \{2, 3, 4, 5, 6\}$
3. $F = \{0, 1\}$
4. the functions $f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}$ are defined as shown in the tables below:

\rightarrow	0	1	2	3	4	5	6	\neg	\wedge	0	1	2	3	4	5	6
0	6	6	6	6	6	6	6	6	0	0	0	0	0	0	0	0
1	0	5	5	5	5	5	6	5	1	0	1	1	1	1	1	1
2	0	1	2	3	4	5	6	2	2	0	1	2	2	2	2	2
3	0	1	1	3	4	5	6	3	3	0	1	2	3	3	3	3
4	0	1	1	1	4	5	6	4	4	0	1	2	3	4	4	4
5	0	1	1	1	0	5	6	1	5	0	1	2	3	4	5	5
6	0	0	0	0	0	0	6	0	6	0	1	2	3	4	5	6

\vee	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	2	3	4	5	6
2	2	2	2	3	4	5	6
3	3	3	3	3	4	5	6
4	4	4	4	4	4	5	6
5	5	5	5	5	5	5	6
6	6	6	6	6	6	6	6

- 5. $K_1 = \{4\}$
- 6. $K_2 = \{3\}$
- 7. $K_3 = \{2\}$
- 8. $a_T = 6$
- 9. $a_t = 5$
- 10. $a_F = 0$
- 11. $a_f = 1$.

Remark 6.9 The following is a Hasse diagram of $M_{DF6.8}$.



Definition 6.10 Consider the following logic $S_{DF6.8}$.

Axioms

- a1. $(A \wedge B) \rightarrow A / (A \wedge B) \rightarrow B$
- a2. $[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$
- a3. $A \rightarrow (A \vee B) / B \rightarrow (A \vee B)$
- a4. $[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C]$
- a5. $[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)]$
- a6. $(A \rightarrow B) \vee (B \rightarrow A)$
- a7. $[A \rightarrow (A \rightarrow B)] \leftrightarrow (A \rightarrow B)$
- a8. $A \leftrightarrow \neg\neg A$
- a9. $(A \wedge \neg B) \rightarrow \neg(A \rightarrow B)$
- a10. $(A \rightarrow B) \rightarrow [(\neg A \rightarrow B) \rightarrow B]$

Rules

- Modus ponens: If A and $A \rightarrow B$, then B .
- Adjunction: If A and B , then $A \wedge B$.
- Suffixing: If $A \rightarrow B$, then $(B \rightarrow C) \rightarrow (A \rightarrow C)$.

- Assertion: If A , then $(A \rightarrow B) \rightarrow B$.
 Contraposition: If $A \rightarrow B$ and $\neg B$, then $\neg A$.
 Reductio: If $A \rightarrow B$ and $A \rightarrow \neg B$, then $\neg A$.

The logic $S_{DF6.8}$ is one of the logics verified by $M_{DF6.8}$. It is not included in RMO (cf. [7] about this logic; also Definition 6.12 and Proposition 6.14 below): a_6 is not a theorem of RMO.

However, we have the following proposition.

Proposition 6.11 *The logic $S_{DF6.8}$ has the svsp and the nlpp.*

Proof By checking that $M_{DF6.8}$ is an sr-matrix and that $S_{DF6.8}$ is verified by it. \square

Next, some simplified matrices. First, the matrix (here in a different notation) for the logic RMO defined in [7].

Definition 6.12 Consider the matrix $M_{DF6.12} = (K, T, F, f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg})$ where

1. $K = \{0, 1, 2, 3, 4, 5, 6, 7\}$
2. $T = \{1, 2, 3, 4, 5, 6, 7\}$
3. $F = \{0\}$
4. the functions $f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}$ are defined as shown in the tables below:

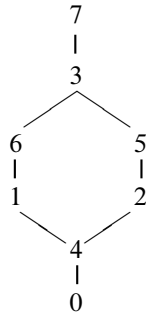
\rightarrow	0	1	2	3	4	5	6	7	\neg	7	\wedge	0	1	2	3	4	5	6	7
0	7	7	7	7	7	7	7	7	7	7	0	0	0	0	0	0	0	0	0
1	0	6	0	3	0	0	6	7	6	6	1	0	1	4	1	4	4	1	1
2	0	0	5	3	0	5	0	7	5	5	2	0	4	2	2	4	2	4	2
3	0	0	0	3	0	0	0	7	4	4	3	0	1	2	3	4	5	6	3
4	0	1	2	3	4	5	6	7	3	3	4	0	4	4	4	4	4	4	4
5	0	0	2	3	0	5	0	7	2	2	5	0	4	2	5	4	5	4	5
6	0	1	0	3	0	0	6	7	1	1	6	0	1	4	6	4	4	6	6
7	0	0	0	0	0	0	0	7	0	0	7	0	1	2	3	4	5	6	7

\vee	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	3	3	1	3	1	7
2	2	3	2	3	2	5	3	7
3	3	3	3	3	3	3	3	7
4	4	1	2	3	4	5	6	7
5	5	3	5	3	5	5	3	7
6	6	6	3	3	6	3	6	7
7	7	7	7	7	7	7	7	7

Moreover,

5. $K_1 = \{2, 5\}$
6. $K_2 = \{1, 6\}$
7. $a_T = \{7\}$
8. $a_F = \{0\}$.

Remark 6.13 The following is a Hasse diagram of $M_{DF6.12}$.



As in the case of $S_{DF6.8}$, we have the following.

Proposition 6.14 *The logic RMO has the svsp and nlpp.*

Proof By checking that $M_{DF6.12}$ is an sr-matrix (use Definition 6.5) and that RMO is verified by it (cf. [7]). \square

Finally, still a couple of examples. The first is Anderson and Belnap’s tables in [1], §22.1.3.

Definition 6.15 Consider the matrix $M_{DF6.15} = (K, T, F, f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg})$ where

1. $K = \{-0, -1, -2, -3, +0, +1, +2, +3\}$
2. $T = \{+0, +1, +2, +3\}$
3. $F = \{-0, -1, -2, -3\}$
4. the functions $f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}$ are defined as in [1], pp. 252–53.

Moreover,

5. $K_1 = \{-1, +1\}$
6. $K_2 = \{-2, +2\}$
7. $a_T = \{+3\}$
8. $a_F = \{-3\}$.

Matrix $M_{DF6.15}$ verifies R . And, as in the preceding examples, an analogue of Proposition 6.11 (or Proposition 6.14) for R (E) is immediate.

Finally, a simple sr-matrix featuring a series of logics not included in RM3.

Definition 6.16 Consider the matrix $M_{DF6.16} = (K, T, F, f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg})$ where

1. $K = \{0, 1, 2, 3\}$
2. $T = \{1, 2, 3\}$
3. $F = \{0\}$
4. the functions $f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}$ are defined as shown in the tables below:

\rightarrow	0	1	2	3	\neg	\wedge	0	1	2	3	\vee	0	1	2	3	
0	1	1	3	3	3	0	0	0	0	0	0	0	0	1	2	3
1	0	1	2	3	1	1	0	1	1	1	1	1	1	1	2	3
2	0	0	2	3	2	2	0	1	2	2	2	2	2	2	2	3
3	0	0	0	1	0	3	0	1	2	3	3	3	3	3	3	3

5. $K_1 = \{2\}$
6. $K_2 = \{1\}$
7. $a_T = 3$
8. $a_F = 0$.

Remark 6.17 The following is a Hasse diagram of $M_{DF6.15}$.



Definition 6.18 Consider the following logic $S_{DF6.16}$. The logic $S_{DF6.16}$ is axiomatized by adding to Routley and Meyer's basic positive logic B_+ (cf. [10]) the following axioms and rules.

Axioms

- a1. $[(A \rightarrow A) \rightarrow B] \rightarrow B$
- a2. $[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$
- a3. $[[[(B \rightarrow B) \rightarrow A] \rightarrow A] \rightarrow (A \rightarrow A)]$
- a4. $A \rightarrow [[[(B \rightarrow B) \rightarrow A] \rightarrow A] \rightarrow A]$
- a5. $[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$
- a6. $(A \rightarrow B) \vee (B \rightarrow A)$
- a7. $A \leftrightarrow \neg\neg A$
- a8. $(A \rightarrow \neg A) \rightarrow \neg A$

Rules

Contraposition: If $A \rightarrow B$ and $\neg B$, then $\neg A$.

Reductio: If $A \rightarrow B$ and $A \rightarrow \neg B$, then $\neg A$.

The following is proved.

Proposition 6.19 *The logic $S_{DF6.16}$ has the svsp and the nlpp.*

Proof As in the precedent examples, by checking that $M_{DF6.16}$ is an sr-matrix (use Definition 6.5) verifying $S_{DF6.16}$. \square

Now, notice that (a3) and (a4) are not provable in RM (they are not provable in RM3).

7 A Note on Relevance and the VSP

Throughout the paper, we have provided a number of logics not included in RM with the vsp, the nlpp, or both properties: our main aim has been to show that there are (maybe interesting) strong logics with these properties far off the spectrum of standard relevant logics. But, on the other hand, some of these logics show that Anderson and Belnap were right when insisting that the vsp is a necessary but by no means sufficient condition of relevance (for a logic to be a relevant logic). Take, for instance, $S_{DF6.16}$. This logic has the svsp and the nlpp, but one of its axioms is

$$(A \rightarrow B) \vee (B \rightarrow A)$$

which does not seem to be acceptable from an intuitive point of view.

In this sense, two interesting alternatives to the vsp have been proposed: the relevant equivalence property and the depth relevance property.

The *relevant equivalence property* (rep) is proposed in a preliminary study by Humberstone and Meyer in [6]. It reads as follows.

(rep) A logic “S will be said to have the relevant equivalence property if any two formulas which are equivalent in S are constructed from exactly the same propositional variables.” ([6], p. 165).

The results in [6] are almost exclusively confined to positive logics, but one of them is worth remarking: there are logics with the rep which lack the vsp, to wit, the $\{\rightarrow, \neg\}$ -fragment of RM.

The *depth relevance property* (drp), on the other hand, is proposed by Brady in [5] (see also [3] §11.1). It reads as follows.

(drp) A logic has the drp if in all theorems of S of the form $A \rightarrow B$, A and B share a variable at the same depth (“the depth of an occurrence of a subformula in a formula A is the degree of nesting “ \rightarrow ” s that is required to “reach” the occurrence of the subformula starting with the formula A ” ([5], p. 64).

The drp is a strong condition. No logic in which any of the following is provable has the drp:

$$\begin{aligned} &(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)] \\ &(q \rightarrow r) \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)] \\ &[(p \rightarrow p) \rightarrow q] \rightarrow q \\ &[p \wedge (p \rightarrow q)] \rightarrow q \\ &[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p \\ &[(p \rightarrow q) \wedge (p \rightarrow \neg q)] \rightarrow \neg p \\ &(p \rightarrow \neg p) \rightarrow \neg p \end{aligned}$$

And there are other well-known wffs producing the same effect. Therefore, standard relevant logics such as T , E , and R are not deep relevant logics. But, on the other hand, in [9] it is shown that depth relevance and loose pieces are compatible: there are deep relevant logics not included in RM. Consequently, each of the properties has its own drawbacks. Actually, the three are compatible with contrainuitive theses. The vsp, with axioms as (a6) in Definition 6.18 above; the rep, with paradoxes of relevance; and, finally, the drp, in addition to being rather strict, is compatible with loose pieces in wff.

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References

- [1] Anderson, A. R., and N. D. Belnap, Jr., *Entailment. The Logic of Relevance and Necessity. Volume I*, Princeton University Press, Princeton, 1975. [Zbl 0323.02030](#). [MR 0406756](#). [223](#), [224](#), [229](#), [233](#), [241](#)
- [2] Avron, A., “Relevant entailment—Semantics and formal systems,” *The Journal of Symbolic Logic*, vol. 49 (1984), pp. 334–42. [Zbl 0586.03017](#). [MR 745365](#). [224](#)
- [3] Brady, R. T., editor, *Relevant Logics and Their Rivals. Vol. II*, Ashgate, Farnham, 2003. [243](#)
- [4] Brady, R. T., “Completeness proofs for the systems RM3 and BN4,” *Logique et Analyse. Nouvelle Série*, vol. 25 (1982), pp. 9–32. [Zbl 0498.03012](#). [MR 666984](#). [232](#)
- [5] Brady, R. T., “Depth relevance of some paraconsistent logics,” *Studia Logica*, vol. 43 (1984), pp. 63–73. [Zbl 0581.03014](#). [MR 782849](#). [243](#)
- [6] Humberstone, L., and R. K. Meyer, “The relevant equivalence property,” *Logic Journal of the IGPL*, vol. 15 (2007), pp. 165–81. [Zbl 1128.03012](#). [MR 2333808](#). [243](#)
- [7] Méndez, J. M., “The compatibility of relevance and mingle,” *Journal of Philosophical Logic*, vol. 17 (1988), pp. 279–97. [Zbl 0651.03016](#). [MR 958621](#). [223](#), [240](#), [241](#)
- [8] Méndez, J. M., “Erratum to: The compatibility of relevance and mingle,” *Journal of Philosophical Logic*, vol. 39 (2010), p. 339. [MR 2645300](#). [223](#)
- [9] Robles, G., and J. M. Méndez, “Deep relevant logics not included in R-mingle,” in preparation. [232](#), [243](#)
- [10] Routley, R., V. Plumwood, R. K. Meyer, and R. T. Brady, *Relevant Logics and Their Rivals. Part I. The Basic Philosophical and Semantical Theory*, Ridgeview Publishing Co., Atascadero, 1982. [Zbl 0579.03011](#). [MR 728950](#). [232](#), [242](#)

Dpto. de Psicología, Sociología y Filosofía
 Universidad de León
 Campus de Vegazana, s/n
 24071 León
 SPAIN
gemmarobles@gmail.com
<http://grobv.unileon.es>

Universidad de Salamanca
 Campus Unamuno, Edificio FES
 37007 Salamanca
 SPAIN
sefus@usal.es
<http://web.usal.es/sefus>