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Weak Theories of Concatenation and Arithmetic

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Abstract We define a new theory of concatenation WTC which is much weaker than Grzegorczyk's well-known theory TC. We prove that WTC is mutually interpretable with the weak theory of arithmetic R. The latter is, in a technical sense, much weaker than Robinson's arithmetic Q, but still essentially undecidable. Hence, as a corollary, WTC is also essentially undecidable.

1 Introduction

The study of concatenation theory and arithmetic goes back to Quine [10] and Tarski [18]. The latter introduced two important axioms for concatenation: associativity of concatenation and the so-called editor axiom.

In 2005, Grzegorczyk [4] introduced a theory of concatenation TC with only two distinguished single-letters and studied this theory from the viewpoint of undecidability. In January 2007, Grzegorczyk and Zdanowski [5] proved that TC is essentially undecidable. But they left the following question open: Is Robinson's arithmetic Q interpretable in TC? Later in 2007, this question was solved positively; that is, Q is interpretable in TC. Hence, these two theories are mutually interpretable, since it is well known that $\mathsf{I}\Delta_0$ interprets TC and Q interprets $\mathsf{I}\Delta_0$ (see Hájek and Pudlák [6] and Nelson [9]).

The interpretability of Q in TC was proved via three different approaches (Sterken and Visser [12, 21], Ganea [3], and Švejdar [16]), which we sketch now. The main difficulty of the interpretability of Q in TC is the translation of the arithmetical product. To overcome this difficulty, interpretability is proved indirectly, that is, via an intermediate theory.

First, we consider the proof by Sterken and Visser. They proved the interpretability of Q in TC via the intermediate theory TC_Q, introduced by Sterken. In her Master's thesis [12], Sterken defined a theory TC_Q of concatenation which is an analogue

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of Q for concatenation theory, and she proved that $\mathsf{TC_Q}$ interprets Q. Then Visser proved in [21] that TC and $\mathsf{TC_Q}$ are mutually interpretable.

Another proof is given by Ganea or Švejdar. They proved the interpretability of Q in TC via Grzegorczyk's variant Q^- of Q, whose addition and multiplication can be nontotal. In Švejdar [15], he proved that Q and Q^- are mutually interpretable by applying Solovay's method of shortening of cuts (about this method, developed in an unpublished letter Solovay [11], see [6]). Hence, to interpret Q in TC, it is enough to interpret Q^- in TC. Subsequently, Ganea [3] and Švejdar [16] constructed this interpretation.

In this paper, we focus on the very weak arithmetical theory R which was introduced in Tarski et al. [17]. It is known that R cannot interpret Q since R is *locally finitely satisfiable*; that is, for any finite subtheory T of R, T has a finite model. Hence, R is much weaker than the Robinson's arithmetic Q. Nonetheless, R is still essentially undecidable [17]. Thus, we first define a new weak theory of concatenation WTC. Then we prove that this theory and R are mutually interpretable. This fact implies that WTC is essentially undecidable.

Here we briefly consider other ways of axiomatizing weak theories of concatenation which are mutually interpretable with R. In fact, in the Appendix A, we show that the alternative theory WTC' is logically equivalent to WTC. For yet another axiomatization, we can consider a weak theory of concatenation without the empty sequence, called $\mathsf{WTC}^{-\varepsilon}$. We conjecture that $\mathsf{WTC}^{-\varepsilon}$ and WTC (and hence R) are mutually interpretable. For this conjecture and the definition of $\mathsf{WTC}^{-\varepsilon}$, see Section 4.3 of this article.

By combining the main result of this article (Theorem 4.1) and the result of Visser [22], we can prove that for each theory T, WTC interprets T if and only if T is locally finitely satisfiable. Here Visser's result is the fact that R interprets T if and only if T is locally finitely satisfiable. In general, it is very difficult to prove that some theory T does not interpret some theory S. Hence, the Visser's result mentioned above is important. Therefore, our rephrasing of Visser's result in the context of the theory of concatenation is significant.

As to a brief overview of the rest of the paper: in Section 2, we introduce the relevant known theories of concatenation and their properties. In Section 3, we introduce our new theory of concatenation WTC and prove that this theory is Σ_1-complete. In Section 4, we prove our main theorem; that is, WTC and R are mutually interpretable. In Subsection 4.1, we provisionally prove that WTC interprets R. The full proof is in Appendix B. In Subsection 4.2, we prove that R interprets WTC. In Subsection 4.3, we consider some corollaries of our main theorem and formulate some questions for future research.

2 The Theories TC, Q, and R

First of all, we define the notion of interpretation.

Definition 2.1 (Interpretation) Let Σ and Ξ be recursive languages of first-order logic. A *relative translation* $\tau : \Sigma \to \Xi$ is a pair $\langle \delta, F \rangle$ such that

(1) δ is a Ξ-formula with one free variable;
(2) F is a mapping from Σ to the Ξ-formulas.

We translate Σ-formulas to Ξ-formulas as follows:

(1) For each n-ary relation symbol R of Σ,

$$(R(x_1, \cdots, x_n))^\tau := F(R)(x_1, \cdots, x_n);$$

(2) $(\varphi \wedge \psi)^\tau := \varphi^\tau \wedge \psi^\tau$ and likewise for other propositional connectives;
(3) $(\forall x \varphi(x))^\tau := \forall x(\delta(x) \to \varphi^\tau)$;
(4) $(\exists x \varphi(x))^\tau := \exists x(\delta(x) \wedge \varphi^\tau)$.

Let S and T be a Σ-theory and a Ξ-theory, respectively. Then S is *interpretable* in T, denoted by $T \rhd S$, if there exists a translation τ from Σ to Ξ such that for each axiom φ of S, T proves φ^τ.

Here we only consider relational languages. If the language has function symbols, then we first translate it to a relational one, by the usual method for eliminating function symbols. For a more precise definition of interpretation, see, for example, Visser [20].

Interpretability has some useful properties.

Proposition 2.2 *Suppose $T \rhd S$.*

(1) *If T is consistent, then so is S;*
(2) *if S is essentially undecidable, then so is T.*

Hence, to prove that a theory is essentially undecidable, it is enough to show that the theory interprets another theory which is already known to be essentially undecidable. Similarly, we can use interpretability to prove the relative consistency of some theories, by interpreting a theory in some consistent theory. We can also see the notion of interpretability as a measure of the strength of theories. Especially, it is important that we can compare the strength of some theories whose languages are *different*.

Next, we consider the definition of the theory of concatenation with two single-letters. Before this, we define standard strings. Let a and b be single-letters. We say that u is a *standard string* over $\{a, b\}$ if u is a finite sequence of the elements of $\{a, b\}$. An empty string is denoted by ε. Then let $\{a, b\}^*$ be a set of empty string ε and all standard strings over $\{a, b\}$.

Next, to represent standard strings in theories, we define the *name* of standard strings as follows: for each $u \in \{a, b\}^*$, we represent u in theories as $\underline{u}$ by rewriting a to α and b to β, and the elements of u associate to the left. Here α and β are (single-letter) constants of the theory. For example, $\underline{abbab} = (((\alpha\beta)\beta)\alpha)\beta$.

The theory TC of concatenation, as defined by Grzegorczyk, is the $(\frown, \alpha, \beta)$-theory with the following axioms:

(TC1) $\forall x\, \forall y\, \forall z\, (x \frown (y \frown z) = (x \frown y) \frown z)$;
(TC2) $\forall x\, \forall y\, \forall u\, \forall v\, [x \frown y = u \frown v \to ((x = u \wedge y = v)$
$\vee (\exists w\, ((x \frown w = u \wedge y = w \frown v) \vee (x = u \frown w \wedge w \frown y = v))))]$;
(TC3) $\forall x\, \forall y\, (x \frown y \neq \alpha)$;
(TC4) $\forall x\, \forall y\, (x \frown y \neq \beta)$;
(TC5) $\alpha \neq \beta$.

This theory has the standard model $(\{a, b\}^* \setminus \{\varepsilon\}; \frown, a, b)$ for texts. The axioms (TC1) and (TC2) were introduced by Tarski. The latter is called the *editor axiom*. It was proved in [5] that TC and its analogue with n's single-letters, $n \geq 2$, are mutually interpretable. By the previous, all of them are essentially undecidable.

Another variant of the concatenation theory is the theory $\mathsf{TC}^{\varepsilon}$ with empty string ε. The latter is the $(\frown, \varepsilon, \alpha, \beta)$-theory with the following axioms:

($\mathsf{TC}^{\varepsilon}$1) $\forall x\,(x^\frown\varepsilon = \varepsilon^\frown x = x)$;
($\mathsf{TC}^{\varepsilon}$2) $\forall x\,\forall y\,\forall z\,(x^\frown(y^\frown z) = (x^\frown y)^\frown z)$;
($\mathsf{TC}^{\varepsilon}$3) $\forall x\,\forall y\,\forall u\,\forall v\,(x^\frown y = u^\frown v \rightarrow$
$\exists w\,((x^\frown w = u \wedge y = w^\frown v) \vee (x = u^\frown w \wedge w^\frown y = v)))$;
($\mathsf{TC}^{\varepsilon}$4) $\alpha \neq \varepsilon \wedge \forall x\,\forall y\,(x^\frown y = \alpha \rightarrow x = \varepsilon \vee y = \varepsilon)$;
($\mathsf{TC}^{\varepsilon}$5) $\beta \neq \varepsilon \wedge \forall x\,\forall y\,(x^\frown y = \beta \rightarrow x = \varepsilon \vee y = \varepsilon)$;
($\mathsf{TC}^{\varepsilon}$6) $\alpha \neq \beta$.

This theory has the standard model $(\{a,b\}^*; \frown, \alpha, \beta, \varepsilon)$ for texts. It was proved in [21] that $\mathsf{TC}^{\varepsilon}$ and TC are mutually interpretable.

In what follows, we abbreviate $x^\frown y$ to xy, and by $\mathsf{TC}^{\varepsilon}$2, we can omit parentheses. Let us consider some facts which are (not) provable in $\mathsf{TC}^{\varepsilon}$. For the proofs of the following propositions and other properties of $\mathsf{TC}^{\varepsilon}$, see [3, 4, 5, 16, 21], Švejdar [13], and Čačić [1].

Definition 2.3 ($\sqsubseteq, \sqsubseteq_{\mathsf{ini}}, \sqsubseteq_{\mathsf{end}}$) We say that x is a *substring* of y if there exist z and z' such that $zxz' = y$, denoted by $x \sqsubseteq y$. We say that x is an *initial string* of y if there exists z such that $xz = y$, denoted by $x \sqsubseteq_{\mathsf{ini}} y$. Similarly, we say that x is an *end string* of y if there exists z such that $zx = y$, denoted by $x \sqsubseteq_{\mathsf{end}} y$.

The proofs of the following proposition may be found in [5] and [16].

Proposition 2.4 $\mathsf{TC}^{\varepsilon}$ *proves the following assertions:*

(1) $\forall x\,(x\alpha \neq \varepsilon \wedge \alpha x \neq \varepsilon)$.
(2) $\forall x\,\forall y\,(xy = \varepsilon \rightarrow x = \varepsilon \wedge y = \varepsilon)$.
(3) $\forall x\,\forall y\,(x\alpha = y\alpha \vee \alpha x = \alpha y \rightarrow x = y)$.
(4) $\forall x\,\forall y\,\forall z\,(xy = z\alpha \rightarrow y = \varepsilon \vee \alpha \sqsubseteq_{\mathsf{end}} y)$.
(5) $\forall x\,\forall y\,(\alpha \sqsubseteq xy \rightarrow \alpha \sqsubseteq x \vee \alpha \sqsubseteq y)$.

In each item, α may be replaced by β.

The following facts express that, even though $x \sqsubseteq y$, we cannot find the *exact* position of x within y. The proof of the following proposition may be found in [13].

Proposition 2.5 $\mathsf{TC}^{\varepsilon}$ *cannot prove the following assertions:*

(1) $\forall x\,\forall y\,\forall z\,(xz = yz \rightarrow x = y)$.
(2) $\forall x\,\neg(\exists y\,(xy = x \wedge y \neq \varepsilon))$.

Item (2) expresses that $\mathsf{TC}^{\varepsilon}$ cannot refute the statement x *is a proper initial string of itself*.

Next we consider some arithmetics and their properties. The well-known Gödel and Rosser's first incompleteness theorem states that Q is essentially undecidable. Here Q is Robinson's arithmetic with language $(+, \cdot, 0, \mathrm{S})$ and axioms as follows.

(Q1) $\forall x\,\forall y\,(\mathrm{S}(x) = \mathrm{S}(y) \rightarrow x = y)$;
(Q2) $\forall x\,(\mathrm{S}(x) \neq 0)$;
(Q3) $\forall x\,(x \neq 0 \rightarrow \exists y(x = \mathrm{S}(y)))$;
(Q4) $\forall x\,(x + 0 = x)$;
(Q5) $\forall x\,\forall y\,(x + \mathrm{S}(y) = \mathrm{S}(x + y))$;
(Q6) $\forall x\,(x \cdot 0 = 0)$;
(Q7) $\forall x\,\forall y\,(x \cdot \mathrm{S}(y) = x \cdot y + x)$.

Here $x \leq y$ is defined as $\exists z\,(x + z = y)$. In [17], the much weaker arithmetic R is defined. The arithmetic R is the $(+, \cdot, \mathrm{S}, 0)$-theory with axioms as follows. For all $n, m \in \omega$,

(R1) $\bar{n} + \bar{m} = \overline{n+m}$;
(R2) $\bar{n} \cdot \bar{m} = \overline{n \cdot m}$;
(R3) $\bar{n} \neq \bar{m}$ for $n \neq m$;
(R4) $\forall x (x \leq \bar{n} \to x = \bar{0} \vee \cdots \vee x = \bar{n})$;
(R5) $\forall x (x \leq \bar{n} \vee \bar{n} \leq x)$.

Here we define $x \leq y \equiv \exists z (z + x = y)$ and $\bar{0} = 0$ and $\overline{n+1} = \mathrm{S}(\bar{n})$. It was proved in [17] that R is also essentially undecidable. Moreover, it was proved by Cobham (e.g., Vaught [19]) that if we drop the axiom scheme (R5) from R, the resulting theory, called R_0 in Jones and Shepherdson [7], and R are mutually interpretable. Hence, R_0 is also essentially undecidable.

It is interesting to note that R_0 is *minimal* essentially undecidable in the following sense: if one omits one of the axiom schemes of R_0, the resulting theory is *not* essentially undecidable. See [7] for other minimal arithmetical theories which are essentially undecidable.

Another important fact is that the theory R is locally finitely satisfiable. Hence, Q is not interpretable in R, since Q is finitely axiomatizable. In 2009, Visser proved that R interprets T if and only if T is locally finitely satisfiable [22]. We will apply this very powerful and important theorem to interpret WTC in R. This is the topic of Section 4.2.

3 The Theory WTC and Σ_1-Completeness

We now define the notions *standard string* and *name* of the standard string over three single-letters in the same way as for two single-letters. Note that the standard strings associate to the left.

Definition 3.1 The $(\frown, \varepsilon, \alpha, \beta, \gamma)$-theory WTC has the following axioms. For each $u \in \{a, b, c\}^*$,

(WTC1) $\forall x \sqsubseteq \underline{u}\, (x \frown \varepsilon = \varepsilon \frown x = x)$;
(WTC2) $\forall x\, \forall y\, \forall z\, [((x \frown y) \frown z \sqsubseteq \underline{u} \vee x \frown (y \frown z) \sqsubseteq \underline{u}) \to (x \frown y) \frown z = x \frown (y \frown z)]$;
(WTC3) $\forall x\, \forall y\, \forall s\, \forall t\, [(x \frown y = s \frown t \wedge x \frown y \sqsubseteq \underline{u}) \to$
$\exists w\, ((x \frown w = s \wedge y = w \frown t) \vee (x = s \frown w \wedge w \frown y = t))]$;
(WTC4) $\alpha \neq \varepsilon \wedge \forall x\, \forall y\, (x \frown y = \alpha \to x = \varepsilon \vee y = \varepsilon)$;
(WTC5) $\beta \neq \varepsilon \wedge \forall x\, \forall y\, (x \frown y = \beta \to x = \varepsilon \vee y = \varepsilon)$;
(WTC6) $\gamma \neq \varepsilon \wedge \forall x\, \forall y\, (x \frown y = \gamma \to x = \varepsilon \vee y = \varepsilon)$;
(WTC7) $\alpha \neq \beta \wedge \beta \neq \gamma \wedge \gamma \neq \alpha$.

Here, since within the system WTC, the associative law does not hold for every triple of strings, we define the relation symbols $\sqsubseteq$, $\sqsubseteq_{\mathsf{ini}}$, and $\sqsubseteq_{\mathsf{end}}$ as follows:

$$x \sqsubseteq y \equiv x = y \vee \exists k\, \exists l\, [k \frown x = y \vee x \frown l = y \vee (k \frown x) \frown l = y \vee k \frown (x \frown l) = y],$$
$$x \sqsubseteq_{\mathsf{ini}} y \equiv x = y \vee \exists l\, (x \frown l = y),$$
$$x \sqsubseteq_{\mathsf{end}} y \equiv x = y \vee \exists k\, (k \frown x = y).$$

Remark 3.2 There are different ways to axiomatize weak theories of concatenation. In Appendix A, we consider another axiomatization WTC' of weak theory of concatenation. Our reasons for our choice of the axioms of WTC are the following:

(1) The theory WTC is a naturally weaker version of TC^ε in the following sense: the axioms (WTC1), (WTC2), and (WTC3) of WTC are weaker versions of the axioms ($\mathrm{TC}^\varepsilon 1$), ($\mathrm{TC}^\varepsilon 2$), and ($\mathrm{TC}^\varepsilon 3$) of TC^ε, respectively. The other axioms of WTC are similar to the axioms ($\mathrm{TC}^\varepsilon 4$), ($\mathrm{TC}^\varepsilon 5$), and ($\mathrm{TC}^\varepsilon 6$) of TC^ε.

(2) While the statement

$$\forall x\,(x \sqsubseteq \underline{u} \leftrightarrow \bigvee_{v \sqsubseteq u} x = \underline{v})$$

is an axiom of WTC$'$, it is provable in WTC; that is, we do not need to add this statement to the axioms of WTC.

Remark 3.3 Here we have to define the theory WTC as a theory with three single-letters. As we will prove in Section 4.3, any two theories WTC_n ($n \geq 2$) are mutually interpretable (see Theorem 4.13). Our choice of WTC makes the proof of the main theorem, Theorem 4.8, as clear as possible.

In WTC, the constant string associates *to the left*. As such, WTC proves the identity of all strings which only differ up to positions of parentheses.

Example 3.4 WTC proves $\alpha((\beta\alpha)\alpha) = (\alpha\beta)(\alpha\alpha)$.

Proof Let $u \in \{a, b, c\}^*$ be $abaa$. Then $\underline{u} = ((\alpha\beta)\alpha)\alpha$. By (WTC2),

$$\underline{u} = (\alpha\beta)(\alpha\alpha). \quad (\dagger)$$

Since $(\alpha\beta)\alpha \sqsubseteq \underline{u}$, by (WTC2), $(\alpha\beta)\alpha = \alpha(\beta\alpha)$. Hence, $\underline{u} = (\alpha(\beta\alpha))\alpha$. By (WTC2), $(\alpha(\beta\alpha))\alpha = \alpha((\beta\alpha)\alpha)$. Therefore,

$$\underline{u} = \alpha((\beta\alpha)\alpha). \quad (\ddagger)$$

Then, by ($\dagger$) and ($\ddagger$), $\alpha((\beta\alpha)\alpha) = (\alpha\beta)(\alpha\alpha)$. ☐

The theory WTC is much weaker than TC, but WTC still proves the following statements.

Proposition 3.5 WTC *proves the following assertions:*

(1) $\forall x\,(x\alpha \neq \varepsilon \wedge \alpha x \neq \varepsilon)$. *The same for* β *and* γ.

(2) $\forall x\,\forall y\,(xy = \varepsilon \rightarrow x = \varepsilon \wedge y = \varepsilon)$.

Proof We reason in WTC. We prove (1) by contradiction. To do this, let us assume that $x\alpha = \varepsilon$ holds (the same argument holds for the other case). Then, by (WTC1), $\beta(x\alpha) = \beta$. Then, by (WTC2),

$$(\beta x)\alpha = \beta. \quad (\dagger)$$

Then, by (WTC5), $\beta x = \varepsilon$ or $\alpha = \varepsilon$. Then, by (WTC4), $\beta x = \varepsilon$ holds. Then, by ($\dagger$), $\beta = (\beta x)\alpha = \varepsilon\alpha = \alpha$. This contradicts (WTC7).

For part (2), let us assume that $xy = \varepsilon$ holds. Then $(xy)\alpha = \alpha$. By (WTC2), $x(y\alpha) = \alpha$. By (WTC4), $x = \varepsilon$ or $y\alpha = \varepsilon$. But by 1, $x = \varepsilon$ holds. Also, from the assumption, $\alpha(xy) = \alpha$ holds. Then we can prove $y = \varepsilon$ by the same arguments as above. ☐

Next we define the notion *good string*.

Definition 3.6 (Good string) We define the formula Good(x) as follows:

$$\mathrm{Good}(x) \equiv \mathrm{ID}(x) \wedge \mathrm{AS}(x) \wedge \mathrm{EA}(x),$$

where

(i) $\mathrm{ID}(x) \equiv \forall s \sqsubseteq x\, (s^\frown\varepsilon = \varepsilon^\frown s = s)$;
(ii) $\mathrm{AS}(x) \equiv \forall s_0\, \forall s_1\, \forall s_2\, [[s_0{}^\frown(s_1{}^\frown s_2) \sqsubseteq x \vee (s_0{}^\frown s_1)^\frown s_2 \sqsubseteq x] \rightarrow s_0{}^\frown(s_1{}^\frown s_2) = (s_0{}^\frown s_1)^\frown s_2]$;
(iii) $\mathrm{EA}(x) \equiv \forall s_0\, \forall s_1\, \forall t_0\, \forall t_1\, [(s_0{}^\frown s_1 = t_0{}^\frown t_1 \wedge s_0{}^\frown s_1 \sqsubseteq x) \rightarrow \exists w\, ((s_0{}^\frown w = t_0 \wedge s_1 = w^\frown t_1) \vee (s_0 = t_0{}^\frown w \wedge w^\frown s_1 = t_1))]$.

If Good(x) holds, then we say that x is a *good string*.

The set of good strings contains all standard texts.

Proposition 3.7 *For each* $u \in \{a, b, c\}^*$, WTC $\vdash$ Good$(\underline{u})$.

Proof For each $u \in \{a, b, c\}^*$, since $\underline{u} \sqsubseteq \underline{u}$ holds inside WTC, we can apply for $\underline{u}$ the axioms (WTC1), (WTC2), and (WTC3) of WTC. Thus Good$(\underline{u})$ holds within WTC. □

We can prove that, if we limit the relation $\sqsubseteq$ to substrings of good strings, the definition of $\sqsubseteq$ inside WTC is equivalent to the definition inside TC. Then we can prove that the good strings are closed under substrings.

Proposition 3.8 WTC *proves the following assertions:*

(1) $\forall x\, (\mathrm{Good}(x) \rightarrow \forall y\, (y \sqsubseteq x \leftrightarrow \exists k\, \exists l\, [(ky)l = x]))$;
(2) $\forall x\, (\mathrm{Good}(x) \rightarrow \mathrm{TR}_\sqsubseteq(x))$, *where*
$$\mathrm{TR}_\sqsubseteq(x) \equiv \forall y\, \forall z\, (y \sqsubseteq x \wedge z \sqsubseteq y \rightarrow z \sqsubseteq x);$$
(3) $\forall x\, (\mathrm{Good}(x) \rightarrow \forall y \sqsubseteq x\ \mathrm{Good}(y))$.

Proof We reason in WTC. For part (1), let x be a good string. For each y, by the definition of $\sqsubseteq$ within WTC, the direction "$\leftarrow$" is trivial. We prove the converse by cases in the definition of $\sqsubseteq$. Let us assume that $y \sqsubseteq x$.

(i) If $y = x$, then put $k = \varepsilon$ and $l = \varepsilon$. Then, ID(x) and $y \sqsubseteq x$ implies $y\varepsilon = \varepsilon y = y$. Thus, $(ky)l = (\varepsilon y)l = yl = y\varepsilon = y$ holds.
(ii) If there exist k and l such that $(ky)l = x$, then there is nothing to prove.
(iii) If there exist k and l such that $k(yl) = x$, then $k(yl) \sqsubseteq x$ and AS(x) imply $(ky)l = k(yl) = x$.
(iv) If there exists k such that $ky = x$, then put $l = \varepsilon$. Then $ky \sqsubseteq x$ and ID(x) imply $(ky)l = (ky)\varepsilon = ky = x$.
(v) If there exists l such that $yl = x$, then we can give an argument analogous to the one used in case (iv).

For part (2), let x be a good string and let y and z be strings such that $y \sqsubseteq x$ and $z \sqsubseteq y$. By part (1), the condition $y \sqsubseteq x$ is equivalent to $(ky)l = x$ for some k and l. We prove (2) by cases in the definition of $z \sqsubseteq y$ as for (1).

(i) If $y = x$, then there is nothing to prove.
(ii) If there exist k' and l' such that $(k'y)l' = x$, then we can prove this case as follows:

$$
\begin{aligned}
x &= [k((k'z)l')]l \\
&= [k(k'(zl'))]l && \text{(by } (k'z)l' \sqsubseteq x \text{ and AS}(x)) \\
&= [(kk')(zl')]l && \text{(by } k(k'(zl')) \sqsubseteq x \text{ and AS}(x)) \\
&= (kk')[(zl')l] && \text{(by } (kk')(zl') \sqsubseteq x \text{ and AS}(x)) \\
&= (kk')[z(l'l)] && \text{(by } (zl')l \sqsubseteq x \text{ and AS}(x))
\end{aligned}
$$

Hence, by the definition of $\sqsubseteq$, $z \sqsubseteq x$ holds.

(iii), (iv), and (v) can be proved by a similar argument in (ii).

For part (3), let x be a good string and let y be a substring of x. Then, by part (2), $\mathrm{TR}_{\sqsubseteq}(x)$ holds. Then this property provides that $\mathrm{ID}(x)$, $\mathrm{AS}(x)$, and $\mathrm{EA}(x)$ imply $\mathrm{ID}(y)$, $\mathrm{AS}(y)$, and $\mathrm{EA}(y)$, respectively. Hence $\mathrm{Good}(y)$ holds. □

Proposition 3.9 *The theory* WTC *proves the following assertions:*

(1) $\forall x\, \forall y\, (((\mathrm{Good}(x\alpha) \wedge x\alpha = y\alpha) \vee (\mathrm{Good}(\alpha x) \wedge \alpha x = \alpha y)) \to x = y)$.
The same for β and γ.

(2) $\forall x\, \forall y\, \forall z\, ((\mathrm{Good}(xy) \wedge xy = z\alpha) \to (y = \varepsilon \vee \alpha \sqsubseteq_{\mathsf{end}} y))$.
The same for β and γ.

(3) $\forall x\, \forall y\, \forall z\, ((\mathrm{Good}(xy) \wedge xy = \alpha z) \to (x = \varepsilon \vee \alpha \sqsubseteq_{\mathsf{ini}} x))$.
The same for β and γ.

Proof We reason in WTC. For part (1), let us assume that $\mathrm{Good}(x\alpha) \wedge x\alpha = y\alpha$. The other case is similar. By $\mathrm{EA}(x\alpha)$, there exists w such that

$$(xw = y \wedge \alpha = w\alpha) \vee (x = yw \wedge w\alpha = \alpha).$$

In both cases, $w = \varepsilon$ holds by (WTC4). Hence, $x = y$ holds by $\mathrm{ID}(x\alpha)$.

For part (2), let us assume that $\mathrm{Good}(xy) \wedge xy = z\alpha$. Then, by $\mathrm{EA}(xy)$, the following assertion holds: there exists w such that

$$(xw = z \wedge y = w\alpha) \vee (x = zw \wedge wy = \alpha).$$

Assume that the first clause holds; then, by $y = w\alpha$, $\alpha \sqsubseteq_{\mathsf{end}} y$ holds. This is what we want to prove. Assume that the second clause holds; then, by $wy = \alpha$, $w = \varepsilon \vee y = \varepsilon$ holds. If $y = \varepsilon$, then there is nothing to prove. If $w = \varepsilon$, then $y = w\alpha = \alpha$ and this implies that $\alpha \sqsubseteq_{\mathsf{end}} y$.

Finally, part (3) follows in the same way as part (2). □

Lemma 3.10 *For each $u, v \in \{a, b, c\}^*$,*

(1) *$u = v$ if and only if* WTC $\vdash \underline{u} = \underline{v}$;

(2) *$u \neq v$ if and only if* WTC $\vdash \underline{u} \neq \underline{v}$.

Proof This is proved via (meta-)induction on the length of $v \in \{a, b, c\}^*$. □

Lemma 3.11 *For each $u \in \{a, b, c\}^*$,* WTC *proves*

$$\forall x\, (x \sqsubseteq \underline{u} \leftrightarrow \bigvee_{v \sqsubseteq u} x = \underline{v}).$$

Proof By the definition of $\sqsubseteq$ inside WTC, the direction "$\leftarrow$" is easy. Thus we only need to prove the converse. This is proved via (meta-)induction on the length of $u \in \{a, b, c\}^*$.

First, let u be an empty string for the base step of the induction. We reason in WTC. Since u is an empty string, $\underline{u} = \varepsilon$. We can easily prove that for all x, if $x \sqsubseteq \varepsilon$, then $x = \varepsilon$. Thus, this lemma holds in this case.

Next, we assume the following: for some $u \in \{a, b, c\}^*$,

$$\forall x \, (x \sqsubseteq \underline{u} \rightarrow \bigvee_{v \sqsubseteq u} x = \underline{v}).$$

Then we have to prove that $\forall x (x \sqsubseteq \underline{u}\alpha \rightarrow \bigvee_{v \sqsubseteq ua} x = \underline{v})$ (the same discussion holds for $\underline{u}\beta$ and $\underline{u}\gamma$). To prove this, let us take x such that $x \sqsubseteq \underline{u}\alpha$. Since $\underline{u}\alpha$ is a good string, the statement $x \sqsubseteq \underline{u}\alpha$ is equivalent to $\exists k \exists l ((kx)l = \underline{u}\alpha)$. Then $\mathsf{EA}((kx)l)$ implies that there exists w such that

$$((kx)w = \underline{u} \wedge l = w\alpha) \vee (kx = \underline{u}w \wedge wl = \alpha).$$

Assume that the first clause $(kx)w = \underline{u} \wedge l = w\alpha$ holds. Then the induction hypothesis implies that $x = \underline{v}$ for some $v \sqsubseteq u$. It is trivial that this v is also a substring of ua. This implies that the induction step holds for this case.

Assume that the second clause $kx = \underline{u}w \wedge wl = \alpha$ holds. Then $wl = \alpha$ implies $w = \varepsilon \vee l = \varepsilon$. If $w = \varepsilon$ holds, then, by $kx = \underline{u}w$, $x \sqsubseteq \underline{u}$ holds, which implies the desired condition. If $l = \varepsilon$ holds, then $kx = \underline{u}\alpha$. If $x = \varepsilon$, then there is nothing to prove. Thus we assume that $x \neq \varepsilon$. By Proposition 3.9(2), this assumption and $kx = \underline{u}\alpha$ imply that $x = x'\alpha$ for some x'. Then, by Proposition 3.9(1), $(kx')\alpha = \underline{u}\alpha$ implies $kx' = \underline{u}$. By the induction hypothesis, since $x' \sqsubseteq \underline{u}$, there exists $v \sqsubseteq u$ such that $x' = \underline{v}$. Then $(k\underline{v})\alpha = \underline{u}\alpha$ holds. Since $\underline{u}\alpha$ is a standard string, the associative law implies $k(\underline{v}\alpha) = \underline{u}\alpha$. Thus, $\underline{v}\alpha \sqsubseteq \underline{u}\alpha$. Since $x = x'\alpha$ and $x' = \underline{v}$, $x = \underline{v}\alpha$. Hence, $x = \underline{v}\alpha$ holds where va is a substring of ua. □

Lemma 3.12 *For each $u, v \in \{a, b, c\}^*$,*

(1) *$u \sqsubseteq v$ if and only if* WTC $\vdash \underline{u} \sqsubseteq \underline{v}$*;*
(2) *$\neg u \sqsubseteq v$ if and only if* WTC $\vdash \neg \underline{u} \sqsubseteq \underline{v}$*.*

Proof This is proved via induction on the length of $v \in \{a, b, c\}^*$. □

Next we consider the Σ_1-completeness of WTC. Here a Σ_1-formula is defined as follows: φ is a Σ_0-formula if the all quantifiers occurring in φ are *bounded*, that is, are the form $\forall x \sqsubseteq t$ or $\exists x \sqsubseteq t$ where t is a term which does not contain x. Then φ is a Σ_1-formula if φ is a Σ_0-formula or the form $\exists x_1 \cdots \exists x_n \, \theta(x_1, \cdots, x_n)$ where θ is a Σ_0-formula and $n \geq 1$.

Theorem 3.13 (Σ_1-completeness of WTC) *The theory* WTC *is Σ_1-complete; that is, for each Σ_1-formula φ, if $\{a, b, c\}^* \vDash \varphi$, then* WTC $\vdash \varphi$*.*

Proof It is enough to prove this statement for Σ_0-formulas. The theorem follows from Lemmas 3.10, 3.11, 3.12. □

4 The Theories WTC and R Are Mutually Interpretable

In this section and Appendix B, we prove our main theorem.

Theorem 4.1 *The theories* WTC *and* R *are mutually interpretable.*

In Subsection 4.1, we show that WTC can interpret R. This is proved in Appendix B, because the proof is complicated and the reader does not need to know the details to understand the rest of this section. In Subsection 4.2, we prove that R interprets WTC by applying Visser's result in [22]. In Subsection 4.3, we consider some corollaries and some open questions.

4.1 The theory WTC interprets R In this subsection, we prove the interpretability of R in WTC. For details of this interpretation, consult Appendix B. Since R and R_0 are mutually interpretable, to interpret R in WTC, it is sufficient to interpret R_0, whose axioms are the first four schemes of R. The difficult part of this interpretation is to translate the multiplication of R_0. The essential idea of this interpretation is from [16]. The translation from R_0 to WTC, excluding the product,

- $0 \Longrightarrow \varepsilon$;
- $\mathrm{S}(x) \Longrightarrow x^\frown\alpha$;
- $x + y = z \Longrightarrow x^\frown y = z$.

Then, in WTC, we can prove the interpreted (R1), (R3), and (R4) by Lemma 3.10 and Lemma 3.11.

In what follows, we will prove some propositions to construct the translation of the product. The key result of this subsection (and this article) is Theorem 4.8.

Lemma 4.2 *The theory* WTC *proves*

$$\forall x\,\forall y\,((\mathrm{Good}(xy) \wedge \alpha \sqsubseteq xy) \to (\alpha \sqsubseteq x \vee \alpha \sqsubseteq y)).$$

The same is true for β and γ.

Proof We reason in WTC. Let us assume that xy is good and α is a substring of xy. Then, by Proposition 3.8(1), $\alpha \sqsubseteq xy$ is equivalent to $\exists k \exists l (k\alpha)l = xy$. Then, by $\mathrm{EA}(xy)$, there exists w such that

$$((k\alpha)w = x \wedge l = wy) \vee (k\alpha = xw \wedge wl = y).$$

If the first clause holds, then $(k\alpha)w = x$ implies $\alpha \sqsubseteq x$. If the second clause holds, by Proposition 3.9(2), $k\alpha = xw$ implies $w = \varepsilon$ or $\alpha \sqsubseteq_{\mathsf{end}} w$. If $w = \varepsilon$, then $k\alpha = x$ implies $\alpha \sqsubseteq x$. If $\alpha \sqsubseteq_{\mathsf{end}} w$, then there exists w' such that $w'\alpha = w$ and $y = wl = (w'\alpha)l$. This means that α is a substring of y. □

Definition 4.3 (Number string) We define the formula $\mathrm{Num}(x)$ as follows:

$$\mathrm{Num}(x) \equiv \forall y\,((y \sqsubseteq x \wedge y \neq \varepsilon) \to \alpha \sqsubseteq_{\mathsf{end}} y).$$

If $\mathrm{Num}(x)$ holds, then we say that x is a *number string*.

Proposition 4.4 *For each $u \in \{a\}^*$,* WTC $\vdash \mathrm{Num}(\underline{u})$.

Proof This is proved via induction on the length of $u \in \{a\}^*$. □

Proposition 4.5 *The theory* WTC *proves the following assertions:*

(1) $\forall x\,((\mathrm{Good}(x) \wedge \mathrm{Num}(x)) \to \forall y \sqsubseteq x\,\mathrm{Num}(y))$;
(2) $\forall x\,\forall y\,((\mathrm{Good}(xy) \wedge \mathrm{Num}(x) \wedge \mathrm{Num}(y)) \to \mathrm{Num}(xy))$.

Proof We reason in WTC. For part (1), let us assume that x is good and $\mathrm{Num}(x)$ holds. Let y be any substring of x. If $y = \varepsilon$, then there is nothing to prove. If $y \neq \varepsilon$, then let z be a substring of y and assume that $z \neq \varepsilon$. By $\mathrm{TR}_{\sqsubseteq}(x)$, $z \sqsubseteq x$ holds, and by $\mathrm{Num}(x)$, $\alpha \sqsubseteq_{\mathsf{end}} z$ holds. This implies $\mathrm{Num}(y)$.

For part (2), let x, y be number strings which satisfy Good(xy). To prove that xy is a number string, let z be a nonempty substring of xy. Then $kzl = xy$ for some k and l. Then, by EA(xy), applying the editor axiom to $k(zl)$, there exists w such that

$$(kw = x \wedge zl = wy) \vee (k = xw \wedge wzl = y).$$

If $w = \varepsilon$, then $z \sqsubseteq y$. Since y is a number string, $\alpha \sqsubseteq_{\text{end}} z$ holds. So let us assume that $w \neq \varepsilon$. Then, if $kw = x \wedge zl = wy$, then, by Proposition 3.8(3), wy is also a good string. Hence, by EA(wy), there exists h which satisfies

$$(zh = w \wedge l = hy) \vee (z = wh \wedge hl = y).$$

If $h = \varepsilon$, then $z \sqsubseteq x$ or $z \sqsubseteq y$ and thus there is nothing to prove. So let us assume that $h \neq \varepsilon$. The first clause implies $z \sqsubseteq x$ and then $\alpha \sqsubseteq_{\text{end}} z$ since x is a number string. The second clause implies $h \sqsubseteq y$. Since $h \neq \varepsilon$, $h = h'\alpha$ for some h'. Because $z = wh$, we find that $z = wh'\alpha$. Finally, if $k = xw \wedge wzl = y$, then $z \sqsubseteq y$ holds. Then, since y is a number string, $\alpha \sqsubseteq_{\text{end}} z$ holds. This completes the proof of this proposition. □

The next lemma states that we can treat the string between β's as a block.

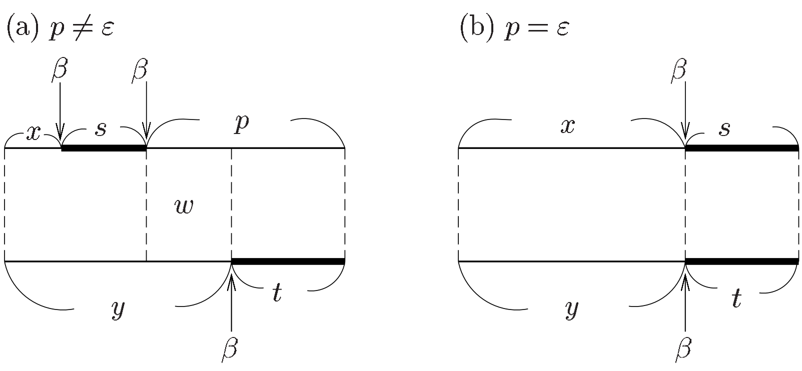
Lemma 4.6 *The theory* WTC *proves the following assertions:*

(1) *for all x, y, s and t, if $x\beta s$ is good, $x\beta s = y\beta t$, and s and t have no occurrences of β, then $x = y$ and $s = t$.*

(2) *for all x, y, s, t and p, if $x\beta s\beta p$ is good, $x\beta s\beta p = y\beta t\beta$, and s and t have no occurrences of β, then either*

  (a) there exists w such that $x\beta s\beta w = y\beta$ and $wt\beta = p$, or

  (b) $x = y$, $s = t$, and $p = \varepsilon$.



Proof For part (1), let us assume that Good$(x\beta s) \wedge (x\beta)s = (y\beta)t$ holds. Then, by EA$(x\beta s)$, there exists w such that

$$((x\beta)w = y\beta \wedge s = wt) \vee (x\beta = (y\beta)w \wedge ws = t).$$

We only consider the first case. In the other case, we can prove similarly. If $w \neq \varepsilon$, then, by Proposition 3.9(2), $w = w'\beta$ for some w'. Then $s = wt = (w'\beta)t$ holds, and this implies that $\beta \sqsubseteq s$. But this contradicts $\neg(\beta \sqsubseteq s)$. Hence, $w = \varepsilon$ and $x\beta = y\beta$ must hold, and by Proposition 3.9(1), $x = y$ holds. Finally, by $s = wt$, we have $s = t$.

For part (2), let us assume that $\mathrm{Good}(x\beta s\beta p) \wedge x\beta s\beta p = y\beta t\beta$ holds. Then, by the editor axiom for $(x\beta s\beta)p = (y\beta)(t\beta)$, there exists w such that

$$((x\beta s\beta)w = y\beta \wedge p = wt\beta) \vee (x\beta s\beta = y\beta w \wedge wp = t\beta).$$

The first case is immediate. In the second case, if $w = \varepsilon$, then there is nothing to prove. Let us assume that $w \neq \varepsilon$. Then, by Proposition 3.9(2), $(x\beta s)\beta = (y\beta)w$ implies that $w = w'\beta$ for some w'. Then

$$t\beta = wp = w'\beta p. \quad (\dagger)$$

Now we prove $p = \varepsilon$ by contradiction. Assume that $p \neq \varepsilon$; then, by Proposition 3.9(2), $(\dagger)$ implies $p = p'\beta$ for some p'. Then, by Proposition 3.9(1), $(\dagger)$ implies $t = w'\beta p'$ and this contradicts $\neg(\beta \sqsubseteq t)$. Thus $p = \varepsilon$ holds.

Therefore, $(\dagger)$ implies that $w = t\beta$. Hence, $x\beta s\beta = y\beta w = y\beta t\beta$. By Proposition 3.9(1), $x\beta s = y\beta t$ holds. By (1) of this lemma, $x = y$ and $s = t$ hold. This completes the proof of this lemma. □

To interpret the multiplication of R, we define a witnessing string for the product.

Definition 4.7 We define the formula $\mathrm{PWitn}(x, y, w)$ which means "w is a witness of the product of x and y" as follows:

(i) $\mathrm{Num}(x) \wedge \mathrm{Num}(y) \wedge \mathrm{Good}(w)$;
(ii) $\beta\gamma\beta \sqsubseteq_{\mathsf{ini}} w$;
(iii) $\exists z\, (\mathrm{Num}(z) \wedge \beta y \gamma z \beta \sqsubseteq_{\mathsf{end}} w)$;
(iv) $\forall p\, \forall z\, ((\mathrm{Num}(z) \wedge p\beta y \gamma z \beta = w) \to \forall z'\, (\mathrm{Num}(z') \to \neg(\beta y \gamma z' \beta \sqsubseteq p\beta)))$;
(v) $\forall p\, \forall q\, \forall s_2\, \forall t_2\, [(\mathrm{Num}(s_2) \wedge \mathrm{Num}(t_2) \wedge p\beta s_2 \gamma t_2 \beta q = w \wedge p \neq \varepsilon)$
$\to (\exists s_1\, \exists t_1\, (\mathrm{Num}(s_1) \wedge \mathrm{Num}(t_1) \wedge s_2 = s_1\alpha \wedge t_2 = t_1 x \wedge \beta s_1 \gamma t_1 \sqsubseteq_{\mathsf{end}} p))]$;
(vi) $\forall p\, \forall q\, \forall s\, \forall t\, ((\mathrm{Num}(s) \wedge \mathrm{Num}(t) \wedge p\beta s \gamma t \beta q = w \wedge q \neq \varepsilon)$
$\to s\alpha\gamma t x \beta \sqsubseteq_{\mathsf{ini}} q)$.

In this way, we can construct a standard witnessing string for the product of standard numbers. Moreover, we can prove the uniqueness of the witnessing by applying the condition (iv) of PWitn. Thus, we can prove the following theorem.

Theorem 4.8 *For each $u, v \in \{a\}^*$, there exists $w \in \{a, b, c\}^*$ such that*

$$\mathsf{WTC} \vdash \mathrm{PWitn}(\underline{u}, \underline{v}, \underline{w}) \wedge \forall w'\, (\mathrm{PWitn}(\underline{u}, \underline{v}, w') \to \underline{w} = w').$$

For the proof of this theorem, see Appendix B. With this theorem, we can define the translation of the product $x \cdot y = z$ in R by formula (1) and prove that this translation is well-defined.

$$\mathrm{M}(x, y, z) \equiv (\exists! w\, \mathrm{PWitn}(x, y, w) \wedge \exists w\, (\mathrm{PWitn}(x, y, w) \wedge \gamma z \beta \sqsubseteq_{\mathsf{end}} w)) \\ \vee [(\neg \exists! w\, \mathrm{PWitn}(x, y, w)) \wedge z = 0]. \quad (1)$$

In this way, we obtain the following corollary.

Corollary 4.9 *The theory* WTC *interprets* R.

4.2 The theory R interprets WTC To interpret WTC in R, we apply Visser's result [22, Theorem 5.1]; that is, for each theory T, it is locally finitely satisfiable if and only if $\mathsf{R} \rhd T$. Thus, we obtain the following theorem.

Theorem 4.10 *The theory* R *interprets* WTC.

Proof One can easily check that WTC is locally finitely satisfiable. By Theorem 5.1 in [22], R interprets WTC. □

Therefore, by Corollary 4.9 and the above theorem, we obtain our main theorem, Theorem 4.1.

4.3 Conclusion and remarks In this section, we consider some corollaries of our main theorem. First, we consider the following assertions.

Corollary 4.11

(1) *The theory* WTC *is essentially undecidable.*
(2) *The theory* WTC *cannot interpret* TC.
(3) *For the notion "*Good*",* WTC *cannot prove that* Good *is closed under concatenation.*

Proof For (1), since R is essentially undecidable, the interpretability of R in WTC implies the essential undecidability of WTC. Since R cannot interpret Q and Q and TC are mutually interpretable, (2) holds. If WTC would prove that $\mathrm{Good}(x)$ is closed under concatenation, then WTC can interpret TC relative to $\mathrm{Good}(x)$. Thus (3) holds. □

By combining Theorem 4.1 and Visser's result in [22], we obtain the following.

Corollary 4.12 *For each theory T,* WTC *interprets T if and only if T is locally finitely satisfiable.*

Proof By [22, Theorem 5.1], the following conditions are equivalent:

(1) R interprets T;
(2) T is locally finitely satisfiable.

Then, by Theorem 4.1, we obtain that WTC interprets T if and only if T is locally finitely satisfiable. □

Theorem 4.13 *Let* WTC_n $(n \geq 2)$ *be weak theories of concatenation* WTC *with n's single-letters (by exchanging some axioms of* WTC *or adding some axioms to* WTC*). Any two theories* WTC_n *$(n \geq 2)$ are mutually interpretable.*

Proof It is enough to prove that for each $n \geq 3$, WTC_2 interprets WTC_n. We prove this by the following two steps:

(1) for each $n \geq 3$, R interprets WTC_n;
(2) WTC_2 interprets R.

For part (1), we can easily check that all theories WTC_n, $n \geq 3$, are locally finitely satisfiable. Then, by Visser's result [22], R interprets WTC_n, $n \geq 3$.

For part (2), we can prove this as follows. We can prove Theorem 4.8 in the case of WTC_2 by replacing β and γ which appear in the proof of Theorem 4.8, by $\beta\beta$ and β, respectively. This is because we can prove the following facts, which are the modified statements of Lemma 4.6:

(1) $\forall x\, \forall y\, \forall s\, \forall t\, [(\mathrm{Good}(x\beta\beta s) \wedge (x\beta\beta)s = (y\beta\beta)t \wedge \neg(\beta\beta \sqsubseteq s) \wedge \neg(\beta\beta \sqsubseteq t)) \rightarrow (x = y \wedge s = t)]$.
(2) $\forall x\, \forall y\, \forall s\, \forall t\, \forall p\, [(\mathrm{Good}(x\beta\beta s\beta k\beta\beta p) \wedge x\beta\beta s\beta k\beta\beta p = y\beta\beta t\beta l\beta\beta \wedge \neg(\beta \sqsubseteq s) \wedge \neg(\beta \sqsubseteq t) \wedge \neg(\beta \sqsubseteq k) \wedge \neg(\beta \sqsubseteq l)) \rightarrow ((\exists w\, (x\beta s\beta\beta k\beta w = y\beta \wedge wt\beta\beta l\beta = p))$

$\vee(x = y \wedge s = t \wedge k = l \wedge p = \varepsilon))]$.

Then, by the transitivity of interpretations, WTC_2 interprets WTC_n for each $n \geq 3$. Thus, any two theories WTC_n $(n \geq 2)$ are mutually interpretable. □

Finally, we formulate some questions. First, we consider an alternative weak theory of concatenation $\mathsf{WTC}^{-\varepsilon}$. The language is $(\frown, \alpha, \beta, \gamma)$ and the axioms are as follows.

Definition 4.14 (The theory $\mathsf{WTC}^{-\varepsilon}$) For each $u \in \{a, b, c\}^* \setminus \{\varepsilon\}$,

(WTC$^{-\varepsilon}$1) $\forall x\, \forall y\, \forall z\, [[x^\frown(y^\frown z) \sqsubseteq \underline{u} \vee (x^\frown y)^\frown z \sqsubseteq \underline{u}] \rightarrow x^\frown(y^\frown z) = (x^\frown y)^\frown z]$;

(WTC$^{-\varepsilon}$2) $\forall x\, \forall y\, \forall s\, \forall t\, [(x^\frown y = s^\frown t \wedge x^\frown y \sqsubseteq \underline{u}) \rightarrow ((x = y) \vee \exists w\, ((x^\frown w = s \wedge y = w^\frown t) \vee (x = s^\frown w \wedge w^\frown y = t)))]$;

(WTC$^{-\varepsilon}$3) $\forall x\, \forall y\, (x^\frown y \neq \alpha)$;

(WTC$^{-\varepsilon}$4) $\forall x\, \forall y\, (x^\frown y \neq \beta)$;

(WTC$^{-\varepsilon}$5) $\forall x\, \forall y\, (x^\frown y \neq \gamma)$;

(WTC$^{-\varepsilon}$6) $\alpha \neq \beta \wedge \beta \neq \gamma \wedge \gamma \neq \alpha$.

We believe the following question can be answered positively.

(Q1) Are WTC and $\mathsf{WTC}^{-\varepsilon}$ mutually interpretable?

Next we consider our results from the point of view of the Tarski degree theory (for the latter, see Friedman [2], Švejdar [14], or Chapter 7 of Lindström [8]). An important goal in this theory is to find *meaningful* theories whose Tarski degree is different from or equal to the known theories. Here, "meaningful" means that the statements of the axioms of the theory are *natural*.

(Q2) Is there some meaningful theory T such that $\mathsf{TC} \rhd T \rhd \mathsf{WTC}$, but $\mathsf{WTC} \not\rhd T$ and $T \not\rhd \mathsf{TC}$?

With regard to this question, the second referee of this paper formulated a theory which is strictly between TC and WTC, with respect to interpretability. However, the theory in question is not very natural. Moreover, the second referee also stated several theories T such that

(i) $\mathsf{TC} \rhd T$ and $T \not\rhd \mathsf{TC}$, and

(ii) T and WTC are incompatible, with respect to interpretability.

One example is the theory of one successor, called S. It is easy to see that S is interpretable in TC. Since S has no finite model, WTC cannot interpret S by Corollary 4.12. Since S is decidable, S cannot interpret WTC.

Appendix A

We define the weak theory WTC' of concatenation as follows. The language of WTC' is the same as that of WTC and the axioms of WTC' are as follows.

Definition 4.15 (The theory WTC') For each $u_0, u_1, u_2, u_3 \in \{a, b, c\}^*$,

(WTC'1) $\underline{u_0}^\frown \varepsilon = \varepsilon^\frown \underline{u_0} = \underline{u_0}$;

(WTC'2) $(\underline{u_0}^\frown \underline{u_1})^\frown \underline{u_2} = \underline{u_0}^\frown(\underline{u_1}^\frown \underline{u_2})$;

(WTC'3) $\underline{u_0}^\frown \underline{u_1} = \underline{u_2}^\frown \underline{u_3} \rightarrow \exists w\, ((\underline{u_0}^\frown w = \underline{u_2} \wedge \underline{u_1} = w^\frown \underline{u_3}) \vee (\underline{u_0} = \underline{u_2}^\frown w \wedge w^\frown \underline{u_1} = \underline{u_3}))$;

(WTC'4) $\alpha \neq \varepsilon \wedge \forall x\, \forall y\, (x^\frown y = \alpha \rightarrow x = \varepsilon \vee y = \varepsilon)$;

(WTC'5) $\beta \neq \varepsilon \wedge \forall x\, \forall y\, (x^\frown y = \beta \rightarrow x = \varepsilon \vee y = \varepsilon)$;

(WTC'6) $\gamma \neq \varepsilon \wedge \forall x\, \forall y\, (x^\frown y = \gamma \rightarrow x = \varepsilon \vee y = \varepsilon)$;

(WTC'7) $\alpha \neq \beta \wedge \beta \neq \gamma \wedge \gamma \neq \alpha$;
(WTC'8) $\forall x\, (x \sqsubseteq \underline{u_0} \to \bigvee_{v \sqsubseteq u_0} x = \underline{v})$.

The axioms (WTC'1), (WTC'2), and (WTC'3) of WTC' are the *standard* version of the axioms ($\mathrm{TC}^{\varepsilon}1$), ($\mathrm{TC}^{\varepsilon}2$), and ($\mathrm{TC}^{\varepsilon}3$) of $\mathsf{TC}^{\varepsilon}$, respectively. The axiom (WTC'8) of WTC' corresponds to the axiom (R4) of R. The other axioms of WTC' are the same with the corresponding axioms of WTC. The definitions of the relations $\sqsubseteq$, $\sqsubseteq_{\mathsf{ini}}$, and $\sqsubseteq_{\mathsf{end}}$ are same with the one of WTC.

It follows that WTC and WTC' are logically equivalent; that is, WTC proves the axioms of WTC', denoted by $\mathsf{WTC} \vdash \mathsf{WTC}'$, and vice versa. In fact, to prove $\mathsf{WTC} \vdash \mathsf{WTC}'$, it suffices to prove $\mathsf{WTC} \vdash$ ($\mathrm{WTC}'8$), but this is already done in Lemma 3.11. To prove $\mathsf{WTC}' \vdash \mathsf{WTC}$, it is enough to show that $\mathsf{WTC}' \vdash$ (WTC$*$) where $* = 1, 2, 3$. But in WTC', we can prove (WTC$*$) from ($\mathrm{WTC}'*$), by applying ($\mathrm{WTC}'8$) to the antecedent of (WTC$*$).

Appendix B

In this appendix, we prove Theorem 4.8. This is done in two steps.

Step 1 (Existence of the witness) For each $u, v \in \{a\}^*$, there exists $w \in \{a, b, c\}^*$ such that $\mathsf{WTC} \vdash \mathrm{PWitn}(\underline{u}, \underline{v}, \underline{w})$.

Proof of Step 1 We prove this fact by the induction on the length of $v \in \{a\}^*$. First, we prove this fact when v is the empty string. In this case, we put $w := bcb$. Then, for this w, we prove that $\mathrm{PWitn}(\underline{u}, \varepsilon, \underline{w})$ holds in WTC by proving each condition of the definition of PWitn. We can easily prove (i), (ii), and (iii) of the definition of PWitn.

For part (iv), let us assume that $p\beta y\gamma z\beta = \beta\gamma\beta$ $(= \underline{w})$. Then we can easily prove that $p = \varepsilon$ holds. Then, for any z', we can prove that $\beta y\gamma z'\beta \sqsubseteq p\beta$ never holds, and this implies that (iv) holds. To prove (v), it is enough to prove that for each p, q, s, and t, if $\mathrm{Num}(s) \wedge \mathrm{Num}(t)$ and $p\beta s\gamma t\beta q = \underline{w}$, then $p = \varepsilon$ holds. We assume this antecedent. Then

$$p\beta s\gamma t\beta q = \beta\gamma\beta. \qquad (\dagger)$$

To prove $p = \varepsilon$ by contradiction, assume that $p \neq \varepsilon$. Then, by Lemma 4.6(2),

$$\begin{cases} \exists h\, (p\beta s\gamma t\beta h = \beta \wedge q = h\gamma\beta), & \text{or} \\ p = \varepsilon \wedge s\gamma t = \gamma \wedge q = \varepsilon. \end{cases}$$

The second case implies $p = \varepsilon$, which contradicts the assumption. We consider the first case. Since $p\beta s\gamma t\beta h = \beta$ for some h, $p\beta = \varepsilon$ or $s\gamma t\beta h = \varepsilon$. If $p\beta = \varepsilon$ holds, then, by Proposition 3.5(2), $\beta = \varepsilon$ holds, which contradicts (WTC5). Hence $s\gamma t\beta h = \varepsilon$ holds. Then, by Proposition 3.5(2), $s\gamma = \varepsilon$ and $t\beta h = \varepsilon$. Thus, both of them yield a contradiction. Therefore, $p = \varepsilon$ holds, and hence (v) is proved. For part (vi), let us assume that for some p, q, s, and t, $\mathrm{Num}(s) \wedge \mathrm{Num}(t)$ and $p\beta s\gamma t\beta q = \beta\gamma\beta$ $(= \underline{w})$ hold. We can prove $q = \varepsilon$ by the same discussion in (v), and this implies that (vi) holds.

Secondly, we prove the induction step. To prove this, let us assume that for some $v \in \{a\}^*$, for each $u \in \{a\}^*$, there exists $w \in \{a, b, c\}^*$ such that

$$\mathsf{WTC} \vdash \mathrm{PWitn}(\underline{u}, \underline{v}, \underline{w}).$$

We have to prove the above for $\underline{v}\alpha$. Fix $u \in \{a\}^*$ and let $w \in \{a,b,c\}^*$ be such that PWitn$(\underline{u}, \underline{v}, \underline{w})$ holds by the induction hypothesis. Then, by (iii) of the definition of PWitn$(\underline{u}, \underline{v}, \underline{w})$, there exist z_0 and w_0 such that $\underline{w} = \underline{w_0}\beta\underline{v}\gamma\underline{z_0}\beta$. Then let $\hat{w} \in \{a,b,c\}^*$ be

$$\hat{w} := w_0 bvcz_0 bvacz_0 ub \ (= wvacz_0 ub).$$

In what follows, we prove that this $\hat{w}$ satisfies PWitn$(\underline{u}, \underline{v}\alpha, \underline{\hat{w}})$ by proving each condition of the definition of PWitn.

We can easily prove (i), (ii), and (iii) of the PWitn. For part (iv), for each p and number string z, let us assume the antecedent of (iv); that is, $p\beta\underline{v}\alpha\gamma z\beta = \underline{w}\,\underline{v}\alpha\gamma\underline{z_0}\,\underline{u}\beta$. Since $z\beta$ and $\underline{z_0}\,\underline{u}\beta$ have no γ, by Lemma 4.6(1), $p\beta\underline{v}\alpha = \underline{w}\,\underline{v}\alpha$. Since $\underline{w} = \underline{w_0}\beta\underline{v}\gamma\underline{z_0}\beta$, $p\beta\underline{v}\alpha = \underline{w_0}\beta\underline{v}\gamma\underline{z_0}\beta\underline{v}\alpha$. Since $\underline{v}\alpha$ has no β, by Lemma 4.6(1), $p = \underline{w_0}\beta\underline{v}\gamma\underline{z_0}$ and hence $p\beta = \underline{w}$. Thus, to prove the consequent of (iv), it is enough to show

$$\forall z' \neg(\beta\underline{v}\alpha\gamma z'\beta \sqsubseteq \underline{w}).$$

We prove this by contradiction. For this, assume that $k\beta\underline{v}\alpha\gamma z'\beta l = \underline{w}$ for some k, l and z'. Then, since $k\beta\underline{v}\alpha\gamma z'\beta l = \underline{w_0}\beta\underline{v}\gamma\underline{z_0}\beta$, Lemma 4.6(2) implies that there exists h such that

$$\begin{cases} (*) \ \ k\beta\underline{v}\alpha\gamma z'\beta h = \underline{w_0}\beta \wedge l = h\underline{v}\gamma\underline{z_0}\beta, \ \text{ or} \\ (**) \ \ k = \underline{w_0} \wedge \underline{v}\alpha\gamma z' = \underline{v}\gamma\underline{z_0} \wedge l = \varepsilon. \end{cases}$$

For case $(*)$, we can prove $k \neq \varepsilon$ as follows: If $k = \varepsilon$, then, $\underline{w} = \beta\underline{v}\alpha\gamma z'\beta h\underline{v}\gamma\underline{z_0}\beta$ holds. But, $\beta\gamma\beta \sqsubseteq_{\mathsf{ini}} \beta\underline{v}\alpha\gamma z'\beta h\underline{v}\gamma\underline{z_0}\beta$ never happens. Thus, $k \neq \varepsilon$.

Then, by (v) of the definition of PWitn$(\underline{u}, \underline{v}, \underline{w})$, there exist number strings s and t such that $\underline{v}\alpha = s\alpha$, $z' = t\underline{u}$, and $\beta s\gamma t \sqsubseteq_{\mathsf{end}} k$. Then, $\underline{v}\alpha = s\alpha$ implies $\underline{v} = s$, and $\beta s\gamma t \sqsubseteq_{\mathsf{end}} k$ implies $k = k'\beta s\gamma t$ for some k'. Hence, $k = k'\beta\underline{v}\gamma t$ holds. Thus, by $(*)$, $\underline{w_0}\beta = (k'\beta\underline{v}\gamma t)\beta\underline{v}\alpha\gamma z'\beta h$ holds. This implies that $\beta\underline{v}\gamma t\beta \sqsubseteq \underline{w_0}\beta$. This contradicts (iv) for the witness $\underline{w}$. For case $(**)$, $\underline{v}\alpha\gamma z' = \underline{v}\gamma$ implies $\underline{v}\alpha = \underline{v}$. But this is impossible since, by finitely many applications of Proposition 3.9(1) (in fact, by metainduction), $\underline{v}\alpha = \underline{v}$ implies $\alpha = \varepsilon$. For parts (v) and (vi), these steps are probably at least as complicated as step (iv). This completes the proof of step 1. □

In the next step, we prove the uniqueness of the witness. In the following, we fix $u, v \in \{a\}^*$ and $w \in \{a,b,c\}^*$ such that $\mathsf{WTC} \vdash$ PWitn$(\underline{u}, \underline{v}, \underline{w})$.

Step 2 (Uniqueness of the witness) $\mathsf{WTC} \vdash \forall w'\,(\text{PWitn}(\underline{u}, \underline{v}, w') \rightarrow \underline{w} = w')$.

To prove step 2, we first prove the following two lemmas. Note that each witnessing string is a good string.

Lemma 4.16

(1) *For each* $k, l \in \{a\}^*$, WTC *proves*

$$\forall w'\,[\text{PWitn}(\underline{u}, \underline{v}, w') \rightarrow \forall p\,(p\beta\underline{k}\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} \underline{w} \rightarrow p\beta\underline{k}\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} w')].$$

(2) *The theory* WTC *proves* $\forall w'\,[\text{PWitn}(\underline{u}, \underline{v}, w') \rightarrow \underline{w} \sqsubseteq_{\mathsf{ini}} w']$.

Proof For part (1), we prove this by the induction on the length of $k \in \{a\}^*$. If $\underline{k} = \varepsilon$, we can easily check that $p\beta\underline{k}\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} \underline{w}$ implies $p = \varepsilon$ and $\underline{l} = \varepsilon$. Thus, by (ii) in the definition of PWitn$(\underline{u}, \underline{v}, w')$, we have $p\beta\underline{k}\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} w'$.

To prove the induction step, we assume that for some $k \in \{a\}^*$, we have, for each $l \in \{a\}^*$,

$$\mathsf{WTC} \vdash \forall p (p\beta\underline{k}\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} \underline{w} \to p\beta\underline{k}\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} w'). \quad (*)$$

For $\underline{k}\alpha$, assume that $p\beta\underline{k}\alpha\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} \underline{w}$ for some $l \in \{a\}^*$. By (ii) of the definition of PWitn, it is easy to show that $p \neq \varepsilon$. Then, by (iv) of PWitn, there exist number strings s and t such that $\underline{k}\alpha = s\alpha \wedge \underline{l} = t\underline{u} \wedge \beta s\gamma t \sqsubseteq_{\mathsf{ini}} p$. Hence, $p\beta = p'\beta s\gamma t\beta$ for some p'. By assumption,

$$p'\beta s\gamma t\beta\underline{k}\alpha\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} \underline{w} \qquad (\dagger)$$

holds. Thus $p'\beta s\gamma t\beta \sqsubseteq_{\mathsf{ini}} \underline{w}$ and therefore, $p'\beta\underline{k}\gamma t\beta \sqsubseteq_{\mathsf{ini}} \underline{w}$. Then, by the induction hypothesis, it follows that $p'\beta\underline{k}\gamma t\beta \sqsubseteq_{\mathsf{ini}} w'$. Hence, $p'\beta\underline{k}\gamma t\beta q = w'$ for some q.

Here we can prove that $q \neq \varepsilon$ as follows. If $q = \varepsilon$, then, by Lemma 4.6(2), we can show that $\underline{k} = \underline{v}$. Then this contradicts the assumption $p\beta\underline{k}\alpha\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} \underline{w}$, (v) and (iv). Hence, $q \neq \varepsilon$. Then we can prove (1) as follows: by (vi) of the definition of PWitn, we can easily prove $p'\beta\underline{k}\gamma t\beta\underline{k}\alpha\gamma t\underline{u}\beta \sqsubseteq_{\mathsf{ini}} w'$. Since $\underline{l} = t\underline{u}$, we have proved $(*)$. This completes the proof of (1) of this lemma.

For part (2), we can easily prove it as a corollary of (1). In fact, since $\mathrm{PWitn}(\underline{u}, \underline{v}, \underline{w})$ holds, there exists some number string z such that $\beta\underline{v}\gamma z\beta \sqsubseteq_{\mathsf{end}} \underline{w}$. Then $\underline{w} = p\beta\underline{v}\gamma z\beta$ for some p. Then, since $z \sqsubseteq \underline{w}$, $z = \underline{u_z}$ for some $u_z \in \{a\}^*$, and hence $\underline{w} = p\beta\underline{v}\gamma\underline{u_z}\beta$. Then, since $p\beta\underline{v}\gamma\underline{u_z}\beta \sqsubseteq_{\mathsf{ini}} \underline{w}$, by part (1) of this lemma, $p\beta\underline{v}\gamma\underline{u_z}\beta \sqsubseteq_{\mathsf{ini}} w'$. This means that $\underline{w} \sqsubseteq_{\mathsf{ini}} w'$. □

The second lemma, provable in WTC, is as follows.

Lemma 4.17 *For each w' with* $\mathrm{PWitn}(\underline{u}, \underline{v}, w')$,

(1) $\forall x\, \forall y\, \forall p\, \forall q\, ((\mathrm{Num}(x) \wedge \mathrm{Num}(y) \wedge p\beta x\gamma y\beta q = w' \wedge q \neq \varepsilon) \to$
$\exists z\, (\mathrm{Num}(z) \wedge \beta\underline{v}\gamma z\beta \sqsubseteq_{\mathsf{end}} \beta q))$;

(2) $\forall x\, \forall y\, \forall p\, \forall q\, ((\mathrm{Num}(x) \wedge \mathrm{Num}(y) \wedge p\beta x\gamma y\beta q = w' \wedge q \neq \varepsilon) \to$
$\forall z'\, (\mathrm{Num}(z') \to \neg(\beta\underline{v}\gamma z'\beta \sqsubseteq p\beta x\gamma y\beta)))$.

Proof We reason in WTC. For part (1), let p and q be such that $p\beta x\gamma y\beta q = w'$ and $q \neq \varepsilon$ hold. By (iii) of the definition of $\mathrm{PWitn}(\underline{u}, \underline{v}, w')$, there exists z such that $\mathrm{Num}(z) \wedge \beta\underline{v}\gamma z\beta \sqsubseteq_{\mathsf{end}} w'$ holds. Thus, $w' = w''\beta\underline{v}\gamma z\beta$ for some w''. Then, by assumption, $p\beta x\gamma y\beta q = w''\beta\underline{v}\gamma z\beta$ holds. Then, by Lemma 4.6(2) and $q \neq \varepsilon$, there exists h such that

$$p\beta x\gamma y\beta h = w''\beta \ \text{ and } \ q = h\underline{v}\gamma z\beta.$$

Here, if $h = \varepsilon$, then $q = \underline{v}\gamma z\beta$. Hence, $\beta q = \beta\underline{v}\gamma z\beta$ and this implies $\beta\underline{v}\gamma z\beta \sqsubseteq_{\mathsf{end}} \beta q$. If $h \neq \varepsilon$, then, by $p\beta x\gamma y\beta h = w''\beta$, there exists h' such that $h = h'\beta$. Since $q = h'\beta\underline{v}\gamma z\beta$, $\beta q = \beta h'\beta\underline{v}\gamma z\beta$ holds. Thus, $\beta\underline{v}\gamma z\beta \sqsubseteq_{\mathsf{end}} \beta q$ holds.

For part (2), let p and q be such that $p\beta x\gamma y\beta q = w'$ and $q \neq \varepsilon$ hold. Then, by (1) of this lemma, there exist q' and a number string z such that

$$\beta q = q'\beta\underline{v}\gamma z\beta. \quad (\dagger\dagger)$$

Hence, $w' = p\beta x\gamma y q'\beta\underline{v}\gamma z\beta$ holds. Now fix some number string z'. To prove (2), we have to prove the following assertion:

$$\neg(\beta\underline{v}\gamma z'\beta \sqsubseteq p\beta x\gamma y\beta). \quad (**)$$

Now, by (iv) of the definition of PWitn($\underline{u}, \underline{v}, w'$), the following assertion holds:

$$\neg(\beta\underline{v}\gamma z'\beta \sqsubseteq p\beta x\gamma y q'\beta). \quad (\dagger\dagger\dagger)$$

Here, if $q' = \varepsilon$, then $\neg(\beta\underline{v}\gamma z'\beta \sqsubseteq p\beta x\gamma y\beta)$ holds, which means $(**)$ holds. If $q' \neq \varepsilon$, then by $(\dagger\dagger)$, $q' = \beta q''$ for some q''. Thus, $(\dagger\dagger\dagger)$ implies $\neg(\beta\underline{v}\gamma z'\beta \sqsubseteq p\beta x\gamma y\beta q''\beta)$. Hence, $(**)$ holds. □

Now we prove step 2 and complete the proof of Theorem 4.8. Recall the statement of step 2:

$$\mathsf{WTC} \vdash \forall w'\, (\mathrm{PWitn}(\underline{u}, \underline{v}, w') \to \underline{w} = w').$$

Proof of Step 2 We reason in WTC. Fix w' such that PWitn($\underline{u}, \underline{v}, w'$) holds. Then, by Lemma 4.16(2), $\underline{w} \sqsubseteq_{\mathsf{ini}} w'$ holds. Here $w' = \underline{w}q$ for some q. To prove $\underline{w} = w'$ by way of contradiction, assume that $q \neq \varepsilon$. Since $\underline{w}$ is a witness for the product of $\underline{u}$ and $\underline{v}$, there exists a number string z_0 such that $\beta\underline{v}\gamma z_0\beta \sqsubseteq_{\mathsf{end}} \underline{w}$. Thus, $p\beta\underline{v}\gamma z_0\beta = \underline{w}$ for some p. Hence, $p\beta\underline{v}\gamma z_0\beta q = w'$. Since $q \neq \varepsilon$, by Lemma 4.17(2), $\neg(\beta\underline{v}\gamma z'\beta \sqsubseteq p\beta\underline{v}\gamma z_0\beta)$ for each number string z'. This is a contradiction. Hence, $q = \varepsilon$. Thus, $\underline{w} = w'$ holds. This completes the proof of step 2. □

Then, by step 1 and step 2, Theorem 4.8 is proved, and this completes the proof of Corollary 4.9.

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