

A Note on Induction, Abstraction, and Dedekind-Finiteness

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Abstract The purpose of this note is to present a simplification of the system of arithmetical axioms given in previous work; specifically, it is shown how the induction principle can in fact be obtained from the remaining axioms, without the need of explicit postulation. The argument might be of more general interest, beyond the specifics of the proposed axiomatization, as it highlights the interaction of the notion of Dedekind-finiteness and the induction principle.

1 Introduction

In previous work we introduced a formalization of arithmetic employing an abstraction operator assigning numbers to predicates, supplemented by a cardinality quantifier—referred to as the “Frege” quantifier (see [3]). This theory, which for the purposes of this note will be called AFQ (Arithmetic with the Frege Quantifier), comprises a system of arithmetical axioms in which the natural numbers are identified with cardinals the set of whose predecessors is Dedekind-finite. The axioms also explicitly include a principle of induction formulated in the form “Every bounded set of natural numbers has a maximum.” However, it turns out that such an axiom is superfluous, in that the principle of induction (in its usual form) can be derived from the remaining axioms. The purpose of this note is to prove this.

Before we get into the details of the derivation of the induction principle, it is useful to review the main ideas behind the proof, as they appear to be of some independent interest. Suppose that, in the Frege-Russell tradition, we identify cardinals with first-order representatives of equinumerosity classes. There are then at least two ways in which we can go about selecting, among the cardinals, those that are to be identified with the natural numbers. On the more straightforward approach, we can identify the natural numbers with the Dedekind-finite cardinals, that is, with representatives of equinumerosity classes whose members are Dedekind-finite. But in a

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slight variation of this approach, we can also identify the natural numbers with cardinals the set of whose *predecessors* in the ordering relation is Dedekind-finite (call such cardinals “predecessor-finite”). Here the ordering relation between cardinals α and β is defined in the standard way by saying that $\alpha < \beta$ if and only if there are sets x and y in the classes represented by α and β , respectively, such that there is an injection from x into y but not the other way around.

The identification of the natural numbers with predecessor-finite cardinals delivers the principle of induction in quite a straightforward way. In fact, suppose that P is any property that holds at zero and is preserved by the successor operation, but still fails at some natural number m^* . Then the set of predecessors of m^* is not Dedekind-finite, and in fact one can explicitly define an injection of the set $\{n : n \leq m^*\}$ into a proper subset of itself. These ideas are best encapsulated in the following semi-formal result. The result is “semi-formal” because we don’t specify the (weak) arithmetical system in which the proof is to be carried out.

Theorem 1.1 *If every bounded class of natural numbers is Dedekind-finite, then induction holds.*

Proof Assume that induction fails in order to show that there is a bounded set S of natural numbers and an injection f from S into a proper subset of S . In particular, if induction fails, there is a class $Q \subseteq \mathbb{N}$ such that $0 \in Q$ and $n \in Q \rightarrow n + 1 \in Q$ for every $n \in \mathbb{N}$, but $m^* \notin Q$ for some $m^* \in \mathbb{N}$. Let

$$S = \{n : n \leq m^*\},$$

so that S is bounded. Then there is an injection $f : S \rightarrow S'$, where S' is a proper subset of S . In fact, define f as follows, for each $n \in S$:

$$f(n) = \begin{cases} n & \text{if } n \in Q; \\ n - 1 & \text{if } n \notin Q. \end{cases}$$

Since $0 \in Q$, the function is well defined. Observe that $n - 1 \leq f(n) \leq n$. In particular, $\text{rng}(f)$ is a subset of S . Moreover, since $m^* \notin Q$, $f(m^*) < m^*$, so $\text{rng}(f)$ is a *proper* subset S' of S . It remains to show that f is injective. Let $0 \leq i < j \leq m^*$ in order to show $f(i) \neq f(j)$. We distinguish two cases:

1. Both $i = j - 1$ and $i \in Q$. By hypothesis $i \in Q \rightarrow i + 1 \in Q$, so also $j \in Q$, whence $f(i) = i < j = f(j)$.
2. Either $i \leq j - 2$ or $i \notin Q$. If the former, then $f(i) \leq i \leq j - 2 < f(j)$; if the latter, $f(i) < i \leq j - 1 \leq f(j)$.

In either case, $f(i) \neq f(j)$. □

The main idea of the proof can be found, albeit in a different context, in Floyd and Beigel [5, pp. 59–60], where a very similar argument is used to show that, over very weak assumptions, the pigeonhole principle implies the axiom of induction. But apparently it has so far escaped attention that one can similarly show that induction follows almost directly from the Dedekind-finiteness of bounded sets of naturals.

In fact, the proof applies more broadly, in that only the more general notion of cardinal number is needed, so that one can show that if every bounded set of cardinals is Dedekind-finite, induction holds. We will come back to the issue of that status of induction in an account of arithmetic at the end, but first, let us show how to reproduce the proof within the context of our theory.

2 Arithmetic with the Frege Quantifier

In order to provide an axiomatization of arithmetic, AFQ introduces a first-order language whose formulas are built-up from predicates and Boolean operators using only an abstraction operator Num and a cardinality quantifier F (the Frege quantifier). The abstraction operator Num binds a single variable in a formula $\varphi(x)$ (possibly including parameters) to return a singular term $\text{Num}_x \varphi(x)$, whose intended referent is the number of objects (in the given domain of quantification) satisfying the formula $\varphi(x)$. For convenience, $\text{Num}_x \varphi(x)$ is also written $\text{Num}_x \varphi(x)$. The quantifier F simultaneously binds variables in two formulas $\varphi(x)$ and $\psi(x)$ (also possibly including parameters) giving a formula $F x(\varphi(x), \psi(x))$, whose intended reading is that “there are no more φ ’s than ψ ’s,” or (in the words of [4]) “for every φ there is a (distinct) ψ .” The resulting language can be given a *standard* interpretation by assuming that $F x(\varphi(x), \psi(x))$ holds if the cardinality of the set $\{x : \varphi(x)\}$ is less than or equal to the cardinality of $\{x : \psi(x)\}$. But it can also be given a *general* interpretation by assuming that models come equipped with a collection of injections between subsets of the domain and stipulating that $F x(\varphi(x), \psi(x))$ holds if there is an injection in the collection witnessing the cardinality claim. The class of injections is assumed to satisfy six closure conditions, denoted by **Cc1–Cc6** in [3]; these conditions are not replicated here, but we just mention that they express natural requirements, such as closure under composition. Moreover, it is immediate to see that the ordinary quantifier $\forall x \varphi(x)$ can then be expressed by saying that there is an injection of the complement of φ into the empty set: $F x(\neg\varphi(x), x \neq x)$ (and dually for \exists).

With the help of these two devices, the Frege quantifier and the abstraction operator, one can then proceed to lay down a set of arithmetical axioms and show that first-order Peano arithmetic, PA, is interpretable in the resulting theory AFQ (on the general semantics for the Frege quantifier and hence, a fortiori, on the standard one as well). For instance, among the axioms there is one that formalizes Hume’s Principle, HP, by saying that $\text{Num}_x \varphi(x) = \text{Num}_x \psi(x)$ if and only if both $F x(\varphi(x), \psi(x))$ and $F x(\psi(x), \varphi(x))$ hold (the correctness of the stipulation being guaranteed by the Schröder-Bernstein theorem). In fact, the equinumerosity (i.e., Hartig’s) quantifier $l x(\varphi(x), \psi(x))$ is definable by the conjunction of $F x(\varphi(x), \psi(x))$ and $F x(\psi(x), \varphi(x))$. And in a similar vein, one can express the notion of Dedekind-finiteness in a natural fashion by saying that the extension of some formula $\varphi(x)$ is finite, written $\text{Fin}_x \varphi(x)$, if and only if for every y such that $\varphi(y)$, there is no injection of $\varphi(x)$ into $\varphi(x) \wedge x \neq y$.

An important principle available in AFQ is the so-called infinitary axiom, which plays a role similar to that of comprehension principles in second-order systems for arithmetic. The axiom says that if a formula $\theta(x, y)$ defines an injection of φ into ψ , then $F x(\varphi(x), \psi(x))$ holds; that is, an injection is available in the model to witness the cardinality claim about φ and ψ .

The abstraction operator Num provides (in the context of HP) a general notion of “cardinal number.” But of course what is needed in order to interpret arithmetic is the more restrictive notion of “natural number.” AFQ comprises an *implicit definition* of natural numbers by means of an axiom, using a primitive one-place predicate \mathbb{N} , to the effect that $\mathbb{N}(x)$ holds if and only if $x = \text{Num}_y(\mathbb{N}(y) \wedge y < x)$ and, moreover, the predicate $\mathbb{N}(y) \wedge y < x$ is Dedekind-finite.

As mentioned, the original formulation of AFQ also includes an explicit version of induction. However, this characterization of the natural numbers as predecessor-finite cardinals is sufficient to yield the standard principle of induction, in the form,

$$\varphi(0) \wedge \forall x \forall y ((\varphi(x) \wedge \text{Succ}(x, y)) \rightarrow \varphi(y)) \rightarrow \forall x \varphi(x), \quad (*)$$

where $\text{Succ}(x, y)$ abbreviates the formula that states that y is the successor of x (the reader is referred to [3] for this and other details).

The proof of this fact amounts to showing that the crucial arithmetical facts more or less tacitly employed in the proof of Theorem 1.1 are available in AFQ, so that the function f in Theorem 1.1 is definable. Once this is established, a crucial application of the “infinitary axiom” ensures that f is available to witness the quantifier F in the statement of Dedekind-finiteness. We begin by recording such facts in a few relatively easy lemmas.

Lemma 2.1 *Suppose $\mathbb{N}(x)$, $\mathbb{N}(y)$, and $x \leq y$, where this last claim is witnessed by a function f . Then if f is not surjective, it follows that $x < y$.*

Proof Since $\mathbb{N}(x)$ and $\mathbb{N}(y)$, the function f can be taken to map the predecessors of x into the predecessors of y . If $x < y$ fails, then there is a g mapping the predecessors of y into the predecessors of x . Then the composition $h = g \circ f$, which exists by one of the closure conditions, **Cc6**, maps the predecessor of x into a proper subset of itself, making the set of predecessors of x Dedekind-infinite, against the hypothesis that $\mathbb{N}(x)$. \square

Lemma 2.2 *Where $\mathbb{N}(x)$ and $\mathbb{N}(y)$, $\text{Succ}(x, y)$ implies $x < y$.*

Proof This is Corollary 7.6 in [3], but it also follows immediately from Lemma 2.1, since $\text{Succ}(x, y)$ implies $x \leq y$ with a non-surjective witnessing function. \square

Lemma 2.3 *Where $\mathbb{N}(x)$: $x \neq 0 \rightarrow \exists! y \text{Succ}(y, x)$.*

Proof Existence is proved by induction in [3], but that strategy is of course not available to us here. However, it is not difficult to establish the lemma directly. If $x \neq 0$ then $\{p : p < x\}$ is not empty, so choose such a $p < x$, and let $y = \text{Num}_z[\mathbb{N}(z) \wedge z < x \wedge z \neq p]$. Then $\text{Succ}(y, x)$ as desired. As for uniqueness, it follows from the fact that Succ is an injection, which is proved as Proposition 7.7, part (2), in [3]. \square

Lemma 2.4 *Where $\mathbb{N}(i)$ and $\mathbb{N}(j)$, if $i < j$ and $\text{Succ}(k, j)$ then $i \leq k$; that is, in more readable notation, if $i < j$ then $i \leq j - 1$.*

Proof In standard fashion, in AFQ, $i < j$ is just an abbreviation for $i \leq j \wedge j \not\leq i$. From $i \leq j$ we have $Fx(x < i, x < j)$ expressing the existence of an injection f from the predecessors of i into those of j . If, moreover, $j \not\leq i$, then there must be $p < j$ such that $p \notin \text{rng } f$ —otherwise, f is onto and f^{-1} witnesses $Fx(x < j, x < i)$ (existence of inverses is guaranteed by one of the closure conditions, **Cc3**). Now if $\text{Succ}(k, j)$, we also have

$$\text{Ix}(x < k, x < j \wedge x \neq p)$$

by the closure conditions **Cc4** and **Cc5**. Therefore, to establish $i \leq k$ suffices to show $Fx(x < i, x < j \wedge x \neq p)$, but f itself witness the truth of this last statement. \square

Lemma 2.5 *If $\mathbb{N}(j)$, $\text{Succ}(k, j)$, $\text{Succ}(l, k)$, and $i < j$, then $i = k \vee i \leq l$; that is, in more readable notation, if $i < j$ then either $i = j - 1$ or $i \leq j - 2$.*

Proof Assume that j is the successor of k and k the successor of l . If $i < j$ then by Lemma 2.4 also $i \leq k$, and the function witnessing this claim either is or is not onto. Therefore, either $i = k$ or (by Lemma 2.1) $i < k$, where the latter implies $i \leq l$, again by Lemma 2.4. So either $i = k$ or $i \leq l$, as desired. \square

We are now ready to state and prove the main result.

Theorem 2.6 *The principle of induction in the form (*) holds in every general model of the axioms A.1–A.4 and B.1 (i.e., omitting the induction principle B.2).*

Proof We follow the outline given in the proof of Theorem 1.1. Suppose $\varphi(0)$ and $\forall y((\varphi(x) \wedge \text{Succ}(x, y)) \rightarrow \varphi(y))$, but $\neg\varphi(m^*)$ for some m^* . Let $\theta(x, y)$ abbreviate the formula,

$$x < m^* \wedge [(x = y \wedge \varphi(x)) \vee (\text{Succ}(y, x) \wedge \neg\varphi(x))].$$

Using the lemmas just proved one can show that θ defines an injection of the predecessors of m^* into a proper subset of itself. By the infinitary axiom one concludes $Fx(x < m^*, x < m^* - 1)$, which contradicts the fact that m^* is predecessor-finite. \square

3 Conclusion: The Status of Induction

The proper status of induction has been debated at length, from the early conflicting views of Frege, Russell, and Poincarè, to [6]. To the extent that the characterization of arithmetic as the theory of predecessor-finite cardinals is regarded as natural, the induction principle would also seem to be, if not analytical of, at least conceptually bundled with, the notion of natural number.

Indeed, what are the alternatives? Any approach to the formalization of arithmetic that falls broadly in the Frege-Russell tradition must implement some device in order to select, from among the cardinals, those that are to serve as natural numbers. There appear to be at least a couple of ways to accomplish this, for instance, by either identifying the natural numbers with Dedekind-finite cardinals or with the “predecessor-finite” ones. And as we have seen, the latter option quite readily delivers a notion of natural number that satisfies the induction schema.

Even explicitly positing a schema (equivalent to the principle of) induction amounts to selecting a subclass of cardinals. The standard Frege-Russell definition of the natural numbers as the smallest class of cardinals containing zero and closed under successor achieves just that, at the cost of a highly impredicative restriction. The two alternatives mentioned in this note would appear to fare marginally better. Let us consider again these two options. The original version of AFQ as published in [3] adopted explicitly the following principle as axiom B.2:

BddMax: *Every bounded set of natural numbers has a maximum.*

This can be seen to imply the standard version of induction over quite weak assumptions (in fact, within the context of AFQ, it does not require the infinitary axiom). Alternatively, one could posit the version implicit in Theorem 1.1:

BddFin: *Every bounded set of natural numbers is Dedekind-finite.*

The marginal advantage of both of these over the Russellian approach is that only quantification over bounded sets is required, as opposed to the employment of absolutely unrestricted quantification over classes. It is not clear whether this fact translates into any clearly identifiable mathematical advantages. But if one has to implement a selection device for the cardinals, then—we submit—one might as well use the notion of predecessor-finite. Then no *separate* stipulation such as **BddMax** or **BddFin** is needed.

The resulting approach is far from current construals of neo-Fregeanism, interpreted as the claim that arithmetic is derivable in second-order logic from Hume's Principle alone. It is far from these because, first of all, AFQ is a first-order theory, albeit one employing a nonstandard quantifier; and second because Hume's Principle plays a secondary role in AFQ. Rather than granting a privileged status to HP, which is hardly a logical principle (see [2]) AFQ abides by a more general interpretation of logicism, according to which cardinality itself *already* has a plausible claim at being a logical notion, and does not require a reduction to HP to show that it does. On this view, it is the Frege quantifier, rather than HP, that carries the logicist banner. This is a topic that exceeds the scope of the present note, but the interested reader might usefully consult [1] for the purpose.

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