Notre Dame Journal of Formal Logic
Volume 53, Number 2, 2012

The Set of Restricted Complex Exponents for Expansions of the Reals

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Abstract We introduce the set of definable restricted complex powers for expansions of the real field and calculate it explicitly for expansions of the real field itself by collections of restricted complex powers. We apply this computation to establish a classification theorem for expansions of the real field by families of locally closed trajectories of linear vector fields.

1 Introduction and Preliminaries

In this paper we extend the notion of the field of exponents for expansions of the field of real numbers, introduced by Miller [10], to allow for complex exponents. In this we seek to better understand the definable functions of the structure $\mathbb{R}^{\mathrm{RE}} := (\overline{\mathbb{R}}, \exp\restriction[0,1], \sin\restriction[0,1])$, in particular the model theoretic implications of expanding o-minimal (and other tame) structures on the real field by such functions. Here "RE" stands for "restricted elementary"; we direct the reader to van den Dries [5] for more information about $\mathbb{R}^{\mathrm{RE}}$ and [6] for basic results in o-minimality.

Throughout this paper, "definable" means first-order definable with parameters and a "term" is a term in the language of the structure being considered. We write $\overline{\mathbb{R}}$ for the ordered field of real numbers $(\mathbb{R}, +, -, \cdot, <, 0, 1)$, and we write $\mathbb{C}$ to mean the field of complex numbers as a definable object in $\overline{\mathbb{R}}$.

For $x > 0$ and $a, b \in \mathbb{R}$, we put $x^{a+ib} = x^a(\cos b \log x + i \sin b \log x)$, where log denotes the real logarithm. For an expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$, we say that $a + ib$ is a *restricted complex exponent* of $\mathfrak{R}$ and that $x^{a+ib}\restriction[1,2]$ is a *restricted complex power* if and only if the restriction $x^{a+ib}\restriction[1,2] : [1,2] \to \mathbb{C}$ is definable in $\mathfrak{R}$ (the interval [1,2] is chosen for the sake of convenience; any compact infinite subinterval of $\mathbb{R}^+$ would work just as well). We write $E(\mathfrak{R})$ to denote the set of restricted

Received October 21, 2010; accepted June 22, 2011; printed May 4, 2012
2010 Mathematics Subject Classification: Primary 03C64; Secondary 34A30,
Keywords: o-minimal, exponents, definability
 10.1215/00294527-1715671

complex exponents for $\mathfrak{R}$, or just E if $\mathfrak{R}$ is clear from context. Whenever convenient, a restricted complex power (or any partially defined function) can be extended to be totally defined by setting it equal to 0 off its domain.

For $a, b \in \mathbb{R}$, notice that $x^a = |x^{a+ib}|$ and $x^{ib} = x^{a+ib}/x^a$, so $a + ib \in E$ if and only if both $a \in E$ and $ib \in E$. Letting $K = E \cap \mathbb{R}$ be the set of real elements of E, we see that K is a subfield of $\mathbb{R}$ and that E is a vector space over K. Note that $E = \mathbb{C}$ if both $\exp \restriction [0, 1]$ and $\sin \restriction [0, 1]$ are definable, and that $E \subseteq \mathbb{R}$ if one of the functions $\exp \restriction [0, 1]$ or $\sin \restriction [0, 1]$ is definable but not the other (in fact, $E = \mathbb{R}$ if $\exp \restriction [0, 1]$ is definable and $\sin \restriction [0, 1]$ is not). Lemma 3.1 and Proposition 3.3 characterize structures for which E is a field.

As our main technical results, we calculate E for certain expansions of $\overline{\mathbb{R}}$. The first of these results is then applied to solve (Theorem 2.5) a problem of Miller [13] for expansions of $\overline{\mathbb{R}}$ by locally closed trajectories of linear vector fields. To state these results, we define an operation on subsets of $\mathbb{C}$ as follows. Fix $Z \subseteq \mathbb{C}$. Let $X = \{\operatorname{Re} z : z \in Z\} \cup \{1\}$ and $Y = \{i \operatorname{Im} z : z \in Z\}$. Let F be the subfield of $\mathbb{R}$ generated by X and the set $\{a/b : b \neq 0 \text{ and } a, b \in \operatorname{Span}_{\mathbb{Q}(X)} Y\}$. Put $V(Z) = F + \operatorname{Span}_F(Y)$. Then we have the following theorems.

Theorem 1.1 *For any $Z \subseteq \mathbb{C}$, the structure $(\overline{\mathbb{R}}, (x^z \restriction [1, 2])_{z \in Z})$ has set of restricted complex exponents $V(Z)$.*

Theorem 1.2 *For any $Z \subseteq \mathbb{C}$, the structure $(\overline{\mathbb{R}}, \sin \restriction [0, 1], (x^z \restriction [1, 2])_{z \in Z})$ has set of restricted complex exponents $V(Z)$ if $Z \subseteq \mathbb{R}$, and has set of restricted complex exponents $\mathbb{C}$ otherwise.*

For $Z \subseteq \mathbb{R}$, Theorem 1.1 is immediate from a result of Bianconi [3]: The set of restricted real exponents of the structure $(\overline{\mathbb{R}}, (x^z)_{z \in Z})$ is $\mathbb{Q}(Z)$ if $Z \subseteq \mathbb{R}$. Our proof for Theorem 1.1 for complex powers is along similar lines as his proof for real powers. We omit the proof of Theorem 1.2, as it is a routine modification of that for Theorem 1.1.

Toward the end of the paper, we demonstrate conditions under which $E(\mathfrak{R})$ and $E(\mathfrak{R}')$ are isomorphic for elementarily equivalent structures $\mathfrak{R}$ and $\mathfrak{R}'$ (Theorem 4.4). The article closes with some generalizations, open problems, and a sketch of how to treat structures over fields other than $\overline{\mathbb{R}}$.

2 Proof of Main Result

Before proving Theorem 1.1, we first need a few facts regarding the algebraic structure of E and the behavior of V.

Lemma 2.1 *Let $\mathfrak{R}$ be an expansion of $\overline{\mathbb{R}}$. The quotient of any two nonzero imaginary elements of E is an element of E. If $E \not\subseteq \mathbb{R}$, then E is a vector space of dimension 2 over the field $E \cap \mathbb{R}$.*

Proof For a nonzero real number a, let $g_a : [1, \exp(2\pi/|a|)) \to \mathbb{S}^1$ be given by $g_a(x) = x^{ia}$, where $\mathbb{S}^1$ denotes the unit circle in $\mathbb{R}^2$. Observe that g_a is a bijection and is definable if and only if $x^{ia} \restriction [1, 2]$ is definable. Suppose $a, b \in E \cap \mathbb{R}$ are nonzero. Then the function $f : [1, \exp(2\pi/|a|)) \to [1, \exp(2\pi/|b|))$ given by $f = g_b^{-1} \circ g_a$ is definable and bijective. A calculation then shows that $f(x) = x^{a/b}$ on the interval $[1, \exp(2\pi/|a|))$. If $[1, 2] \subseteq [1, \exp(2\pi/|a|))$, then $f \restriction [1, 2]$ is the function $x^{a/b} \restriction [1, 2]$. If not, apply the multiplicative property of powers to definably extend f to the interval $[1,2]$. That E is a vector space of dimension 2 over $E \cap \mathbb{R}$ now follows. $\square$

Lemma 2.2 *The operation $Z \mapsto V(Z)$ is an abstract closure operation on subsets of $\mathbb{C}$. A set Z is V-closed if and only if $Z \cap \mathbb{R}$ is a field and Z is a vector space over $Z \cap \mathbb{R}$ of either dimension 1 or 2. For any expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$, $E(\mathfrak{R})$ is V-closed.*

Proof Let $Z, Z' \subseteq \mathbb{C}$. It is clear that $Z \subseteq V(Z)$ and that $Z \subseteq Z'$ implies $V(Z) \subseteq V(Z')$. We have shown that $V(Z) \cap \mathbb{R}$ is a field, $V(Z)$ is a vector space over $V(Z) \cap \mathbb{R}$, and that $V(Z)$ is closed under taking quotients of nonzero imaginary elements (and hence always of dimension 1 or 2). It is clear from construction that V fixes exactly the sets satisfying these properties. $\square$

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $Z \subseteq \mathbb{C}$ and $\mathfrak{R} = (\overline{\mathbb{R}}, (x^z \restriction [1,2])_{z \in Z})$. We show that $V(Z) = E$. We have that $V(Z) \subseteq E$ by Lemma 2.2 and that $Z \subseteq E$. For the opposite inclusion we give details only for the special case $Z = \{i\alpha, i\beta\}$, where the set $\{\alpha, \beta\} \subseteq \mathbb{R}$ is algebraically independent. We do this to avoid difficulties due solely to notation; the general case is obtained by clerical modification.

Suppose toward a contradiction that $V(Z) \neq E$, and let $a, b \in \mathbb{R}$ be such that $a + ib \in E \setminus V(Z)$ and $x^{a+ib} \restriction [1,2]$ is definable. Since $x^{a+ib} \restriction [1,2]$ is definable if and only if $x^a \restriction [1,2]$ and $x^{ib} \restriction [1,2]$ are definable, we assume without loss of generality that either $a = 0$ or $b = 0$. By Lemma 2.1, $x^{ib} \restriction [1,2]$ is definable if and only if $x^{b/\alpha} \restriction [1,2]$ is definable, so we assume without loss of generality that $b = 0$.

It is convenient from a technical standpoint to work with a structure interdefinable with $\mathfrak{R}$ in which the function symbols represent functions that are real analytic on $\mathbb{R}$. For $\omega \in \mathbb{R}$, define the functions $f_\omega, f_{i\omega}, c_\omega, s_\omega : \mathbb{R} \to \mathbb{C}$ as follows:

$$c_\omega(x) = \cos\Big(\omega \log\Big(\frac{2x^2+1}{x^2+1}\Big)\Big), s_\omega(x) = \sin\Big(\omega \log\Big(\frac{2x^2+1}{x^2+1}\Big)\Big)$$

$$f_\omega(x) = \Big(\frac{2x^2+1}{x^2+1}\Big)^\omega, f_{i\omega}(x) = c_\omega(x) + i s_\omega(x)$$

Let $\mathfrak{R}' = (\overline{\mathbb{R}}, c_\alpha, s_\alpha, c_\beta, s_\beta, \alpha, \beta)$. That $\mathfrak{R}'$ is interdefinable with $\mathfrak{R}$ is clear, and the functions $c_\alpha, s_\alpha, c_\beta, s_\beta : \mathbb{R} \to \mathbb{R}$ are real analytic. By [4], $\mathfrak{R}$ is o-minimal, hence so is $\mathfrak{R}'$. The structure $\mathfrak{R}'$ is both existentially and universally interdefinable with the structure

$$(\overline{\mathbb{R}}, \exp(\alpha \arctan(x)), \exp(\beta \arctan(x)), \alpha, \beta),$$

which is model complete by Wilkie [16, First Main Theorem], so $\mathfrak{R}'$ is model complete. Since $x^a \restriction [1,2]$ is definable in $\mathfrak{R}$, we have that f_a is definable in $\mathfrak{R}$ and hence definable in $\mathfrak{R}'$. By basic facts about real closed fields, there is a term $F(x, y, w)$ such that $y = f_a(x)$ if and only if $\exists w(F(x, y, w) = 0)$. By Wilkie desingularization [16, Corollary 3.2], there are positive integers p, q and terms $g_{k,j}$ in the language of $\mathfrak{R}'$ in which the functions $f_{i\alpha}$ and $f_{i\beta}$ are only applied to variables such that $y = f_a(x)$ if and only if

$$\exists w \bigvee_{j=1}^{p} \Big(\Big(\bigwedge_{k=0}^{q} g_{k,j}(x, y, w) = 0\Big) \wedge \det\Big(\frac{\partial(g_{0,j}, \ldots, g_{q,j})}{\partial(y, w_1, \ldots, w_q)}\Big) \neq 0\Big).$$

Without loss of generality, we assume that this q is minimal over all such nonsingular systems, even if terms from $(\overline{\mathbb{R}}, (f_z)_{z\in V(Z)}, (z)_{z\in V(Z)})$ are allowed. Then $q > 0$ since $a \notin V(Z)$. We assume that f_a is defined at 0 by the first system of equations, that is $y = f_a(0)$ if and only if

$$\exists w\Big(\Big(\bigwedge_{k=0}^{q} g_{k,1}(0,y,w) = 0\Big) \wedge \det\Big(\frac{\partial(g_{0,1},\dots,g_{q,1})}{\partial(y,w_1,\dots,w_q)}\Big)(0,y,w) \neq 0\Big).$$

By o-minimality and the Implicit Function Theorem, this formula defines $f_a(x)$ for x in an open interval I containing 0 such that there are real analytic functions $w_1,\dots,w_q$ with $\bigwedge_{k=0}^{q} g_{k,1}(x, f_a(x), w_1(x),\dots,w_q(x)) = 0$ for all $x \in I$. Notice that $g_{k,1}(x,y,w)$ is a polynomial in terms z, $c_\alpha(z)$, $s_\alpha(z)$, $c_\beta(z)$, and $s_\beta(z)$, where z can be any unary term occurring in x, y, or w. Let C be the ring generated by series of the form $z, s_\alpha(z), s_\beta(z)$ for $z = x, f_a(x), w_1(x),\dots,w_q(x)$. Since c_ω and s_ω are algebraically dependent, the above shows that

$$\operatorname{trdeg}_{\mathbb{C}}(C) < 3(q+2) - (q+1) = 2q+5.$$

The remainder of the proof consists of deriving a contradictory lower bound for $\operatorname{trdeg}_{\mathbb{C}}(C)$. For ease of notation, put $\bar{w}_0(x) = (2x^2+1)/(x^2+1)$, $\bar{f}_a = \bar{w}_0 \circ f_a$ and $\bar{w}_j = \bar{w}_0 \circ w_j$ for $j > 0$. Consider the set W made up of the formal series $\log \bar{f}_a(x) - \log \bar{f}_a(0)$, $i\alpha(\log \bar{f}_a(x) - \log \bar{f}_a(0))$, and $i\beta(\log \bar{f}_a(x) - \log \bar{f}_a(0))$, along with the series

1. $\log \bar{w}_j(x) - \log \bar{w}_j(0)$, $j = 0,\dots,q$,
2. $i\alpha(\log \bar{w}_j(x) - \log \bar{w}_j(0))$, $j = 0,\dots,q$,
3. $i\beta(\log \bar{w}_j(x) - \log \bar{w}_j(0))$, $j = 0,\dots,q$.

We assume that for each j we have $w_j(0) > 0$. Each of the series in W has zero constant term, and each series converges on a neighborhood of zero.

Suppose toward a contradiction that the set W is $\mathbb{Q}$-linearly dependent. Each element of W is either real or purely imaginary, so any $\mathbb{Q}$-linear dependence relation that holds among the elements of W must hold among the real elements and the imaginary elements of W separately. Suppose that the set of imaginary elements of W is $\mathbb{Q}$-linearly dependent. Then the set of formal series $W' := \{\log \bar{f}_a(x) - \log \bar{f}_a(0); \log \bar{w}_j(x) - \log \bar{w}_j(0) : j = 0,\dots,q\}$ is $\mathbb{Q}(\alpha/\beta)$-linearly dependent. The linear dependence relation witnessing this must involve nontrivially at least one of the terms $\log \bar{w}_j(x) - \log \bar{w}_j(0)$, $j \geq 1$, and without loss of generality we take $j = q$. Solving this dependence relation for $\log \bar{w}_q(x)$ and exponentiating, we present the function $\bar{w}_q(x)$ as $\bar{w}_q(x) = c \cdot h(\bar{f}_a(x), \bar{w}_0(x),\dots,\bar{w}_{q-1}(x))$, where $c \in \mathbb{R}$ and the function $h(\bar{u}, \bar{w}_0,\dots,\bar{w}_{q-1})$ is a product of the terms $\bar{u}, \bar{w}_0,\dots,\bar{w}_{q-1}$ raised to powers from $\mathbb{Q}(\alpha/\beta)$. We see that $h(\bar{u}, \bar{w}_0,\dots,\bar{w}_{q-1})$ is a term in the language of $\mathfrak{R}'$, and that $c_\alpha(z), s_\alpha(z), c_\beta(z)$, and $s_\beta(z)$, $z = h(\bar{u}, \bar{w}_0,\dots,\bar{w}_{q-1})$, are equal to polynomials in terms $(f_\zeta(z'))_{\zeta\in V(Z)}$, where z' ranges over u and the unary terms occurring in w (as before we denote $\bar{x} = w_0(x)$). Thus, we may replace w_q by $\bar{w}_0^{-1}(c \cdot h(\bar{u}, \bar{w}_0,\dots,\bar{w}_{q-1}))$ in the system of equations defining f_a. Expand the determinant $\partial(g_{0,1},\dots,g_{q,1})/\partial(y,w_1,\dots,w_q)$ by minors along the qth column. Since

$$\frac{\partial(g_{0,1},\dots,g_{q,1})}{\partial(y,w_1,\dots,w_q)}(f_a(0), w_0(0),\dots,w_q(0)) \neq 0,$$

we have that one of these minors is nonsingular at 0, and hence in a neighborhood of 0. Delete the corresponding equation and note that $w_q'(0) \neq 0$; it is a calculus problem to check that the resulting system is nonsingular, a contradiction. Thus the set of imaginary elements of W is $\mathbb{Q}$-linearly independent. A nontrivial $\mathbb{Q}$-linear dependence relation among the real elements of W would yield a nontrivial $\mathbb{Q}(\alpha/\beta)$-linear dependence relation on W', so W itself is $\mathbb{Q}$-linearly independent.

By Ax [1], we have $\operatorname{trdeg}_{\mathbb{C}}(\mathbb{C}[W \cup \{e^w : w \in W\}]) \geq 1 + 3(q+2) = 3q+7$. The set W is $\mathbb{C}$-linearly dependent, however, and hence algebraically dependent over $\mathbb{C}$. Since $\dim_{\mathbb{C}}(W) \leq q+1$, the difference between the transcendence degrees of $\mathbb{C}[W \cup \{e^w : w \in W\}]$ and $\mathbb{C}[\{e^w : w \in W\}]$ is at most $q+1$. Thus,

$$\operatorname{trdeg}_{\mathbb{C}}(\mathbb{C}[\{e^w : w \in W\}]) \geq 3q+7-(q+1) = 2q+6.$$

The rings $\mathbb{C}[\{e^w : w \in W\}]$ and C have the same transcendence degree over $\mathbb{C}$, so

$$2q+6 \leq \operatorname{trdeg}_{\mathbb{C}}(C) \leq 2q+5.$$

This is a contradiction, so f_a is not definable, and hence $E = V(Z)$. □

2.1 Unbounded domains Theorems 1.1 and 1.2 were stated and proved for powers defined on bounded intervals, but in some cases we are able to extend the domains of the powers considered to unbounded intervals while still maintaining those theorems' conclusions. For real powers, we do this by modifying the proof of Theorem 1.1 in a straightforward manner (see [3]) to see that for $Z \subseteq \mathbb{C}$ and $Z' \subseteq Z \cap \mathbb{R}$, the structure $(\overline{\mathbb{R}}, (x^z \restriction [1,2])_{z \in Z}, (x^r)_{r \in Z'})$ has set of restricted complex exponents $V(Z)$. For imaginary powers, we see below that the situation is more interesting.

Proposition 2.3 *Let $Z \subseteq \mathbb{C}$ and let $\omega \in \mathbb{R}$ such that $i\omega \in Z$. Let $\mathfrak{R} = (\overline{\mathbb{R}}, x^{i\omega}, (x^z \restriction [1,2])_{z \in Z})$. Then $E(\mathfrak{R}) = V(Z)$.*

Proof By [12, Theorem 4], if a function is definable on the interval $[1,2]$ in $\mathfrak{R}$, then a restriction of this function to some open subinterval of $[1,2]$ is definable in the structure $(\overline{\mathbb{R}}, x^{i\omega} \restriction [1,2], (x^z \restriction [1,2])_{z \in Z})$. Apply Theorem 1.1 to this structure, noting that $i\omega \in Z$. □

Even though E behaves nicely for an expansion by real powers or a single imaginary power, the case of two or more imaginary exponents is very different. To put the following result in context, recall that the sets definable in the structure $(\mathbb{R}, \mathbb{Z})$ form what is called the real projective hierarchy, which is considered wild from a model theoretic point of view (see Kechris [9], for instance). For any $n \in \mathbb{N}$, each Borel subset of $\mathbb{R}^n$ is definable in $(\mathbb{R}, \mathbb{Z})$, and as a consequence, global complex exponentiation is definable in $(\mathbb{R}, \mathbb{Z})$. In particular, $E((\mathbb{R}, \mathbb{Z})) = \mathbb{C}$.

Proposition 2.4 *Let $\alpha, \beta \in \mathbb{R}$ be $\mathbb{Q}$-linearly independent; let $\mathfrak{R} = (\overline{\mathbb{R}}, x^{i\alpha}, x^{i\beta})$. Then $\mathbb{Z}$ is definable in $\mathfrak{R}$.*

Proof For any nonzero real ω, the set $G_\omega := \{x > 0 : x^{i\omega} \in \mathbb{R}^+\}$ is a cyclic multiplicative subgroup of $\overline{\mathbb{R}}$, hence discrete. As α and β are $\mathbb{Q}$-linearly independent, $G_\alpha \cdot G_\beta$ is dense in $\mathbb{R}^+$. By Hieronymi [8], $\mathbb{Z}$ is definable. □

2.2 Linear vector fields We now apply Theorem 1.1 to the study of expansions of $\overline{\mathbb{R}}$ by trajectories of linear vector fields as in [13]. Before this, we require a few definitions. A *linear vector field* is an $\mathbb{R}$-linear transformation $F : \mathbb{R}^n \to \mathbb{R}^n$ for some $n \geq 1$, and a enumeratesolution of F is a differentiable function $\gamma\colon I \to \mathbb{R}^n$ defined on a nontrivial interval $I \subseteq \mathbb{R}$ such that $\gamma'(t) = F(\gamma(t))$ for all $t \in I$. A enumeratetrajectory of F is the image of a solution, viewed as a set with neither parameterization nor orientation. A trajectory is enumeratelocally closed if it is the intersection of an open set and a closed set. The following is a classification, up to interdefinability, of expansions of $\overline{\mathbb{R}}$ by families of locally closed trajectories of linear vector fields.

Theorem 2.5 *Let $\mathcal{G}$ be a collection of locally closed trajectories of linear vector fields. Let $\mathcal{Z}$ denote the image of $Z \mapsto V(Z)\colon \mathcal{P}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})$.*

1. *If $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{G}})$ does not define* $\exp\restriction[0,1]$*, then it is interdefinable with exactly one of the following:*
   (a) $(\overline{\mathbb{R}}, (x^a)_{a\in K}, (x^z\restriction[1,2])_{z\in Z})$ *for some subfield K of $\mathbb{R}$ such that $K \subseteq Z \in \mathcal{Z}$,*
   (b) $(\overline{\mathbb{R}}, x^{ib}, (x^z\restriction[1,2])_{z\in Z})$ *for some nonzero $b \in \mathbb{R}$ such that $ib \in Z \in \mathcal{Z}$.*
2. *If $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{G}})$ defines* $\exp\restriction[0,1]$*, but neither* $\sin\restriction[0,1]$ *nor* $\exp$*, then it is interdefinable with $(\overline{\mathbb{R}}, \exp\restriction[0,1], (x^a)_{a\in K})$ for some subfield K of $\mathbb{R}$.*
3. *If $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{G}})$ defines both* $\exp\restriction[0,1]$ *and* $\sin\restriction[0,1]$*, but not* $\exp$*, then it is interdefinable with exactly one of the following:*
   (a) $(\mathbb{R}^{\mathrm{RE}}, (x^a)_{a\in K})$ *for some subfield K of $\mathbb{R}$,*
   (b) $(\mathbb{R}^{\mathrm{RE}}, x^{ib})$ *for some nonzero $b \in \mathbb{R}$.*
4. *If $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{G}})$ defines* $\exp$*, then it is interdefinable with exactly one of the following:*
   (a) $(\overline{\mathbb{R}}, \exp)$*,*
   (b) $(\overline{\mathbb{R}}, \exp, \sin\restriction[0,1])$*,*
   (c) $(\overline{\mathbb{R}}, \mathbb{Z})$*.*

Proof A locally closed trajectory Γ of a linear vector field either has compact closure or infinite logarithmic length. Write $\mathcal{G} = \mathcal{H} \cup \mathcal{K}$, where every $\Gamma \in \mathcal{H}$ has infinite logarithmic length and every $\Gamma \in \mathcal{K}$ has compact closure. By [13], $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{H}})$ is interdefinable with exactly one of the following:

1. $(\overline{\mathbb{R}}, (x^r)_{r\in K})$ for some subfield K of $\mathbb{R}$,
2. $(\overline{\mathbb{R}}, x^{ib})$ for some nonzero real b,
3. $(\overline{\mathbb{R}}, \exp)$,
4. $(\overline{\mathbb{R}}, \mathbb{Z})$.

An examination of the proof shows that $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{K}})$ is interdefinable with at least one of the following:

1. $(\overline{\mathbb{R}}, (x^z\restriction[1,2])_{z\in Z})$ for some $Z \in \mathcal{Z}$,
2. $(\overline{\mathbb{R}}, \exp\restriction[0,1])$,
3. $\mathbb{R}^{\mathrm{RE}}$.

By [2], $\mathbb{R}^{\mathrm{RE}}$ and $(\overline{\mathbb{R}}, \exp\restriction[0,1])$ are not interdefinable, and by Theorem 1.1, there is no $Z \in \mathcal{Z}$ such that $(\overline{\mathbb{R}}, (x^w\restriction[1,2])_{z\in Z})$ is interdefinable with $(\overline{\mathbb{R}}, \exp\restriction[0,1])$ or $\mathbb{R}^{\mathrm{RE}}$. As $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{G}})$ can be written as the expansion $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{H}}, (\Gamma)_{\Gamma\in\mathcal{K}})$, and so $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{G}})$ is interdefinable with an expansion of one of the structures on the first list by the definable functions of one of the structures on the second list. The claim follows from basic properties of restricted complex powers, Proposition 2.3 and the comments immediately preceding it, and [11]. $\square$

The classes of structures given in the statement of Theorem 2.5 are distinct by Theorem 1.1 and [2, 3]. As mentioned in [13, Section 3], after suitable normalization it is possible to calculate the field K and the set Z (when appropriate) in terms of the of eigenvalues of the vector fields of which the sets in $\mathcal{G}$ are trajectories.

3 Complex Powers on Subsets of $\mathbb{C}$

We proceed to study the situation of complex powers defined on open subsets of $\mathbb{C}$. Bianconi showed in [3] that any holomorphic (complex differentiable) function definable on an open subset of $\mathbb{C}$ in the structure $(\overline{\mathbb{R}}, (x^z)_{z \in Z})$, where $Z \subseteq \mathbb{R}$, is definable in $\overline{\mathbb{R}}$. Thus, in an expansion of $\overline{\mathbb{R}}$ by real power functions, all holomorphic definable functions are semialgebraic. This is not always the case when restricted imaginary powers are involved.

Lemma 3.1 *Let $\mathfrak{R}$ be an expansion of $\overline{\mathbb{R}}$. Let Y be the set of imaginary elements of $E(\mathfrak{R})$. The following are then equivalent:*

1. *The set E is a field.*
2. *The set $Y \setminus \{0\}$ is closed under the taking of multiplicative inverses.*
3. *The square of each element of Y lies in E.*
4. *Either $Y = \{0\}$ or there are two nonzero elements of Y such that their product lies in E.*

Proof This equivalence is clear if $E \subseteq \mathbb{R}$. Otherwise, let $i\alpha \in Y$ be nonzero and put $K = E \cap \mathbb{R}$. Notice that $E = K + i\alpha K$ is a vector space of dimension 2 over K. Multiplication by a nonzero element of Y induces a nonsingular linear transformation on $\mathbb{C}$. If one such linear transformation preserves E, then all such linear transformations preserve E. This shows that the conditions listed above are equivalent. $\square$

Proposition 3.2 *Let $Z \subseteq \mathbb{C}$ and let $\mathfrak{R} = (\overline{\mathbb{R}}, (x^z \restriction [1, 2])_{z \in Z})$.*

1. *If there exists $\omega \in \mathbb{R}$ such that $V(Z)$ contains both $i\omega$ and i/ω, then the function $x^{i\omega} \restriction [1, 2]$ extends definably to a holomorphic function on some nonempty open subset of $\mathbb{C}$.*
2. *If $V(Z)$ contains a nonzero imaginary element, then for any real $\omega \in V(Z)$, the function $x^{\omega} \restriction [1, 2]$ extends definably to a holomorphic function on some nonempty open subset of $\mathbb{C}$.*

Proof For (1), suppose that $\omega > 0$ and let $f : [1, \exp(\pi/\omega)) \to \mathbb{S}^1$ be given by $f(x) = x^{i\omega} \restriction [1, \exp(\pi/\omega))$. Notice that f is a bijection, its inverse is a branch of the analytic function $z^{1/i\omega} = x^{-i/\omega}$ on $\mathbb{S}^1$, and f is definable if and only if $i\omega \in V(Z)$. Suppose $i\omega \in V(Z)$ and let $S = \{re^{i\theta} : 1 \le r < \exp(\pi/\omega), 0 \le \theta < \pi\}$. We define the function $\tilde{f} : S \to \mathbb{C}$ by

$$\tilde{f}(z) = \frac{f(|z|)}{f^{-1}(z/|z|)}.$$

Observe that $\tilde{f}$ is an analytic continuation of $x^{i\omega} \restriction [1, \exp(\pi/\omega))$ to S and that $\tilde{f}$ is definable.

The proof of (2) is similar: given $ia \in V(Z)$ with $a \in \mathbb{R}$ nonzero, conjugate $x^{\omega} \restriction [1,2]$ by $x^{ia} \restriction [1,2]$ to define z^{ω} on a nontrivial arc of the unit circle. Then proceed as above to define z^{ω} on a nonempty open subset of the plane. $\square$

Remark. If $\mathfrak{R}$ is o-minimal, then by cell decomposition, algebraic properties of power functions, and some basic results in complex analysis, the preceding proof can be extended to allow S to be any bounded definable subset of $\mathbb{C}$ which lies within some compact, simply connected subset of $\mathbb{R}^2 \setminus \{0\}$.

Combining Lemmas 2.2 and 3.1 with Proposition 3.2, we have the following characterization.

Proposition 3.3 *The set E is a field if and only if either $E \subseteq \mathbb{R}$ or each restricted complex power definable on a nonempty open interval extends definably to a holomorphic function on a nonempty open subset of the plane.*

Combined with Proposition 2.3, this yields a curious dichotomy.

Corollary 3.4 *Let $\omega \in \mathbb{R}$. A branch of $z^{i\omega}$ is definable on some nonempty open subset of $\mathbb{C}$ in the structure $(\overline{\mathbb{R}}, x^{i\omega})$ if and only if ω is either rational or quadratic irrational.*

Proof By Proposition 2.3, the set of restricted complex exponents for the structure $(\overline{\mathbb{R}}, x^{i\omega})$ is $V(\{i\omega\}) = \mathbb{Q} + i\omega\mathbb{Q}$, which is a field if and only if ω is either rational or quadratic irrational. Apply Proposition 3.3. $\square$

4 Invariance of E under Elementary Equivalence

Fix an expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$. We show that $E(\mathfrak{M})$ and $E(\mathfrak{R})$ are isomorphic when $\mathfrak{R}$ is an o-minimal expansion of $\overline{\mathbb{R}}$ that does not define $\exp \restriction [0,1]$ and $\mathfrak{M}$ is elementarily equivalent to $\mathfrak{R}$. We start with this lemma.

Lemma 4.1 *Let $I \subseteq \mathbb{R}^+$ be a nonempty open interval. The following are equivalent:*

1. *$\mathfrak{R}$ defines* $\exp z \restriction I^2$.
2. *$\mathfrak{R}$ defines $x^{iy} \restriction I^2$.*
3. *$\mathfrak{R}$ defines* $\exp \restriction I$ *and* $\sin \restriction I$.

Proof The equivalence (1) $\Leftrightarrow$ (3) is clear, so we consider the implication (2) $\Rightarrow$ (3). Since $x^{iy} = (\cos(y \log x), \sin(y \log x))$, this follows from o-minimality via partial derivatives. $\square$

With this, we turn our attention to definable families of functions. We show that a definable family of functions can contain only finitely many restricted complex powers unless there is a nonempty open interval such that $\exp \restriction I$ is definable.

Proposition 4.2 *Suppose that no restriction of* $\exp z$ *to a nonempty open subset of $\mathbb{C}$ is definable. Let $A \subseteq \mathbb{R}^n$, let I be a nonempty open interval, and let $f : A \times I \to \mathbb{R}$ be definable. Then the set*

$$\Omega := \{\omega \in \mathbb{R} : \exists a \in A, \forall x \in I, f(a,x) = x^{i\omega}\}$$

is finite. If $\mathfrak{R}$ does not define $\exp \restriction J$ *for any nonempty open interval J, then the set*

$$B := \{b \in \mathbb{R} : \exists a \in A, \forall x \in I, f(a,x) = x^{b}\}$$

is finite.

Proof Suppose toward a contradiction that Ω is infinite. By o-minimality, Ω must contain a nonempty open subinterval J. Thus the map $x^{iy} : I \times J \to \mathbb{R}^2$ is definable, a contradiction. The proof that B is finite if $\mathfrak{R}$ does not define $\exp \restriction J$ is similar. $\square$

Corollary 4.3

1. *Suppose that no restriction of* $\exp z$ *to a nonempty open subset of* $\mathbb{C}$ *is definable and that* $\omega \in \mathbb{R}$ *is such that* $i\omega \in E$. *Let* I *be a nonempty, open,* $\emptyset$*-definable interval such that* $x^{i\omega} \restriction I$ *is definable. Then the restricted power* $x^{i\omega} \restriction I$ *is* $\emptyset$*-definable.*
2. *Suppose that* $\exp \restriction J$ *is not definable for any open interval* J *and that* $b \in \mathbb{R}$ *is such that* $b \in E$. *Let* I *be a nonempty, open,* $\emptyset$*-definable interval such that* $x^b \restriction I$ *is definable. Then the restricted power* $x^b \restriction I$ *is* $\emptyset$*-definable.*

Proof For (1), suppose that $x^{i\omega} \restriction I$ is not $\emptyset$-definable. By standard o-minimal arguments (see [10], for instance) there is a function $f : A \times I \to \mathbb{R}$ such that $f(a, x) = x^{i\omega}$ for $x \in I$. The set $\{\alpha \in \mathbb{R} : \exists a \in A : f(a, x) = x^{\alpha}$ for $x \in I\}$ is a definable subset of $\mathbb{R}$, and so must be a finite union of intervals. If one of these intervals is infinite, then the set Ω as defined in Proposition 4.2 is infinite. Thus A must be a finite set, a contradiction to the assumption that $x^{i\omega} \restriction I$ is not $\emptyset$-definable. The proof for (2) is similar. $\square$

Subject to the condition that $\exp \restriction J$ is not definable for any nonempty open interval J, the proof of Corollary 4.3 shows that definable restricted complex powers $x^z \restriction I$ are $\emptyset$-definable up to parameters used to define I. Over $\overline{\mathbb{R}}$ we can always take $[1, 2] \subseteq I$, so all restricted complex powers are actually $\emptyset$-definable.

To state the final theorem of this section, we need a more general definition of the set of restricted complex exponents. Let $\mathfrak{R}$ be an o-minimal expansion of a real closed field with underlying set R, and write C to mean the algebraic closure of R considered as a definable object in $\mathfrak{R}$. A *restricted complex power* is a partial multiplicative homomorphism $f : I \to C$ such that $I \subseteq R$ is a nontrivial interval and $1 \in I$. By o-minimality, f is differentiable in a neighborhood of 1. We say that the restricted complex exponent of f is the value $f'(1)$, and we define the *set of restricted complex exponents* E in the obvious way. These definitions are equivalent to our earlier definitions if $R = \mathbb{R}$. If $\mathfrak{R}$ is elementarily equivalent to an o-minimal expansion of $\overline{\mathbb{R}}$, then the results of this section apply with no additional argument required, and we have the following theorem.

Theorem 4.4 *Suppose* $\mathfrak{R}$ *is an o-minimal expansion of* $\overline{\mathbb{R}}$ *that does not define* $\exp \restriction [0, 1]$ *and suppose that* $\mathfrak{M}$ *is elementarily equivalent to* $\mathfrak{R}$. *Let* M *be the underlying set of* $\mathfrak{M}$. *Then* $M \cap E(\mathfrak{M})$ *and* $\mathbb{R} \cap E(\mathfrak{R})$ *are isomorphic as ordered fields. If* $E(\mathfrak{R})$ *is a field, then the isomorphism extends to a field isomorphism of* $E(\mathfrak{M})$ *and* $E(\mathfrak{R})$. *Otherwise, the isomorphism extends to an isomorphism of* $E(\mathfrak{M})$ *and* $E(\mathfrak{R})$ *as vector spaces over* $\mathbb{R} \cap E(\mathfrak{R})$.

5 Optimality

5.1 O-minimal expansions of real closed fields Many of the results in this article can be stated and proved for o-minimal expansions of real closed fields. Let $\mathfrak{R}$ be an o-minimal expansion of a real closed field with underlying set R, and write C to mean the algebraic closure of R considered as a definable object in $\mathfrak{R}$. The proofs of the results regarding the behavior of V on subsets of C and the definability of powers on open subsets of C carry over in an obvious way, while our proof for Theorem 1.1 depended on results from [16] that only apply to expansions of $\overline{\mathbb{R}}$ (Foster [7] contains a generalization of [16]). The invariance results for E generalize with a little work. If $\mathfrak{R}$ supports abstract exponential and arctangent functions, that is, definably continuous functions $f, g : R \to R$ such that $f(x + y) = f(x)f(y)$ and $g(x) + g(1/x) = c \in R$, then the proofs given above work modulo an argument at the end involving quantification over parameters and the existence of prime models for o-minimal structures (Pillay and Steinhorn [14]). If not, then a more delicate treatment as in [11] can be used.

5.2 Other expansions of $\overline{\mathbb{R}}$ For at least one other kind of expansion of o-minimal expansion of $\overline{\mathbb{R}}$, Theorem 1.1 generalizes. Let $\mathfrak{R}$ be an expansion of $\overline{\mathbb{R}}$, and for $z \in \mathbb{R}$, put $g_z(x, y) = (\cos(z \arccos x), \sin(z \arcsin y))$. Let $E'(\mathfrak{R})$ be the set of all $z \in \mathbb{R}$ such that there is a nontrivial arc S of the unit circle and the restriction $g_z \restriction S$ is definable in $\mathfrak{R}$. If a nontrivial restricted imaginary power is definable, then an examination of the proof of Proposition 3.2 shows that $E' = E$. Otherwise, argument along the lines of our proof of Theorem 1.1 yields the following result.

Proposition 5.1 *Let $Z, Z' \subseteq \mathbb{R}$, and put $S = \{e^{i\theta} : 0 \leq \theta < \pi/4\} \subseteq \mathbb{C}$. Then*

1. $E'((\overline{\mathbb{R}}, \exp\restriction[0,1], g_{z'} \restriction S)_{z' \in Z'})) = V(Z')$,
2. $E'((\overline{\mathbb{R}}, (x^z \restriction[1,2])_{z\in Z}, (g_{z'} \restriction S)_{z' \in Z'})) = V(Z')$,
3. $E((\overline{\mathbb{R}}, (x^z \restriction[1,2])_{z\in Z}, (g_{z'} \restriction S)_{z' \in Z'})) = V(Z)$.

How else might Theorem 1.1 generalize? Let $\mathfrak{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$ such that neither $\sin\restriction[0,1]$ nor $\exp\restriction[0,1]$ is definable. Though the structure $(\mathfrak{R}, (x^z \restriction[1,2])_{z\in Z})$ is known to be o-minimal and exponentially bounded as a consequence of Speissegger [15], it is not known at this time if $(\mathfrak{R}, (x^z \restriction[1,2])_{z\in Z})$ must be polynomially bounded, let alone what its set of restricted complex exponents is. In fact, it is not known if $E((\mathfrak{R}, (x^z \restriction[1,2])_{z\in Z})) = V(E(\mathfrak{R}) \cup Z)$ holds, even assuming that $E((\mathfrak{R}, (x^z \restriction[1,2])_{z\in Z})) \neq \mathbb{C}$ and that $\mathfrak{R}$ is an expansion of $\overline{\mathbb{R}}$ by restrictions of analytic functions.

Acknowledgments

I thank Chris Miller, my Ph.D. supervisor, for help in preparing this article. I thank Lou van den Dries for comments.

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