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Notes on the Model Theory of DeMorgan Logics

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Abstract We here make preliminary investigations into the model theory of DeMorgan logics. We demonstrate that Łoś's Theorem holds with respect to these logics and make some remarks about standard model-theoretic properties in such contexts. More concretely, as a case study we examine the fate of Cantor's Theorem that the classical theory of dense linear orderings without endpoints is $\aleph_0$-categorical, and we show that the taking of ultraproducts commutes with respect to previously established methods of constructing nonclassical structures, namely, Priest's Collapsing Lemma and Dunn's Theorem in 3-Valued Logic.

1 Semantics for DeMorgan Logics

We may suppose that the fundamental component to a logic λ is the relation $\vdash_{\lambda}$ that holds between sets of formulas and sets of formulas, indicating that the latter is *derivable* from the former. As each logic λ that we will be invoking is sound and complete, we may consider the relation $\models_{\lambda}$ associated with each λ and define it semantically. In so doing, we will sufficiently define the logic itself.

The logics upon which we herein focus are the classical predicate calculus CL, the paraconsistent (inconsistency-tolerant) logics LP and RM_3, the paracomplete (incompleteness-tolerant) logics K_3 and $\mathsf{Ł}_3$, and the paraconsistent and paracomplete logic FDE. For a discussion of these logics' origins and philosophical motivations, we refer the reader to Priest [8]. These logics may be thought of as, to extend the nomenclature of Field [2], *DeMorgan* logics, insofar as for each logic λ in this class

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the DeMorgan Laws hold. This motivates our referencing the class of the aforementioned logics as $\mathfrak{Dem}$. Formally, the following conditions hold in each $\lambda \in \mathfrak{Dem}$:

$$\neg(\varphi \wedge \psi) \mathrel{=\!\!\!|\!\!\models_{\lambda}} \neg\varphi \vee \neg\psi, \text{ and}$$
$$\neg(\varphi \vee \psi) \mathrel{=\!\!\!|\!\!\models_{\lambda}} \neg\varphi \wedge \neg\psi,$$

where $\mathrel{=\!\!\!|\!\!\models_{\lambda}}$ represents interderivability with respect to logic λ. It is our task in this précis to provide an account of the relation $\models_{\lambda}$ for the logics in $\mathfrak{Dem}$. We begin by making syntactic considerations.

Definition 1.1 A *signature* is an ordered set $\sigma = (\mathbf{C}, \mathbf{F}, \mathbf{R}, \sigma')$ of sets of symbols $\mathbf{C}, \mathbf{F}, \mathbf{R}$ and a function $\sigma' : \mathbf{F} \cup \mathbf{R} \to \mathbb{N}$ mapping function and relation symbols to their intended arity. In this paper, we include the identity symbol as a member of $\mathbf{R}$ for any signature σ.

Each signature determines a *language*, $\mathcal{L}_\sigma$, built up recursively. First, a set of *terms* may be constructed by the following procedure:

1. all variables $x, y, \ldots$ and constants $c \in \mathbf{C}$ are terms;
2. for $n > 0$, if each t_i of n-tuple $\vec{t}$ is a term and $\sigma'(f) = n$ for an $f \in \mathbf{F}$, then $f(\vec{t})$ is a term.

With the terms recursively defined, we may construct $\mathcal{L}_\sigma$.

Definition 1.2 A language $\mathcal{L}_\sigma$ is the smallest set such that for all $n > 0$, n-tuple of terms $\vec{t}$, and all $R \in \mathbf{R}$ such that $\sigma'(R) = n$, $R(\vec{t}) \in \mathcal{L}_\sigma$ and closed under the following:

1. if $\varphi \in \mathcal{L}_\sigma$, then $\neg\varphi \in \mathcal{L}_\sigma$;
2. if $\varphi, \psi \in \mathcal{L}_\sigma$, then $(\varphi \circ \psi) \in \mathcal{L}_\sigma$, where $\circ \in \{\vee, \wedge, \to\}$;
3. if $\varphi \in \mathcal{L}_\sigma$ and x is a variable, then $\mathsf{Q}x\varphi \in \mathcal{L}_\sigma$, where $\mathsf{Q} \in \{\forall, \exists\}$.

We now give a characterization of each $\models_{\lambda}$. Following Mortensen [3], we'll provide for each logic (a) a Hasse diagram $\mathcal{H}_\lambda$ taking as nodes a set $\mathcal{S}_\lambda$ of truth values, (b) definitions of the connectives and quantifiers with respect to the Hasse diagram, (c) a set $\nabla_\lambda \subset \mathcal{S}_\lambda$ of *designated values*, and (d) a function $v : \mathcal{L} \to \mathcal{S}_\lambda$ mapping formulas to truth values.

Let $\mathcal{S}_{\mathsf{CL}} = \{\mathsf{T}, \mathsf{F}\}$, $\mathcal{S}_{\mathsf{LP}} = \mathcal{S}_{\mathsf{RM}_3} = \{\mathsf{T}, \mathsf{B}, \mathsf{F}\}$, $\mathcal{S}_{\text{Ł}_3} = \mathcal{S}_{\mathsf{K}_3} = \{\mathsf{T}, \mathsf{N}, \mathsf{F}\}$, and $\mathcal{S}_{\mathsf{FDE}} = \{\mathsf{T}, \mathsf{N}, \mathsf{B}, \mathsf{F}\}$. We consider the following Hasse diagram $\mathcal{H}$ in Figure 1. Let each $\mathcal{H}_\lambda = \mathcal{H} \upharpoonright \mathcal{S}_\lambda$ represent an ordering on the truth values associated with λ. On each of these lattices, let $\sqcup$ denote *join* and $\sqcap$ denote *meet*.

We may now give definitions for the connectives and quantifiers by means of their associated truth functions $f_\circ^\lambda : \mathcal{S}_\lambda \to \mathcal{S}_\lambda$.

1. For all $\lambda \in \mathfrak{Dem}$, $f_\neg^\lambda(\mathsf{T}) = \mathsf{F}$ and $f_\neg^\lambda(\mathsf{F}) = \mathsf{T}$.
2. For $\lambda \in \{\mathsf{RM}_3, \mathsf{LP}, \mathsf{FDE}\}$, $f_\neg^\lambda(\mathsf{B}) = \mathsf{B}$.
3. For $\lambda \in \{\mathsf{K}_3, \text{Ł}_3, \mathsf{FDE}\}$, $f_\neg^\lambda(\mathsf{N}) = \mathsf{N}$.
4. For all $\lambda \in \mathfrak{Dem}$, $f_\vee^\lambda(x, y) = x \sqcup y$, where $\sqcup$ is defined on $\mathcal{H}_\lambda$ and $x, y \in \mathcal{S}_\lambda$.
5. For all $\lambda \in \mathfrak{Dem}$, $f_\wedge^\lambda(x, y) = x \sqcap y$, where $\sqcap$ is defined on $\mathcal{H}_\lambda$ and $x, y \in \mathcal{S}_\lambda$.
6. For $\lambda \in \{\mathsf{CL}, \mathsf{K}_3, \mathsf{LP}, \mathsf{FDE}\}$, $f_\to^\lambda(x, y) = f_\vee^\lambda(f_\neg^\lambda(x), y)$, where $x, y \in \mathcal{S}_\lambda$.
7. For $\lambda \in \{\text{Ł}_3, \mathsf{RM}_3\}$, we consult the truth tables in Figure 2.

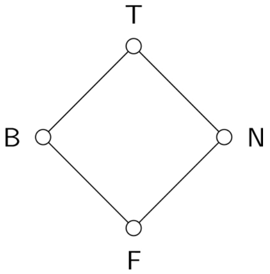

Figure 1 Hasse Diagram $\mathscr{H}$.

$f^{Ł_3}_{\rightarrow}$	T	N	F
T	T	N	F
N	T	T	N
F	T	T	T

$f^{\mathsf{RM}_3}_{\rightarrow}$	T	B	F
T	T	F	F
B	T	B	F
F	T	T	T

Figure 2 Truth tables for $f^{Ł}_{\rightarrow}$ and $f^{\mathsf{RM}_3}_{\rightarrow}$.

Finally, we give sets of *designated values* ∇_λ for each $\lambda \in \mathfrak{Dem}$. These are the truth values that intuitively imply that the evaluated formula *holds*. Let $\nabla_{\mathsf{CL}} = \nabla_{\mathsf{K}_3} = \nabla_{Ł_3} = \{\mathsf{T}\}$, $\nabla_{\mathsf{RM}_3} = \nabla_{\mathsf{LP}} = \nabla_{\mathsf{FDE}} = \{\mathsf{T}, \mathsf{B}\}$. It can be checked that in each case $\nabla_\lambda \subset \mathcal{S}_\lambda$.

From here, we can introduce structures and truth in a model. A structure gives an interpretation to a signature.

Definition 1.3 A *structure* is an ordered set $\mathfrak{A} = (A, \mathbf{C}^{\mathfrak{A}}, \mathbf{F}^{\mathfrak{A}}, \mathbf{R}^{\mathfrak{A}+}, \mathbf{R}^{\mathfrak{A}-})$, where A is a *universe* of elements, $\mathbf{C}^{\mathfrak{A}} \subseteq A$ is a set of interpretations of *constants*, $\mathbf{F}^{\mathfrak{A}}$ is a set of interpretations of function symbols, and $\mathbf{R}^{\mathfrak{A}+}$ and $\mathbf{R}^{\mathfrak{A}-}$ are, respectively, sets of *positive* and *negative* interpretations of relation symbols. By the definition of signature, the symbol $=$ is a member of $\mathbf{R}$, and we define $=^{\mathfrak{A}+}$ as $\{(x, x) : x \in A\}$; that is, equality has the intended, positive interpretation.

Any closed term t then has an interpretation $t^{\mathfrak{A}}$ in $\mathfrak{A}$.

1. If $t = c$ for some $c \in \mathbf{C}$, then $t^{\mathfrak{A}} = c^{\mathfrak{A}}$.
2. If $t = f(\vec{s})$ for some n-ary $f \in \mathbf{F}$ and n-tuple of closed terms $\vec{s}$, then $t^{\mathfrak{A}} = f^{\mathfrak{A}}(s_0^{\mathfrak{A}}, \ldots, s_{n-1}^{\mathfrak{A}})$.

In order to ensure that in discussing some structure or other, it is capable of determining a λ-interpretation, we introduce the notion of *permissibility* with respect to a logic λ. A structure $\mathfrak{A}$ is *consistent* if for all n-ary R (including equality), $R^{\mathfrak{A}+} \cap R^{\mathfrak{A}-} = \varnothing$ (inconsistent otherwise), and *complete* if for all R (including equality), $R^{\mathfrak{A}+} \cup R^{\mathfrak{A}-} = A^n$ (incomplete otherwise). The class of consistent, complete structures is permissible for all $\lambda \in \mathfrak{Dem}$, the class of inconsistent structures is permissible for LP, RM$_3$, and FDE, and the class of incomplete structures is permissible for K$_3$, Ł$_3$, and FDE.

Finally, in order to give an accurate account of the quantifiers and talk about an element or tuple of elements *satisfying* a formula, we introduce the following.

Definition 1.4 The *named counterpart* of a structure $\mathfrak{A}$, hereafter $(\mathfrak{A}, A)$, is the structure gotten from $\mathfrak{A}$ by adding a constant $\underline{a}$ for each element $a \in A$.

Each structure permissible with respect to a logic λ then gives an interpretation of the *sentences* (formulas with no free variables) of the language. For an atomic formula $R(\vec{t})$ (including identities of the form $s = t$) and a structure $\mathfrak{A}$ permissible with respect to λ,

$$u_{\lambda}^{\mathfrak{A}}(R(\vec{t})) = \begin{cases} \mathsf{T} & \text{if } \vec{t}^{\mathfrak{A}} \in R^{\mathfrak{A}+} \text{ and } \vec{t}^{\mathfrak{A}} \notin R^{\mathfrak{A}-} \\ \mathsf{F} & \text{if } \vec{t}^{\mathfrak{A}} \notin R^{\mathfrak{A}+} \text{ and } \vec{t}^{\mathfrak{A}} \in R^{\mathfrak{A}-} \\ \mathsf{B} & \text{if } \vec{t}^{\mathfrak{A}} \in R^{\mathfrak{A}+} \text{ and } \vec{t}^{\mathfrak{A}} \in R^{\mathfrak{A}-} \\ \mathsf{N} & \text{if } \vec{t}^{\mathfrak{A}} \notin R^{\mathfrak{A}+} \text{ and } \vec{t}^{\mathfrak{A}} \notin R^{\mathfrak{A}-} \end{cases}.$$

It is easy to check that if a structure is permissible with respect to λ, the semantic constraints will ensure that no atoms will be given a truth value not a member of $\mathcal{S}_{\lambda}$.

Using the evaluations of atoms as a basis, $u_{\lambda}^{\mathfrak{A}}$ can be recursively defined according to the following conditions:

$$\begin{aligned} u_{\lambda}^{\mathfrak{A}}(\neg\varphi) &= f_{\neg}^{\lambda}(u_{\lambda}^{\mathfrak{A}}(\varphi)) \\ u_{\lambda}^{\mathfrak{A}}(\varphi \vee \psi) &= f_{\vee}^{\lambda}(u_{\lambda}^{\mathfrak{A}}(\varphi), u_{\lambda}^{\mathfrak{A}}(\psi)) \\ u_{\lambda}^{\mathfrak{A}}(\varphi \wedge \psi) &= f_{\wedge}^{\lambda}(u_{\lambda}^{\mathfrak{A}}(\varphi), u_{\lambda}^{\mathfrak{A}}(\psi)) \\ u_{\lambda}^{\mathfrak{A}}(\varphi \rightarrow \psi) &= f_{\rightarrow}^{\lambda}(u_{\lambda}^{\mathfrak{A}}(\varphi), u_{\lambda}^{\mathfrak{A}}(\psi)) \\ u_{\lambda}^{\mathfrak{A}}(\forall x\varphi(x)) &= glb\{u_{\lambda}^{(\mathfrak{A},A)}(\varphi(\underline{a})) : a \in A\} \\ u_{\lambda}^{\mathfrak{A}}(\exists x\varphi(x)) &= lub\{u_{\lambda}^{(\mathfrak{A},A)}(\varphi(\underline{a})) : a \in A\}. \end{aligned}$$

We now are equipped to provide a definition of truth in a model.

Definition 1.5 For a structure $\mathfrak{A}$ permissible with respect to a logic $\lambda \in \mathfrak{Dem}$, $\mathfrak{A} \models_{\lambda} \varphi$ if and only if $u_{\lambda}^{\mathfrak{A}}(\varphi) \in \nabla_{\lambda}$.

This leads immediately to a definition of consequence between formulas modulo each $\lambda \in \mathfrak{Dem}$ by claiming that $\varphi \models_{\lambda} \psi$ if and only if for every structure $\mathfrak{A} \models_{\lambda} \varphi$, also $\mathfrak{A} \models_{\lambda} \psi$. Furthermore, given structure $\mathfrak{A}$, we can speak of an n-tuple $\vec{a} \in A^n$ *satisfying* an n-ary formula φ in logic λ by the condition that $\mathfrak{A} \models_{\lambda} \varphi(\vec{a})$ if and only if $(\mathfrak{A}, A) \models_{\lambda} \varphi(\underline{\vec{a}})$.

Granted the above definition, we may also note the following equivalences between the claim that $\mathfrak{A} \models_{\lambda} \varphi$ and natural language:

$$\begin{gathered} \mathfrak{A} \models_{\lambda} \varphi \vee \psi \text{ iff } \mathfrak{A} \models_{\lambda} \varphi \text{ or } \mathfrak{A} \models_{\lambda} \psi \\ \mathfrak{A} \models_{\lambda} \varphi \wedge \psi \text{ iff } \mathfrak{A} \models_{\lambda} \varphi \text{ and } \mathfrak{A} \models_{\lambda} \psi \\ \mathfrak{A} \models_{\lambda} \forall x\varphi(x) \text{ iff for all } a \in A, \mathfrak{A} \models_{\lambda} \varphi(a) \\ \mathfrak{A} \models_{\lambda} \exists x\varphi(x) \text{ iff for some } a \in A, \mathfrak{A} \models_{\lambda} \varphi(a). \end{gathered}$$

This connection can be easily confirmed by glancing at the truth functions for the connectives and quantifiers, but will enable us to argue about model-theoretic properties in plain language in the following.[1]

Given a definition of truth in a model, we may generalize some typical model-theoretic definitions that will come into play in the following.

Definition 1.6 The *theory* $\mathrm{Th}^{\lambda}(\mathfrak{A})$ of a structure $\mathfrak{A}$ with respect to a logic λ is the set of sentences true in $\mathfrak{A}$ with respect to λ. Formally, $\mathrm{Th}^{\lambda}(\mathfrak{A}) = \{\varphi : \mathfrak{A} \models_{\lambda} \varphi\}$.

We define a notion of isomorphism that holds for all DeMorgan logics.

Definition 1.7 Two structures $\mathfrak{A}, \mathfrak{B}$ are *isomorphic* ($\mathfrak{A} \cong \mathfrak{B}$) if and only if there is a one-to-one correspondence h such that for all constant symbols c, $c^{\mathfrak{B}} = h(c^{\mathfrak{A}})$, for all function symbols f, $f^{\mathfrak{B}}(h(\vec{a}^{\mathfrak{A}})) = h(f^{\mathfrak{A}}(\vec{a}^{\mathfrak{A}}))$, and for all relation symbols $\vec{a}^{\mathfrak{A}} \in R^{\mathfrak{A}+}$ if and only if $h(\vec{a}^{\mathfrak{A}}) \in R^{\mathfrak{B}+}$ and $\vec{a}^{\mathfrak{A}} \in R^{\mathfrak{A}-}$ if and only if $h(\vec{a}^{\mathfrak{A}}) \in R^{\mathfrak{B}-}$.

Such a generalization of isomorphism should be intuitively correct; for one, that $\mathfrak{A} \cong \mathfrak{B}$ implies that $\mathfrak{A} \equiv \mathfrak{B}$, that is, that $\mathrm{Th}^{\lambda}(\mathfrak{A}) = \mathrm{Th}^{\lambda}(\mathfrak{B})$. Furthermore, we easily see that $\cong$ is an equivalence relation on structures. With these definitions in hand, we proceed to some more concrete observations.

2 Generalizing Łoś's Theorem to the Case of $\mathfrak{Dem}$

We define a *product structure* $\prod_{i \in I} \mathfrak{A}_i$ in the following manner. First, the elements $a^{\Pi\mathfrak{A}} \in \prod_{i \in I} A_i$ are those functions taking arguments i from I and returning as value an element from A_i. Tuples of such elements $\vec{a}$ of arity m are to be thought of as a sequence of such functions $(a_0, \ldots, a_{m-1})$ so that $\vec{a}(i) = (a_0(i), \ldots, a_{m-1}(i))$. Constants $c^{\Pi\mathfrak{A}}$ denote the element $a \in \prod_{i \in I} A_i$ such that $c^{\mathfrak{A}_i} = a(i)$ for all $i \in I$. Function symbols are interpreted as $f(\vec{a})^{\Pi\mathfrak{A}} = b^{\Pi\mathfrak{A}}$ such that $\mathfrak{A}_i \models_{\lambda} b(i) = f^{\mathfrak{A}_i}(\vec{a}(i))$ for all $i \in I$. Relation symbols R are interpreted as having both extension and anti-extension; for a tuple $\vec{a} \in \prod_{i \in I} A_i$, we say that $\prod_{i \in I} \mathfrak{A}_i \models_{\lambda} R(\vec{a})$ iff $\vec{a}(i) \in R^{\mathfrak{A}_i +}$ for all $i \in I$.

We define the *reduced product* of $\prod_{i \in I} \mathfrak{A}_i$ modulo a filter $\mathcal{U} \subset \wp(I)$, which we hereafter call $\mathfrak{A}^{\natural}$,[2] in the following manner. We first define an equivalence relation $\sim_{\mathcal{U}}$ by dictating that any two elements $a, b \in \prod_{i \in I} A_i$ are equivalent modulo $\sim_{\mathcal{U}}$ if and only if $\{i : \mathfrak{A}_i \models_{\lambda} a(i) = b(i)\} \in \mathcal{U}$. The universe $\prod_{i \in I} A_i / \mathcal{U}$ thus comprises equivalence classes $\{b : a \sim_{\mathcal{U}} b\} = [a]$. Constants of this structure c are interpreted as $c^{\mathfrak{A}\natural} = a \in A^{\natural}$ such that $\{i : \mathfrak{A}_i \models_{\lambda} c^{\mathfrak{A}_i} = a(i)\} \in \mathcal{U}$. Relation symbols R, including $=$,[3] are interpreted, again, as having both extension and anti-extension. n-ary relation symbol R and an n-tuple $\vec{a} \in (A^{\natural})^n$, $\mathfrak{A}^{\natural} \models_{\lambda} R(\vec{a})$ iff $\{i : \vec{a}(i) \in R^{\mathfrak{A}_i +}\} \in \mathcal{U}$, or, alternately, iff $\{i : \mathfrak{A}_i \models_{\lambda} R(\vec{a})\} \in \mathcal{U}$. For the same R and $\vec{a}$, $\mathfrak{A}^{\natural} \models_{\lambda} \neg R(\vec{a})$ iff $\{i : \mathfrak{A}_i \models_{\lambda} \neg R(\vec{a})\} \in \mathcal{U}$, to include equational sentences of the form $\neg(a = b)$.

Łoś's Theorem in the classical case is the theorem that for any family of structures $\{\mathfrak{A}_i\}$, indexed by set I, and ultrafilter $\mathcal{U} \subset \wp(I)$, the following holds for all sentences φ:

$$\prod_{i \in I} \mathfrak{A}_i / \mathcal{U} \models_{\mathsf{CL}} \varphi \quad \text{iff} \quad \left\{ i : \mathfrak{A}_i \models_{\mathsf{CL}} \varphi \right\} \in \mathcal{U}.$$

Łoś's Theorem is useful classically, as controlling the properties of the ultraproduct in many cases reduces to a careful selection of the ultrafilter. In the case of the logics of $\mathfrak{Dem}$, the typical methods of constructing new models have the limitation of only either ensuring that some class of formulas are satisfied or preventing some class of formulas from being satisfied. The theorem in this context carries the benefit of not only determining which formulas are found in $\mathrm{Th}^{\lambda}(\mathfrak{A}^{\natural})$, but also determining which formulas are *not* in the theory. The present task, then, is to demonstrate that the theorem extends to the logics currently in question.

Theorem 2.1 *For any class of structures $\{\mathfrak{A}_i\}$ permissible with respect to a logic $\lambda \in \mathfrak{Dem}$, index I, and ultrafilter $\mathcal{U} \subset \wp(I)$, $\prod_{i \in I} \mathfrak{A}_i / \mathcal{U} \models_{\lambda} \varphi$ if and only if $\{i : \mathfrak{A}_i \models_{\lambda} \varphi\} \in \mathcal{U}$ for the $\rightarrow$-free fragment of $\mathcal{L}$, that is, those formulas $\varphi \in \mathcal{L}$ that contain no occurrences of the symbol $\rightarrow$.*

Proof A brief sketch: taking the literals as basis step, we proceed inductively by first showing the result holds for connectives $\vee$ and $\wedge$ and then demonstrating its holding in the case of the quantifiers. We then provide an argument for the theorem holding for $\neg$ by cases, in essence, running through the negations of formulas of these forms. We then merely define $\rightarrow$ by means of the previous connectives for the logics $\mathsf{FDE}, \mathsf{K}_3, \mathsf{LP}$, and CL. As the truth function associated with the connective $\rightarrow$ is not definable in terms of these connectives in the case of $\mathsf{Ł}_3$ or RM_3, we'll have to treat these logics separately.

Call $\{i : \mathfrak{A}_i \models_{\lambda} \varphi\}$ the *Boolean extension* of φ (hereafter $\|\varphi\|$) and assume that $\|\varphi\| \in \mathcal{U}$ and $\|\psi\| \in \mathcal{U}$. Then by the finite intersection property, or *fip*, $\|\varphi\| \cap \|\psi\| \in \mathcal{U}$. A cursory glance at Figure 1 reveals that $\|\varphi\| \cap \|\psi\| = \|\varphi \wedge \psi\|$, and so $\|\varphi \wedge \psi\| \in \mathcal{U}$. Similarly, assume that $\|\varphi \wedge \psi\| \in \mathcal{U}$; both $\|\varphi \wedge \psi\| \subseteq \|\varphi\|$ and $\|\varphi \wedge \psi\| \subseteq \|\psi\|$ hold. Since $\mathcal{U}$ is closed under supersets, it follows that $\|\varphi\| \in \mathcal{U}$ and $\|\psi\| \in \mathcal{U}$. Supposing that Łoś's Theorem holds for φ and ψ, then we may see that $\mathfrak{A}^{\natural} \models_{\lambda} \varphi \wedge \psi$ if and only if $\mathfrak{A}^{\natural} \models_{\lambda} \varphi$ and $\mathfrak{A}^{\natural} \models_{\lambda} \psi$ if and only if $\|\varphi\| \in \mathcal{U}$ and $\|\psi\| \in \mathcal{U}$ if and only if $\|\varphi \wedge \psi\| \in \mathcal{U}$.

Now we demonstrate that this holds for disjunction as well. Assume that either $\|\varphi\| \in \mathcal{U}$ *or* $\|\psi\| \in \mathcal{U}$. We know that $\|\varphi\| \subseteq \|\varphi\| \cup \|\psi\|$ and $\|\psi\| \subseteq \|\varphi\| \cup \|\psi\|$, so as $\mathcal{U}$ is closed under supersets in either case $\|\varphi\| \cup \|\psi\| \in \mathcal{U}$. Finally, $\|\varphi\| \cup \|\psi\| = \|\varphi \vee \psi\|$ so the latter is also a member of the ultrafilter. Now, assume that $\|\varphi \vee \psi\| \in \mathcal{U}$; this is equivalent to the hypothesis that $\|\varphi\| \cup \|\psi\| \in \mathcal{U}$. Now, either $\|\varphi\| \in \mathcal{U}$ or $\|\varphi\| \notin \mathcal{U}$. If the former holds, we've established that $\|\varphi \vee \psi\| \in \mathcal{U}$ implies that $\|\varphi\| \in \mathcal{U}$. If the latter holds, then by maximality of $\mathcal{U}$, $I \smallsetminus \|\varphi\| \in \mathcal{U}$. By the finite intersection property and the hypothesis, then, $(\|\varphi\| \cup \|\psi\|) \cap (I \smallsetminus \|\varphi\| \in \mathcal{U})$, which, by distributivity, entails that $(\|\varphi\| \cap (I \smallsetminus \|\varphi\|)) \cup (\|\psi\| \cap (I \smallsetminus \|\varphi\|)) \in \mathcal{U}$, which is equivalent to $\|\psi\| \cap (I \smallsetminus \|\varphi\|) \in \mathcal{U}$. Of course, $\|\psi\| \cap (I \smallsetminus \|\varphi\|) \subseteq \|\psi\|$ and by the upward closure of $\mathcal{U}$, $\|\psi\| \in \mathcal{U}$. Hence, if $\|\varphi \vee \psi\| \in \mathcal{U}$, then either $\|\varphi\| \in \mathcal{U}$ or $\|\psi\| \in \mathcal{U}$. Again, if we assume Łoś's Theorem holds for φ and ψ, then it follows that $\mathfrak{A}^{\natural} \models_{\lambda} \varphi \vee \psi$ if and only if $\mathfrak{A}^{\natural} \models_{\lambda} \varphi$ or $\mathfrak{A}^{\natural} \models_{\lambda} \psi$ if and only if $\|\varphi\| \in \mathcal{U}$ or $\|\psi\| \in \mathcal{U}$ if and only if $\|\varphi \vee \psi\| \in \mathcal{U}$.

Suppose that $\|\exists x \varphi(x)\| \in \mathcal{U}$. Then for each $j \in \|\exists x \varphi(x)\|$, $\mathfrak{A}_j \models_{\lambda} \varphi(a^{\mathfrak{A}_j})$ for some element $a^{\mathfrak{A}_j} \in A_j$. Let $b \in \prod A_i$ be such that $b \restriction \|\exists x \varphi(x)\|$ maps i to a witness of φ in A_i, and allow the value to be arbitrary otherwise. Then $\|\exists x \varphi(x)\| = \|\varphi(b(i))\|$, and hence the latter is likewise in $\mathcal{U}$. Likewise, if, for some

$b' \in \prod A_i$, $\|\varphi(b'(i))\| \in \mathcal{U}$, we note that as at any i such that $\mathfrak{A}_i \models_{\lambda} \varphi(b'(i))$ it follows that $\mathfrak{A}_i \models_{\lambda} \exists x\varphi(x)$ and hence $\|\exists x\varphi(x)\| \subseteq \|\varphi(b'(i))\|$, ensuring that the latter, by upward closure of $\mathcal{U}$, is likewise in the ultrafilter. Again, if the theorem holds for $\varphi(x)$, then $\mathfrak{A}^{\natural} \models_{\lambda} \exists x\varphi(x)$ if and only if there is a $b \in A^{\natural}$ such that $\mathfrak{A}^{\natural} \models_{\lambda} \varphi(b)$ if and only if $\|\varphi(b(i))\| \in \mathcal{U}$ if and only if $\|\exists x\varphi(x)\| \in \mathcal{U}$. An analogous argument provides the result for universally quantified formulas.

Finally, we look at negation by an argument by cases. For a formula $\neg\varphi$, φ is either a negation, a conjunction, a disjunction, or a quantified formula. In the former case, if $\varphi = \neg\psi$ for some ψ, then we note that $\mathfrak{B} \models_{\lambda} \neg\neg\psi$ if and only if $\mathfrak{B} \models_{\lambda} \psi$. Hence $\|\neg\neg\psi\| \in \mathcal{U}$ if and only if $\|\psi\| \in \mathcal{U}$. Thus $\mathfrak{A}^{\natural} \models_{\lambda} \neg\neg\psi$ if and only if $\mathfrak{A}^{\natural} \models_{\lambda} \psi$ if and only if $\|\psi\| \in \mathcal{U}$ if and only if $\|\neg\neg\psi\| \in \mathcal{U}$.

In the cases of connectives $\vee$, $\wedge$, we appeal to the fact that DeMorgan's Laws hold in each $\lambda \in \mathfrak{Dem}$. Thus, assuming that the theorem holds for all subformulas and their negations, $\|\neg(\varphi \vee \psi)\| = \|\neg\varphi \wedge \neg\psi\|$. So $\mathfrak{A}^{\natural} \models_{\lambda} \neg(\varphi \vee \psi)$ if and only if $\mathfrak{A}^{\natural} \models_{\lambda} \neg\varphi \wedge \neg\psi$ if and only if $\|\neg\varphi \wedge \neg\psi\| \in \mathcal{U}$ if and only if $\|\neg(\varphi \vee \psi)\| \in \mathcal{U}$. Analogous reasoning gives us the result for formulas $\neg(\varphi \wedge \psi)$.

Finally, we look at the case of quantified formulas. We note that quantifier interchange is valid in all $\lambda \in \mathfrak{Dem}$, and assuming the result for all formulas of lesser complexity, we note that $\mathfrak{A}^{\natural} \models_{\lambda} \neg\exists x\varphi(x)$ if and only if $\mathfrak{A}^{\natural} \models_{\lambda} \forall x\neg\varphi(x)$ if and only if $\|\forall x\neg\varphi(x)\| \in \mathcal{U}$ if and only if $\|\neg\exists x\varphi(x)\| \in \mathcal{U}$. A similar argument secures the result for negated universal quantifiers as well.

This establishes that Łoś's Theorem holds for the $\rightarrow$-free fragments of the logics in $\mathfrak{Dem}$. □

Theorem 2.2 *For any class of structures $\{\mathfrak{A}_i\}$ permissible with respect to the logic, index I, and ultrafilter $\mathcal{U} \subset \wp(I)$, $\prod_{i\in I} \mathfrak{A}_i/\mathcal{U} \models_{\mathsf{FDE,LP,K_3,CL}} \varphi$ if and only if $\{i : \mathfrak{A}_i \models_{\mathsf{FDE,LP,K_3,CL}} \varphi\} \in \mathcal{U}$ for arbitrary φ.*

Proof In the case of FDE, LP, and K3 (as well as CL), $\mathfrak{B} \models_{\mathsf{FDE,LP,K_3,CL}} \varphi \rightarrow \psi$ if and only if $\mathfrak{B} \models_{\mathsf{FDE,LP,K_3,CL}} \neg\varphi \vee \psi$, and so Łoś's Theorem can be demonstrated for formulas of this form by definition. □

The converse of Łós's Theorem states that for a reduced product $\mathfrak{A}^{\natural}$, $\mathfrak{A}^{\natural} \not\models_{\lambda} \varphi$ if and only if $\|\varphi\| \notin \mathcal{U}$, which we recall is in general a *different* claim than that $\mathfrak{A}^{\natural} \models_{\lambda} \neg\varphi$ if and only if $\|\neg\varphi\| \in \mathcal{U}$. This means that for a truth-functional connective *in virtue of its truth functionality* Łós's Theorem may yet be established. If we can define the truth function associated with a connective inductively in terms of $\models_{\lambda}$ and $\not\models_{\lambda}$, then we can inductively prove Łós's Theorem. We focus first on Ł3.

Theorem 2.3 *For any class of structures $\{\mathfrak{A}_i\}$ permissible with respect to Ł3, index I, and ultrafilter $\mathcal{U} \subset \wp(I)$, $\prod_{i\in I} \mathfrak{A}_i/\mathcal{U} \models_{\text{Ł}_3} \varphi$ if and only if $\{i : \mathfrak{A}_i \models_{\text{Ł}_3} \varphi\} \in \mathcal{U}$.*

Proof Note that given a structure $\mathfrak{A}$,

$$\begin{aligned}\mathfrak{A} \models_{\text{Ł}_3} \varphi \rightarrow \psi \quad \text{iff} \quad &\text{(a) } \mathfrak{A} \models_{\text{Ł}_3} \neg\varphi, \text{ or}\\ &\text{(b) } \mathfrak{A} \models_{\text{Ł}_3} \psi, \text{ or}\\ &\text{(c) } \mathfrak{A} \not\models_{\text{Ł}_3} \varphi, \mathfrak{A} \not\models_{\text{Ł}_3} \neg\varphi, \mathfrak{A} \not\models_{\text{Ł}_3} \psi, \text{ and } \mathfrak{A} \not\models_{\text{Ł}_3} \neg\psi.\end{aligned}$$

We first translate into the context of an ultraproduct by examining $\|\varphi \rightarrow \psi\|$, the set of indices of structures $\mathfrak{A}_i$ such that the sentence holds in $\mathfrak{A}_i$. We note that $\{i : \mathfrak{A}_i \not\models_{\text{Ł}_3} \varphi\} = I \setminus \{i : \mathfrak{A}_i \models_{\text{Ł}_3} \varphi\}$ and may translate conditions (a)–(c). More precisely, the translation of the above point is that $i \in \|\varphi \rightarrow \psi\|$ if and only if $i \in \|\neg\varphi\| \cup \|\psi\| \cup (I \setminus (\|\varphi\| \cup \|\neg\varphi\| \cup \|\psi\| \cup \|\neg\psi\|))$, and so these sets are equal.

Left-to-right, suppose that Łoś's Theorem has been shown to hold for all subformulas of $\varphi \rightarrow \psi$ and their negations and that $\mathfrak{A}^\natural \models_{\text{Ł}_3} \varphi \rightarrow \psi$. Then at least one of conditions (a)–(c) holds of $\mathfrak{A}^\natural$. Suppose that condition (a) holds; then, ex hypothesi, $\mathfrak{A}^\natural \models_{\text{Ł}_3} \neg\varphi$ implies that $\|\neg\varphi\| \in \mathcal{U}$. We note that $\|\neg\varphi\| \in \mathcal{U} \subseteq \|\varphi \rightarrow \psi\|$, and as $\mathcal{U}$ is closed under supersets, $\|\varphi \rightarrow \psi\| \in \mathcal{U}$ as well. Analogous reasoning gives a similar result for condition (b). Finally, we consider the case in which condition (c) holds; in this case, we may appeal to the contrapositive form of the theorem and the hypothesis. $\mathfrak{A}^\natural \not\models_{\text{Ł}_3} \varphi$ implies that $\|\varphi\| \notin \mathcal{U}$, $\mathfrak{A}^\natural \not\models_{\text{Ł}_3} \neg\varphi$ implies that $\|\neg\varphi\| \notin \mathcal{U}$, and so forth. Since $\mathcal{U}$ is maximal, this implies that $I \setminus \|\varphi\| \in \mathcal{U}$, $I \setminus \|\neg\varphi\| \in \mathcal{U}$, $I \setminus \|\psi\| \in \mathcal{U}$, and $I \setminus \|\neg\psi\| \in \mathcal{U}$, and by DeMorgan's laws, this implies that $I \setminus (\|\varphi\| \cup \|\neg\varphi\| \cup \|\psi\| \cup \|\neg\psi\|) \in \mathcal{U}$. Again, though, this set has been observed to be a subset of $\|\varphi \rightarrow \psi\|$, and by upward closure we deduce that $\|\varphi \rightarrow \psi\| \in \mathcal{U}$. As cases (a)–(c) exhaust the conditions under which $\varphi \rightarrow \psi$ is true in $\mathfrak{A}^\natural$, we've demonstrated the left-to-right half of the theorem.

Right-to-left, suppose that $\|\varphi \rightarrow \psi\| \in \mathcal{U}$ and that the theorem has been shown to hold for subformulas and their negations. Note again that $\|\varphi \rightarrow \psi\| = \|\neg\varphi\| \cup \|\psi\| \cup (I \setminus (\|\varphi\| \cup \|\neg\varphi\| \cup \|\psi\| \cup \|\neg\psi\|))$. As $\mathcal{U}$ is maximal, if an element is equal to a finite union of sets, then at least one of these sets is also an element of $\mathcal{U}$; hence, the hypothesis yields the result that either $\|\neg\varphi\| \in \mathcal{U}$, $\|\psi\| \in \mathcal{U}$, or $(I \setminus (\|\varphi\| \cup \|\neg\varphi\| \cup \|\psi\| \cup \|\neg\psi\|)) \in \mathcal{U}$. In the first two cases, Łoś's Theorem ensures that either $\mathfrak{A}^\natural \models_{\text{Ł}_3} \neg\varphi$ or $\mathfrak{A}^\natural \models_{\text{Ł}_3} \psi$, respectively. Both cases, of course, ensure that $\mathfrak{A}^\natural \models_{\text{Ł}_3} \varphi \rightarrow \psi$. In the latter case, we note that this is equivalent to stating that $\|\varphi\| \notin \mathcal{U}$ and $\|\neg\varphi\| \notin \mathcal{U}$ and $\|\psi\| \notin \mathcal{U}$ and $\|\neg\psi\| \notin \mathcal{U}$. Appealing once more to the holding of the contraposition of Łoś's Theorem to subformulas of $\varphi \rightarrow \psi$ and their negations, we see that this implies that $\mathfrak{A}^\natural \not\models_{\text{Ł}_3} \varphi$ and $\mathfrak{A}^\natural \not\models_{\text{Ł}_3} \neg\varphi$ and $\mathfrak{A}^\natural \not\models_{\text{Ł}_3} \psi$ and $\mathfrak{A}^\natural \not\models_{\text{Ł}_3} \neg\psi$, satisfying condition (c), which is sufficient to establish that $\mathfrak{A}^\natural \models_{\text{Ł}_3} \varphi \rightarrow \psi$.

Finally, in the additional case for negation, note that $\mathfrak{A}^\natural \models_{\text{Ł}_3} \neg(\varphi \rightarrow \psi)$ if and only if $\mathfrak{A}^\natural \models_{\text{Ł}_3} \varphi$ and $\mathfrak{A}^\natural \models_{\text{Ł}_3} \neg\psi$; that is, $\|\neg(\varphi \rightarrow \psi)\| = \|\varphi\| \cap \|\neg\psi\|$. Thus, assuming that Łoś's Theorem holds for all formulas of lesser complexity, $\mathfrak{A}^\natural \models_{\text{Ł}_3} \neg(\varphi \rightarrow \psi)$ if and only if $\mathfrak{A}^\natural \models_{\text{Ł}_3} \varphi$ and $\mathfrak{A}^\natural \models_{\text{Ł}_3} \neg\psi$ if and only if $\|\varphi\| \in \mathcal{U}$ and $\|\neg\psi\| \in \mathcal{U}$. As $\mathcal{U}$ has the fip and is maximal, this is equivalent to stating that $\|\varphi\| \cap \|\neg\psi\| \in \mathcal{U}$, which we've established is equivalent to stating that $\|\neg(\varphi \rightarrow \psi)\| \in \mathcal{U}$. This completes the cases for negation, and hence the induction for Łoś's Theorem for Ł3. □

Theorem 2.4 *For any class of structures* $\{\mathfrak{A}_i\}$ *permissible with respect to* RM_3*, index* I*, and ultrafilter* $\mathcal{U} \subset \wp(I)$*,* $\prod_{i \in I} \mathfrak{A}_i / \mathcal{U} \models_{\mathsf{RM}_3} \varphi$ *if and only if* $\{i : \mathfrak{A}_i \models_{\mathsf{RM}_3} \varphi\} \in \mathcal{U}$.

Proof We make the following observations about the interpretation of the logical connective $\rightarrow$ in RM_3:

$$\begin{array}{lll} \mathfrak{A} \models_{\mathsf{RM}_3} \varphi \rightarrow \psi & \text{iff} & \text{(a) } \mathfrak{A} \not\models_{\mathsf{RM}_3} \varphi, \text{ or} \\ & & \text{(b) } \mathfrak{A} \not\models_{\mathsf{RM}_3} \neg\psi, \text{ or} \\ & & \text{(c) } \mathfrak{A} \models_{\mathsf{RM}_3} \varphi, \mathfrak{A} \models_{\mathsf{RM}_3} \neg\varphi, \mathfrak{A} \models_{\mathsf{RM}_3} \psi, \text{ and } \mathfrak{A} \models_{\mathsf{RM}_3} \neg\psi. \end{array}$$

Translating, this implies that $i \in \|\varphi \rightarrow \psi\|$ if and only if $i \in (I \setminus \|\varphi\|) \cup (I \setminus \|\neg\psi\|) \cup (\|\varphi\| \cap \|\neg\varphi\| \cap \|\psi\| \cap \|\neg\psi\|)$.

Left-to-right, we assume that the theorem has been established for formulas of lesser complexity and that $\mathfrak{A}^\natural \models_{\mathsf{RM}_3} \varphi \rightarrow \psi$. Then at least one of conditions (a)–(c) holds. In the cases of (a) and (b), ex hypothesi, $\|\varphi\| \notin \mathcal{U}$ or $\|\neg\psi\| \notin \mathcal{U}$, respectively, and by maximality, $I \setminus \|\varphi\| \in \mathcal{U}$ or $I \setminus \|\neg\psi\| \in \mathcal{U}$. Both these sets are subsets of $\|\varphi \rightarrow \psi\|$, and hence $\|\varphi \rightarrow \psi\| \in \mathcal{U}$ by upward closure. In case (c), φ and ψ are both true and false in $\mathfrak{A}^\natural$, which tells us that $\|\varphi\|$, $\|\neg\varphi\|$, $\|\psi\|$, $\|\neg\psi\|$ are all members of $\mathcal{U}$. By the fip, their intersection is also in $\mathcal{U}$, and as this is a subset of $\|\varphi \rightarrow \psi\|$, by upward closure, so too is it a member of $\mathcal{U}$. Right-to-left follows a similar adaptation of the Ł_3 case.

For the case of negation, note that $\mathfrak{A}^\natural \models_{\mathsf{RM}_3} \neg(\varphi \rightarrow \psi)$ if and only if $\mathfrak{A}^\natural \models_{\mathsf{RM}_3} \varphi$ and $\mathfrak{A}^\natural \not\models_{\mathsf{RM}_3} \psi$. Supposing that the theorem holds for subformulas and their negations, we infer that $\mathfrak{A}^\natural \models_{\mathsf{RM}_3} \neg(\varphi \rightarrow \psi)$ if and only if $\mathfrak{A}^\natural \models_{\mathsf{RM}_3} \varphi$ and $\mathfrak{A}^\natural \not\models_{\mathsf{RM}_3} \psi$, if and only if , in turn, $\|\varphi\| \cap (I \setminus \|\psi\|) \in \mathcal{U}$. But this set is equivalent to $\|\neg(\varphi \rightarrow \psi)\|$, and hence the foregoing is equivalent to the claim that the Boolean extension of the formula is in $\mathcal{U}$. Thus the case of $\rightarrow$ and its negation are covered, completing the induction. □

As an application, we may use Łoś's Theorem to demonstrate that in any of these logics, the model-theoretic properties of *inconsistency* and *incompleteness* are not general first-order; that is, the class of inconsistent structures is not axiomatizable in a first-order language. In a finite signature, of course, inconsistency is first-order; for finitely many relation symbols P_i (indexed by a finite set I), the sentence $\sigma_{\lightning} = \bigvee_{i \in I} \exists \vec{x}_i (P_i(\vec{x}_i) \wedge \neg P_i(\vec{x}_i))$, where $\vec{x}_i$ and P_i are of identical arity, $\mathfrak{A} \models_{\mathsf{LP}} \sigma_{\lightning}$ if and only if $\mathfrak{A}$ is inconsistent. When moving to a signature of cardinality $\kappa \geq \aleph_0$, however, such a $\sigma_{\lightning}$ is not well-formed, as it will have κ-many disjuncts. This does not, however, tell that no such $\sigma_{\lightning}$ exists; such a sentence for each signature may indeed exist, though it would be a consequence of such a property. Łoś's Theorem, however, speaks against the existence of any such sentence, or set of sentences.

Theorem 2.5 *The structural property of being inconsistent in an infinite signature is not general first-order; that is, there is no sentence* $\sigma_{\lightning}$ *that axiomatizes the class of inconsistent structures, nor is there an infinite set of sentences that does so.*

Proof We may take a family of inconsistent structures $\{\mathfrak{A}_i : i \in \kappa\}$ with infinite signature $\sigma = (A, \{P_j : j \in \kappa\})$ with $A = \{a\}$ such that the extension of P_j in a model $\mathfrak{A}_i$ is $P_j^{\mathfrak{A}_i+} = \{a^{\mathfrak{A}_i}\}$ if $i = j$ and $P_j^{\mathfrak{A}_i+} = \varnothing$ otherwise, and the *anti*-extension P_j in a model $\mathfrak{A}_i$ is $P_j^{\mathfrak{A}_i-} = \{a^{\mathfrak{A}_i}\}$ for all i, j.

Now it is immediate that each structure is inconsistent; in general, $\mathfrak{A}_i \models_{\mathsf{FDE,LP,RM_3}} \exists x(P_i x \wedge \neg P_i x)$. Suppose that there exists a first-order sentence $\sigma_{\lightning}$ that axiomatizes the class of inconsistent structures. Just as in the canonical proof that the property of a field's having finite characteristic is not first-order, one can make use of Łós's Theorem to demonstrate that $\sigma_{\lightning}$ is not general first-order. We first consider the reduced product $\mathfrak{A}^{\natural} = \prod_{i\in\kappa} \mathfrak{A}_i/\mathcal{U}$, where $\mathcal{U}$ is nonprincipal. Noting that $\prod_{1\in\kappa} A_i$ is a singleton, it follows that the domain $A^{\natural} = \{\lambda i.a^{\mathfrak{A}_i}\}$, that is, the function mapping each index i to the element $a \in A_i$.

It is clear that this structure is first-order consistent. Consider the diagram: for no relation symbol P_j are both $P_j^{\mathfrak{A}^{\natural}}$ and $\neg P_j^{\mathfrak{A}^{\natural}}$ satisfied. By Łós's Theorem, $\mathfrak{A}^{\natural} \models_{\mathsf{FDE,LP,RM_3}} P_j(\lambda i.a^{\mathfrak{A}_i})$ if and only if $\|P_j(\lambda i.a^{\mathfrak{A}_i}(i))\| \in \mathcal{U}$. But for any candidate P_j, the set of structures that make true this formula is either empty or a singleton; both are precluded from inclusion in $\mathcal{U}$. Thus although $\|\neg P_j(a(i))\|$ is always κ, and hence a member of $\mathcal{U}$, a contradiction between two atomic formulas is true at only a singleton in the power set. Furthermore, any inconsistent formula φ constructed from such contradictions is finite in length, and as such $\|\varphi\|$ is finite and hence not contained in $\mathcal{U}$. As the theory is determined by the diagram, that the diagram is consistent ensures that the theory of the structure is (a) classical and (b) nontrivial.

That the theory is first-order consistent means that $\prod \mathfrak{A}_i/\mathcal{U} \not\models_{\mathsf{FDE,LP,RM_3}} \sigma_{\lightning}$. But by hypothesis, all $\mathfrak{A}_i \models_{\mathsf{FDE,LP,RM_3}} \sigma_{\lightning}$, which Łós's Theorem tells us is impossible. □

Analogous reasoning over a similarly artificial set of incomplete structures yields that there is no first-order sentence σ_{Inc} that holds of all structures with incomplete theories.

Theorem 2.6 *The structural property of being incomplete in an infinite signature is not general first-order, that is, there is no sentence* σ_{Inc} *that axiomatizes the class of incomplete structures, nor is there an infinite set of sentences that does so.*

Proof Consider the family $\{\mathfrak{B}_i : i \in \omega\}$ such that $P_j^{\mathfrak{B}_i+} = \varnothing$ for all i, j and $P_j^{\mathfrak{B}_i-} = \varnothing$ if $i = j$ and $P_j^{\mathfrak{B}_i-} = \{b^{\mathfrak{B}_i}\}$ otherwise. By slightly amending the argument, it follows that $\mathfrak{B}^{\natural} \not\models_{\mathsf{FDE,K_3,Ł_3}} \sigma_{\mathrm{Inc}}$ and hence the property of a structure's having a complete theory is not first-order. □

As a further result, we may apply a simple, model-theoretic proof of compactness for the logics $\lambda \in \mathfrak{Dem}$, due to Malcev, desirable as no reference to syntax is required. We refer the reader to the elegant presentation of Malcev's proof in Rothmaler [10] and note that the proof immediately applies to all $\lambda \in \mathfrak{Dem}$ without any generalization. We can now move on to make some comments about categoricity in the context of $\mathfrak{Dem}$.

3 Categoricity and Cantor's Theorem

In this section we wish to explore the general case of Cantor's Theorem and make some notes about categoricity with respect to logics in $\mathfrak{Dem}$.

Theorem 3.1 *For any language $\mathcal{L}$, every set of $\mathcal{L}$-sentences (to include $\mathcal{L}$ itself) has an* LP*-model (alternately,* RM_3*-model,* FDE*-model).*

Proof Consider a structure in the signature of $\mathcal{L}$, $\mathfrak{A}^{\mathcal{L}}$, in which $A^{\mathcal{L}}$ is a singleton $\{a\}$ and for all c, $c^{\mathfrak{A}^{\mathcal{L}}} = a$, for all f, $f^{\mathfrak{A}^{\mathcal{L}}}(\vec{a}) = a$ and for all R, $R^{\mathfrak{A}^{\mathcal{L}}+} = R^{\mathfrak{A}^{\mathcal{L}}-} = A^{\mathcal{L}}$. We proceed by induction on complexity of formulas that $\mathfrak{A}^{\mathcal{L}} \models_{\mathsf{LP},\mathsf{RM}_3,\mathsf{FDE}} \mathcal{L}$.

We use as the base case literals (equational formulas, atoms, and their negations) and immediately see that all literals in $\mathcal{L}$ are true in $\mathfrak{A}^{\mathcal{L}}$ (as well as false). The values of all constants and all functions denote a, and both $\mathfrak{A}^{\mathcal{L}} \models_{\mathsf{LP},\mathsf{RM}_3,\mathsf{FDE}} a = a$ and $\mathfrak{A}^{\mathcal{L}} \models_{\mathsf{LP},\mathsf{RM}_3,\mathsf{FDE}} a \neq a$; hence, all equational formulas are both true and false. Similarly, for any term t, $\mathfrak{A}^{\mathcal{L}} \models_{\mathsf{LP},\mathsf{RM}_3,\mathsf{FDE}} R(t)$ and $\mathfrak{A}^{\mathcal{L}} \models_{\mathsf{LP},\mathsf{RM}_3,\mathsf{FDE}} \neg R(t)$, and so all literals are both true and false.

For connectives, if φ, ψ are both true and false, then by consulting Figures 1 and 2 we see that $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, and $\varphi \rightarrow \psi$ are likewise both true and false. Similarly, appealing to the interpretation of the quantifiers, if $\varphi(\vec{a})$ is both true and false, then $\forall\vec{x}\varphi(\vec{x})$ and $\exists\vec{x}\varphi(\vec{x})$ are both true and false as well.

This procedure exhausts $\mathcal{L}$ and hence we reason that $\mathfrak{A}^{\mathcal{L}} \models_{\mathsf{LP},\mathsf{RM}_3,\mathsf{FDE}} \mathcal{L}$. □

By compactness, the foregoing gives the result that every set of sentences has a model in these logics. This does not say, of course, that every set of sentences has a model in which all and only those sentences is true.[4] $\mathfrak{A}^{\mathcal{L}}$ is a peculiar beast.

Theorem 3.2 *For a language $\mathcal{L}$, $\mathfrak{A}^{\mathcal{L}}$ is up to isomorphism the unique model of $\mathcal{L}$.*

Proof We consider first the universe $A^{\mathcal{L}}$. Since $\mathfrak{A}^{\mathcal{L}} \models_{\mathsf{LP},\mathsf{RM}_3,\mathsf{FDE}} \forall x, y[x = y]$, then by the truth conditions for equational formulas we see that $A^{\mathcal{L}}$ is a singleton. Hence the only function from one model of $\mathcal{L}$ to another is one-to-one. Interpretations of constants and the value of any argument of the interpretation of function symbols must be that element of the domain, and as all n-ary relations are both true and false of that element (or the n-tuple of that element), their extensions and anti-extensions will be identical. □

We can look at these results to examine the plight of Cantor's Theorem that DLO_{--} is $\aleph_0$-categorical. [3] provides an explicit construction that demonstrates that the result does not hold for RM_3 and Priest's Collapsing Lemma may be appealed to in order to provide an explicit construction for which Cantor fails in LP (and hence FDE as well). As we'll require the Collapsing Lemma shortly, we'll briefly give an example of the applicability in the case of Cantor.

Given a consistent structure $\mathfrak{A}$, we may define a congruence relation $\sim$ on A such that for any n-ary $f^{\mathfrak{A}}$ and $\vec{a}, \vec{b} \in A^n$, if $\vec{a} \sim \vec{b}$, then $f^{\mathfrak{A}}(\vec{a}) \sim f^{\mathfrak{A}}(\vec{b})$. We then partition A into $A^{\sim}$, consisting of the classes $[a] = \{b \in A : b \sim a\}$ and define interpretations of constants and function symbols in the following way:

$c^{\mathfrak{A}^\sim} = [a]$ such that $c^{\mathfrak{A}} \in [a]$ and $f^{\mathfrak{A}^\sim}([a_0], \ldots, [a_n]) = [a_{n+1}]$ if and only if for some $b_0 \in [a_0], \ldots, b_{n+1} \in [a_{n+1}]$, $f^{\mathfrak{A}}(b_0, \ldots, b_n) = b_{n+1}$. Furthermore, we may interpret each m-ary relation symbol R so that for its extension, $([a_0], \ldots, [a_m]) \in R^{\mathfrak{A}^\sim +}$ if and only if for some $b_0 \in [a_0], \ldots, b_m \in [a_m]$, $b_0, \ldots, b_m \in R^{\mathfrak{A}+}$, and for its anti-extension, $([a_0], \ldots, [a_m]) \in R^{\mathfrak{A}^\sim -}$ if and only if for some $b_0 \in [a_0], \ldots, b_m \in [a_m]$, $b_0, \ldots, b_m \in R^{\mathfrak{A}-}$. Collecting these interpretations together, we define $\mathfrak{A}^\sim = (A^\sim, \mathbf{C}^{\mathfrak{A}^\sim}, \mathbf{F}^{\mathfrak{A}^\sim}, \mathbf{R}^{\mathfrak{A}^\sim +}, \mathbf{R}^{\mathfrak{A}^\sim -})$ and call it the *collapse* of $\mathfrak{A}$ modulo $\sim$.

Theorem 3.3 (Collapsing Lemma) $\mathrm{Th}^{\mathsf{LP}}(\mathfrak{A}^\sim) \supseteq \mathrm{Th}^{\mathsf{CL}}(\mathfrak{A})$.

Proof We refer the reader to [6]. □

Corollary 3.4 *The classical theory* DLO_{--} *is not* $\aleph_0$*-categorical with respect to the class of* LP*-structures, nor is it categorical in any cardinality.*

Proof We take the classical model of DLO_{--} and produce two structures, $\mathbb{Q}^{\sim 1}$ and $\mathbb{Q}^{\sim 2}$ such that $|\mathbb{Q}^{\sim 1}| = |\mathbb{Q}^{\sim 2}| = \aleph_0$ and $\mathbb{Q}^{\sim 1}, \mathbb{Q}^{\sim 2} \models_{\mathsf{LP}} \mathsf{DLO}_{--}$ but $\mathbb{Q}^{\sim 1} \not\cong \mathbb{Q}^{\sim 2}$. Consider the classical set of linearly ordered rationals $(\mathbb{Q}, <)$. Consider two intervals defined by parameters $(a, b), (c, d) \subset \mathbb{Q}$ such that $\mathbb{Q} \models_{\mathsf{CL}} b <^{\mathbb{Q}} c$. We then define two equivalence relations, $\sim^1$ and $\sim^2$ such that for $e, f \in \mathbb{Q}$, $e \sim^1 f$ if and only if $e = f$ or $e, f \in (a, b)$, and $e \sim^2 f$ if and only if $e \sim^1 f$ or $e, f \in (c, d)$. We then consider the collapsed structures $\mathbb{Q}^{\sim 1}$ and $\mathbb{Q}^{\sim 2}$. As both are gotten through congruence relations on a countable structure, they are at most countably infinite, and as $[b, c]$ is a proper subset of each, they are at least countably infinite. Furthermore, by the Collapsing Lemma, each structure makes true $\mathrm{Th}^{\mathsf{CL}}((\mathbb{Q}, <))$, and hence they both model DLO_{--}.

Now suppose that there is an isomorphism $h : \mathbb{Q}^{\sim 2} \cong \mathbb{Q}^{\sim 1}$. We note that $(a, b)^{\sim 2}$ and $(c, d)^{\sim 2}$ each are single, discrete elements in the former—that is, there are no elements between $a^{\sim 2}$, $(a, b)^{\sim 2}$, and $b^{\sim 2}$, and likewise for $(c, d)^{\sim 2}$—and that the image of h under each would likewise have to pick out a discrete element in the latter. But there is only one such point in $\mathbb{Q}^{\sim 1}$ that each could be mapped to, and hence $h((a, b)^{\sim 2}) = h((c, d)^{\sim 2}) = (a, b)^{\sim 1}$. But as h is bijective, this would imply that $\mathbb{Q}^{\sim 2} \models_{\mathsf{LP}} (a, b)^{\sim 2} = (c, d)^{\sim 2}$, which in fact fails.

As DLO_{--} is not classically κ-categorical for any uncountable κ, the classical witnesses of the failure of categoricity in each such cardinality, as they are permissible for LP, serve to generalize this result for uncountable cardinalities in LP. □

We can also examine the fate of Cantor's theorem in the logics K_3 and Ł$_3$. We first establish a result about categoricity of classical theories with respect to these logics.

Theorem 3.5 *If some theory T classically is categorical in some cardinal κ and has no finite models, then it is κ-categorical in both* K_3 *and* Ł$_3$.

Proof Consider such a T. By the Łoś-Tarski test, it is a complete theory, so for every model of T $\mathfrak{A}$, n-ary relation symbol R, and n-tuple $\vec{a} \in A^n$, $\vec{a} \in R^{\mathfrak{A}+} \cup R^{\mathfrak{A}-}$. It follows that the only K_3 and Ł$_3$ models of T are the classical, consistent ones. But ex hypothesi, T was classically κ-categorical, and so any two such structures of cardinality κ will be isomorphic. □

From this, we may observe the following.

Corollary 3.6 *The classical theory* DLO_{--} *is* $\aleph_0$*-categorical with respect to the class of* K_3*- and* $\mathsf{Ł}_3$*-structures.*

Proof Immediate from the theorem. □

4 Some Commutative Properties of Ultraproducts

We now outline a failed strategy to weigh in on an open problem of Priest [7] and left open by Paris and Pathmanathan [4]—whether every countably infinite LP-model of PA is the collapse of a consistent model of arithmetic, or an elementary substructure thereof. A negative answer to this problem was initially the target, motivating the investigation of Łoś's Theorem. The strategy is modestly outlined; the structures it generates may be interesting, even if they do not solve the problem. We then go into more detail about *why* the strategy fails, as its failure is due to model theoretic theorems interesting in their own right.

Priest [6] introduces inconsistent, finite models of arithmetic, with which we shall here concern ourselves. The so-called cycle models make true all sentences of Peano Arithmetic PA though, of course, it may likewise make the *negations* of some sentences $\varphi \in \mathsf{PA}$ as well. These models are gotten by the Collapsing Lemma, generated by means of congruence relations $\sim^{n,p}$ for natural numbers n, p. We partition the set $\mathbb{N}$, the universe of the structure $(\mathbb{N}, S^{\mathbb{N}}, +^{\mathbb{N}}, \times^{\mathbb{N}}, <^{\mathbb{N}+}, <^{\mathbb{N}-})$, modulo $\sim^{n,p}$ by claiming that for $a, b \in \mathbb{N}$, $a \sim^{n,p} b$ if and only if both $a, b < n$ and $a = b$ or both $a, b \geq n$ and $a \equiv_p b$. We shall hereafter refer to the structure $\mathbb{N}^{\sim^{n,p}}$ as $\mathfrak{A}^n_p$ for some n, p, as the general composition of these structures is bipartite: an initial segment of (consistent) elements of length n, followed by a single cycle of period p.

We briefly describe a structure $\prod_{i\in\omega} \mathfrak{A}^n_i/\mathcal{U}$, where $\mathcal{U}$ is a nonprincipal ultrafilter on $\wp(\omega)$. Such a structure looks like a single "tag-end" of length n, extended by an ω^*-block on one end and an ω-block on the other. Beyond the limits of each end of this block lies an undifferentiated "sea" of further ζ-blocks of nonstandard elements; these blocks are not meaningfully orderable, as any element of any particular block is both greater than and less than the elements of every other block. It is most convenient to think of such as structure as a densely ordered cycle of $\mathfrak{c}$-many ζ-blocks, but these blocks may just as well be interwoven among each other, or stacked atop one another, or worse.

The conjecture forwarded in earlier drafts of this paper was that ultraproducts of such structures could be generated that were not the collapse of any classical model of PA. It is not clear that for any element a in a classical nonstandard model ${}^*\mathbb{N}$ that such a $\prod_{i\in\omega} \mathfrak{A}^n_i/\mathcal{U}$ is the collapse of ${}^*\mathbb{N}$ modulo $\sim^{1,a}$. But this isn't to say that there is no such collapse; Theorem 4.1 shows that there is always such a collapse, albeit not a simple one.

Theorem 4.1 *For any collection of collapsed* LP*-models* $\{\mathfrak{A}_i^{\sim_i}\}$ *indexed by a set* I *and an ultrafilter* $\mathcal{U} \subset \wp(I)$*, there exists a collapse* $\sim_I$ *such that* $\prod_{i\in I} \mathfrak{A}_i^{\sim_i}/\mathcal{U} \cong (\prod_{i\in I} \mathfrak{A}_i/\mathcal{U})^{\sim_I}$*, that is, collapsing and taking ultrapowers commute.*

Proof We continue to denote $\prod_{i\in I} \mathfrak{A}_i^{\sim_i}/\mathcal{U}$ by $\mathfrak{A}^{\natural}$, while denoting $\prod_{i\in I} \mathfrak{A}_i/\mathcal{U}$ by $\mathfrak{A}^{\flat}$ and $(\prod_{i\in I} \mathfrak{A}_i/\mathcal{U})^{\sim_I}$ by $\mathfrak{A}^{\flat\sim}$. A few remarks about notation: when helpful, a subscript will be placed by an element a, for example, $a_{\sim_{\mathcal{U}}}$, to reinforce that a is an equivalence class modulo that relation. More often than not, the domain from which the element is drawn will provide the context and such subscripts will be suppressed. Furthermore, our abbreviations for the ultraproducts omit mention of the ultrafilter whence they are constructed. It is important to bear in mind that $\mathcal{U}$ is taken to be common to all structures; by this we can transport facts about one into the other. Finally, when dealing with an n-tuple of elements, we use $\vec{a} \in \vec{b}$ to mean that for all $j < n, a(j) \in b(j)$.

We define $\sim_I$ by claiming that for two $a, b \in A^\flat$, $a \sim_I b$ if for some $a' \in a$ and $b' \in b$, $\|a'(i) \sim_i b'(i)\| \in \mathcal{U}$. We must first demonstrate that this is an equivalence relation. To demonstrate reflexivity, we note that ex hypothesi, $\sim_i$ is a congruence relation for all $i \in I$. This being the case, $\|a'(i) \sim_i a'(i)\| = I$ and is hence a member of $\mathcal{U}$ for any a. To demonstrate transitivity, we suppose that $a \sim_I b$ and $b \sim_I c$, and hence that for some $a' \in a$, $b' \in b$, and $c' \in c$ both $\|a'(i) \sim_i b'\| \in \mathcal{U}$ and $\|b'(i) \sim_i c'(i)\| \in \mathcal{U}$. Since $\sim_i$ is assumed to be transitive, at every i in the intersection of these sets $a'(i) \sim_i c'(i)$ holds. Hence $\|a'(i) \sim_i b'\| \cap \|b'(i) \sim_i c'(i)\| \subseteq \|a'(i) \sim_i c'(i)\|$. By the fip, $\|a'(i) \sim_i b'\| \cap \|b'(i) \sim_i c'(i)\| \in \mathcal{U}$, and by upward closure of $\mathcal{U}$, $\|a'(i) \sim_i c'(i)\| \in \mathcal{U}$, and thus $a \sim_I c$. Finally, to demonstrate symmetry, we merely note that as all $\sim_i$ are symmetric, $\|a'(i) \sim_i b'(i)\| = \|b'(i) \sim_i a'(i)\|$, and hence $a \sim_I b$ implies $b \sim_I a$.

More tricky is that $\sim_I$ is a congruence relation, that is, that if for an n-ary function symbol f and n-tuples $\vec{a}, \vec{b} \in (A^\flat)^n$, $\vec{a} \sim_I \vec{b}$ implies $f^{\mathfrak{A}^\flat}(\vec{a}) \sim_I f^{\mathfrak{A}^\flat}(\vec{b})$. By assumption for all $j < n$ $\|a_j(i) \sim b_j(i)\| \in \mathcal{U}$. By the finiteness of n and the fip, $\cap_{j<n}\|a_j(i) \sim b_j(i)\| \in \mathcal{U}$ as well. Since all $\sim_i$ are congruence relations, $\cap_{j<n}\|a_j(i) \sim b_j(i)\| \subseteq \|f^{\mathfrak{A}_i}(\vec{a}(i)) \sim_i f^{\mathfrak{A}_i}(\vec{b}(i))\|$, and by upward closure, the latter is a member of $\mathcal{U}$. But $f^{\mathfrak{A}^\flat}(\vec{a}) = \{c : \|f^{\mathfrak{A}_i}(\vec{a}(i)) = c\| \in \mathcal{U}\}$, similarly for $f^{\mathfrak{A}^\flat}(\vec{b})$, and so this is just to say that a representative from each class are equivalent modulo $\sim_i$ at almost all i's, that is, $f^{\mathfrak{A}^\flat}(\vec{a}) \sim_I f^{\mathfrak{A}^\flat}(\vec{b})$.

We submit as candidate isomorphism the function h that maps $a \in \mathfrak{A}^\natural$ to the $b_{\sim I} \in \mathfrak{A}^{\flat\sim}$ such that there exists a $b_{\sim\mathcal{U}} \in b_{\sim\mathcal{I}}$, a $b' \in b_{\sim\mathcal{U}}$ and an $a' \in a_{\sim\mathcal{U}}$ such that $\|b'(i) \in a'(i)\| \in \mathcal{U}$.

To prove injectivity of h, suppose that $h(a) = h(b)$. Then there is an $a' \in a$ and a $b' \in b$ and element $c \in \prod_{i \in I} A_i$ such that $\|c(i) \in a'(i)\| \in \mathcal{U}$ and $\|c(i) \in b'(i)\| \in \mathcal{U}$. This implies that $\|c(i) \in a'(i)\| \cap \|c(i) \in b'(i)\| \in \mathcal{U}$, and hence that $a'(i)$ and $b'(i)$ share a member at almost all indices. We recognize that $a'(i)$ and $b'(i)$ denote equivalence classes modulo $\sim_i$ and so reason that $\|a'(i) = b'(i)\| \in \mathcal{U}$. This, of course, implies that $a = b$.

To demonstrate surjectivity of h, consider an arbitrary $a \in A^{\flat\sim}$, an arbitrary $a' \in a$, and an arbitary $a'' \in a'$. At each index, $a''(i)$ picks out an element $a''(i) \in A_i$, and there is an equivalence class $b'(i) \in A_i^{\sim i}$ of which $a''(i)$ is a member. Furthermore, consider the function b' mapping each i to the equivalence class $b'(i) \ni a''(i)$ for all i. $b' \in \prod_{i \in I} A_i^{\sim i}$ and, as $\sim_{\mathcal{U}}$ partitions this domain, is thus a member of some $b \in A^\natural$. The selection of b ensures that $h(b) = a$, and as a was chosen arbitrarily, this implies surjectivity of h. By the foregoing, we conclude that h is bijective.

We want to show that h is not only a bijection, but is an isomorphism. We begin with constants. In order to examine $c^{\mathfrak{A}^{\flat\sim}}$, we first note that $c^{\mathfrak{A}^\flat} = \{a \in \prod_{i \in I} A_i :$

$\|a(i) = c^{\mathfrak{A}_i}\| \in \mathcal{U}\}$. So $c^{\mathfrak{A}^{\flat\sim}} = \{b \in A^\flat : b \sim_I c^{\mathfrak{A}^\flat}\}$, or, alternately, $\{b \in A^\flat : \exists b' \in b$ such that $\|b'(i) \sim_i c^{\mathfrak{A}_i}(i)\| \in \mathcal{U}\}$. Consider $c^{\mathfrak{A}^\natural} = \{a \in \prod_{i \in I} A_i^{\sim i} : \|a(i) = c^{\mathfrak{A}_i^{\sim i}}\| \in \mathcal{U}\}$; we define $h(c^{\mathfrak{A}^\natural}) = \{b \in A^\flat : \exists b' \in b$ such that $\|b'(i) \in c^{\mathfrak{A}_i^{\sim i}}(i)\| \in \mathcal{U}\}$. We recognize, however, $c^{\mathfrak{A}_i^{\sim i}}(i)$ as the class of elements of A_i collapsed modulo $\sim_i$ and reason that $b'(i) \in c^{\mathfrak{A}_i^{\sim i}}(i)$ if and only if $b'(i) \sim_i c^{\mathfrak{A}_i}(i)$. So $\{b \in A^\flat : \exists b' \in b$ such that $\|b'(i) \sim_i c^{\mathfrak{A}_i}(i)\| \in \mathcal{U}\} = \{b \in A^\flat : \exists b' \in b$ such that $\|b'(i) \in c^{\mathfrak{A}_i^{\sim i}}(i)\| \in \mathcal{U}\}$, that is, $c^{\mathfrak{A}^{\flat\sim}} = h(c^{\mathfrak{A}^\natural})$.

Next, for an n-ary function symbol f and n-tuple $\vec{a} \in (A^\natural)^n$, we must demonstrate that $f^{\mathfrak{A}^{\flat\sim}}(h(\vec{a})) = h(f^{\mathfrak{A}^\natural}(\vec{a}))$. Consider $h(\vec{a})$. h maps this n-tuple to the equivalence class $\{\vec{b} \in (A^\flat)^n : \exists \vec{b}' \in \vec{b}$ such that $\forall j < n \ \|b'_j(i) \in a_j(i)\| \in \mathcal{U}\}$. We may then ask what the extension of $f^{\mathfrak{A}^{\flat\sim}}(h(\vec{a}))$ is; to that we answer that we may choose a representative $\vec{c} \in h(\vec{a})$ and consider that $f^{\mathfrak{A}^{\flat\sim}}(h(\vec{a}))$ will be equal to the class of all $d \in A^\flat$ such that $d \sim_I f^{\mathfrak{A}^\flat}(\vec{c})$, or $\{d \in A^\flat : \exists d' \in d$ such that $\|d'(i) \sim_i f^{\mathfrak{A}_i}(\vec{a})(i)\| \in \mathcal{U}\}$. Of course, since $\sim_i$ is a congruence relation for all i, $d'(i) \sim_i f^{\mathfrak{A}_i}(\vec{a})(i)$ if and only if $d'(i) \in f^{\mathfrak{A}_i^{\sim i}}(\vec{a})(i)$, and so we may rewrite this as $\{d \in A^\flat : \exists d' \in d$ such that $\|d'(i) \in f^{\mathfrak{A}_i^{\sim i}}(\vec{a})(i)\| \in \mathcal{U}\}$. Now we may finally turn our attention toward $h(f^{\mathfrak{A}^\natural}(\vec{a}))$ and note that this is the very same set. As $f^{\mathfrak{A}^\natural}(\vec{a})$ is the set of all elements of $\prod_{i \in I} \mathfrak{A}_i^{\sim i}$ that are almost everywhere equal to $f^{\mathfrak{A}_i^{\sim i}}(\vec{a})(i)$, $h(f^{\mathfrak{A}^\natural}(\vec{a}))$ is the set of elements of $\mathfrak{A}^\flat$ such that they are almost everywhere a *member* of this element. Thus $h(f^{\mathfrak{A}^\natural}(\vec{a})) = \{d \in A^\flat : \exists d' \in d$ such that $\|d'(i) \in f^{\mathfrak{A}_i^{\sim i}}(\vec{a})(i)\| \in \mathcal{U}\} = f^{\mathfrak{A}^{\flat\sim}}(h(\vec{a}))$, and we establish identity.

Finally, we simply demonstrate, that for any n-ary literal R or $\neg R$, that an n-tuple $\vec{a} \in R^{\mathfrak{A}^\natural +}$ if and only if $h(\vec{a}) \in R^{\mathfrak{A}^{\flat\sim} +}$ (alternately, $\vec{a} \in R^{\mathfrak{A}^\natural -}$ if and only if $h(\vec{a}) \in R^{\mathfrak{A}^{\flat\sim} -}$). For left-to-right, suppose that $\vec{a} \in R^{\mathfrak{A}^\natural +}$; this implies that $\|\vec{a}(i) \in R^{\mathfrak{A}_i^{\sim i} +}\| \in \mathcal{U}$. Now $\vec{a}(i) \in R^{\mathfrak{A}_i^{\sim i} +}$ if and only if there exists a $\vec{a}'(i) \in \vec{a}(i)$ such that $\vec{a}' \in R^{\mathfrak{A}_i +}$, and hence this is equivalent to stating that $\|\vec{a}'(i) \in R^{\mathfrak{A}_i +}\| \in \mathcal{U}$. This in turn implies that for the equivalence class $\vec{a}'_{\sim_\mathcal{U}} \ni \vec{a}'$, $\vec{a}'_{\sim_\mathcal{U}} \in R^{\mathfrak{A}^\flat +}$, and in turn that for a $\vec{a}'_{\sim_I} \ni \vec{a}'_{\sim_\mathcal{U}}$, $\vec{a}'_{\sim_I} \in R^{\mathfrak{A}^{\flat\sim} +}$. But we immediately may recognize $\vec{a}'_{\sim_I} = h(\vec{a})$.

Right-to-left, we suppose that $h(\vec{a}) \in R^{\mathfrak{A}^{\flat\sim} +}$. $h(\vec{a})$ is the class of all elements of $\vec{a}' \in R^{\mathfrak{A}^\flat +}$ such that there exists an $\vec{a}'' \in \vec{a}'$ such that $\|\vec{a}''(i) \in \vec{a}(i)\| \in \mathcal{U}$. Ex hypothesi, we know that $\vec{a}' \in R^{\mathfrak{A}^\flat +}$ and hence that $\|\vec{a}'' \in R^{\mathfrak{A}_i +}\| \in \mathcal{U}$. So at almost all $\mathfrak{A}_i$, $\vec{a}''(i) \in R^{\mathfrak{A}_i +}$. But at each such i, we have a collapsed model modulo $\sim_i$, and we reason that $\vec{a}(i) \ni \vec{a}''$ and $\vec{a}(i) \in R^{\mathfrak{A}_i^{\sim i} +}$ at each such i. Hence $\|\vec{a}(i) \in R^{\mathfrak{A}_i^{\sim i} +}\| \in \mathcal{U}$, and we conclude that $\vec{a} \in R^{\mathfrak{A}^\natural +}$. The above proof obviously applies in the case of the anti-extension of R as well. Given the definition of $\sim_I$ and h, we conclude that h is an isomorphism. □

Corollary 4.2 *For any ultraproduct of collapsed models of arithmetic* $\mathfrak{A}^\natural$*, there exists a classical nonstandard model of arithmetic* $^*\mathbb{N}$ *and a collapsing relation* $\sim$ *such that* $\mathfrak{A}^\natural \cong (^*\mathbb{N})^\sim$.

Proof Immediate from the theorem. □

Such a result can be had for other methods of constructing nonclassical models more general than collapsing. In [1], Dunn offers a technique for the construction of 3-valued structures from consistent structures. His presentation formally differs from ours, and we in a sense bifurcate his result into an LP case and a K_3 case.[5]

Taking a pair of consistent structures $\mathfrak{A}, \mathfrak{A}'$ and a surjective, operation-preserving homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$, we define the *inconsistent* structure determined by h, $\tilde{3}\mathfrak{A}$, by $\tilde{3}A = A' = \{h(a) : a \in A\}$, $c^{\tilde{3}\mathfrak{A}} = h(c^{\mathfrak{A}})$, $f^{\tilde{3}\mathfrak{A}}(\vec{b}) = h(f^{\mathfrak{A}}(\vec{a}))$, where $a_0 \in h^{-1}[b_0], \ldots, a_{n-1} \in h^{-1}[b_{n-1}]$, and for n-ary $R^{\tilde{3}\mathfrak{A}+} = \{(b_0, \ldots, b_{n-1}) \in B^n : \exists b'_0 \in h^{-1}[b_0], \ldots, b'_{n-1} \in h^{-1}[b_{n-1}], (b'_0, \ldots, b'_{n-1}) \in R^{\mathfrak{A}+}\}$ and $R^{\tilde{3}\mathfrak{A}-} = \{(b_0, \ldots, b_{n-1}) \in B^n : \exists b'_0 \in h^{-1}[b_0], \ldots, b'_{n-1} \in h^{-1}[b_{n-1}], (b'_0, \ldots, b'_{n-1}) \in R^{\mathfrak{A}-}\}$.

Given the identical homomorphism, we generate an *incomplete* structure $\check{3}\mathfrak{A}$ by retaining the universe and interpretations of constants and function symbols while defining an n-ary $R^{\check{3}\mathfrak{A}+} = \{(b_0, \ldots, b_{n-1}) \in B^n : \forall b'_0 \in h^{-1}[b_0], \ldots, b'_{n-1} \in h^{-1}[b_{n-1}], (b'_0, \ldots, b'_{n-1}) \in R^{\mathfrak{A}+}\}$ and $R^{\check{3}\mathfrak{A}-} = \{(b_0, \ldots, b_{n-1}) \in B^n : \forall b'_0 \in h^{-1}[b_0], \ldots, b'_{n-1} \in h^{-1}[b_{n-1}], (b'_0, \ldots, b'_{n-1}) \in R^{\mathfrak{A}-}\}$. The intuition is that in the LP interpretation, $\tilde{3}\mathfrak{A}$ makes true R of some element b if and only if of something in its preimage under h is R true in $\mathfrak{A}$; in the K_3 interpretation, R is true of b in $\check{3}\mathfrak{A}$ if and only if R is true of *everything* in its preimage under h.

Dunn offers a preservation theorem with respect to such constructions, which we split as follows.

Theorem 4.3 (Dunn for LP) *For a structure $\tilde{3}\mathfrak{A}$ determined by an operational, surjective homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$, $\mathfrak{A} \models_{\mathsf{CL}} \varphi(\vec{a})$ only if $\tilde{3}\mathfrak{A} \models_{\mathsf{LP}} \varphi(h(\vec{a}))$.*

Proof We refer the reader to [1]. ☐

Theorem 4.4 (Dunn for K_3) *For a structure $\check{3}\mathfrak{A}$ determined by an operational, surjective homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$, $\check{3}\mathfrak{A} \models_{\mathsf{K}_3} \varphi(h(\vec{a}))$ only if $\mathfrak{A} \models_{\mathsf{CL}} \varphi(\vec{a})$.*

Proof We refer the reader to [1]. ☐

Referring to alternately $\tilde{3}\mathfrak{A}$ or $\check{3}\mathfrak{A}$ as $3\mathfrak{A}$ when the permissibility of the structure is irrelevant, we offer the following.

Theorem 4.5 *For an index I and a class of either inconsistent or incomplete structures $\{3\mathfrak{A}_i\}$ such that each is determined by a function $h_i : \mathfrak{A}_i \to \mathfrak{B}_i$, and an ultraproduct $(3\mathfrak{A})^\natural = \prod_{i \in I} 3\mathfrak{A}_i/\mathcal{U}$, there exists a function $h : \mathfrak{A}^\natural \to \mathfrak{B}^\natural$ such that the structure determined by this function, $3(\mathfrak{A}^\natural) = 3(\prod_{i \in I} \mathfrak{A}_i/\mathcal{U})$, is identical to $(3\mathfrak{A})^\natural$, that is, $(3\mathfrak{A})^\natural = 3(\mathfrak{A}^\natural)$.*

Proof We offer as candidate operational homomorphism $h : \mathfrak{A}^\natural \to \mathfrak{B}^\natural$ the function $h(a) = \{b' \in \prod_{i \in I} B_i : \forall a' \in a, \|b'(i) = h_i(a'_i(i))\| \in \mathcal{U}\}$. We must show that h is surjective and operation-preserving. First, for an arbitrary $b \in B^\natural$ and a member $b' \in b$, as ex hypothesi all h_i are surjective, there exists some $a'_i \in A_i$ such that $h_i(a'_i) = b'_i$ for all i. Let a be an equivalence class of elements of $\prod_{i \in I} A_i$ equivalent to $a' : i \mapsto a_i$ modulo $\mathcal{U}$. As all $a'' \in a$ are equal to a' at almost all indices, $\|h_i(a''_i) = b'_i\| \in \mathcal{U}$, and as for all $b'' \in b$, b'' is equal to b' at almost all indices $\|h_i(a'_i) = b''_i\| \in \mathcal{U}$. We quickly see that the condition holds for any $a'' \in a$, $b'' \in b$, and thus there exists a preimage of b under h.

h is also an operational homomorphism. We now must establish that $h(f^{\mathfrak{A}^\natural}(a_0,\dots,a_{n-1})) = f^{\mathfrak{B}^\natural}(h(a_0),\dots,h(a_{n-1}))$. First, we expand the former to see that it is $\{b' \in \prod_{i\in I} B_i : \|b'(i) = h_i(f^{\mathfrak{A}^\natural}(a_0,\dots,a_{n-1})(i))\| \in \mathcal{U}\}$. But this is just the set $\{b' \in \prod_{i\in I} B_i : \|b'(i) = h_i(f^{\mathfrak{A}_i}(a_0(i),\dots,a_{n-1}(i)))\| \in \mathcal{U}\}$, as $\mathfrak{A}^\natural$ and $\mathfrak{B}^\natural$ are reduced modulo the same filter. We may expand the latter as $\{b' \in \prod_{i\in I} B_i : \|b'(i) = f^{\mathfrak{B}^\natural}(h(a_0(i)),\dots,h_i(a_{n-1}(i)))(i)\| \in \mathcal{U}\}$, or by similar reasoning, $\{b' \in \prod_{i\in I} B_i : \|b'(i) = f^{\mathfrak{B}_i}(h_i(a_0(i)),\dots,h_i(a_{n-1}(i)))\| \in \mathcal{U}\}$. But ex hypothesi, for all i, h_i preserves operations, so $h_i(f^{\mathfrak{A}_i}(a_0(i),\dots,a_{n-1}(i))) = f^{\mathfrak{B}_i}(h_i(a_0(i)),\dots,h_i(a_{n-1}(i)))$ at each i. Thus the two sets are identical.

The foregoing establishes that h is a surjective, operational homomorphism and thus determines a structure $3(\mathfrak{A}^\natural)$. We now observe that $(3A)^\natural = 3(A^\natural)$. By the manner of construction due to Dunn, $3A_i = B_i$, and hence $(3A)^\natural$ is $\prod_{i\in I} B_i$ reduced modulo $\mathcal{U}$. By the equivalent construction of $3(\mathfrak{A}^\natural)$, we note that $3(A^\natural) = B^\natural$, which is just $\prod_{i\in I} B_i$ reduced modulo $\mathcal{U}$.

This isn't, of course, enough; we must ensure that $id_{B^\natural}$ also preserves interpretations. First, for a constant c, we ensure that $c^{(3\mathfrak{A})^\natural} = c^{3(\mathfrak{A}^\natural)}$. Now, $c^{(3\mathfrak{A})^\natural} = \{b \in \prod_{i\in I} 3A_i : \|b(i) = c^{3\mathfrak{A}_i}\| \in \mathcal{U}\}$. Noticing that $c^{3\mathfrak{A}_i}$ picks out the $b' \in B_i$ such that $b' = h_i(c^{\mathfrak{A}_i})$, we rewrite this as $\{b \in \prod_{i\in I} 3A_i : \|b(i) = h_i(c^{\mathfrak{A}_i})\| \in \mathcal{U}\}$. As for all $i \in I$, $3A_i = B_i$, we further rewrite this as $\{b \in \prod_{i\in I} B_i : \|b(i) = h_i(c^{\mathfrak{A}_i})\| \in \mathcal{U}\}$. Since $c^{3(\mathfrak{A}^\natural)} = h(c^{\mathfrak{A}^\natural})$, this is the set $\{b \in \prod_{i\in I} B_i : \forall a \in c^{\mathfrak{A}^\natural}, \|b(i) = a(i)\| \in \mathcal{U}\}$. Now an $a \in c^{\mathfrak{A}^\natural}$ if and only if $\|a(i) = h_i(c^{\mathfrak{A}_i})\| \in \mathcal{U}$, and so we may rewrite this element as $\{b \in \prod_{i\in I} B_i : \|b(i) = h_i(c^{\mathfrak{A}_i})\| \in \mathcal{U}\}$, which establishes identity.

Furthermore, we must demonstrate that $f^{(3\mathfrak{A})^\natural}(\vec{b}) = f^{3(\mathfrak{A}^\natural)}(\vec{b})$. Fix $\vec{a} \in A^\natural$ such that $\exists \vec{a}' \in \vec{a}, \vec{b}' \in \vec{b}$ such that $\|a'_0 \in h^{-1}[b_0],\dots,a'_{n-1} \in h^{-1}[b_{n-1}]\| \in \mathcal{U}$. We immediately may expand $f^{(3\mathfrak{A})^\natural}(\vec{b})$ as the set of b' such that $b'(i)$ is almost everywhere equal to $f^{3\mathfrak{A}_i}(\vec{b}(i))$, or $\{b' \in \prod_{i\in I} B_i : \|b'(i) = f^{3\mathfrak{A}_i}(\vec{b}(i))\| \in \mathcal{U}\}$. Now at all $i \in I$, $f^{3\mathfrak{A}_i}(\vec{b}(i)) = h_i(f^{\mathfrak{A}_i}(\vec{a}'(i)))$, so we rephrase this as $\{b' \in \prod_{i\in I} B_i : \|b'(i) = h_i(f^{\mathfrak{A}_i}(\vec{a}'(i)))\| \in \mathcal{U}\}$. But by selection of a' and the definition of h, $\|b'(i) = h_i(f^{\mathfrak{A}_i}(\vec{a}'(i)))\| \in \mathcal{U}$ if and only if $\|b'(i) = h(f^{\mathfrak{A}^\natural}(\vec{a})(i))\| \in \mathcal{U}$, so we rewrite this as $\{b' \in \prod_{i\in I} B_i : \|b'(i) = h(f^{\mathfrak{A}^\natural}(\vec{a}))(i)\| \in \mathcal{U}\}$. Since every member of $h(f^{\mathfrak{A}^\natural}(\vec{a}))$ is equal to every other almost everywhere, we recognize this as $h(f^{\mathfrak{A}^\natural}(\vec{a}))$, which is equal to $f^{3(\mathfrak{A}^\natural)}(\vec{b})$. Thus the two are equal.

To demonstrate identity between the structures, we must treat the interpretation of relation symbols. We now split the cases of the inconsistent and incomplete structures determined by h.

In the LP case, an element $(b_0,\dots,b_{n-1}) \in R^{(\tilde{3}\mathfrak{A})^\natural +}$ if and only if $\{i : (b_0(i),\dots,b_{n-1}(i)) \in R^{\tilde{3}\mathfrak{A}_i +}\} \in \mathcal{U}$. This holds if and only if $\{i : \exists a'_0(i) \in h_i^{-1}[b_0(i)],\dots,a'_{n-1}(i) \in h_i^{-1}[b_{n-1}(i)]$ such that $(a'_0(i),\dots,a'_{n-1}(i)) \in R^{\mathfrak{A}_i +}\} \in \mathcal{U}$, which is equivalent to stating that $\{a'' \in \prod_{i\in I} A_i : a'_0 = a''_0 \wedge \cdots \wedge a'_{n-1} = a''_{n-1}\} \in R^{\mathfrak{A}^\natural +}$. This, finally, is equivalent to claiming that there exists $\vec{a} \in A^\natural$ such that $h(\vec{a}) = \vec{b}$ and $\vec{a} \in R^{\mathfrak{A}^\natural +}$, which is equivalent to stating that $\vec{b} \in R^{\tilde{3}(\mathfrak{A}^\natural)+}$. Analogous reasoning establishes the result for the anti-extension of R.

More easily, for K_3-structures so determined, $\vec{b} \in R^{(\check{3}\mathfrak{A})^{\natural}+}$ is again those $\vec{b}' \in (\prod_{i \in I} B_i)^n$ such that $\{i : (b'_0(i), \ldots, b'_{n-1}(i)) \in R^{\check{3}\mathfrak{A}_i+}\} \in \mathcal{U}$. This will hold if and only if every $\vec{a}' \in (\prod_{i \in I} A_i)^n$ such that $a'_0 \in h^{-1}[b'_0], \ldots, a'_{n-1} \in h^{-1}[b'_{n-1}]$ is a member of $R^{\mathfrak{A}_i+}$. This implies that all $\vec{a} \in A^{\natural}$ such that $\vec{a}' \in \vec{a}$ are members of $R^{\mathfrak{A}^{\natural}+}$. But the set of such $\vec{a}$ is $h^{-1}[\vec{b}]$, and so $\vec{b} \in R^{\check{3}(\mathfrak{A}^{\natural})}$. The argument for the anti-extension is again identical. Given that the structures $3(\mathfrak{A}^{\natural})$ and $(3\mathfrak{A})^{\natural}$ are determined by the extensions of their respective interpretations of symbols, we may conclude that the two are identical. □

5 Concluding Remarks

From this point, I hope that we've gotten generalizations of a few fundamental techniques and seen some applications suggesting that mathematics set upon a DeMorgan-logical landscape is something that warrants study. A few concluding notes concerning future directions of such a study might be in order to further stress its worth.

One motivation may be made apparent by an analogy with the reverse mathematics program. Reverse mathematics, rather than investigating what pre-set-theoretic mathematical fruits are gotten given particular set-theoretic assumptions, works backward, attempting to reveal what set-theoretic assumptions (e.g., comprehension axiomata) are necessary and sufficient in order to secure those fruits. Inasmuch as particular mathematical theorems hold for certain DeMorgan logics and fail for others, we may hope that an analogous investigation may be made in determining what logical properties are necessary to secure particular mathematical theorems. It is, for example, my suspicion that logics intermediate between, for example, LP and CL could be generated to mark with precision those points at which particular theorems hold or fail, by adding additional inference rules. For example, by adding the schemata $\varphi, \neg\varphi \models \psi$, where φ is, for example, of some bounded complexity, to LP would produce such an intermediate logic that could potentially provide, say, an account of how much inconsistency a particular theorem can "handle." Certainly there are such "intermediate points."

To wit, regarding Cantor's Theorem, in the minimally inconsistent logic LP_m introduced in Priest [5], since the structure $(\mathbb{Q}, <)$ is trivially the minimally inconsistent model of DLO_{--} of cardinality $\aleph_0$, the theorem holds in LP_m. Hence the theorem fails at some intermediate point between LP and LP_m. The question of *where* it fails in this spectrum is the question of *how much* logical apparatus- what sort of logical presuppositions- are requisite in order to secure the result.[6] Just as we cannot, for example, prove Łoś's Theorem without the axiom of choice, there is some logical assumption made classically that underwrites Cantor. If one may discover the precise location in this spectrum of logics at which some theorem fails this constitutes evidence that there is a correlation between, perhaps, some structural rule of that logic and the success or failure of that theorem.

It also may be hoped that transfer properties between the class of structures of a nonclassical logic and the class of classical structures could be established. Such transfer properties have the potential to provide facts about classical mathematical theories. Just, for instance, as the fruits of non-standard analysis may be applied to standard analysis without the theorist accepting the accompanying ontology, so might we hope that transfer principles could very well provide useful, classical results.

Finally, [3] suggests a "special case hypothesis" that classical mathematics is a special case of a broader swarth of mathematics. This, it seems, goes beyond hypothesis to being a truism. The structures we herein describe are there, and in virtue of being describable, deserve study. We can play the pragmatist and outline strategies to entice the working mathematician, but the truth is that DeMorgan logics have a model theory- surreal and curious as it may be- and its existence alone is sufficient to warrant its study. Even if one is inclined to think of it as teratology, monstrosities yet fall under the purview of the science as a whole. Regardless, the above introduces the use of ultraproducts as a viable method of constructing nonclassical models and establishes their "nice" properties with respect to previously established techniques for constructing models in these logics. Motivations aside, this provides another tool in the nonclassical logician's armamentarium.

Notes

1. That the semantics for the logics herein considered translate so swiftly to natural language constitutes prima facie evidence that they withstand the scrutiny of, for example, Quine's maxim that a "change of logic" is a "change of subject" in Quine [9].

2. The "chromatic" notation for ultraproducts is borrowed from Schoutens [11], though we will not retain its particular algebraic purpose.

3. Though, of course, $=$ is privileged in its positive extension.

4. Note that this result preempts the typical proof of the upward Löwenheim-Skolem Theorem. Given a structure $\mathfrak{A}$, typically, one merely adds κ-many formulas of the form $c_i \neq c_j$ for every $i, j \in \kappa$ to the theory $\mathrm{Th}(\mathfrak{A})$. By compactness, this has a model, and by the inclusion of the set $\{c_i \neq c_j : i, j \in \kappa\}$, it will have a model of cardinality greater than or equal to κ. One then uses the downward theorem to establish the existence of a model of cardinality κ. The problem is obvious; while classically, that $\mathfrak{A}' \models_{\mathsf{CL}} \{c_i \neq c_j : i, j \in \kappa\}$ implies that $\mathfrak{A}' \not\models_{\mathsf{CL}} c_i = c_j$ for all $i, j \in \kappa$, ensuring that $|\mathfrak{A}'| > \kappa$. But in FDE, RM$_3$, and LP, such an inference is unwarranted; that $\mathfrak{A}' \models_{\mathsf{LP},\mathsf{RM}_3,\mathsf{FDE}} \{c_i \neq c_j : i, j \in \kappa\} \cup \{c_i = c_j : i, j \in \kappa\}$ is possible while $|\mathfrak{A}'| \lneq \kappa$.

5. Dunn doesn't use these names; he mentions the "Łukasiewicz logic" but only presents the matrices for negation and conjunction with a third truth value neuter (N). Dunn states that this can be either read as "both true and false" or "neither true nor false"; the interpretation is, of course, central in our presentation. The result thus splits, with the former applying in LP and the latter applying to K$_3$.

6. Priest has pointed out that the serviceability of LP_m in such an endeavor is strained as its implication relation is not closed under uniform substitution. Nevertheless, this fact does illustrate that there *are* well-defined, intermediate points that may serve in such an investigation.

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