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Fractal Patterns in Reasoning

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Abstract This paper is the third and final one in a sequence of three. All three papers emphasize that a proposition can be justified by an infinite regress, on condition that epistemic justification is interpreted probabilistically. The first two papers showed this for one-dimensional chains and for one-dimensional loops of propositions, each proposition being justified probabilistically by its precursor. In the present paper we consider the more complicated case of two-dimensional nets, where each "child" proposition is probabilistically justified by two "parent" propositions. Surprisingly, it turns out that probabilistic justification in two dimensions takes on the form of Mandelbrot's iteration. Like so many patterns in nature, probabilistic reasoning might in the end be fractal in character.

1 Introduction

The concept of a *regressus ad infinitum* has afflicted many branches of philosophy, and epistemology is no exception. As Bonjour remarks: "Considerations with respect to the regress argument [are] perhaps the most crucial in the entire theory of knowledge" (Bonjour [3], p. 18).

The epistemological regress problem traditionally takes the form of an epistemic chain in which (a belief in) a proposition E_0 is justified by (a belief in) E_1, which in turn is justified by (a belief in) E_2, and so on. Since the chain does not have a final link from which the justification springs, it seems that there can be no justification for E_0 at all. In the words of Ginet:

> Inference cannot *originate* justification, it can only *transfer* it from premises to conclusion. And so it cannot be that, if there actually occurs justification, it is all inferential. ... [T]here can be no justification to be transferred unless ultimately something else, something other than the inferential relation, does create justification. (Ginet [5], p. 148; emphasis by the author)

In earlier papers we have shown that this problem only occurs when epistemic justification is seen as a form of deductive inference, where each E_n is deductively

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inferred from E_{n+1}. In such a classical infinite regress, the target proposition E_0 can never receive a definite truth value. However, once we assume that each E_n is only *made probable* by E_{n+1}, the resolution of the problem is at hand. For an infinite probabilistic regress *can* confer a definite (unconditional) probability value on the target proposition E_0. If this probability is greater than one half, then E_0 is said to be (probabilistically) justified, since in that case E_0 is more likely to be true than to be false.[1] In that sense a probabilistic regress can justify a proposition whereas a traditional, nonprobabilistic regress cannot. This applies not only to probabilistic regresses that have the form of one-dimensional chains (see Atkinson and Peijnenburg [1]), but also to probabilistic regresses that take the shape of one-dimensional loops (see Atkinson and Peijnenburg [2]).

The matter becomes even more interesting if we replace the infinite one-dimensional structure (a chain or a loop) by an infinite, many-dimensional probabilistic network. As Fumerton has rightly observed, the regress problem is not confined to concerns about our ability to complete a single infinite chain; rather it manifests itself in all its overwhelming complexity when we realize that infinite regresses actually "mushroom out" in many different directions (Fumerton [4], p. 57). In the present paper we show that this proliferating pattern, however intricate it may seem, can nevertheless be held in check. First, it turns out that a many-dimensional probabilistic network generally converges to a unique unconditional probability value for the target proposition E_0. This means that E_0 can receive a well-defined justification not only from a single infinite chain, but also from a complicated infinite network. Second, we found that such a network follows from exactly the same recursion as does the famous Mandelbrot set. The only requirement for obtaining these two surprising results is the condition of probabilistic support; that is, each E_n is made probable by one other proposition (in the case of a one-dimensional structure) or by more than one proposition (in the case of a many-dimensional net).

We will proceed as follows. In Section 2 we describe an example of probabilistic justification by a one-dimensional epistemic chain, showing how the latter can yield a well-defined probability for the target proposition E_0. This example draws upon our first paper (Atkinson and Peijnenburg [1]). In Section 3 we present a more complicated example of justification, namely, one that has the form of a two-dimensional net. In Section 4 we explain the intimate relation of this net to the Mandelbrot fractal. In Section 5, we then argue that epistemic justification still exhibits a generalized Mandelbrot structure even if it fans out in more than two dimensions.

2 A One-Dimensional Probabilistic Chain

Imagine that we are trying to develop a medicine for a certain disease and that we want to know whether or not a particular bacterium has a certain hereditary trait T. Bacteria reproduce asexually, so just one parent, the "mother," produces a child, the "daughter." Suppose that we have bred several batches of bacteria, each batch growing out of one single primordial ancestor. This primordial ancestor might have T or might lack T—we do not know. However, we do know that a T-daughter is more likely to have a mother with T than a mother without T.

We now randomly select from our batches one bacterium that we call Barbara-0. We do not know whether Barbara-0 has T, nor do we know whether the primordial ancestor in her batch has the trait. Let E_0 be the proposition that Barbara-0 has T.

E_1 is the proposition that her mother, Barbara-1, has T, E_2 that her grandmother, Barbara-2, has T, and so on. Since E_0 is more probable if E_1 is true than if E_1 is false, it is the case that

$$\mathrm{P}(E_0|E_1) > \mathrm{P}(E_0|\neg E_1). \tag{1}$$

Formula (1) is the condition of probabilistic support. In our example it holds for any two bacteria, Barbara-n and Barbara-$(n+1)$:

$$\mathrm{P}(E_n|E_{n+1}) > \mathrm{P}(E_n|\neg E_{n+1}).$$

For the time being, however, we will only talk about Barbara-0 and Barbara-1, and thus about propositions E_0 and E_1; it is easy enough to keep in mind that what we say about the pair E_0 and E_1 also goes for E_1 and E_2, for E_2 and E_3, and so on. The unconditional probabilities $\mathrm{P}(E_0)$ and $\mathrm{P}(E_1)$ are related by the rule of total probability:

$$\mathrm{P}(E_0) = \mathrm{P}(E_0|E_1)\ \mathrm{P}(E_1) + \mathrm{P}(E_0|\neg E_1)[1 - \mathrm{P}(E_1)]. \tag{2}$$

With

$$\alpha = \mathrm{P}(E_0|E_1) \quad \text{and} \quad \beta = \mathrm{P}(E_0|\neg E_1)$$

the condition of probabilistic support (1) becomes

$$1 > \alpha > \beta > 0, \tag{3}$$

where we have now explicitly excluded the extreme values 1 and 0.[2] Equation (2) may now be rewritten in the form,

$$\mathrm{P}(E_0) = \beta + (\alpha - \beta)\ \mathrm{P}(E_1). \tag{4}$$

Next let us make the condition of probabilistic support quantitative by supposing that the probability that a T-daughter has a T-mother is 0.99, and the probability that a T-daughter has a T-less mother is 0.02. Thus,

$$1 > \alpha = 0.99 > \beta = 0.02 > 0. \tag{5}$$

Given (5), what is the value of $\mathrm{P}(E_0)$; in other words, what is the unconditional probability that the randomly selected Barbara-0 has trait T? The answer depends not only on whether the primordial ancestor of Barbara-0 has T, but also on the *distance* between this primordial ancestor and Barbara-0. If the primordial ancestor of Barbara-0 is simply her mother, viz. Barbara-1, then the distance is at its smallest. In this case the matter is straightforward: $\mathrm{P}(E_0)$ is 0.99 if Barbara-1 has T, it is 0.02 if she lacks T, and it is a value in between those two numbers if it is uncertain whether Barbara-1 has T. If Barbara-0 has two ancestors, the distance is a little bit greater. We must now iterate formula (4), in the sense that we must substitute "$\beta+(\alpha-\beta)\ \mathrm{P}(E_2)$" for "$\mathrm{P}(E_1)$":

$$\mathrm{P}(E_0) = \beta + (\alpha - \beta)[\beta + (\alpha - \beta)\ \mathrm{P}(E_2)], \tag{6}$$

where we are assuming that α and β are known and keep their values throughout the chain. With this assumption, we find that $\mathrm{P}(E_0)$ is 0.9803 if E_2 is true, and 0.0394 if E_2 is false. The greater the distance is between Barbara-0 and her primeval parent, the more often will (4) have to be iterated, and the smaller will be the influence of the primeval parent on Barbara-0. If the primal ancestor is very far away, then it hardly matters for the value of $\mathrm{P}(E_0)$ whether she has T or lacks T. The difference between the two extreme values of $\mathrm{P}(E_0)$ will be tiny, and $\mathrm{P}(E_0)$ will mainly be determined by the joint conditional probabilities that separate Barbara-0 from her original ancestor. For instance, if Barbara-0 has one hundred and fifty ancestors, so that the chain

starts with Barbara-150, then the contribution of E_{150} to the unconditional probability of E_0 will be rather small. Indeed, if E_{150} is false, so that Barbara-150 does not have T, the probability of E_0 ("Barbara-0 has T") is 0.660, which is only marginally different from 0.670—the number that we find when we assume that Barbara-150 has T.

What happens if the number of ancestors of Barbara-0 is infinite? We can now give a clear answer to this question: then it does not matter whether the infinitely remote ancestor has T or lacks T. In this infinite case, the primordial mother has completely disappeared from the picture. All the probabilistic justification for E_0 now comes from the conditional probabilities, and none comes from the infinitely distant *Urmutter*. This does not imply, as Ginet and many others have thought, that there is no justification at all. On the contrary. Although all the justification for E_0 is inferential, E_0 is still probabilistically justified. We are able to compute a uniquely determined value for $\mathrm{P}(E_0)$, the unconditional probability of E_0, even though the justification consists of an infinite chain of conditional probabilities; with the quantities chosen for α and β in (5), we find that $\mathrm{P}(E_0)$ equals $\frac{2}{3}$. In this case E_0 has been probabilistically justified, since E_0 has been shown to be probably true—indeed, any threshold of acceptance between $\frac{1}{2}$ and $\frac{2}{3}$ may be adopted in this case.[3]

3 A Two-Dimensional Probabilistic Net

It might be objected that our result in the previous section is based on an unrealistic simplification. For real epistemic justification is of course much richer and much more complicated than a one-dimensional chain. As Fumerton has observed, "infinite regresses are mushrooming out in... different directions," so anybody who "worr[ies] about the possibility of completing one infinitely long chain of reasoning, ... should be downright depressed about the possibility of completing an infinite number of infinitely long chains of reasoning" (Fumerton [4], p. 57).

In the present section we deal with these worries. We explain what happens when we replace the infinite one-dimensional probabilistic chain by an infinite probabilistic network in more than one dimension. Our investigation reveals that things are not as grim as Fumerton suggests, and that the situation is on the contrary extremely intriguing. For first, a many-dimensional network of conditional probabilities generally yields a definite unconditional probability for the target proposition E_0. And second, this network leads to an iteration that is precisely the same as Mandelbrot's recursion.

We start by considering a two-dimensional net, where a proposition is linked probabilistically to two others. In the next section we show that this net is directly related to the Mandelbrot set. At the end of this paper, in Section 5, we will sketch what happens when we extend the net to more than two dimensions.

A two-dimensional net could serve as a model for the propagation of genetic traits under sexual reproduction, in which the traits of a child are related probabilistically to those of both mother and father. Let $\mathrm{P}(E_0)$ again be the unconditional probability that Barbara-0 has trait T. However, this time Barbara-0 is an organism with two parents, a father and a mother. For the purpose of fixing ideas it will prove convenient to talk about sexual reproduction and about fathers and mothers, but we should bear in mind that the formalism is much more general.

Since Barbara-0 stems from two parents, the probability that she has T is related to her mother *and* to her father. Rather than two reference classes (the mother having or not having T), we now have four: both the mother and the father have T, neither of them has it, the father has T but the mother does not, and the mother has T but the father does not. The corresponding four conditional probabilities can be represented as follows:

$$\begin{aligned}\alpha &= \mathrm{P}(E_0|E_1(\mathrm{f})\ \&\ E_1(\mathrm{m}))\\ \beta &= \mathrm{P}(E_0|\neg E_1(\mathrm{f})\ \&\ \neg E_1(\mathrm{m}))\\ \gamma &= \mathrm{P}(E_0|E_1(\mathrm{f})\ \&\ \neg E_1(\mathrm{m}))\\ \delta &= \mathrm{P}(E_0|\neg E_1(\mathrm{f})\ \&\ E_1(\mathrm{m})),\end{aligned}$$

where α means "the probability that Barbara-0 has T, given that her father has it, $E_1(\mathrm{f})$, and that her mother has it, $E_1(\mathrm{m})$," and where β, γ, and δ are given analogous readings. In this case, the analogue of the rule of total probability (2) is

$$\begin{aligned}\mathrm{P}(E_0) = \alpha\ \mathrm{P}(E_1(\mathrm{f})\ \&\ E_1(\mathrm{m})) + \beta\ \mathrm{P}(\neg E_1(\mathrm{f})\ \&\ \neg E_1(\mathrm{m})) + \\ \gamma \mathrm{P}(E_1(\mathrm{f})\ \&\ \neg E_1(\mathrm{m})) + \delta\ \mathrm{P}(\neg E_1(\mathrm{f})\ \&\ E_1(\mathrm{m})).\end{aligned} \tag{7}$$

[4]

To iterate the two-dimensional Equation (7), as we did with the one-dimensional (2) and (4), we would now need more complicated relations for the unconditional probabilities appearing in this expression. It is no longer sufficient to consider $\mathrm{P}(E_1)$ and replace it by "$\beta + (\alpha - \beta)\ \mathrm{P}(E_2)$," for now we are dealing with the probability of a conjunction of two parents, $\mathrm{P}(E_1(\mathrm{f})\ \&\ E_1(\mathrm{m}))$. Each of these parents has two parents, so we encounter in fact the probabilities of conjunctions of four individuals (the four "grandparents"). This can be continued further and further, involving more and more progenitors. Such seething complication is the very essence of how many natural systems work, but it is difficult to express the full complexity in iterated versions of (7).

Fortunately, however, we can make simplifying assumptions. Here we will work under three simplifications (which we will relax in Section 5).

1. *Independence* The probabilities for the occurrence of the trait T in females and in males are independent of one another in any of the n generations: $\mathrm{P}(E_n(\mathrm{f})\ \&\ E_n(\mathrm{m})) = \mathrm{P}(E_n(\mathrm{f})) \times (\mathrm{P}(E_n(\mathrm{m}))$. This assumption seems reasonable when we are dealing with sexual reproduction in a large population where sibling impregnation is taboo.
2. *Gender symmetry* The probability of the occurrence of the trait T is the same for females and for males in any of the n generations: $\mathrm{P}(E_n(\mathrm{f})) = \mathrm{P}(E_n(\mathrm{m}))$. This implies that we only consider inheritable traits which are gender-independent, such as having blue eyes or being red-haired (and not, for example, having breast cancer or being taller than two meters). With this assumption "f" and "m" can be dropped, and in combination with the first assumption we obtain $\mathrm{P}(E_n(\mathrm{f})\ \&\ E_n(\mathrm{m})) = \mathrm{P}(E_n)\mathrm{P}(E_n) = \mathrm{P}^2(E_n)$.
3. *Uniformity* The conditional probabilities are the same in any of the n generations. In other words, α, β, γ, and δ remain the same throughout the net.

Together these assumptions enable us to simplify (7) to

$$\mathrm{P}(E_0) = \alpha\ \mathrm{P}^2\ (E_1) + \beta\ \mathrm{P}^2\ (\neg E_1) + (\gamma + \delta)\ \mathrm{P}(E_1)\mathrm{P}(\neg E_1). \tag{8}$$

This expression equates $P(E_0)$ to the sum of three terms, each of which reflects the probability that Barbara-0 has T, given that both her parents have it, or that neither parent has it, or that only one parent has it. Equation (8) can be written in the form

$$P(E_0) = \beta + (\gamma + \delta - 2\beta)\, P(E_1) + (\alpha + \beta - \gamma - \delta)\, P^2(E_1) \tag{9}$$

(see the Appendix). In the special case that

$$\alpha + \beta = \gamma + \delta, \tag{10}$$

the third term of (9) is zero, and what remains is the same as Equation (4). In other words, in the special case (10) the two-dimensional quadratic Equation (9) reduces to the one-dimensional linear form (4).

When the special equality (10) does *not* hold, this reduction is impossible; then the P^2 term describes an essentially new situation. It is precisely when P^2 does not disappear that probabilistic justification exhibits the same structure as the Mandelbrot set. For in that case, as we will explain in the next section, the quadratic relation (9) turns out to be equivalent to Mandelbrot's famous fractal generating recursion.

4 The Mandelbrot Set

Some thirty years ago Mandelbrot introduced his celebrated iteration:

$$q_{n+1} = c + (q_n)^2, \tag{11}$$

where c and q are complex numbers (Mandelbrot [6]). Starting with $q_0 = 0$, the iteration (11) gives us successively

$$\begin{aligned} q_1 &= c, \\ q_2 &= c + c^2, \\ q_3 &= c + (c + c^2)^2, \\ q_4 &= c + \{c + (c + c^2)^2\}^2, \text{ and so on.} \end{aligned}$$

For many values of c, the iteration will diverge, allowing q_n to grow beyond any bound as n becomes larger and larger. For example, if $c = 1$ we obtain

$$\begin{aligned} q_1 &= 1, \\ q_2 &= 2, \\ q_3 &= 5, \\ q_4 &= 26, \text{ and so on.} \end{aligned}$$

But if, for instance, $c = 0.1$, then q_n does not diverge, and in this case actually converges to the number $0.11271\ldots$. Taken together, all the values of c for which the iteration (11) does *not* diverge form the Mandelbrot set, which can be visualized in the well-known picture shown in Figure 1.

The black area contains the points that belong to the Mandelbrot set. Each point corresponds to a complex number, c, being the ordered pair of the Cartesian coordinates, (x, y). The edge of the Mandelbrot set forms the boundary between those values of c that are members of the Mandelbrot set and those that are not. This boundary, the "Mandelbrot fractal," has the property of being infinitely structured in a remarkable way: no matter how far you zoom in on it, you will always find a new structure that is similar to, although not completely identical with, the Mandelbrot set itself.

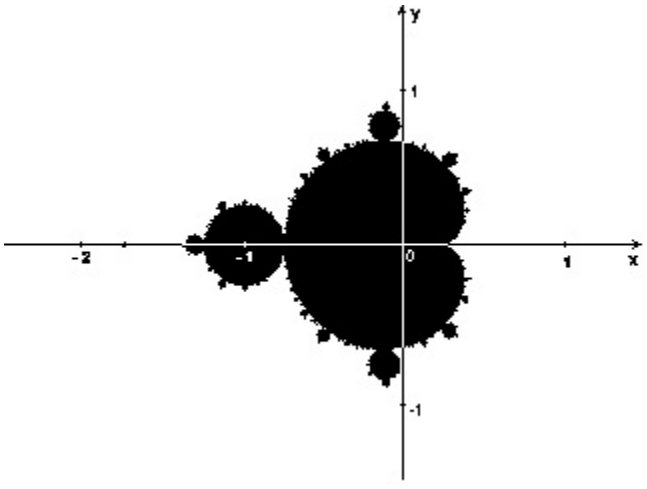

Figure 1 The Mandelbrot set

Our aim in this section is to demonstrate that, when (10) does not hold, the quadratic relation (9) is equivalent to the Mandelbrot iteration (11). As it turns out, c will be a function of the conditional probabilities α, β, γ, and δ alone, and will thus be a known quantity. The q's, on the other hand, will be directly related to the unconditional probabilities; these are unknown and their values are to be determined through the iteration.

First it will prove convenient to define

$$\varepsilon = \frac{1}{2}(\gamma + \delta),$$

which is the mean conditional probability that the target—in our case Barbara-0—has the trait T, given that only one of the parents has T. Equation (9) now becomes

$$\mathrm{P}(E_0) = \beta + 2(\varepsilon - \beta)\ \mathrm{P}(E_1) + (\alpha + \beta - 2\varepsilon)\ \mathrm{P}^2(E_1),$$

or more generally,

$$\mathrm{P}(E_n) = \beta + 2(\varepsilon - \beta)\ \mathrm{P}(E_{n+1}) + (\alpha + \beta - 2\varepsilon)\ \mathrm{P}^2(E_{n+1}). \tag{12}$$

At first sight, this iteration may not look very much like the Mandelbrot form (11). In the latter we go upward as it were, starting from q_n and then counting to q_{n+1}, whereas in (12) we start with $P(E_{n+1})$ and iterate downward to $\mathrm{P}(E_n)$. Moreover, (12) is about conditional and unconditional probabilities, and thus about real numbers between zero and one, whereas (11) is an uninterpreted formula involving complex numbers. On closer inspection, however, we see that there is an important similarity between (11) and (12). For both are quadratic expressions: the former contains $(q_n)^2$ and the latter $\mathrm{P}^2(E_{n+1})$.

In order to transform (12) into (11) we introduce a linear mapping that serves to remove from (12) the term $2(\varepsilon - \beta)\ \mathrm{P}(E_{n+1})$, and also the coefficient $(\alpha + \beta - 2\varepsilon)$. The unique linear mapping that does the trick, $\mathrm{P}(E_n) \rightarrow q_n$, is defined by

$$q_n = (\alpha + \beta - 2\varepsilon)\ \mathrm{P}(E_n) - \beta + \varepsilon. \tag{13}$$

On substituting (12) for $P(E_n)$ in (13) we obtain a formula that can be rewritten as

$$q_n = \varepsilon(1-\varepsilon) - \beta(1-\alpha) + (q_{n+1})^2. \tag{14}$$

The details of the transition from (12) and (13) to (14) can be found in the Appendix. Now define

$$c = \varepsilon(1-\varepsilon) - \beta(1-\alpha). \tag{15}$$

Note that c involves only the conditional probabilities, α, β, and ε, and so is an invariant quantity during the execution of the iteration. On the other hand, q_n also contains the unconditional probability, $P(E_n)$, which we seek to evaluate through the iteration. With the definition (15), Equation (14) becomes

$$q_n = c + (q_{n+1})^2. \tag{16}$$

Evidently (16) is very similar to the standard Mandelbrot iteration (11). There is only one cosmetic difference to which we already alluded: instead of an iteration upward from $n = 0$, the iteration in (16) proceeds from a large n value, corresponding to the primeval parents, down to the target child proposition at $n = 0$. Of course, this difference has no significance for the iteration as such.

We are now in a position to take advantage of some of the lore that has accumulated about the Mandelbrot iteration. *Some* but not all: epistemic justification as we discuss it here deals with probabilities and those are real numbers, rather than complex ones, so we must concentrate on the real subset of the complex numbers c in (15), namely, those for which $c = (x, 0)$, corresponding to the x-axis in Figure 1. It should be noted that, when c is real, all the q_n are automatically real (cf. the explicit expressions for the first few n-values, just after Equation (11)).

It is known that the real interval $-2 \leq c \leq \frac{1}{4}$ lies within the Mandelbrot set, but not all of these values correspond to an iteration that converges to a unique limiting value. However, let us now impose the condition of probabilistic support with exclusion of zero and one, namely,

$$1 > \alpha > \beta > 0. \tag{3′}$$

Then we can show from (15) that $-\frac{1}{4} < c < \frac{1}{4}$ (see the Appendix). In this domain the Mandelbrot iteration is known to converge to a unique limit. If α, β, and ε are such that this limit corresponds to a value of $P(E_0)$ which is greater than a half (or more generally greater than some agreed threshold), then E_0 has been probabilistically justified.

Although (3′) resembles (3), which was the requirement of probabilistic support for the one-dimensional chain, it should be realized that α and β do not have quite the same meanings in the two contexts. For the one-dimensional chain, $\alpha > \beta$ means that the probability of the child's having trait T is greater if the mother has it than if the mother does not have it. For the two-dimensional net, however, $\alpha > \beta$ means that the probability of the child's having trait T is greater if both of her parents have it than if neither of them do. It is interesting in this case that the probability of the child's having T if only one of her parents has T plays no role: ε may have any value between one and zero, including zero itself, for (3′) is a sufficient condition that $c > -\frac{1}{4}$.

5 Concluding Remarks

Present day epistemology is suffused with the idea that justification comes in degrees, but the implication has yet to be fully understood. It is that a proposition can still have a fixed probability, not only when it is justified by an infinite one-dimensional chain, but also by an infinite two-dimensional network. Moreover, this network is generated by the same recursion that produces the Mandelbrot set in the complex plane. True, we have only to do with the real line between $-\frac{1}{4}$ and $+\frac{1}{4}$, and not with the complex plane (where the remarkable fractal structure is apparent). But the point is that the *algorithm* which produces our sequence of probabilities, and that which generates the Mandelbrot fractal, are exactly the same. Note, incidentally, that the real domain extends to $x = \frac{1}{4}$, which is on the edge of the Mandelbrot set; that is, it is a point in the fractal itself.

We have used three simplifying assumptions in proving this, viz., those of gender symmetry, independence, and uniformity. There are, however, strong indications that essentially similar results also hold when these assumptions are dropped. Imagine a situation in which the probabilities are different for males and females, as is the case if we consider, for example, the property of being more than two meters tall. Then there will be two quadratic iterations, one for $\mathrm{P}(E_n(\mathrm{f}))$ and one for $\mathrm{P}(E_n(\mathrm{m}))$. Each of these is related to $\mathrm{P}(E_{n+1}(\mathrm{f}))$ as well as $\mathrm{P}(E_{n+1}(\mathrm{m}))$. This means that the quadratic iterations are coupled, so the fixed points will satisfy quartic rather than quadratic equations. The latter, however, is just a technical complication, for it is still possible to find a domain in which the iterations converge. The relation is in fact a generalized Mandelbrot iteration, being of fourth order, rather than second order, and analogous results obtain. This indicates that the assumption of gender symmetry is not necessary for the argument that probabilistic justification has a Mandelbrot structure.

The same goes for the assumption that the parents are independent. Clearly, if the parent probabilities depend on one another, we may have to include into the equation grandparents, and perhaps great-grandparents, which of course complicates matters considerably. However, in general terms it means nothing more than that the final fixed-point equations will be of order even higher than four. Again a generalized Mandelbrot-style iteration will hold sway, and again domains of convergence exist.

Furthermore, in many situations the conditional probabilities may not be uniform, changing from generation to generation. In those cases the iteration will become considerably more involved. We have seen that for the one-dimensional chain it proved possible to write down explicitly the result of concatenating an arbitrary number of steps. It is true that for a two-dimensional net this would be very cumbersome. However, with the use of a fixed-point theorem it is relatively simple to give conditions under which convergence once more occurs under changing values of α, β, γ, and δ.

What will happen when the network has more dimensions than two? The answer is straightforward: then the iterations and the fixed-point equations are of progressively higher and higher order, necessitating computer programs for their calculation, but the picture remains essentially the same. The probabilities are determined by polynomial recurrent expressions, and there is always a domain in which they are uniquely determined.

We conclude that probabilistic epistemic justification has a structure that gives rise to a generalized Mandelbrot recursion, and this still holds when we abandon

our three simplifying assumptions, and work in more than two dimensions. Like so many other patterns in nature, our reasoning may well have an intriguing relation to simple algorithms that can generate a fractal form.

Appendix

(i) Derivation of (9) from

(8) $\mathrm{P}(E_0) = \alpha\ \mathrm{P}^2(E_1) + \beta\ \mathrm{P}^2(\neg E_1) + (\gamma + \delta)\ \mathrm{P}(E_1)\ \mathrm{P}(\neg E_1).$

Demonstration:

$$\begin{aligned}\mathrm{P}(E_0) &= \alpha\mathrm{P}^2(E_1) + \beta[1 - \mathrm{P}(E_1)]^2 + (\gamma + \delta)\mathrm{P}(E_1)[1 - \mathrm{P}(E_1)] \\ &= \alpha\mathrm{P}^2(E_1) + \beta[1 - 2\mathrm{P}(E_1) + \mathrm{P}^2(E_1)] + (\gamma + \delta)[\mathrm{P}(E_1) - \mathrm{P}^2(E_1)].\end{aligned}$$

Collect all terms involving $\mathrm{P}(E_1)$ together, and likewise all terms involving $\mathrm{P}^2(E_1)$:

(9) $\mathrm{P}(E_0) = \beta + (\gamma + \delta - 2\beta)\ \mathrm{P}(E_1) + (\alpha + \beta - \gamma - \delta)\ \mathrm{P}^2(E_1).$

(ii) Derivation of (14) from

(12) $\mathrm{P}(E_n) = \beta + 2(\varepsilon - \beta)\ \mathrm{P}(E_{n+1}) + (\alpha + \beta - 2\varepsilon)\ \mathrm{P}^2(E_{n+1}).$
(13) $q_n = (\alpha + \beta - 2\varepsilon)\ \mathrm{P}(E_n) - \beta + \varepsilon.$

Demonstration:

$$\begin{aligned}q_n &= (\alpha + \beta - 2\varepsilon)\ \mathrm{P}(E_n) - \beta + \varepsilon \\ &= (\alpha + \beta - 2\varepsilon)\ [\beta + 2(\varepsilon - \beta)\ \mathrm{P}(E_{n+1}) + (\alpha + \beta - 2\varepsilon)\ \mathrm{P}^2(E_{n+1})] - \beta + \varepsilon \\ &= (\alpha + \beta - 2\varepsilon)\ \beta + 2(\varepsilon - \beta)[(\alpha + \beta - 2\varepsilon)\mathrm{P}(E_{n+1})] + \\ &\qquad\qquad [(\alpha + \beta - 2\varepsilon)\mathrm{P}(E_{n+1})]^2 - \beta + \varepsilon.\end{aligned}$$

However, after replacing n by $n + 1$ in (13), we see that

$$[(\alpha + \beta - 2\varepsilon)\mathrm{P}(E_{n+1})] = \beta - \varepsilon + q_{n+1},$$

and therefore

$$\begin{aligned}q_n &= (\alpha + \beta - 2\varepsilon)\ \beta + 2(\varepsilon - \beta)(\beta - \varepsilon + q_{n+1}) + (\beta - \varepsilon + q_{n+1})^2 - \beta + \varepsilon \\ &= \alpha\beta - \varepsilon^2 + (q_{n+1})^2 - \beta + \varepsilon\end{aligned}$$

and so

(14) $q_n = \varepsilon(1 - \varepsilon) - \beta(1 - \alpha) + (q_{n+1})^2.$

(iii) To show that $-\frac{1}{4} < c < \frac{1}{4}$, if $0 \leq \varepsilon \leq 1$ and $0 < \beta < \alpha < 1$, where

(15) $c = \varepsilon(1 - \varepsilon) - \beta(1 - \alpha).$

Demonstration:

(a) $c < \varepsilon(1 - \varepsilon) = \frac{1}{4} - (\frac{1}{2} - \varepsilon)^2 \leq \frac{1}{4}$;
(b) $c \geq -\beta(1 - \alpha) > -\alpha(1 - \alpha) = (\frac{1}{2} - \alpha)^2 - \frac{1}{4} \geq -\frac{1}{4}.$

Notes

1. Often a threshold of acceptance (dependent on the context and greater than one-half) is introduced, and E_0 is said to be probabilistically justified if the probability that E_0 is true is greater than this threshold.

2. We assume for convenience that α and β remain unchanged from bacterial generation to generation. This assumption of uniformity simplifies the calculation, but is not essential.

3. Our claim that "all the probabilistic justification for E_0 now comes from the conditional probabilities" might engender the question how the conditional probabilities themselves are to be justified. (We thank an anonymous referee for having raised this issue.) Although this question is not the subject of the present paper (which only deals with the probabilistic justification of the target proposition E_0), our first, tentative, answer would be that the conditional probabilities are justified by further probabilities, where the latter can be conditional or unconditional. This answer will not satisfy the convinced foundationalist, but we are not sure we understand what sort of answer *would* satisfy her. Of course, one could define "justification" as something that can only come from a source that itself is in no need of justification. But such a definition would, we think, come close to a *petitio*.

4. Of course, in this two-dimensional case Barbara-0 also has a gender, and can in turn become either a father or a mother. In the first case the statement that Barbara-0 has T must be written as $E_0(\mathrm{f})$, in the second case as $E_0(\mathrm{m})$. Since we are going to assume that the presence of T is independent of the gender—see the main text—we will suppress Barbara-0's gender and continue writing E_0.

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