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Elementary Cuts in Saturated Models of Peano Arithmetic

James H. Schmerl

Abstract A model $\mathcal{M} = (M, +, \times, 0, 1, <)$ of Peano Arithmetic (PA) is boundedly saturated if and only if it has a saturated elementary end extension $\mathcal{N}$. The ordertypes of boundedly saturated models of PA are characterized and the number of models having these ordertypes is determined. Pairs $(\mathcal{N}, M)$, where $\mathcal{M} \prec_{\text{end}} \mathcal{N} \models \text{PA}$ for saturated $\mathcal{N}$, and their theories are investigated. One result is: If $\mathcal{N}$ is a κ-saturated model of PA and $\mathcal{M}_0, \mathcal{M}_1 \prec_{\text{end}} \mathcal{N}$ are such that $\aleph_1 \leq \min(\text{cf}(M_0), \text{dcf}(M_0)) \leq \min(\text{cf}(M_1), \text{dcf}(M_1)) < \kappa$, then $(\mathcal{N}, M_0) \equiv (\mathcal{N}, M_1)$.

1 Introduction

Let λ be an infinite cardinal number. A linearly ordered set $(A, <)$ is λ-*dense* if whenever X, Y are (possibly empty) subsets of A such that $X < Y$ and $|X|, |Y| < \lambda$, then there is $a \in A$ such that $X < a < Y$. Then $(A, <)$ is a dense linear order without endpoints (i.e., a model of DLO) if and only if it is $\aleph_0$-dense.

Consider an arbitrary structure $\mathfrak{A} = (A, \ldots)$ in some countable language. As usual, $\mathfrak{A}$ is λ-saturated if and only if whenever $C \subseteq A$, $|C| < \lambda$, and $\Sigma(v)$ is a set of unary formulas $\varphi(v)$ with parameters from C that is finitely realized in $\mathfrak{A}$, then $\Sigma(v)$ is realized in $\mathfrak{A}$. Also, $\mathfrak{A}$ is saturated if and only if it is $|A|$-saturated. If $(A, <)$ is a model of DLO, then $(A, <)$ is λ-saturated if and only if $(A, <)$ is λ-dense.

Let $\mathcal{M}$ be a nonstandard model of PA. Define an equivalence relation on the set of nonstandard elements of $\mathcal{M}$ by: x, y are equivalent if and only if either $x = y + n$ or $y = x + n$ for some $n < \omega$. Let $[x]$ be the equivalence class to which x belongs, and then let $[M]$ be the set of all equivalence classes. The ordered set $([M], <)$ is the *reduced ordered set* of $\mathcal{M}$, and it is a model of DLO. Its ordertype ρ is the *reduced ordertype* of $\mathcal{M}$, which is the unique ordertype ρ such that $(M, <)$ has ordertype

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$\omega + \mathbb{Z}\cdot\rho$. For any infinite λ, $(M, <)$ is λ-saturated if and only if $([M], <)$ is λ-dense. Pabion [7] proved the following remarkable theorem concerning ordertypes of λ-saturated models.

Theorem 1.1 (Pabion's Theorem) *Let λ be an uncountable cardinal and $\mathcal{M}$ be a model of* PA. *Then $\mathcal{M}$ is λ-saturated if and only if $(M, <)$ is λ-saturated.*

In other words, $\mathcal{M}$ is λ-saturated if and only if its reduced ordered set is λ-dense. In particular, $\mathcal{M}$ is saturated if and only if its reduced ordertype is η_κ for some uncountable κ. Here, η_κ, if it exists, is the unique ordertype of a κ-dense ordered set of cardinality κ. Thus, if κ is uncountable and $\mathcal{M}, \mathcal{N}$ are elementarily equivalent, uncountable models of PA each having reduced ordertype η_κ, then they are isomorphic.

We say that a model $\mathcal{M}$ of PA is *boundedly λ-saturated* if, whenever $b \in C \subseteq M$, $|C| < \lambda$, $\Sigma(v)$ is a set of unary formulas $\varphi(v)$ with parameters from C that is finitely realized in $\mathcal{M}$, and the formula $v < b$ is in $\Sigma(v)$, then $\Sigma(v)$ is realized in $\mathcal{M}$. Clearly, every λ-saturated model of PA is boundedly λ-saturated, and if $\mathcal{N}$ is boundedly λ-saturated and $\mathcal{M} \prec_{\mathsf{end}} \mathcal{N}$, then $\mathcal{M}$ is boundedly λ-saturated. An uninteresting example is the standard model, which is boundedly λ-saturated for all λ. However, if $\mathcal{M}$ is nonstandard and boundedly $\aleph_0$-saturated, then $|M| \geq 2^{\aleph_0}$. If $\mathcal{M}$ is boundedly $|M|$-saturated, then it is *boundedly saturated*.

The following theorem gives an alternate characterization of boundedly saturated models of PA.

Theorem 1.2 *If $\mathcal{M}$ is a nonstandard model of* PA*, then $\mathcal{M}$ is boundedly saturated if and only if there is a saturated $\mathcal{N}$ such that $\mathcal{M} \prec_{\mathsf{end}} \mathcal{N}$.*

Proof Suppose $\mathcal{N}$ is saturated and $\kappa = |N|$. Then $\mathcal{N}$ is κ-saturated, so that if $\mathcal{M} \prec_{\mathsf{end}} \mathcal{N}$, then $\mathcal{M}$ is boundedly κ-saturated. Since $|M| \leq \kappa$, then $\mathcal{M}$ is boundedly saturated.

For the converse, suppose $\mathcal{M}$ is nonstandard and boundedly saturated of cardinality κ. Then $\kappa \geq 2^{\aleph_0}$. Let $a \in M$ be nonstandard. Then $(\{[x] \in [M] : x < [a]\}, <)$ is a saturated model of DLO of cardinality κ, so that η_κ exists and κ is uncountable and regular. Therefore, there is a saturated $\mathcal{N} \equiv \mathcal{M}$ such that $|N| = \kappa$. By a back-and-forth construction, we will obtain an elementary map $f : \mathcal{M} \longrightarrow \mathcal{N}$ that is onto a proper initial segment of $\mathcal{N}$.

Let $\mathcal{N}_0 \prec_{\mathsf{end}} \mathcal{N}$ be such that $\mathcal{N}_0 \cong \mathcal{N}$, the existence of which follows from the resplendency of $\mathcal{N}$. Assume that both M and N_0 are well-ordered with ordertype κ. When referring to the first element of a subset of M or of N_0, we will mean with respect to these well orderings.

We will obtain an increasing sequence $\langle f_i : i < \kappa\rangle$ of elementary maps $f_i : X_i \longrightarrow \mathcal{N}_0$, where $X_i \subseteq M$ and $|X_i| < \kappa$. To get started, let $a \in M$ be nonstandard, $b \in N_0$ realizing the same type as a, $X_0 = \{a\}$ and $f_0(a) = b$. If $i < \kappa$ is a limit ordinal, then let $X_i = \bigcup\{X_j : j < i\}$ and $f_i = \bigcup\{f_j : j < i\}$.

For the case of a successor ordinal, suppose we already have $f_i : X_i \longrightarrow N_0$. Let $a \in M$ be the first element of $M\backslash X_i$, and then, by the saturation of $\mathcal{N}_0$, let $b \in N_0$ be such that $f_i \cup \{\langle a, b\rangle\}$ is an elementary map. Next, let $b' \in N_0$ be the first element in $\{y \in N_0 : y < f_i(x), x \in X_i\}\backslash\{f_i(x) : x \in X_i \cup \{a\}\}$, and then, by the bounded saturation of $\mathcal{M}$, let $a' \in M$ be such that $f_i \cup \{\langle a, b\rangle, \langle a', b'\rangle\}$ is an elementary map. Let $X_{i+1} = X_i \cup \{a, a'\}$ and $f_{i+1} = f_i \cup \{\langle a, b\rangle, \langle a', b'\rangle\}$. Then $f = \bigcup\{f_i : i < \kappa\}$ is as required. □

Thus, if $\mathcal{M}$ is boundedly saturated and $|M| = \kappa$, then η_κ exists. As is well known, η_κ exists if and only if κ is regular and $2^\lambda \leq \kappa$ whenever $\lambda < \kappa$. We assume until the end of Section 4 that

κ is an uncountable cardinal and η_κ exists.

In particular, κ is regular.

2 Boundedly Saturated Models

This section begins with a generalization of Pabion's Theorem to boundedly λ-saturated models. The proof involves nothing beyond the proof of Pabion's Theorem. I recommend that the reader look at the proof of Pabion's Theorem in [6] and then verify that, with the obvious adjustments, it also proves Theorem 2.1. A linearly ordered set $(A, <)$ is *boundedly λ-dense* if whenever X, Y are nonempty subsets of A such that $X < Y$ and $|X|, |Y| < \lambda$, then there is $a \in A$ such that $X < a < Y$.

Theorem 2.1 *Let λ be an uncountable cardinal, and let $\mathcal{M}$ be a model of* PA. *Then $\mathcal{M}$ is boundedly λ-saturated if and only if its reduced ordered set is boundedly λ-dense.*

If $\mathcal{M}$ is a model of PA, then its cofinality $\mathrm{cf}(\mathcal{M})$ is the least cardinality λ of an unbounded subset $X \subseteq M$. More generally, if $\mathcal{N}$ is a model of PA and $I \subseteq N$ a cut (that is, a nonempty, proper initial segment closed under successors), then $\mathrm{cf}(I)$ is the least cardinality λ of a cofinal $X \subseteq I$, and its downward cofinality $\mathrm{dcf}(I)$ is the least cardinality μ of a downward cofinal $X \subseteq N \backslash I$ (that is, such that whenever $I < y \in N$, then there is $x \in X$ such that $x < y$). If $\mathrm{dcf}(I) = \mu$ and $\mathrm{cf}(I) = \lambda$, we will refer to I as a (μ, λ)-cut. If I is a (λ, λ)-cut for some λ, then it is *balanced*, and it is *unbalanced* otherwise.

It follows from Theorem 1.2 of Section 1 that all boundedly saturated models of PA having cardinality κ have reduced ordertype $\eta_\kappa \cdot \lambda$, where $\lambda = \mathrm{cf}(\mathcal{M})$. We will look at the seemingly wider class of models of PA having reduced ordertype $\eta_\kappa \cdot \gamma$, where $|\gamma| \leq \kappa$, or, what is equivalent, having reduced ordertype $\eta_\kappa \cdot (\mu^* + \lambda)$, where $\mu, \lambda \leq \kappa$ are regular cardinals.

Theorem 2.2 *If $\mathcal{M}$ is a nonstandard model of* PA, *then the following are equivalent:*

1. *(1) $\mathcal{M}$ is boundedly saturated and $|M| = \kappa$;*
2. *(2) there is an ordertype γ such that $0 < |\gamma| \leq \kappa$ and $\mathcal{M}$ has reduced order type $\eta_\kappa \cdot \gamma$;*
3. *(3) there is a regular infinite cardinal $\lambda \leq \kappa$ such that $\mathcal{M}$ has reduced ordertype $\eta_\kappa \cdot \lambda$.*

Proof (3) $\Rightarrow$ (2) is trivial.

(2) $\Leftrightarrow$ (1) follows from Theorem 2.1 since an ordered set has ordertype $\eta_\kappa \cdot \gamma$ for some γ where $0 < |\gamma| \leq \kappa$ if and only if it is boundedly κ-dense and has cardinality κ.

(2) $\Rightarrow$ (3) Suppose that $\mathcal{M}$ has reduced ordertype $\eta_\kappa \cdot \gamma$. Then $|M| = \kappa$ and $\mathcal{M}$ has reduced ordertype $\eta_\kappa \cdot (\mu^* + \lambda)$, where $\mu = \mathrm{dcf}(\omega)$ and $\lambda = \mathrm{cf}(M)$. Let $a \in M$ be nonstandard. Then the cut $I = \{y \in M : y < a - n, n < \omega\}$ is a $(\mu, \aleph_0)$-cut. Thus, the initial segment $\{[x] \in [M] : [x] < [a]\}$ of $[M]$ has cofinality μ, so $\mu = \kappa$. Thus, $\mathcal{M}$ has reduced ordertype $\eta_\kappa \cdot \lambda$. □

3 How Many Boundedly Saturated Models?

Theorem 2.2 characterizes the boundedly saturated models of PA having cardinality κ as those having ordertype $\eta_\kappa \cdot \lambda$, where $\lambda \le \kappa$ is regular. The following theorem gives the number of models up to isomorphism having ordertype $\eta_\kappa \cdot \lambda$.

Theorem 3.1 *Suppose that T is a consistent completion of* PA.

(1) Up to isomorphism, there are exactly $2^{\aleph_0}$ models of T having reduced ordertype $\eta_\kappa \cdot \omega$.

(2) If $\lambda \le \kappa$ is an uncountable regular cardinal, then, up to isomorphism, there is exactly one model of T having reduced ordertype $\eta_\kappa \cdot \lambda$.

We will give a more refined version of Theorem 3.1 for which some definitions are needed. Suppose $\mathcal{M}$ is a model of PA. For $a \in M$, the *gap* containing a is the set $\operatorname{gap}(a)$ consisting of those $b \in M$ such that for any elementary cut $I \subseteq_{\mathsf{end}} M$, $a \in I$ if and only if $b \in I$. The set of gaps is a partition of M into convex sets. If $\mathcal{M}$ has a last gap, then $\mathcal{M}$ is *short*, and $\mathcal{M}$ is *tall* if it is not short. The *gaptype* of $\operatorname{gap}(a)$ is the set of types realized by elements of $\operatorname{gap}(a)$. If $\mathcal{M}$ is boundedly $\aleph_0$-saturated, then the set of its gaptypes is a partition of the set of its 1-types, and there are $2^{\aleph_0}$ distinct gaptypes. (This last fact follows from Theorem 4.5 or can be seen as a consequence of [6, Exercise 3.6.9].)

Theorem 3.2 *Suppose $\mathcal{M}_0, \mathcal{M}_1$ are boundedly saturated models of* PA *having cardinality κ. Then, $\mathcal{M}_0 \cong \mathcal{M}_1$ if and only if $\mathcal{M}_0 \equiv \mathcal{M}_1$ and one of the following holds:*

(1) $\mathcal{M}_0, \mathcal{M}_1$ are short and their last gaptypes are the same;

(2) $\mathcal{M}_0, \mathcal{M}_1$ are tall and $\operatorname{cf}(\mathcal{M}_0) = \operatorname{cf}(\mathcal{M}_1)$.

Proof Clearly, if $\mathcal{M}_0 \cong \mathcal{M}_1$, then $\mathcal{M}_0 \equiv \mathcal{M}_1$ and either (1) or (2) holds. We prove the converse. Suppose that $\mathcal{M}_0 \equiv \mathcal{M}_1$.

Right now, we will prove the converse only in the special case that $\operatorname{cf}(\mathcal{M}_0) = \operatorname{cf}(\mathcal{M}_1) = \kappa$. In this case, it is clear that both $\mathcal{M}_0$ and $\mathcal{M}_1$ are saturated, and then that $\mathcal{M}_0 \cong \mathcal{M}_1$. All the remaining cases will follow immediately from Theorem 3.3. $\square$

Theorem 3.2 answers the question of how many boundedly saturated models $\mathcal{M}$ of T there are, where $|M| = \kappa$ and $\operatorname{cf}(\mathcal{M}) = \lambda$. We next consider the number of pairs $(\mathcal{N}, M)$, where $\mathcal{M} \prec_{\mathsf{end}} \mathcal{N}$ and $\mathcal{N}$ is saturated. Some more definitions are needed.

If $\mathcal{M} \prec_{\mathsf{end}} \mathcal{N}$, we will say that the elementary cut M is *short* or *tall* if $\mathcal{M}$ is short or tall. We will also say that the elementary cut M is *coshort* if $N \backslash M$ has a first gap and that it is *cotall* otherwise. Clearly, if M is short, then $\operatorname{cf}(M) = \aleph_0$, and if M is coshort, then $\operatorname{dcf}(M) = \aleph_0$. (In fact, if $a \in N$, then $\operatorname{Scl}(a) \cap \operatorname{gap}(a)$ is a countable set that is both cofinal and downward cofinal in $\operatorname{gap}(a)$. Here, $\operatorname{Scl}(a)$, the Skolem closure of a, is the set of those $x \in N$ that are definable in $\mathcal{N}$ using only the parameter a.)

There are various ways that the term "λ-homogeneous" is defined in the literature. We will be using the one where a structure $\mathfrak{A}$ is λ-*homogeneous* if, whenever $\gamma < \lambda$ and $\langle a_i : i < \gamma \rangle$, $\langle b_i : i < \gamma \rangle$ are sequences from A such that $\operatorname{tp}(\langle a_i : i < \gamma \rangle) = \operatorname{tp}(\langle b_i : i < \gamma \rangle)$ (or, equivalently, $f_0 = \{\langle a_i, b_i \rangle : i < \gamma\}$ is a partial elementary map), then there is an automorphism f of $\mathfrak{A}$ such that $f \supseteq f_0$. If $\mathfrak{A}$ has cardinality λ and is λ-homogeneous, then it is *homogeneous*. As is well known, every saturated structure is homogeneous.

Theorem 3.3 *Suppose $\mathcal{M}_0, \mathcal{M}_1 \prec_{\mathrm{end}} \mathcal{N} \models \mathsf{PA}$ are such that $\mathcal{N}$ is saturated of cardinality κ. Then $(\mathcal{N}, M_0) \cong (\mathcal{N}, M_1)$ if one of the following hold:*

(1) $\mathcal{M}_0, \mathcal{M}_1$ are short and their last gaps have the same gaptype;
(2) $\mathcal{M}_0, \mathcal{M}_1$ are coshort and the first gaps of $N\backslash M_0$ and $N\backslash M_1$ have the same gaptype;
(3) $\mathcal{M}_0, \mathcal{M}_1$ are tall and $\mathrm{cf}(M_0) = \mathrm{cf}(M_1) < \kappa$;
(4) $\mathcal{M}_0, \mathcal{M}_1$ are cotall and $\mathrm{dcf}(M_0) = \mathrm{dcf}(M_1) < \kappa$.

Proof If $\mathcal{M}_0$ is standard, then it is short and so only (1) applies. Since $\mathcal{M}_1$ has the same gaptype as $\mathcal{M}_0$, it also is standard, so $(\mathcal{N}, M_0) \cong (\mathcal{N}, M_1)$. So now assume that both $\mathcal{M}_0, \mathcal{M}_1$ are nonstandard.

In each case, the idea is to use the homogeneity of $\mathcal{N}$ to obtain an automorphism f that extends a particular partial elementary map f_0 with $|\mathrm{dom}(f_0)| < \kappa$ that is chosen so as to assure that $f[M_0] = M_1$.

(1) Let a be in the last gap of M_0, and then let b in the last gap of M_1 realize the same type as a does. Let $f_0 = \{\langle a, b\rangle\}$.
(2) Let a be in the first gap of $N\backslash M_0$, and then let b in the first gap of $N\backslash M_1$ realize the same type as a. Let $f_0 = \{\langle a, b\rangle\}$.

For (3) and (4), let $p(x)$ be a minimal type realized in $\mathcal{N}$. (See Chapter 3.2 and especially Theorem 3.2.10 of [6].)

(3) Let $\lambda = \mathrm{cf}(\mathcal{M}_0) = \mathrm{cf}(\mathcal{M}_1) < \kappa$. Let $\langle a_i : i < \lambda\rangle$ and $\langle b_i : i < \lambda\rangle$ be increasing sequences of elements realizing $p(x)$ that are cofinal in M_0 and M_1, respectively. Let $f_0 = \{\langle a_i, b_i\rangle : i < \lambda\}$.
(4) Let $\mu = \mathrm{dcf}(\mathcal{M}_0) = \mathrm{dcf}(\mathcal{M}_1) < \kappa$. Let $\langle a_i : i < \mu\rangle$ and $\langle b_i : i < \mu\rangle$ be decreasing sequences of elements realizing $p(x)$ that are downward cofinal in $N\backslash M_0$ and $N\backslash M_1$, respectively. Let $f_0 = \{\langle a_i, b_i\rangle : i < \mu\}$. □

Note that Theorem 3.3 fails to cover the situation when $\mathrm{cf}(M_0) = \mathrm{cf}(M_1) = \mathrm{dcf}(M_0) = \mathrm{dcf}(M_1) = \kappa$; that is, both M_0, M_1 are balanced cuts. Theorem 4.6 implies that there are at least $2^{\aleph_0}$ different isomorphism types of $(\mathcal{N}, M)$, where $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N}$ is a (κ, κ)-cut.

Question 3.4 *Up to isomorphism, how many pairs $(\mathcal{N}, M)$, where $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N}$ and M is balanced, are there?*

4 Theories of Pairs, I

In this and the next section we consider the possible theories of pairs $(\mathcal{N}, M)$, where $\mathcal{N}$ is a saturated model of PA and $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N}$. We begin with a simple lemma.

Lemma 4.1 *Suppose that $\mathcal{N}$ is saturated and $I \subseteq_{\mathrm{end}} \mathcal{N}$ is an unbalanced cut. Then, ω is uniformly definable in $(\mathcal{N}, I)$.*

Proof Just observe that $a \in \omega$ if and only if there is no x such that $I \cap \{(x)_n : n < a\}$ is cofinal in I or $\{(x)_n : n < a\}\backslash I$ is downward cofinal in $N\backslash I$. This definition is independent of $\mathcal{N}$ and I. (The referee has pointed out that by slightly modifying this proof we can show that $\mathrm{cf}^{\mathcal{N}}(I) = \omega$. See [6, p. 181] for the definition.) □

Theorem 4.2 *There are sentences $\sigma_1, \sigma_2, \sigma_3$ such that whenever $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N} \models \mathsf{PA}$, where $\mathcal{N}$ is saturated and $\mathcal{M}$ is unbalanced, then*

(1) $(\mathcal{N}, M) \models \sigma_1$ *iff* $\mathcal{M}$ *is short;*
(2) $(\mathcal{N}, M) \models \sigma_2$ *iff* $\mathrm{cf}(M) = \aleph_0$*;*
(3) $(\mathcal{N}, M) \models \sigma_3$ *iff* $\mathrm{dcf}(M) = \aleph_0$*.*

Proof (Essentially from Smoryński [8], Theorem 2.11) Since $\mathcal{M}$ is unbalanced, by Lemma 4.1, ω is uniformly definable in $(\mathcal{N}, M)$. Thus, there are σ_2, σ_3 such that whenever $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N}$ is unbalanced, then $(\mathcal{N}, M) \models \sigma_2$ if and only if $\mathrm{cf}(M) = \aleph_0$, and $(\mathcal{N}, M) \models \sigma_3$ if and only if $\mathrm{dcf}(M) = \aleph_0$. Also, there is then a uniform way to define satisfaction in $\mathcal{M}$, from which it is easy to get a sentence σ_1 such that $(\mathcal{N}, M) \models \sigma_1$ if and only if $\mathcal{M}$ is short. □

At first glance, it might appear that Theorem 4.2 should have a part (4) asserting that $(\mathcal{N}, M) \models \sigma_4$ if and only if $\mathcal{M}$ is coshort. However, this is not so. The following theorem is a very slight variant of Theorem 6.1 of Kossak and Kotlarski [4]; the proof in [4] also proves Theorem 4.3.

Theorem 4.3 (Kossak and Kotlarski) *Suppose that* $\mathcal{N}$ *is recursively saturated. Let* $b, c \in N$ *be such that* $b > \mathrm{Scl}(0)$ *and* $\mathrm{gap}((c)_i) > \mathrm{gap}((c)_{i+1})$ *for all* $i < \omega$*, and then let* $\mathcal{M}_0 = \inf(\mathrm{gap}(b))$ *and* $\mathcal{M}_1 = \inf\{(c)_i : i < \omega\}$*. Then* $(\mathcal{N}, M_0) \equiv (\mathcal{N}, M_1)$*.*

Corollary 4.4 *Suppose that* $\mathcal{N}$ *is saturated, and* $\mathcal{M}_0, \mathcal{M}_1 \prec_{\mathrm{end}} \mathcal{N}$ *are such that* $\mathrm{dcf}(M_0) = \mathrm{dcf}(M_1) = \aleph_0$*. Then* $(\mathcal{N}, M_0) \equiv (\mathcal{N}, M_1)$*.*

Proof Since $\mathcal{N}$ is saturated, there is $b \in N$ such that either $b > \mathrm{Scl}(0)$ and either $M_0 = \inf(\mathrm{gap}(b))$ or $\mathrm{gap}((b)_i) > \mathrm{gap}((b)_{i+1})$ for all $i < \omega$ and $M_0 = \inf\{(b)_i : i < \omega\}$. There is $c \in N$ that determines M_1 in an analogous way. Apply Theorem 4.3 to get that $(\mathcal{N}, M_0) \equiv (\mathcal{N}, M_1)$. □

Theorem 4.5 *Suppose that* $\mathcal{N} \models \mathsf{PA}$ *is saturated. Then there are* $2^{\aleph_0}$ *different theories of* $(\mathcal{N}, M)$*, where* $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N}$ *is short.*

Proof There are recursive sequences $\langle \varphi_i(v) : i < \omega \rangle$ and $\langle \theta_i(v) : i < \omega \rangle$ of formulas such that whenever T is a consistent completion of PA and $I \subseteq \omega$, then there is a unique complete 1-type $p_I(v) \supseteq T \cup \{\varphi_i(v) : i < \omega\} \cup \{\theta_i(v) : i \in I\} \cup \{\neg\theta_i(v) : i \in \omega \backslash I\}$. Moreover, each $p_I(v)$ is a minimal type and, whenever $I \neq J$, then $p_I(v)$, $p_J(v)$ are independent (that is, they cannot be realized in the same gap of a model of T).

Here is a sketch of the construction of the $\varphi_i(v)$ and $\theta_i(v)$, leaving to the reader the verification that they are as required. Let $\langle \alpha_i(x, y) : i < \omega \rangle$ be a recursive list of all 2-ary formulas in the language of PA, and let $\langle t_i(x) : i < \omega \rangle$ be a recursive list of all Skolem terms in the single variable x. We define $\varphi_i(v), \theta_i(v)$ inductively on i. We use the following notation: if $J \subseteq i < \omega$, $\varphi(v)$ is any formula, and $\theta_0(v), \theta_1(v), \ldots, \theta_{i-1}(v)$ have already been defined, then we let

$$\varphi^{J,i}(v) = \varphi(v) \wedge \bigwedge_{j \in J} \theta_j(v) \wedge \bigwedge_{j \in i \backslash J} \neg\theta_j(v) .$$

Let $\varphi_0(v)$ be $v = v$. Suppose we already have $\varphi_i(v)$ and $\theta_0(v), \ldots, \theta_{i-1}(v)$.

We first obtain $\varphi_{i+1}(v)$. Let $\varphi_{i+1}(v)$ be such that PA proves

$$\forall v[\varphi_{i+1}(v) \longrightarrow \varphi_i(v)] \tag{1}$$

and

$$\forall uv[\varphi_{i+1}(u) \wedge \varphi_{i+1}(v) \wedge u < v \longrightarrow t_i(u) < v]\,, \tag{2}$$

and such that for all $J \subseteq i$, PA proves

$$\forall w \exists v > w[\varphi^{J,i}_{i+1}(v)] \tag{3}$$

and

$$\forall v_0 < v_1 < v_2\big[\big(\bigwedge_{k<2} \varphi^{J,i}_{i+1}(v_k)\big) \longrightarrow \big(\alpha_i(v_0, v_1) \leftrightarrow \alpha_i(v_1, v_2)\big)\big]\,. \tag{4}$$

Next, define $\theta_i(v)$ to be the formula

$$\bigwedge_{J \subseteq i} \big(\varphi^{J,i}_{i+1}(v) \longrightarrow \text{“}|\{u : u < v \wedge \varphi^{J,i}_{i+1}(v)\}| \text{ is even”}\big)\,. \tag{5}$$

Let $I \subseteq \omega$. It follows from sentences (1) and (3) that $p_I(v)$ is a consistent type. Sentence (4) guarantees that this type is 2-indiscernible and, hence, complete. Because of sentence (3), this type is unbounded and, therefore, minimal. (See [6, Chap. 3].) From (2) we get that distinct $I, J \subseteq \omega$ yield independent minimal types.

Since the sequences $\langle \varphi_i(v) : i < \omega \rangle$ and $\langle \theta_i(v) : i < \omega \rangle$ are recursive, there is a formula $\delta_0(v)$ in the language of $(\mathcal{N}, \omega)$ that defines the set of those $a \in N$ that realizes some $p_I(v)$.

Now, let $a \in N$ realize $p_I(v)$, and let $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ be short with gap(a) being its last gap. Since ω is uniformly definable in $(\mathcal{N}, M)$, there is a formula $\delta_1(v)$ in the language of $(\mathcal{N}, M)$ (independent of M and I) that defines a in $(\mathcal{N}, M)$. Thus, there is a formula $\beta(x)$ (also in the language of $(\mathcal{N}, M)$ and independent of M and I) that defines I in $(\mathcal{N}, M)$. □

Theorems 3.2(1) and 4.2(1) suggest the question of whether there are short $\mathcal{M}_0, \mathcal{M}_1 \prec_{\text{end}} \mathcal{N}$, where $\mathcal{N}$ is saturated of cardinality κ, such that $(\mathcal{N}, M_0) \equiv (\mathcal{N}, M_1)$ and $(\mathcal{N}, M_0) \not\cong (\mathcal{N}, M_1)$. The almost equivalent question for countable recursively saturated $\mathcal{N}$ was originally asked by Smoryński [8] and recently repeated in [5].

Theorem 4.6 *Suppose that $\mathcal{N} \models$ PA is saturated. Then there are $2^{\aleph_0}$ different theories of $(\mathcal{N}, M)$, where $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ and M is balanced.*

Proof By Theorem 4.5 (or, according to [3] and [8]), there are $2^{\aleph_0}$ different theories of $(\mathcal{N}', M')$, where $\mathcal{M}' \prec_{\text{end}} \mathcal{N}' \equiv \mathcal{N}$. For each such $(\mathcal{N}', M')$, let $(\mathcal{N}'', M'') \equiv (\mathcal{N}', M')$ be saturated of cardinality $\kappa = |N|$ so that M'' is a (κ, κ)-cut. Then $\mathcal{N}'' \cong \mathcal{N}$, so we can arrange that $\mathcal{M}'' \prec_{\text{end}} \mathcal{N} = \mathcal{N}''$. □

The case of unbalanced cuts will be considered in the next section.

5 Theories of Pairs, II

The main result of this section is Theorem 5.5 concerning unbalanced elementary cuts. This section comprises two subsections, the first of which discusses some preliminary combinatorial results and the second of which contains the main result and its proof.

5.1 Some combinatorics For this subsection, let $\mathcal{N}$ be a fixed κ-saturated model of PA. Also, let $\lambda < \kappa$ be a regular, uncountable cardinal. Let M_0, M_1 be (not necessarily elementary) cuts of $\mathcal{N}$ such that $\mathrm{cf}(M_0) = \mathrm{dcf}(M_1) = \lambda$. In the next subsection, where we apply the results of this subsection, we will be interested only in elementary cuts M_0, M_1. By not requiring the cuts M_0, M_1 to be elementary in this subsection, we will avoid having to repeat some arguments.

Suppose $k < \omega$. If $H \subseteq N$, then $[H]^k = \{\langle x_0, x_1, \ldots, x_{k-1}\rangle \in H^k : x_0 < x_1 < \cdots < x_{k-1}\}$. If $S \subseteq N^k$, then H is *homogeneous* for S if either $[H]^k \subseteq S$ or $[H]^k \cap S = \varnothing$.

More generally, suppose that $\mathcal{S}$ is a set of relations on N; that is, for every $S \in \mathcal{S}$ there is $k < \omega$ such that $S \subseteq N^k$. Then, $H \subseteq N$ is *homogeneous* for $\mathcal{S}$ if H is homogeneous for each $S \in \mathcal{S}$. If H is homogeneous for $\mathcal{S}$, then a function $\chi : \mathcal{S} \longrightarrow \{0, 1\}$ is an *$\mathcal{S}$-character* of H if, for each $S \in \mathcal{S}$ such that $S \subseteq N^k$, $\chi(S) = 1$ if and only if $[H]^k \subseteq S$. If H is an infinite homogeneous set for $\mathcal{S}$, then it has a unique $\mathcal{S}$-character.

Lemma 5.1 *Let $\alpha \leq \lambda$ be a limit ordinal. For each $\nu < \alpha$, let S_ν be a definable relation on N, and then for each $\mu \leq \alpha$, let $\mathcal{S}_\mu = \{S_\nu : \nu < \mu\}$. For each $\nu < \alpha$, let $X_\nu \subseteq_{\mathsf{cof}} M_0$ be a homogeneous set for $\mathcal{S}_\nu$ such that whenever $\nu < \mu < \alpha$, then the $\mathcal{S}_\nu$-character for X_ν is the same as the $\mathcal{S}_\nu$-character for X_μ. Then, there is $X \subseteq_{\mathsf{cof}} M_0$ such that X is homogeneous for $\mathcal{S}_\alpha$ and, whenever $\nu < \alpha$, then the $\mathcal{S}_\nu$-character of X is the same as $\mathcal{S}_\nu$-character of X_ν.*

Proof Using the κ-saturation of $\mathcal{N}$, get an increasing sequence $\langle z_\mu : \mu < \lambda\rangle$ that is cofinal in M_0 such that for each $\mu < \lambda$ and $\nu < \alpha$ there is $x \in X_\nu$ such that $z_\mu < x < z_{\mu+1}$. Making use of κ-saturation, we can get $X = \{x_\mu : \mu < \lambda\}$ such that

(1) $z_\mu < x_\mu < z_{\mu+1}$ for all $\mu < \lambda$;
(2) X is homogeneous for $\mathcal{S}_\alpha$;
(3) for each $\nu < \alpha$, the $\mathcal{S}_\nu$-character of X is the same as the $\mathcal{S}_\nu$-character of X_ν.

The x_μs must satisfy a set of λ formulas in the variables v_μ ($\mu < \lambda$) allowing the parameters z_μ ($\mu < \lambda$). The set of formulas is easily seen to be finitely satisfiable. By the κ-saturation of $\mathcal{N}$, such x_μs can then be obtained. This X is as required. □

Lemma 5.1 has a dual version.

Lemma 5.2 *Let $\alpha \leq \lambda$ be a limit ordinal. For each $\nu < \alpha$, let S_ν be a definable relation on N, and then for each $\mu \leq \alpha$, let $\mathcal{S}_u = \{S_\nu : \nu < \mu\}$. For each $\nu < \alpha$, let $X_\nu \subseteq N$ be a homogeneous set for $\mathcal{S}_\nu$ such that whenever $\nu < \mu < \alpha$, then $M_1 = \inf(X_\nu)$ and the $\mathcal{S}_\nu$-character for X_ν is the same as the $\mathcal{S}_\nu$-character for X_μ. Then, there is $X \subseteq N$ such that X is homogeneous for $\mathcal{S}_\alpha$, $M_1 = \inf(X)$ and, whenever $\nu < \alpha$, then the $\mathcal{S}_\nu$-character of X is the same as $\mathcal{S}_\nu$-character of X_ν.*

Proof This lemma can proved in the same way that Lemma 5.1 was. However, instead, we will "reflect" to deduce it from Lemma 5.1.

Let $d > M_1$. For each relation $S \subseteq N^n$ let $S' = \{\langle x_0, x_1, \ldots, x_{n-1}\rangle \in [0, d]^n : \langle d - x_0, d - x_1, \ldots, d - x_{n-1}\rangle \in S\}$. In particular, if $A \subseteq N$, then $A' = \{x \leq d : d - x \in A\}$. For each $\nu \leq \alpha$, let $\mathcal{S}'_\nu = \{S'_\mu : \mu < \nu\}$. Let $M_1^* = (N \backslash M_1)'$. Thus, M_1^* is a cut of N and $\mathrm{cf}(M_1^*) = \lambda$. Also, $X'_\nu \subseteq_{\mathsf{cof}} M_1^*$ is homogeneous for $\mathcal{S}'_\nu$ and the $\mathcal{S}'_\nu$-character for X'_ν is the same as the $\mathcal{S}'_\nu$-character for X'_μ whenever

$\nu < \mu < \alpha$. Apply Lemma 5.1 to get $Y \subseteq_{\mathrm{cof}} M_1^*$ that is homogeneous for $\mathcal{S}'_\alpha$ such that the $\mathcal{S}'_\nu$-character of Y is the same as the $\mathcal{S}'_\nu$-character of X'_ν. Then, $X = Y'$ is as required. □

Lemma 5.3 *Suppose $n < \omega$ and $R \subseteq N^n$ is definable. Let $\mathcal{S}$ be a set of definable relations on N such that $|\mathcal{S}| < \lambda$. Let $X \subseteq_{\mathrm{cof}} M_0$ be homogeneous for $\mathcal{S}$. Then there is $H \subseteq_{\mathrm{cof}} M_0$ that is homogeneous for $\mathcal{S} \cup \{R\}$ such that the $\mathcal{S}$-character of H is the same as the $\mathcal{S}$-character of X.*

Proof The proof is by induction on n. If $n = 0$, then let $H = X$. If $n = 1$, then let $H = X \cap R$ or let $H = X \backslash R$, whichever is a cofinal subset of M_0. Now let $n > 1$ and assume that the lemma holds for all smaller values.

We can assume that $X = \{x_\alpha : \alpha < \lambda\}$, where $x_\alpha < x_\beta$ whenever $\alpha < \beta < \lambda$. Inductively, we will obtain $\langle y_\alpha : \alpha < \lambda\rangle$ such that for every $\alpha < \lambda$:

(1) $x_\alpha \leq y_\alpha \in M_0$;
(2) whenever $\nu < \alpha$, the $y_\nu < y_\alpha$.

Having y_α, let $R_\alpha = \{\langle a_1, a_2, \ldots, a_{n-1}\rangle \in N^{n-1} : \langle y_\alpha, a_1, a_2, \ldots, a_{n-1}\rangle \in R\}$. Along with the y_αs, we will also obtain $\langle Y_\alpha : \alpha < \lambda\rangle$ such that for every $\alpha < \lambda$,

(3) $y_\alpha < Y_\alpha$ and $Y_\alpha \subseteq_{\mathrm{cof}} M_0$;
(4) $\{y_\nu : \nu \leq \alpha\} \cup Y_\alpha$ is homogeneous for $\mathcal{S}$ and the $\mathcal{S}$-character of $\{y_\nu : \nu \leq \alpha\} \cup Y_\alpha$ is the same as the $\mathcal{S}$-character of X;
(5) whenever $\nu \leq \alpha$, then Y_α is homogeneous for R_ν and the $\{R_\nu\}$-character of Y_α is the same as the $\{R_\nu\}$-character of Y_ν.

$\alpha = 0$: Let $y_0 = x_0$ and apply the inductive hypothesis to get Y_0. Pick some $y_\alpha \in Y_\beta$ such that $y_\alpha > \max(y_\beta, x_\alpha)$. Since Y_β is homogeneous for $\mathcal{S} \cup \{R_\nu : \nu \leq \beta\}$, we can apply the inductive hypothesis to get $Y_\alpha \subseteq_{\mathrm{cof}} M$ such that $y_\alpha < Y_\alpha$, Y_α is homogeneous for $\mathcal{S} \cup \{R_\nu : \nu \leq \alpha\}$ and the $(\mathcal{S} \cup \{R_\nu : \nu \leq \beta\})$-character of Y_α is the same as the $(\mathcal{S} \cup \{R_\nu : \nu \leq \beta\})$-character of Y_β.

α is a limit ordinal: For each $\nu < \alpha$, $Y_\nu \subseteq_{\mathrm{cof}} M_0$, Y_ν is homogeneous for $\mathcal{S} \cup \{S_\mu : \mu \leq \nu\}$ and the $\mathcal{S}$-character of Y_ν is the same as the $\mathcal{S}$-character of X. Furthermore, if $\mu < \nu < \alpha$, then the R_μ-character of Y_ν is the same as the R_μ-character of Y_μ. Thus, we can apply Lemma 5.1 to get $Y \subseteq_{\mathrm{cof}} M_0$ that is homogeneous for $\mathcal{S} \cup \{R_\nu : \nu < \alpha\}$ such that whenever $\nu < \alpha$, then the $(\mathcal{S} \cup \{R_\mu : \mu \leq \nu\})$-character of Y is the same as the $(\mathcal{S} \cup \{R_\mu : \mu \leq \nu\})$-character of Y_ν. Pick some $y_\alpha \in Y$ such that $y > x_\alpha$. We can now apply the inductive hypothesis to get $Y_\alpha \subseteq_{\mathrm{cof}} M$ such that $y_\alpha < Y_\alpha$, Y_α is homogeneous for $\mathcal{S} \cup \{R_\nu : \nu \leq \alpha\}$ and, for every $\nu < \alpha$, the $(\mathcal{S} \cup \{R_\mu : \nu \leq \beta\})$-character of Y_α is the same as the $(\mathcal{S} \cup \{R_\mu : \mu \leq \nu\})$-character of Y_ν.

Let $H_0 = \{y_\alpha : \alpha < \lambda$ and $[Y_\alpha]^{n-1} \subseteq R_\alpha\}$ and $H_1 = \{y_\alpha : \alpha < \lambda$ and $y_\alpha \notin H_0\}$. Then $H_0 \cup H_1 = \{y_\alpha : \alpha < \lambda\} \subseteq_{\mathrm{cof}} M_0$. Let $H \in \{H_0, H_1\}$ so that $H \subseteq_{\mathrm{cof}} M_0$. Then, it is clear that H is homogeneous for $\mathcal{S} \cup \{R\}$ and the $\mathcal{S}$-character of H is the same as the $\mathcal{S}$-character of X. □

Lemma 5.3 easily implies the following dual version of it using the same type of reflection that was used to deduce Lemma 5.2 from Lemma 5.1.

Lemma 5.4 *Suppose $n < \omega$ and $R \subseteq N^n$ is definable. Let $\mathcal{S}$ be a set of definable relations on N such that $|\mathcal{S}| < \lambda$. Let $X \subseteq N$ such that* $\inf(X) = M_1$ *and X is homogeneous for $\mathcal{S}$. Then there is $H \subseteq N$ such that* $\inf(H) = M_1$*, H is is* *homogeneous for $\mathcal{S} \cup \{R\}$ and the $\mathcal{S}$-character of H is the same as the $\mathcal{S}$-character of X.*

5.2 Unbalanced elementary cuts The following theorem is the main result of this section. Note that it is concerned with unbalanced cuts with uncountable cofinalities. Unbalanced cuts having a countable cofinality were considered in Theorem 4.2, Corollary 4.4, and Theorem 4.6.

Theorem 5.5 *Suppose that $\mathcal{N}$ is a saturated model of* PA *of cardinality κ and $\mathcal{M}_0, \mathcal{M}_1 \prec_{\mathrm{end}} \mathcal{N}$ are unbalanced such that* $\mathrm{cf}(M_0), \mathrm{cf}(M_1)$, $\mathrm{dcf}(M_0)$, $\mathrm{dcf}(M_1) \geq \aleph_1$. *Then* $(\mathcal{N}, M_0) \equiv (\mathcal{N}, M_1)$.

Theorem 5.5 is a consequence of the even stronger Theorem 5.6 for which some definitions are needed. Consider two structures $\mathfrak{A} = (A, \ldots)$ and $\mathfrak{B} = (B, \ldots)$ for the same finite relational language. Thus, whenever $X \subseteq A$ and $Y \subseteq B$, then $\mathfrak{A}|X$ and $\mathfrak{B}|Y$ are substructures of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. A function $f : X \longrightarrow Y$ is a *partial isomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ if $X \subseteq A$, $Y \subseteq B$ and f is an isomorphism from $\mathfrak{A}|X$ to $\mathfrak{B}|Y$.

Given an ordinal α, we define the game $\mathbb{G}^\alpha(\mathfrak{A}, \mathfrak{B})$, which is the Ehrenfeucht-Fraïssé game of length α. This game is played between Players I and II who each make α moves, playing alternately. In the νth round of play, Player I goes first, choosing either some $a_\nu \in A$ or some $b_\nu \in B$, and then Player II makes a choice from the other set to produce the pair $\langle a_\nu, b_\nu \rangle \in A \times B$. If, after α moves, $f = \{\langle a_\nu, b_\nu \rangle : \nu < \alpha\}$ is a partial isomorphism, then II wins. Otherwise, I wins. We define $\mathfrak{A} \equiv_\alpha \mathfrak{B}$ if Player II has a winning strategy for $\mathbb{G}^\alpha(\mathfrak{A}, \mathfrak{B})$. We mention three basic facts:

(1) $\mathfrak{A} \equiv \mathfrak{B}$ iff $\mathfrak{A} \equiv_n \mathfrak{B}$ for each $n < \omega$;
(2) $\mathfrak{A} \cong \mathfrak{B}$ iff $\mathfrak{A} \equiv_\kappa \mathfrak{B}$ for some (every) $\kappa \geq |A|, |B|$;
(3) $\mathfrak{A} \equiv_{\infty,\omega} \mathfrak{B}$ iff $\mathfrak{A} \equiv_\omega \mathfrak{B}$.

When considering this game for models of PA, we will modify the language of PA to consist of just two 3-ary relation symbols denoting $+$ and $\times$ in order to render models of PA as relational structures.

Theorem 5.6 *Suppose that $\mathcal{N}_0, \mathcal{N}_1$ are κ-saturated models of* PA *such that $\mathcal{N}_0 \equiv \mathcal{N}_1$. Suppose that $\mathcal{M}_0 \prec_{\mathrm{end}} \mathcal{N}_0$ and $\mathcal{M}_1 \prec_{\mathrm{end}} \mathcal{N}_1$ are elementary cuts such that* $\aleph_1 \leq \lambda = \min(\mathrm{cf}(M_0), \mathrm{dcf}(M_0)) \leq \min(\mathrm{cf}(M_1), \mathrm{dcf}(M_1)) < \kappa$. *Then* $(\mathcal{N}, M_0) \equiv_\lambda (\mathcal{N}, M_1)$.

Proof Let κ be the least possible, so κ is regular. Without loss of generality, we can assume that $\lambda \leq \mathrm{cf}(M_0), \mathrm{dcf}(M_1) < \kappa$.

Some more definitions are needed.

We will be concerned with expansions of models $\mathcal{N}$ of PA having the form $(\mathcal{N}, a_\nu)_{\nu<\alpha}$, where α is an ordinal and each $a_\nu \in N$. A subset $I \subseteq N$ is indiscernible for $(\mathcal{N}, a_\nu)_{\nu<\alpha}$ if and only if it is homogeneous for the set of all relations on N definable using only parameters from $\{a_\nu : \nu < \alpha\}$. If I is an infinite set of indiscernibles for $(\mathcal{N}, a_\nu)_{\nu<\alpha}$, then we let $\Phi(I, \langle a_\nu : \nu < \alpha \rangle)$ be the set of formulas $\varphi(x_0, x_1, \ldots, x_{m-1}; \overline{u})$, where $\overline{u}$ is a tuple of variables form $\{u_\nu : \nu < \alpha\}$ such that for some (equivalently: every) $c_0 < c_1 < \cdots < c_{m-1}$ from I, $\mathcal{M} \models \varphi(c_0, c_1, \ldots, c_{m-1}; \overline{a})$. (Here it is to be understood that $\overline{a}$ is the tuple obtained from $\overline{u}$ by replacing each u_ν with a_ν.)

We will prove Theorem 5.6 by exhibiting Player II's winning strategy for the game $\mathbb{G}^\lambda((\mathcal{N}_0, M_0), (\mathcal{N}_1, M_1))$. When Player II makes her αth move (so as to produce the pair $\langle a_\alpha, b_\alpha\rangle$), she also chooses sets I_α and J_α so that the following hold:

(1) $I_\alpha \subseteq_{\mathsf{cof}} M_0$ is indiscernible for $(\mathcal{N}_0, a_\nu)_{\nu\le\alpha}$;
(2) $J_\alpha \subseteq N_1$ is such that $\inf(J_\alpha) = M_1$ and J_α is indiscernible for $(\mathcal{N}_1, b_\nu)_{\nu\le\alpha}$;
(3) $\Phi(I_\alpha, \langle a_\nu : \nu \le \alpha\rangle) = \Phi(J_\alpha, \langle b_\nu : \nu \le \alpha\rangle)$;
(4) whenever $\nu < \alpha$, then $\Phi(I_\nu, \langle a_\mu : \mu \le \nu\rangle) \subseteq \Phi(I_\alpha, \langle a_\mu : \mu \le \alpha\rangle)$;
(5) $|I_\alpha|, |J_\alpha| < \kappa$.

We will check that Player II can always make her αth move. Only the case in which Player I picks some $a_\alpha \in M_0$ will be considered: the dual case when he picks $b_\alpha \in M_1$ instead can be handled by using the dual lemmas.

$\alpha = 0$: Let $I \subseteq N_0$ and $J \subseteq N_1$ be such that I is indiscernible for $\mathcal{N}_0$, J is indiscernible for $\mathcal{N}_1$, $I \subseteq_{\mathsf{cof}} M_0$, $\inf(J) = M_1$, and $\Phi(I; \varnothing) = \Phi(J; \varnothing)$. To get such I and J, first let $p(x)$ be a minimal type realized in $\mathcal{N}_0$, and then let $I = \{a \in M_0 : a$ realizes $p(x)\}$ and $J = \{b \in N_1 \backslash M_1 : b$ realizes $p(x)\}$.

Player I plays a_0. By repeated applications of Lemmas 5.3 followed by one application of Lemma 5.1, we can get $I_0 \subseteq_{\mathsf{cof}} M_0$ that is indiscernible for $(\mathcal{N}_0, a_0)$ such that $|I_0| < \kappa$ and $\Phi(I, \varnothing) \subseteq \Phi(I_0, \langle a_0\rangle)$. Here are the details.

Let $\mathcal{S}_0$ be the set of relations on N that are $\varnothing$-definable. Thus, I is homogeneous for $\mathcal{S}$. Let $\{S_i : i < \omega\}$ be the set of those relations on N that are definable using only the parameter a_0, and then let $\mathcal{S}_j = \mathcal{S} \cup \{S_i : i < j\}$ for $j < \omega$. Let $X_0 = I$. Inductively, use Lemma 5.3 to get $X_{j+1} \subseteq_{\mathsf{cof}} M_0$ that is homogeneous for $\mathcal{S}_{j+1}$ such that the $\mathcal{S}_j$-character of X_{j+1} is the same as the $\mathcal{S}_j$-character of X_j. Then use Lemma 5.1 to get $I_0 \subseteq_{\mathsf{cof}} M_0$ that is homogeneous for $\mathcal{S} \cup \{S_j : j < \omega\}$ such that for each $j < \omega$, the $\mathcal{S}_j$-character of I_0 is the same as the $\mathcal{S}_j$-character of X_j. In particular, the $\mathcal{S}$-character of I_0 is the same as the $\mathcal{S}$-character of I. Clearly, I_0 is indiscernible for $(\mathcal{N}, a_0)$. By replacing I_0 by a subset of itself, if needed, we can get $|I_0| < \kappa$.

Let $J_0 \subseteq J$ be such that $|J_0| < \kappa$ and $\inf(J_0) = M_1$. Since $\mathcal{N}_1$ is κ-saturated, it is easy get b_0 such that J_0 is indiscernible for $(\mathcal{N}_1, b_0)$ and $\Phi(I_0, \langle a_0\rangle) = \Phi(J_0, \langle b_0\rangle)$.

$\alpha = \beta + 1$: Player I plays a_α. By repeated applications of Lemmas 5.1 and 5.3, we can get $I_\alpha \subseteq_{\mathsf{cof}} M_0$ that is indiscernible for $(\mathcal{N}_0, a_\nu)_{\nu\le\alpha}$ such that $|I_\alpha| < \kappa$ and $\Phi(I_\alpha, \langle a_\nu : \nu \le \beta\rangle) = \Phi(I_\beta, \langle a_\nu : \nu \le \beta\rangle)$. The details, which are much like those in the $\alpha = 0$ case, will be omitted. Let $J_\alpha = J_\beta$. Since $\mathcal{N}_1$ is κ-saturated, it is easy get b_α such that J_α is indiscernible for $(\mathcal{N}_1, b_\nu)_{\nu\le\alpha}$ and $(\Phi(I_\alpha, \langle a_\nu : \nu \le \alpha\rangle) = \Phi(J_\alpha, \langle b_\nu : \nu \le \alpha\rangle)$.

α is a limit ordinal: Player I plays a_α. At this point we have $\langle (a_\nu, b_\nu) : \nu < \alpha\rangle$ and we have $\langle I_\nu : \nu < \alpha\rangle$ and $\langle J_\nu : \nu < \alpha\rangle$. By Lemmas 5.3 and 5.4, we can get I and J such that $I \subseteq_{\mathsf{cof}} M_0$, $J \subseteq N_1$, $\inf(J) = M_1$, I is indiscernible for $(\mathcal{N}_0, a_\nu)_{\nu<\alpha}$ and J is indiscernible for $(\mathcal{N}_1, b_\nu)_{\nu<\alpha}$. By repeated applications of Lemmas 5.1 and 5.3, we can get $I_\alpha \subseteq_{\mathsf{cof}} M_0$ such that $|I_\alpha| < \kappa$, I_α is indiscernible for $(\mathcal{N}_0, a_\nu)_{\nu\le\alpha}$ and $\Phi(I, \langle a_\nu : \nu < \alpha\rangle) \subseteq \Phi(I_\alpha, \langle a_\nu : \nu \le \alpha\rangle)$. Let $J_\alpha \subseteq J$ be such that $\inf(J_\alpha) = M_1$ and $|J_\alpha| < \kappa$. Since $\mathcal{N}_1$ is κ-saturated, it is easy to get b_α such that J_α is indiscernible for $(\mathcal{N}_1, b_\nu)_{\nu\le\alpha}$ and $\Phi(I_\alpha, \langle a_\nu : \nu \le \alpha\rangle) = \Phi(J_\alpha, \langle b_\nu : \nu \le \alpha\rangle)$.

This completes the verification that Player II has a winning strategy, thereby finishing the proof of the theorem. □

6 Epilogue

The original purpose for writing this paper was to rectify some statements made in [1]. The following two results are stated without proof in [1, Props. 5.1 and 5.3], where it is said they are proved in [2]; in fact, neither appears [2].

> If $\alpha \leq \kappa$ and $\beta \leq \kappa^+$ are infinite cardinals, then there is a model of PA having reduced ordertype $\eta_\kappa \cdot (\alpha^* + \beta)$.
> If T is a completion of PA and T is not True Arithmetic (TA), then T has at least 4 pairwise nonisomorphic models having reduced ordertype $\eta_\kappa \cdot \omega$ (and 3 if $T = \mathsf{TA}$).

The first of the above statements is false whenever $\alpha < \kappa$ as shown by Theorem 2.2.

It is "suggested" in [1] that every completion $T \supseteq \mathsf{PA}$ has 2^κ nonisomorphic models having reduced ordertype $\eta_\kappa \cdot \omega$, and it is also "suggested" that if $\beta \leq \kappa$ is an uncountable regular cardinal, then every completion $T \supseteq \mathsf{PA}$ has, up to isomorphism, exactly one model having reduced ordertype $\eta_\kappa \cdot \beta$. Theorem 3.1 gives the actual number of models in each case.

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Department of Mathematics
University of Connecticut
Storrs CT 06269
USA
james.schmerl@uconn.edu