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Dp-Minimality: Basic Facts and Examples

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Abstract We study the notion of dp-minimality, beginning by providing several essential facts about dp-minimality, establishing several equivalent definitions for dp-minimality, and comparing dp-minimality to other minimality notions. The majority of the rest of the paper is dedicated to examples. We establish via a simple proof that any weakly o-minimal theory is dp-minimal and then give an example of a weakly o-minimal group not obtained by adding traces of externally definable sets. Next we give an example of a divisible ordered Abelian group which is dp-minimal and not weakly o-minimal. Finally we establish that the field of p-adic numbers is dp-minimal.

1 Introduction

In this note we study many basic properties of dp-minimality as well as developing several fundamental examples. Dp-minimality—see Definition 2.3—was introduced by Shelah in [19] as possibly the strongest of a family of notions implying that a theory does not have the independence property—for which see [20]. The study of dp-minimality beyond Shelah's original work was continued by Onshuus and Usvyatsov in [16] focusing primarily on the stable case and by the second author in [11] primarily in the case of theories expanding the theory of divisible ordered Abelian groups. Our goal in this paper is to provide many basic foundational facts on dp-minimality as well as to explore concrete examples of dp-minimality in the ordered context as well as the valued field context. Much of our motivation arises out of the program of attempting to explore the impact of abstract model theoretic notions, such as dp-minimality, in concrete situations such as ordered model theory on the reals or the study of valued fields. As such, this note may be seen as providing some ground work for further study in this direction—for example, Simon [22] has recently shown that an infinite definable subset of a dp-minimal divisible ordered Abelian group must have interior.

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We give a brief outline of the content of our note. Section 2 is dedicated to definitions and providing several relevant background facts on dp-minimality. Many of these facts are inherent in [19] but we isolate them here and provide straightforward proofs. Section 3 is a brief discussion of the relationship of dp-minimality to some other minimality notions. In Section 4 we focus on weak o-minimality (for which see [14]) and algebraically closed valued fields (for which see [18]). We show that weakly o-minimal theories as well as the theory of algebraically closed valued fields are dp-minimal by showing that, more generally, VC-minimal theories (for which see [2]) are dp-minimal. We also provide an example of a weakly o-minimal group which is not obtained by expanding an o-minimal structure by convex sets. Work in Section 3 as well as results from [11] indicate that a dp-minimal theory expanding that of divisible ordered Abelian groups has some similarity to a weakly o-minimal theory and we may naturally ask whether any such theory is weakly o-minimal. Section 5 provides a negative answer via an example arising from the valued field context. Our final section is dedicated to showing that the theory of the p-adic field is dp-minimal.

2 Basic Facts on DP-Minimality

We develop several basic facts about dp-minimality. The vast majority of the material found below is implicit in Shelah's paper [19], but typically in the more general context of strong dependence. We provide proofs of these various facts for clarity and ease of exposition. Recall the following definition.

Definition 2.1 Fix a structure $\mathfrak{M}$. An ICT *pattern* in $\mathfrak{M}$ consists of a pair of formulas $\varphi(x, \overline{y})$ and $\psi(x, \overline{y})$ and sequences $\{\overline{a}_i : i \in \omega\}$ and $\{\overline{b}_i : i \in \omega\}$ from M so that, for all $i, j \in \omega$, the following is consistent:

$$\varphi(x, \overline{a}_i) \wedge \psi(x, \overline{b}_j) \wedge \bigwedge_{l \neq i} \neg\varphi(x, \overline{a}_l) \wedge \bigwedge_{k \neq j} \neg\psi(x, \overline{b}_k).$$

Remark 2.2 Definition 2.1 should more formally be referred to as an ICT pattern of *depth two* but in this paper we only consider such ICT patterns and thus we omit this extra terminology.

Definition 2.3 A theory T is said to be *dp-minimal* if in no model $M \models T$ is there an ICT pattern.

It is often very convenient to use the following definition and fact.

Definition 2.4 We say two sequences $\{\overline{a}_i : i \in I\}$ and $\{\overline{b}_j : j \in J\}$ are *mutually indiscernible* if $\{\overline{a}_i : i \in I\}$ is indiscernible over $\bigcup_{j \in J} \overline{b}_j$ and $\{\overline{b}_j : j \in J\}$ is indiscernible over $\bigcup_{i \in I} \overline{a}_i$. We call an ICT pattern mutually indiscernible if the witnessing sequences are mutually indiscernible.

Fact 2.5 T is dp-minimal if and only if in no model $\mathfrak{M} \models T$ is there a mutually indiscernible ICT pattern.

Proof This is a simple application of compactness and Ramsey's theorem. □

Before continuing we should mention an alternative characterization of dp-minimality. To this end we have the following definition.

Definition 2.6 A theory T is said to be *inp-minimal* if there is *no* model of $\mathfrak{M}$ of T formulas $\varphi(x, \overline{y})$ and $\psi(x, \overline{y})$, natural numbers k_0 and k_1, and sequences $\{\overline{a}_i : i \in \omega\}$ and $\{\overline{b}_i : i \in \omega\}$ so that $\{\varphi(x, \overline{a}_i) : i \in \omega\}$ is k_0-inconsistent, $\{\psi(x, \overline{b}_j) : j \in \omega\}$ is k_1-inconsistent and for any $i, j \in \omega$ the formula $\varphi(x, \overline{a}_i) \wedge \psi(x, \overline{b}_j)$ is consistent.

With this definition we have the following fact.

Fact 2.7 ([16], Lemma 2.11) If T does not have the independence property and is inp-minimal then T is dp-minimal.

The next fact is extremely useful in showing that a theory is not dp-minimal. It is key in establishing the relationship between dp-minimality and indiscernible sequences that follows.

Fact 2.8 Let T be a complete theory and let $\mathfrak{C}$ be a monster model for T. Suppose there are formulas $\varphi_0(x, \overline{y})$ and $\varphi_1(x, \overline{y})$ and mutually indiscernible sequences $\{\overline{a}_i : i \in \omega\}$ and $\{\overline{b}_i : i \in \omega\}$ so that

$$\varphi_0(x, \overline{a}_0) \wedge \neg\varphi_0(x, \overline{a}_1) \wedge \varphi_1(x, \overline{b}_0) \wedge \neg\varphi_1(x, \overline{b}_1)$$

is consistent. Then T is not dp-minimal.

Proof By compactness there are mutually indiscernible sequences $\overline{a}_i$ for $i \in \mathbb{Z}$ and $\overline{b}_i$ for $i \in \mathbb{Z}$ and c so that

$$\models \varphi_0(c, \overline{a}_0) \wedge \neg\varphi_0(c, \overline{a}_1) \wedge \varphi_1(c, \overline{b}_0) \wedge \neg\varphi_1(c, \overline{b}_1).$$

By applying compactness and Ramsey's theorem we assume that $\{\overline{a}_i : i < 0\}$ and $\{\overline{a}_i : i > 1\}$ are both indiscernible over $\bigcup_{i \in \mathbb{Z}} \overline{b}_i \cup \{c\}$ as well as that $\{\overline{b}_i : i < 0\}$ and $\{\overline{b}_i : i > 1\}$ are both indiscernible over $\bigcup_{i \in \omega} \overline{a}_i \cup \{c\}$. Let $\overline{d}_i = \overline{a}_{2i}\frown\overline{a}_{2i+1}$ and $\overline{e}_i = \overline{b}_{2i}\frown\overline{b}_{2i+1}$ for $i \in \mathbb{Z}$. Note that these two sequences are mutually indiscernible. Let $\psi_0(x, \overline{y}_0\overline{y}_1)$ be $\varphi_0(x, \overline{y}_0) \leftrightarrow \neg\varphi_0(x, \overline{y}_1)$ and $\psi_1(x, \overline{y}_0\overline{y}_1)$ be $\varphi_1(x, \overline{y}_0) \leftrightarrow \neg\varphi_1(x, \overline{y}_1)$. Then $\mathfrak{C} \models \psi_0(c, \overline{d}_0) \wedge \psi_1(c, \overline{e}_0)$ and if $i \neq 0$, then $\models \neg\psi_0(c, \overline{d}_i) \wedge \neg\psi_1(c, \overline{e}_i)$ by the indiscernibility assumptions. It follows that ψ_0, ψ_1, $\{\overline{d}_i : i \in \omega\}$ and $\{\overline{e}_i : i \in \omega\}$ witness that T is not dp-minimal. □

Using the identical proof as above we show the following.

Fact 2.9 The following are equivalent.

1. There are formulas $\varphi_i(\overline{x}, \overline{y})$ for $1 \le i \le N$ and sequences $\overline{a}^i_j$ for $1 \le i \le N$ and $j \in \omega$ so that for every $\eta : \{1, \ldots, N\} \to \omega$ the type
$$\bigwedge_{1 \le i \le N} \varphi_i(\overline{x}, \overline{a}^i_{\eta(i)}) \wedge \bigwedge_{1 \le i \le N} \bigwedge_{j \neq \eta(i)} \neg\varphi_i(\overline{x}, \overline{a}^i_j)$$
is consistent.
2. There are formulas $\psi_i(\overline{x}, \overline{y})$ for $1 \le i \le N$ and mutually indiscernible sequences $\{a^i_j : j \in \omega\}$ for $1 \le i \le N$ so that the type
$$\bigwedge_{1 \le i \le N} \psi_i(\overline{x}, \overline{a}^i_0) \wedge \bigwedge_{1 \le i \le N} \neg\psi_i(\overline{x}, \overline{a}^i_1)$$
is consistent

As in the case of the independence property and strong dependence we have a characterization of dp-minimality in terms of splitting indiscernible sequences.

Fact 2.10 The following are equivalent for a theory T.

1. T is dp-minimal.
2. If $\{\overline{a}_i : i \in I\}$ is an indiscernible sequence and c is an element then there is a partition of I into finitely many convex sets $I_0, \ldots, I_n$, at most two of which are infinite, so that, for any $1 \leq l \leq n$, if $i, j \in I_l$, then $\mathrm{tp}(\overline{a}_i/c) = \mathrm{tp}(\overline{a}_j/c)$.
3. If $\{\overline{a}_i : i \in I\}$ is an indiscernible sequence and c is an element then there is a partition of I into finitely many convex sets $I_0, \ldots, I_n$, at most two of which are infinite, so that, for any $1 \leq l \leq n$, the sequence $\{\overline{a}_i : i \in I_l\}$ is indiscernible over c.

Proof

(1) $\Rightarrow$ (3) Suppose T is dp-minimal and for contradiction suppose that there is an indiscernible sequence (which for notational simplicity we assume consists of singletons) $\{a_i : i \in I\}$ and an element c witnessing the failure of (3). Without loss of generality, we assume that I is a sufficiently saturated dense linear order without endpoints. By [1, Corollary 6] there is an initial segment $I_0 \subseteq I$ and a final segment $I_1 \subseteq I$ so that the sequences $\{a_i : i \in I_0\}$ and $\{a_i : i \in I_1\}$ are indiscernible over c. We may choose I_0 to be maximal in the sense that for no convex J with $I_0 \subset J \subseteq I$ is $\{a_j : j \in J\}$ indiscernible over c and similarly for I_1. If $I \setminus (I_0 \cup I_1)$ is finite then (3) holds so $I \setminus (I_0 \cup I_1)$ is infinite and thus contains an interval.

Let J_0 and J_1 be disjoint convex sets so that $I_0 \subset J_0 \subset I$ and $I_1 \subset J_1 \subset I$. We find $j_1^* < \cdots < j_n^* \in J_0$ and a formula $\varphi(x, y_1, \ldots, y_n)$ so that $\neg\varphi(c, a_{i_1}, \ldots a_{i_n})$ holds for all $i_1 < \cdots < i_n \in I_0$ but $\varphi(c, a_{j_1^*}, \ldots, a_{j_m^*})$ holds. We choose a sequence of n-tuples $\overline{d}_i$ for $i \in \omega$ as follows: let $\overline{d}_0 = a_{j_1^*}, \ldots, a_{j_n^*}$. If $i > 0$, let $\overline{d}_i = a_{i_1}, \ldots, a_{i_n}$ where $i_1, \ldots, i_n$ is any increasing sequence of elements with $i_k \in I_0$ for all $1 \leq k \leq n$ and so that $i_n < \min\{l \in I : a_l \in \overline{d}_{i-1}\}$. Note that $\{\overline{d}_i : i \in \omega\}$ is indiscernible. Similarly, we may find $k_{j_1}^* < \cdots < k_{j_m}^* \in J_1$ and a formula $\psi(x, y_1, \ldots, y_m)$ so that $\neg\psi(c, a_{i_1}, \ldots, a_{i_m})$ holds for all $i_1 < \cdots < i_m \in I_1$ but $\psi(c, a_{k_1^*}, \ldots, a_{k_m^*})$ holds.

We build a sequence of m-tuples $\{\overline{e}_i : i \in \omega\}$ as follows: $\overline{e}_0 = a_{k_1^*}, \ldots, a_{k_m^*}$. For $i > 0$, let $\overline{e}_i = a_{i_1}, \ldots, a_{i_m}$ where $i_1, \ldots, i_m$ is any increasing sequence from I_1 and $i_1 > \max\{l \in I : a_l \in \overline{e}_{i-1}\}$. Notice that $\{\overline{d}_i : i \in \omega\}$ and $\{\overline{e}_j : j \in \omega\}$ are mutually indiscernible sequences. Thus the formulas $\varphi(x, \overline{y}_1, \ldots, \overline{y}_n)$ and $\psi(x, \overline{y}_1, \ldots, \overline{y}_m)$ and the sequences $\overline{d}_i, \overline{e}_i$ for $i \in \omega$ form a configuration as found in Fact 2.8 and T is not dp-minimal.

(3) $\Rightarrow$ (2) Immediate.

(2) $\Rightarrow$ (1) For contradiction suppose there are formulas $\varphi(x, \overline{y})$ and $\psi(x, \overline{y})$ and mutually indiscernible sequences $\{\overline{a}_i : i \in 3 \times \omega\}$ and $\{\overline{b}_i : i \in 3 \times \omega\}$ (lexicographically ordered) which witness the failure of dp-minimality. Such sequences must exist by compactness. For $i \in 3 \times \omega$, let $\overline{c}_i = \overline{a}_i{}^\frown\overline{b}_i$. Note this is an indiscernible sequence. Pick d realizing the type

$$\varphi(x, \overline{a}_\omega) \wedge \psi(x, \overline{b}_{2\times\omega}) \wedge \bigwedge_{i \neq \omega} \neg\varphi(x, \overline{a}_i) \wedge \bigwedge_{j \neq 2\times\omega} \neg\psi(x, \overline{b}_j).$$

$$\text{If } i \neq \omega \text{ or } 2\times\omega, \text{ then } \models \neg\varphi(d,\overline{a}_i)\wedge\neg\psi(d,\overline{b}_i),$$
$$\text{if } i=\omega, \text{ then } \models \varphi(d,\overline{a}_\omega)\wedge\neg\psi(d,\overline{b}_\omega), \text{ and}$$
$$\text{if } i=2\times\omega, \text{ then } \models \neg\varphi(d,\overline{a}_{2\times\omega})\wedge\psi(d,\overline{b}_{2\times\omega}).$$

It follows that the indiscernible sequence $\{\overline{c}_i : i \in 3\times\omega\}$ cannot be decomposed into convex subsets of which at most two are infinite so that the type over d is constant on each convex set. Hence (2) fails. □

We use the preceding fact to provide a characterization of dp-minimality which allows us to consider formulas with more than one free variable in a variant of Definition 2.3 as well as to consider sets of parameters of arbitrary size in a variant of Fact 2.10.

Fact 2.11 For any theory T the following are equivalent.

1. T is dp-minimal.
2. For all $n\in\omega$ if $\{\overline{a}_i : i\in I\}$ is an indiscernible sequence and $\overline{c}$ is an n-tuple from the universe then there is a partition of I into convex sets $I_1,\dots,I_l$ of which at most 2^n are infinite so that if $1\leq l\leq n$ and $i,j\in I_l$ then $\text{tp}(\overline{a}_i/\overline{c})=\text{tp}(\overline{a}_j/\overline{c})$.
3. For all $n\in\omega$ if $\{\overline{a}_i : i\in I\}$ is an indiscernible sequence and $\overline{c}$ is an n-tuple from the universe then there is a partition of I into convex sets $I_1,\dots,I_l$ of which at most 2^n are infinite so that if $1\leq l\leq n$ the sequence $\{\overline{a}_i : i\in I_l\}$ is indiscernible over $\overline{c}$.
4. For all $n\in\omega$ there is no sequence of formulas $\varphi_1(\overline{x},\overline{y}),\dots,\varphi_{2^n}(\overline{x},\overline{y})$ with $|\overline{x}|=n$ and sequences $\{\overline{a}_i^j : i\in\omega\}$ with $1\leq j\leq 2^n$ so that for any $\eta:\{1,\dots,2^n\}\to\omega$ the type
$$\bigwedge_{1\leq k\leq 2^n}\varphi_k(\overline{x},\overline{a}^k_{\eta(k)})\wedge\bigwedge_{1\leq k\leq 2^n}\bigwedge_{\{t\in\omega:\eta(k)\neq t\}}\neg\varphi_k(\overline{x},\overline{a}^k_t)$$
is consistent.

Proof The equivalence of (1), (2), and (3) is immediate using Fact 2.10 and the fact that dp-minimality is preserved upon naming constants. (4) ⇒ (1) is trivial. To finish we show that (2) ⇒ (4) for which we essentially repeat the proof that (2) ⇒ (1) from Fact 2.10. If (4) fails we find formulas $\varphi_1(\overline{x},\overline{y}),\dots,\varphi_{2^n}(\overline{x},\overline{y})$ with $|\overline{x}|=n$ and, using Fact 2.9 together with compactness, mutually indiscernible sequences $\{\overline{a}_i^j : i\in(n+1)\times\omega\}$ (here $(n+1)\times\omega$ is lexicographically ordered) witnessing this failure. Let $\overline{b}_i=\overline{a}_i^1{}^\frown\cdots{}^\frown\overline{a}_i^{2^n}$ for $i\in(n+1)\times\omega$ and note that this is an indiscernible sequence. Pick $\overline{c}$ realizing
$$\bigwedge_{1\leq j\leq 2^n}\varphi_j(\overline{x},\overline{a}^j_{j\times\omega})\wedge\bigwedge_{1\leq j\leq 2^n}\bigwedge_{\substack{\alpha\in(n+1)\times\omega\\ \alpha\neq j\times\omega}}\neg\varphi_j(\overline{x},\overline{a}^j_\alpha).$$
Then the sequence $\overline{b}_i$ and $\overline{d}$ witness the failure of (2). □

Given a dp-minimal theory T it is reasonable to ask if the bound in Fact 2.11(4) may be improved from 2^n to $n+1$. This holds in the stable case and we sketch the proof.

Fact 2.12 If T is dp-minimal and stable then there is no sequence of formulas
$$\varphi_1(\overline{x},\overline{y}),\dots,\varphi_{n+1}(\overline{x},\overline{y})$$

with $|\overline{x}| = n$ and sequences $\{\overline{a}_i^j : i \in \omega\}$ with $1 \leq j \leq n$ so that for any $\eta : \{1, \ldots, n+1\} \to \omega$ the type

$$\bigwedge_{1 \leq k \leq n+1} \varphi_k(\overline{x}, \overline{a}^k_{\eta(k)}) \wedge \bigwedge_{1 \leq k \leq n+1} \bigwedge_{\{t \in \omega : \eta(k) \neq t\}} \neg\varphi_k(\overline{x}, \overline{a}^k_t)$$

is consistent.

Sketch of proof By [16, Theorem 3.5] T is stable and dp-minimal if and only if every 1-type has weight 1. Thus if T is stable and dp-minimal every n-type has weight at most n. Apply [16, Lemma 2.3 and Lemma 2.11] and the result follows. □

This fact also holds for weakly o-minimal T (we defer discussion of this to Section 3). In fact, it is tempting to restate the result with stable replaced by rosy and use þ-weight as in [17] but at the moment it is not clear if the result follows. Overall we do not know whether these improved bounds hold for a general dp-minimal theory. (This has recently been established in [13].)

We finish with two facts which are useful in studying specific examples. The first of these is particularly useful when studying theories which admit some type of cell decomposition—namely, the p-adics.

Fact 2.13 Suppose that $\varphi(x, \overline{y})$ and $\psi(x, \overline{y})$ are formulas witnessing that a theory T is not dp-minimal. Further suppose that $\varphi(x, \overline{y})$ is $\varphi_1(x, \overline{y}) \vee \cdots \vee \varphi_n(x, \overline{y})$. Then for some $1 \leq l \leq n$ $\varphi_l(x, \overline{y})$ and $\psi(x, \overline{y})$ witness that T is not dp-minimal.

Proof There are mutually indiscernible sequences $\{\overline{a}_i : i \in \mathbb{Z}\}$ and $\{\overline{b}_i : i \in \mathbb{Z}\}$ so that, for any $i^*, j^* \in \mathbb{Z}$, the type

$$\varphi(x, \overline{a}_{i^*}) \wedge \psi(x, \overline{b}_{j^*}) \wedge \bigwedge_{i \neq i^*} \neg\varphi(x, \overline{a}_i) \wedge \bigwedge_{j \neq j^*} \neg\psi(x, \overline{b}_j)$$

is consistent. For each $i^* \in \mathbb{Z}$ there is $l(i^*) \in \{1, \ldots, n\}$ so that

$$\varphi_{l(i^*)}(x, \overline{a}_{i^*}) \wedge \psi(x, \overline{b}_0) \wedge \bigwedge_{i \neq i^*} \neg\varphi(x, \overline{a}_i) \wedge \bigwedge_{j \neq j^*} \neg\psi(x, \overline{b}_j)$$

is consistent. Thus for some infinite $I \subseteq \mathbb{Z}$ and $1 \leq l^* \leq n$ if $i \in I$ then $l(i) = l^*$. It follows that the formulas $\varphi_{l^*}(x, \overline{y})$ and $\psi(x, \overline{y})$ together with the sequences $\{\overline{a}_i : i \in I\}$ and $\{\overline{b}_i : i \in \mathbb{Z}\}$ witness that T is not dp-minimal. □

Our final fact shows that a counterexample to dp-minimality may always be found which uses only a single formula rather than two as in Definition 2.3.

Fact 2.14 Suppose that T is not dp-minimal. Then there is a formula $\theta(x, \overline{y})$ and sequences $\{\overline{c}_i : i \in \omega\}$ and $\{\overline{d}_i : i \in \omega\}$ so that for any $i \neq j$ the type

$$\theta(x, \overline{c}_i) \wedge \theta(x, \overline{d}_j) \wedge \bigwedge_{k \neq i} \neg\theta(x, \overline{c}_k) \wedge \bigwedge_{l \neq j} \neg\theta(x, \overline{d}_l)$$

is consistent.

Proof Suppose that the formulas $\varphi(x, \overline{y})$ and $\psi(x, \overline{y})$ and the sequences

$$\{\overline{a}_i : i \in \omega + \omega\} \text{ and } \{\overline{b}_j : j \in \omega + \omega\}$$

witness that T is not dp-minimal. Let $\theta(x, \overline{y}_1, \overline{y}_2)$ be $\varphi(x, \overline{y}_1) \vee \psi(x, \overline{y}_2)$. For $i \in \omega$, let $\overline{c}_i$ be $\overline{a}_i \overline{b}_i$ and, for $j \in \omega$, let $\overline{d}_j$ be $\overline{a}_{\omega+j} \overline{b}_{\omega+j}$. We claim this is as the fact requires. Fix $i, j \in \omega$. There is α realizing the type

$$\varphi(x, \overline{a}_i) \wedge \psi(x, \overline{b}_{\omega+j}) \wedge \bigwedge_{k \neq i} \neg\varphi(x, \overline{a}_k) \wedge \bigwedge_{l \neq \omega+j} \neg\psi(x, \overline{b}_l).$$

Thus α realizes $\theta(x, \overline{c}_i)$ and $\theta(x, \overline{d}_j)$. If $k \neq i$, then α realizes $\neg\varphi(x, \overline{a}_i)$ and $\neg\psi(x, \overline{b}_i)$ and hence realizes $\neg\theta(x, \overline{c}_i)$. Finally, if $l \neq j$ then α realizes

$$\neg\varphi(x, \overline{a}_{\omega+l}) \text{ and } \neg\psi(x, \overline{b}_{\omega+l})$$

and hence $\neg\theta(x, \overline{d}_j)$ as desired. $\square$

3 Relationship with Similar Notions

In this brief section we examine the relationship between dp-minimality and various other strong forms of dependence. We begin with the notion of VC-density as studied in [3]. For the ensuing definition we fix $\Delta(\overline{x}, \overline{y})$ a finite set of formulas where we consider $\overline{y}$ as the parameter variables. If A is a set of $|\overline{y}|$-tuples we write $S^{\Delta}(A)$ for the set of complete Δ-types with parameters from A.

Definition 3.1 A theory T has VC-*density one* if for any finite set of formulas $\Delta(\overline{x}, \overline{y})$ there is a constant C so that for any finite set A of $|\overline{y}|$-tuples $\left|S^{\Delta}(A)\right| \leq C\,|A|^{|\overline{x}|}$.

For example, in [3], it is shown that any weakly o-minimal theory and any quasi o-minimal theory with definable bounds (for which see [5]) has VC-density one. We have a strong relationship between VC-density one and dp-minimality.

Proposition 3.2 *If T has* VC-*density one, then T is dp-minimal.*

Proof Suppose that T is not dp-minimal. Apply Fact 2.14 to find a formula $\varphi(x, \overline{y})$ and sequences $\overline{a}_i$ and $\overline{b}_j$ as described there. For $N \in \mathbb{N}$, let

$$A_N = \{\overline{a}_i : i \leq N\} \cup \{\overline{b}_j : j \leq N\}.$$

By the failure of dp-minimality we immediately see that $S^{\{\varphi\}}(A_N) \geq \frac{1}{4}\,|A_N|^2$ for all N. Thus T does not have VC-density one. $\square$

Note that the above proof only requires that we have that $\left|S^{\Delta}(A)\right| \leq C\,|A|$ for Δ consisting of formulas of the form $\varphi(x, \overline{y})$, that is, with only one free variable.

Corollary 3.3 *Any weakly o-minimal theory as well as any quasi o-minimal theory with definable bounds is dp-minimal.*

We give a short direct proof of the dp-minimality of any weakly o-minimal theory in Section 4. Also under the assumption of VC-density one we obtain the improved bounds in Fact 2.11(4) as discussed in Section 2. With the same proof as in Proposition 3.2 we show the following.

Proposition 3.4 *If T has* VC-*density one there is no sequence of formulas*

$$\varphi_1(\overline{x}, \overline{y}), \ldots, \varphi_{n+1}(\overline{x}, \overline{y})$$

with $|\overline{x}| = n$ *and sequences* $\{\overline{a}_i^j : i \in \omega\}$ *with* $1 \leq j \leq n$ *so that for any* $\eta : \{1, \dots, n+1\} \to \omega$ *the type*

$$\bigwedge_{1 \leq k \leq n+1} \varphi_k(\overline{x}, \overline{a}^k_{\eta(k)}) \wedge \bigwedge_{1 \leq k \leq n} \bigwedge_{\{t \in \omega : \eta(k) \neq t\}} \neg\varphi_k(\overline{x}, \overline{a}^k_t)$$

is consistent.

We consider the notion of VC-minimality as introduced in [2].

Definition 3.5 Fix a theory T and a monster model $\mathfrak{C} \models T$. T is VC-*minimal* if there is a family of formulas Φ of the form $\varphi(x, \overline{y})$ (where the length of $\overline{y}$ is allowed to vary) so that

1. if $\varphi(x, \overline{y}), \psi(x, \overline{y}) \in \Phi$, and $\overline{a}, \overline{b} \in C$, then one of
   (a) $\varphi(C, \overline{a}) \subseteq \psi(C, \overline{b})$,
   (b) $\psi(C, \overline{b}) \subseteq \varphi(C, \overline{a})$,
   (c) $\neg\varphi(C, \overline{a}) \subseteq \psi(C, \overline{b})$,
   (d) $\psi(C, \overline{b}) \subseteq \neg\varphi(C, \overline{a})$;
2. if $X \subseteq C$ is definable then there are a finite collection of $\varphi_i(x, \overline{y})$ from Φ and tuples $\overline{a}_i \in C$ so that X is a Boolean combination of the sets $\varphi_i(C, \overline{a}_i)$.

In [2] Adler cites the theory of algebraically closed valued fields (ACVF), for which see [18], and any weakly o-minimal theory [14] as typical examples of VC-minimal theories. Adler also claims without proof that any VC-minimal theory is dp-minimal. As we are interested in verifying that, in particular, weakly o-minimal as well as ACVF are dp-minimal we will provide a proof of this fact which we defer to Section 4.

The implication that VC-minimality implies dp-minimality may not be reversed as the following proposition shows.

Proposition 3.6 *Let* $\mathcal{L}$ *be the language consisting of unary predicates* P_i *with* $i \in \omega_1$. *For any finite* $I \subseteq \omega_1$ *and* $J \subseteq \omega_1$ *with* $I \cap J = \varnothing$, *let* $\sigma_{I,J,n}$ *be the sentence*

$$\exists^{\geq n}(\bigwedge_{i \in I} P_i x \wedge \bigwedge_{j \in J} \neg P_j x).$$

Let $T = \{\sigma_{I,J,n} :$ *for all* $I, J, n\}$. T *is complete, has quantifier elimination, is dp-minimal (in fact has* VC-*density one), but is not* VC-*minimal.*

Proof We sketch a proof and leave the details to the interested reader. Completeness and quantifier elimination for T are elementary. Quantifier elimination and Fact 2.10(2) yield that T is dp-minimal. For the non-VC minimality of T use quantifier elimination to show that there is no family Φ of formulas $\varphi(x, \overline{y})$ satisfying the compatibility conditions in the definition of VC-minimality and so that any $P_i(x)$ can be obtained by a finite Boolean combination of sets defined by instances of Φ. □

Thus we have an example of a theory which has VC-density one (and hence is dp-minimal) but is not VC-minimal. We conjecture that the theory of the p-adics is also not VC-minimal but have been unable to verify this. We do not have an example of a theory which is dp-minimal but does not have VC-density one.

4 Weakly O-Minimal Theories and ACVF

As alluded to in the previous section, here we show that any VC-minimal theory is dp-minimal which allows us to conclude that any weakly o-minimal theory is dp-minimal and that ACVF, or in fact any C-minimal theory (see [15]), is dp-minimal. Dp-minimality for weakly o-minimal theories first appears in [16, Corollary 3.8] where it is shown that any weakly o-minimal theory obtained via "Shelah expansion"—which we discuss below—is dp-minimal. We show via an example that not all weakly o-minimal theories may be obtained in this way. The dp-minimality of weakly o-minimal theories is also established in [3] by indirect arguments. Before proving that VC-minimality implies dp-minimality we need some more background on VC-minimality. Suppose that T is VC-minimal witnessed by a family Φ. We say that Φ is *directed* if for any $\varphi(x,\overline{y}), \psi(x,\overline{y}) \in \Phi$ and any $\overline{a}, \overline{b} \in \mathfrak{C}^{|\overline{y}|}$ (here $\mathfrak{C}$ is a monster model) one of

(i) $\varphi(C,\overline{a}) \subseteq \psi(C,\overline{b})$,
(ii) $\psi(C,\overline{b}) \subseteq \varphi(C,\overline{a})$,
(iii) $\varphi(C,\overline{a}) \cap \psi(C,\overline{b}) = \varnothing$.

We need the following fact.

Fact 4.1 ([2], Proposition 6) If T is VC-minimal, then after potentially naming constants there is a directed family Φ witnessing VC-minimality.

We can now establish that VC-minimal theories are dp-minimal, as originally remarked by Adler in [2].

Theorem 4.2 *A* VC-*minimal theory is dp-minimal.*

Proof By Fact 4.1 we may assume that T is directed VC-minimal since dp-minimality is preserved under naming or deleting constants. Fix Φ a directed VC-minimal instantiable family. By Proposition 3.2 and the subsequent remark, in order to establish that T is dp-minimal it suffices to show that if $\Delta = \{\psi_1(x,\overline{y}), \ldots, \psi_m(x,\overline{y})\}$ is a finite family of $\mathcal{L}$-formulas, then there is a constant $N \in \mathbb{R}$ so that if $A \subseteq C^{|\overline{y}|}$ is finite then $S^{\Delta}(A) \leq N\,|A|$. Fix Δ; by compactness there is a finite subset $\Phi_0 \subseteq \Phi$, $\Phi_0 = \{\varphi_1(x,\overline{z}), \ldots, \varphi_n(x,\overline{z})\}$ so that any instance of a formula in Δ is a finite Boolean combination of instances of formulas in Φ_0. (Note that we assume every formula in Φ_0 has the same set of parameter variables which we may simply achieve by padding.) There is $M \in \mathbb{N}$ so that any instance of a formula in Δ is equivalent to a Boolean combination of at most M instances of elements of Φ_0. Thus if $A \subset C^{|\overline{y}|}$ is finite there is $B \subset C^{|\overline{z}|}$ with $|B| \leq M\,|A|$ so that if $p \in S^{\Delta}(A)$ there is $q \in S_{\Phi_0}(B)$ with $q \vdash p$. Thus we may assume that $\Delta \subseteq \Phi$. If $p \in S^{\Delta}(A)$ for A finite so that p contains at least one positive instance of an element of Δ then there is $\psi_i(x,\overline{a}) \in p$ so that if $1 \leq j \leq m$ and $\overline{b} \in A$ then $\psi_j(x,\overline{b}) \in p$ if and only if $\psi_i(C,\overline{a}) \subseteq \psi_j(C,\overline{b})$. Hence $\left|S^{\Delta}(A)\right| \leq |\Delta|\,|A| + 1$ and the result follows. □

Corollary 4.3 *Any weakly o-minimal theory is dp-minimal and any C-minimal theory is dp-minimal. In particular, the theories of algebraically closed valued fields and real closed valued fields (for which see [7]) are dp-minimal.*

Proof By [2] weakly o-minimal theories and C-minimal theories are VC-minimal and algebraically closed valued fields are the archetypical example of a C-minimal theory whereas real closed valued fields are weakly o-minimal. □

Corollary 4.3 requires that the *theory* T be weakly o-minimal. We ask if the theory of a weakly o-minimal structure $\mathfrak{M}$ is dp-minimal—recall that a weakly o-minimal structure need not have weakly o-minimal theory. We do not know of an example of a weakly o-minimal structure whose theory is not dp-minimal. Given the close relationship between a theory being weakly o-minimal and elimination of $\exists^{\infty}$ (for which see [6, Section 2]) we are led to ask about the relationship between dp-minimality and elimination of $\exists^{\infty}$. For example, by [11, Lemma 3.3], any dp-minimal theory expanding that of divisible ordered Abelian groups must eliminate $\exists^{\infty}$. However, in full generality this implication is false. Consider the theory T of an equivalence relation with infinitely many infinite classes together with a finite class of size n for each $n \in \mathbb{N}$. It is straightforward to verify that T is dp-minimal; for example, we may use Theorem 4.2 by simply noting that the VC-minimality of T is witnessed by the equivalence relation and equality. T obviously does not eliminate $\exists^{\infty}$. Of course, the converse is also false; the random graph eliminates $\exists^{\infty}$ but is not dp-minimal.

As mentioned earlier, Onshuus and Usvyatsov [16, Corollary 3.8] prove that the theory of a Shelah expansion of an o-minimal structure is dp-minimal. The Shelah expansion is constructed by beginning with a structure $\mathfrak{M}$ and an $|M|^{+}$-saturated elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ and expanding $\mathfrak{M}$ to $\mathfrak{M}^{*}$ by adding predicates for all sets of the form $X \cap M^{m}$ where $X \subseteq N^{m}$ is $\mathfrak{N}$-definable. It follows by results in [4] that if $\mathfrak{M}$ is o-minimal then $\mathfrak{M}^{*}$ has weakly o-minimal theory. Notice of course that the theory of any reduct of $\mathfrak{M}^{*}$ must also be weakly o-minimal and dp-minimal.

To complete the picture regarding weak o-minimality and dp-minimality we give an example of a structure $\mathfrak{M} = \langle M, <, \dots \rangle$ which is a model of the theory of dense linear orderings, has weakly o-minimal theory, and so that no model elementary equivalent to $\mathfrak{M}$ may be obtained as a reduct of a Shelah expansion. The candidate for an example of a weakly o-minimal theory which cannot be obtained via a Shelah expansion is given by Example 2.6.2 in [14]. In particular, we consider the structure, $\mathfrak{M} = \langle \mathbb{Z} \times \mathbb{Q}, <, f \rangle$ where $<$ is the lexicographic order and $f : \mathbb{Z} \times \mathbb{Q} \to \mathbb{Z} \times \mathbb{Q}$ is the function $f(x, y) = (-x, y)$. As noted in [14], it follows easily that this structure has weakly o-minimal theory. Notice that f is a locally increasing function but for any $(x, y) \in \mathbb{Z} \times \mathbb{Q}$ if $w > x$ then $f((w, v)) < f((x, y))$. This is the feature of this structure we exploit to establish.

Proposition 4.4 *Let $\mathfrak{M}_0 \preceq \mathfrak{N}$ be o-minimal structures. In the structure induced by $\mathfrak{N}$ on M_0, there is no definable function $f : M_0 \to M_0$ so that $\langle M_0, <, f \rangle \models \mathrm{Th}(\mathfrak{M})$.*

Proof Without loss of generality, $\mathfrak{N}$ is $|M_0|$-saturated. By [21, Section 1] it suffices to show that for no definable $X \subset (N)^2$ is $X \cap M_0$ the graph of a function f so that $\langle M_0, <, f \rangle \models \mathrm{Th}(\mathfrak{M})$. Suppose for contradiction that such an X exists. By the o-minimality of $\mathfrak{N}$ we decompose X into cells $C_1, \dots, C_n$. By [21, Section 1] the structure induced on M_0 by $\mathfrak{N}$ is weakly o-minimal and thus for some $1 \le i \le n$ and for some $a \in M_0$ the graph of $f \upharpoonright [a, \infty)$ is contained in $C_i \cap M_0$. There are now two equally simple cases. Either C_i is the graph of a continuous (without loss of generality) monotone function or C_i is the region between the graphs of two continuous functions g_1 and g_2 which we may also assume to be monotone.

Suppose that C_i is the graph of g. Thus $g(a) = f(a)$. Pick $b > a$ so that $f(b) > f(a)$ (note such a b must exist). We must have that $g(b) = f(b)$ and hence

g is increasing. Yet we may also pick c so that $c > a$ and $f(c) < f(a)$ and hence since $g(c) = f(c)$, g must be decreasing, a contradiction.

Now suppose that C_i is the region between g_1 and g_2. First we claim that g_2 must be decreasing. Otherwise, for $b > a$ with $f(b) < f(a)$ we would have that $g_2(b) \geq g_2(a) > f(a) > f(b) > g_1(b)$; hence the points $(b, f(a))$ and $(b, f(b))$ lie in $M_0 \cap C_i$ contradicting that this intersection should be the graph of a function. A similar argument shows that for all $c \in M_0$ if $c > f(a)$ then $g_2(a) \leq c$. Pick $b > a$ so that $f(b) > f(a)$; thus $g_2(a) < f(b)$, whence $g_2(b) < f(b)$, which is impossible. □

5 A Complicated DP-Minimal Divisible Ordered Abelian Group

In this section we focus on theories T which extend that of dense linear ordering and so necessarily contain a symbol $<$. A reasonable question arising out of the work found in [11] is whether every dp-minimal T expanding the theory of divisible ordered Abelian groups must be weakly o-minimal. In this section we show via an example that the answer to this question is "no."

For the example let $\mathbb{R}((x^{\mathbb{R}}))$ be the field of generalized power series with real coefficients, real exponents, and well-ordered supports. For $a \in \mathbb{R}((x^{\mathbb{R}}))$ write $v(a)$ for its valuation. We wish to consider only the additive ordered structure of the field augmented with a new relation. Let $\mathcal{L}$ be the language consisting of a binary function $+$, a binary relation $<$, a constant 0, unary functions s_q for $q \in \mathbb{Q}$, a unary predicate P, and binary predicates R_n for each natural number $n \in \omega$ (including 0).

Let $\mathfrak{R}$ be the $\mathcal{L}$ structure with universe $\mathbb{R}((x^{\mathbb{R}}))$ where $+, <, 0$ are interpreted in the obvious way. Interpret the s_q as multiplication by q for each $q \in \mathbb{Q}$. Interpret P as

$$\{x \in \mathbb{R}((x^{\mathbb{R}})) : v(x) \in \mathbb{Z}, v(x) < 0\} \cup \{x \in \mathbb{R}((x^{\mathbb{R}})) : v(x) \geq 0\}.$$

Let R_0 be the equivalence relation of being in the same connected component of P or of $\mathbb{R}((x^{\mathbb{R}})) \setminus P$. For each $n > 0$, let R_n be the set of all pairs (x, y) so that either $x \in P$ and there is a sequence $x = x_0 < x_1 < \cdots < x_n = y$ such that $x_i \in P$ if and only if i is even, but there is no such sequence $x = x_0 < x_1 < \cdots < x_{n+1} = y$, or else $x \notin P$ and there is a sequence $x = x_0 < x_1 < \cdots < x_n = y$ such that $x_i \in P$ if and only if i is odd, but there is no sequence $x = x_0 < x_1 < \cdots < x_{n+1} = y$. Notice that the R_n are definable in the language with just the group structure and P. We add them for quantifier elimination.

Our first goal is to axiomatize $\mathrm{Th}(\mathfrak{R})$ and to show this theory has quantifier elimination. To this end we describe a theory T we intend to show axiomatizes $\mathrm{Th}(\mathfrak{R})$. T consists of

(1) the usual axioms for an ordered divisible Abelian group in the language $\{+, <, 0\}$, and s_q denotes scalar multiplication by q;
(2) $0 \in P$;
(3) $x \in P$ if and only if $-x \in P$;
(4) P and $\neg P$ are open sets;
(5) if $x \leq y$ and the interval $[x, y] \subseteq P$, then for any $\mathbb{Q}$-linear combination z of x and y with positive coefficients, z lies in the same connected component of P as x and y;

(5′) if $x \leq y$ and the interval $[x, y] \subseteq \neg P$, then any $\mathbb{Q}$-linear combination z of x and y with positive coefficients, z lies in the same connected component of $\neg P$ as x and y;

(6) R_0 is a symmetric relation, and if $x \leq y$, then $R_0(x, y)$ holds if and only if the interval $[x, y]$ lies entirely within P or entirely within $\neg P$;

(7) for any x, y and positive $n < \omega$, $R_n(x, y)$ holds if and only if $x < y$ and there are exactly n "alternations of P" between x and y; more precisely, either
   (a) $x \in P$ and there is a sequence $x = x_0 < x_1 < \cdots < x_n = y$ such that $x_i \in P$ if and only if i is even, but there is no such sequence $x = x_0 < x_1 < \cdots < x_{n+1} = y$, or else
   (b) $x \notin P$ and there is a sequence $x = x_0 < x_1 < \cdots < x_n = y$ such that $x_i \in P$ if and only if i is odd, but there is no sequence $x = x_0 < x_1 < \cdots < x_{n+1} = y$;

(8) for any positive x there is a y such that $R_1(x, y)$;

(8') for any positive x there is a y such that $R_1(y, x)$.

For $M \models T$ and $a \in M$ we introduce some useful notation. If $a \in P(M)$ (respectively, $\neg P(M)$) and C is the convex component of $P(M)$ (or $\neg P(M)$) containing a, then $[a] = C \cup -C$. We say $[a] \leq [b]$ if $|a| \leq |b|$, and let $[a]_{\leq} = \bigcup_{[b]\leq[a]}[b]$. So the content of axiom 8 is that the induced ordering on the classes $[a]$ is a discrete ordering with a left endpoint $[0]$ but no right endpoint.

Lemma 5.1 *Suppose $M \models T$ and $a, b \in M$.*

1. *If $[a] < [b]$, then $a + b \in [b]$.*
2. *If $[a] = [b]$, then $a + b \in [a]_{\leq}$.*
3. *$[a]_{\leq}$ is closed under $\mathbb{Q}$-linear combinations.*

Proof (1) If $0 < a < b$, then $a + b > b$; so if $a + b \notin [b]$, then by axiom 5, $2b < a + b$. But this implies that $b < a$, contradiction. If $a < 0 < b$, then $0 < -a < b$; so $0 < a + b < b$. So if $a + b \notin [b]$, then by axiom 5 again, $2a + 2b < b$, and $b < 2(-a)$, a contradiction to axiom 5. The other two cases are similar.

(2) Without loss of generality, $|a| \leq |b|$. If $0 < a \leq b$, then $0 < a + b \leq 2b \in [b]$, by axiom 5. If $a < 0 < b$, then $0 < a + b < b$, and so $a + b \in [b]_{\leq} = [a]_{\leq}$. The other cases are similar.

(3) For any nonzero $q \in \mathbb{Q}$, $qa \in [a]$ by axioms 3 and 5; the rest follows by (2). □

Proposition 5.2 *T is complete and has quantifier elimination.*

Proof We prove both statements simultaneously by a back-and-forth argument: suppose that M and N are ω-saturated models of T, $\overline{a} = (a_0, \ldots, a_{n-1}) \subseteq M$, $\overline{b} = (b_0, \ldots, b_{n-1}) \subseteq N$, and $\mathrm{tp}_{\mathrm{qf}}(\overline{a}, M) = \mathrm{tp}_{\mathrm{qf}}(\overline{b}, N)$. Then for every $a' \in M$, we show that there is some $b' \in N$ such that $\mathrm{tp}_{\mathrm{qf}}((\overline{a}, a'), M) = \mathrm{tp}_{\mathrm{qf}}((\overline{b}, b'), N)$. We do this by cases.

Case A a' is in the $\mathbb{Q}$-linear span of $\overline{a}$. Say $a' = \Sigma_{i<n} s_{q_i}(a_i)$.

Let $b' \in N$ be the corresponding $\mathbb{Q}$-linear combination of $\overline{b}$. If we pick $i < n$ such that $q_i \neq 0$ and $[a_i]$ is as large as possible, then by Lemma 5.1, $[a'] = [a_i]$, and similarly $[b'] = [b_i]$. The equality of the quantifier-free types now follows directly.

Case B a' is not in the $\mathbb{Q}$-linear span of $\overline{a}$ but there is an element c in the $\mathbb{Q}$-linear span of $\overline{a}$ such that $[a'] = [c]$.

Without loss of generality, $a' > 0$. Let

$$S_0 = \left\{x \in \mathrm{Span}_{\mathbb{Q}}(\overline{a}) \cap [c] : x < a'\right\}$$

and let

$$S_1 = \left\{x \in \mathrm{Span}_{\mathbb{Q}}(\overline{a}) \cap [c] : a' < x\right\}.$$

For $\ell = 0, 1$, let $\langle d_i^\ell : i < \omega\rangle$ be an enumeration of S_ℓ. Since $\mathrm{tp}_{\mathrm{qf}}(\overline{a}, M) = \mathrm{tp}_{\mathrm{qf}}(\overline{b}, N)$, we can take corresponding sets T_0 and T_1 in N, with corresponding enumerations $\langle e_i^\ell : i < \omega\rangle$.

Let $f_i^0 = a' - d_i^0$ and let $f_i^1 = d_i^1 - a'$ (these are always positive points). Then, using the fact that the $[\cdot]$-classes in both M and N are discrete linear orderings, we can pick elements $g_i^\ell \in N$ such that

I. if there is any $x \in \mathrm{Span}_{\mathbb{Q}}(\overline{a})$ such that $[x] = [f_i^\ell]$ (or $R_n(x, f_i^\ell)$, or $R_n(f_i^\ell, x)$), then let $y \in \mathrm{Span}_{\mathbb{Q}}(\overline{b})$ be the corresponding element and pick g_i^ℓ such that $[g_i^\ell] = [y]$ (or $R_n(y, g_i^\ell)$, or $R_n(g_i^\ell, y)$);

II. if $x \in \mathrm{Span}_{\mathbb{Q}}(\overline{a})$ and $[x] < [f_i^\ell]$ (or $[f_i^\ell] < [x]$), then let $y \in \mathrm{Span}_{\mathbb{Q}}(\overline{b})$ be the corresponding element, and we require that $[y] < [g_i^\ell]$ (or $[g_i^\ell] < [y]$).

Claim 5.3 *There is an element $b' \in N$ satisfying the conditions,*

1. *b' is in the same $[\cdot]$-class as any element of T_0 or T_1,*
2. *$T_0 < b' < T_1$,*
3. *$[b' - e_i^\ell] = [g_i^\ell]$ for any $i < \omega$ and $\ell = 0, 1$,*
4. *$b' \notin \mathrm{Span}_{\mathbb{Q}}(\overline{b})$.*

(Note that one of T_0 or T_1 may be empty, so half of the second condition may be vacuous.)

Proof First assume T_0 and T_1 are nonempty, for simplicity. One case is where $\left\{[g_i^0] : i < \omega\right\}$ and $\left\{[g_i^1] : i < \omega\right\}$ have least elements—then call these elements $[g_i^0]$ and $[g_j^1]$, and without loss of generality $[g_i^0] \leq [g_j^1]$. If $[g_i^0] < [g_j^1]$ and $[g_i^0]^+$ denotes the positive elements of this class, then pick some $b' \in e_i^0 + [g_i^0]^+$ such that

(i) $b' \notin \mathrm{Span}_{\mathbb{Q}}(\overline{b})$,

(ii) $T_0 < b' < T_1$, and

(iii) for any k such that $a' - d_k^0 \in [f_i^0]$, $b' \in e_k^0 + [g_i^0]^+$.

(This is always possible since $[g_i^0]^+$ is infinite, being closed under scaling by positive elements of $\mathbb{Q}$, and using compactness and ω-saturation.) Using Lemma 5.1, it follows that for any $k < \omega$ and $\ell = 0, 1$, $[b' - e_k^\ell] = [g_k^\ell]$. On the other hand, if $[g_i^0] = [g_i^1]$, then in picking the element b' as above, we can ensure in addition that $b' \in e_j^1 - [g_i^1]^+$, and that for every k such that $d_k^1 - a' \in [f_i^1]$, $b' \in e_k^1 - [g_i^1]^+$; this is enough to ensure that b' satisfies the properties we want.

If $\left\{[g_i^0] : i < \omega\right\}$ and $\left\{[g_i^1] : i < \omega\right\}$ have no least elements, then we can use compactness and ω-saturation to pick $b' \in N$ such that for every $i < \omega$, $b' \in e_i^0 + [g_i^0]^+$ and $b' \in e_i^1 - [g_i^1]^+$, and it automatically follows that $b' \notin \mathrm{Span}_{\mathbb{Q}}(\overline{b})$. The "mixed case" (one of these sets has a least element, the other does not) is handled similarly.

Finally, if T_0 is empty, note that T_1 cannot have a least element (since if $x \in T_1$ then $\frac{1}{2}x$ must be as well), so the usual compactness argument ensures there is a b' in the right $[\cdot]$-class such that $b' < T_1$. As above, we can also ensure that $[b' - e_i^1] = g_i^1$ for each $i < \omega$. The case where $T_1 = \varnothing$ is symmetric. □

With b' as above, Lemma 5.1 ensures that $\mathrm{tp}_{\mathrm{qf}}(\overline{b}, b') = \mathrm{tp}_{\mathrm{qf}}(\overline{a}, a')$.

Case C Cases A and B fail, but there is some $i < n$ such that $a' \in [a_i]_{\leq}$.

Choose $c_0, c_1 \in \mathrm{Span}_{\mathbb{Q}}(\overline{a})$ such that $[c_0] < [a'] < [c_1]$ but the "distances" are minimized; that is, if possible, there is some positive m such that $R_m(c_0, a')$ holds (and similarly for c_1), and such a number m is minimized. (Note that since $a' \neq 0$, we have $[0] < [a']$, so such a c_0 always exists, although it may be zero.) There are subcases: for instance, if $R_{m_0}(c_0, a')$ and $R_{m_1}(a', c_1)$ hold, then we just need to pick $b' \in N$ such that for the corresponding $d_0, d_1 \in \mathrm{Span}_{\mathbb{Q}}(\overline{b})$, $R_{m_0}(d_0, b')$ and $R_{m_1}(b', d_1)$ hold. On the other hand, if there is no such m_0 and no such m_1, then compactness and ω-saturation yield a corresponding $b' \in N$. The final subcase (a' is a finite distance from one of the c_i but not the other) is handled in the same way, noting that axiom 8 ensures that if there are infinitely many components of $P(N)$ between d_0 and d_1, then there are always elements $b' \in N$ such that $R_k(d_0, b')$ or $R_k(b', d_1)$, for any $k < \omega$.

Case D Previous cases fail, but there is some $i < n$ and some positive m such that $R_m(a_i, |a'|)$ holds.

Choose i and m such that m is minimal. By Lemma 5.1, the truth values of $R_k(c, a')$ and $R_k(a', c)$ for any $c \in \mathrm{Span}_{\mathbb{Q}}(\overline{a})$ are now uniquely determined. By axiom 8, there is a corresponding $b' \in N$.

Case E The only remaining case is that $|a'|$ is greater than every $|a_i|$ and $R_m(a_i, |a'|)$ fails for every possible i and every $m < \omega$. Then by Lemma 5.1, the only additional information needed to determine $\mathrm{tp}_{\mathrm{qf}}((\overline{a}, a'), M)$ is the sign of a'. A corresponding $b' \in N$ exists by axiom 8 and ω-saturation. □

Corollary 5.4 *If $\mathfrak{M} \models T$ and $X \subset M$ is infinite and definable then X has interior yet T is not weakly o-minimal.*

Proof This follows immediately from the quantifier elimination. □

Corollary 5.5 *T does not have the independence property.*

Proof Let I be indiscernible over $\varnothing$ with uncountable cofinality. Without loss of generality, I is increasing, and there are three possibilities: either all elements of I are in the same $[\cdot]$-class, or else we have some $n < \omega$ such that $R_n(a_i, a_{i+1})$ holds for every $a_i \in I$, or else neither of these hold and there are infinitely many $[\cdot]$-classes between two adjacent elements of I. Suppose that A is any finite set. Then there are only finitely many $[\cdot]$-classes in $\mathrm{dcl}(A)$ (by Lemma 5.1), and in either of the three cases, it is straightforward to check using quantifier elimination that some cofinal subsequence of I is indiscernible over A. □

Corollary 5.6 *T is dp-minimal.*

Proof Suppose that $t(x, \overline{y})$ is any term in T. Since the only function symbols in the language of T are $+$ and the unary function symbols s_q, and these are assumed to satisfy the usual commutative, associate, and distributive laws, it follows that for some $q \in \mathbb{Q}$ and some term $s(\overline{y})$,

$$T \vdash \forall x \forall \overline{y} \left[t(x, \overline{y}) = s_q(x) + s(\overline{y}) \right].$$

If $\overline{a}, \overline{b} \in M \models T$, then there is some $c \in P(M)$ such that $[c] > [s(\overline{a})]$ and $[c] > [s(\overline{b})]$. By Lemma 5.1, as long as $q \neq 0$, $M \models P(t(c, \overline{a})) \wedge P(t(c, \overline{b}))$. So for any $t(x, \overline{y})$ that depends nontrivially on x,

$$T \vdash \forall \overline{y}_0 \forall \overline{y}_1 \exists x \left[P(t(x, \overline{y}_0)) \wedge P(t(x, \overline{y}_1)) \right].$$

The same argument works with $\neg P$ or $\neg R_n$ in place of P, and for finite conjunctions of such formulas.

This means that if $\{\varphi(x; \overline{a}_i) : i < \omega\}$ is a k-inconsistent sequence of formulas in T, each of which is a conjunction of atomic formulas and negated atomic formulas, then each $\varphi(x; \overline{a}_i)$ can be assumed to be a conjunction of formulas of the following three forms:

$$t_0(x; \overline{a}_i) < t_1(x; \overline{a}_i),$$
$$\neg \left(t_0(x; \overline{a}_i) < t_1(x; \overline{a}_i) \right), \text{ or}$$
$$R_n(t_0(x; \overline{a}_i), t_1(x; \overline{a}_i)).$$

So each $\varphi(x; \overline{a}_i)$ is a finite union of convex sets. From here we can argue as in the weakly o-minimal case to show that T is inp-minimal. □

Thus we have a nonweakly o-minimal theory T extending the theory of divisible ordered Abelian groups which is dp-minimal. Notice though that Corollary 5.4 indicates that infinite definable subsets in models of T are not too complicated; namely, they must have interior. Simon [22] has recently shown that this must be the case, specifically an infinite definable subset of a dp-minimal divisible ordered Abelian group must have interior.

6 DP-Minimality of the p-adics

In this section our main goal is to show that if $\mathbb{Q}_p$ is a p-adic field, then $\mathrm{Th}(\mathbb{Q}_p)$ is dp-minimal. Indeed, we show the following stronger result: if K is a expansion of $\mathbb{Q}_p$ such that $\mathrm{Th}(K)$ is a p-minimal theory (e.g., $K = \mathbb{Q}_p$ or $K = \mathbb{Q}_p^{\mathrm{an}}$, for p-minimality see [12]), then $\mathrm{Th}(K)$ is dp-minimal. The main technical portion of our proof is written in a more general context than that of p-minimal fields in the hopes that the ideas may be generalized to other valued fields.

We begin by establishing some notation and fixing the context for our work. In what follows, $\mathbb{N}$ denotes the positive integers. Let K be a valued field, considered as a one-sorted structure in the language $\mathcal{L}_{\mathrm{vf}} = \mathcal{L}_{\mathrm{ring}} \cup \{v(x) \leq v(y)\}$. Write $v : K^\times \to \Gamma$ for the valuation and value group. We allow the possibility that K has extra structure beyond $\mathcal{L}_{\mathrm{vf}}$. We assume that K is very saturated. Furthermore, we assume that K is elementarily equivalent to a structure K_0 such that the value group of K_0 is $\mathbb{Z}$. For example, K_0 is either $\mathbb{Q}_p$ or $\mathbb{Q}_p^{\mathrm{an}}$, and K is a elementary extension of K_0. Let R be the valuation ring of K and let $\mathfrak{m}$ be its maximal ideal.

Fix $k \in \mathbb{N}$. Note that $1 + \mathfrak{m}^k$ is a definable subgroup of $K^\times$. Adapting notation from Hrushosvki, we write $RV_k(K)$ for the quotient $K^\times/(1 + \mathfrak{m}^k)$. Let $\pi_k : K^\times \to RV_k(K)$ be the quotient map. For notational convenience, we define $\pi_k(0)$ to be a new element ∞ which we adjoin to $RV_k(K)$.

Proposition 6.1 *For all $z \in K$ and all $x, y \in K \setminus \{z\}$, $\pi_k(x - z) = \pi_k(y - z)$ if and only if $v(x - y) \geq v(y - z) + k$.*

Proof Since the claim is first-order, it suffices to verify it in K_0:

$$\begin{aligned} \pi_k(x - z) = \pi_k(y - z) &\iff \frac{x - z}{y - z} - 1 \in \mathfrak{m}^k \\ &\iff \frac{x - y}{y - z} \in \mathfrak{m}^k \\ &\iff v(x - y) \geq v(y - z) + k. \end{aligned}$$

Note for the last equivalence we use the fact that the value group of K_0 is $\mathbb{Z}$. □

The following definition isolates a property of cells in $\mathbb{Q}_p$. (See Definition 6.10 below for the definition of a cell.)

Definition 6.2 The formula $\varphi(x; y_0, y_1, \ldots, y_l)$ is *cell-like* if there is $k \in \mathbb{N}$ such that $\pi_k(x - y_0) = \pi_k(x' - y_0')$ implies $\varphi(x; y_0, y_1, \ldots, y_l) \leftrightarrow \varphi(x'; y_0', y_1, \ldots, y_l)$. We call y_0 the *center* of $\varphi(x; y_0, \ldots, y_n)$.

When $\varphi(x, \vec{y})$ is a cell-like formula and S is the set defined by $\varphi(x, d)$, we also say that S is cell-like, and we refer to d_0 as the center of S. In Proposition 6.12 below, we verify that a cell in $\mathbb{Q}_p$ is indeed cell-like. We say that an ICT pattern is cell-like if both formulas in the ICT pattern are cell-like.

Suppose we have an ICT pattern with formulas $\varphi_1(x, \vec{y})$, $\varphi_2(x, \vec{y})$ and corresponding parameter sequences $\langle d_{1,j} \rangle_{j \in \mathbb{N}}$, $\langle d_{2,j} \rangle_{j \in \mathbb{N}}$. Then, for notational convenience, we let i range over $\{1, 2\}$, we write $S_{i,j}$ for the definable set $\{x \in K : \varphi_i(x, d_{i,j})\}$, and we write $\langle S_{i,j} \rangle_{i,j}$ to refer to the ICT pattern itself. We now state our main technical result on cell-like ICT patterns.

Proposition 6.3 *Let K be as above, and assume further that K has a finite residue field. If $\langle S_{i,j} \rangle_{i,j}$ is a cell-like* ICT *pattern in K, then there is $c \in K$ and there is a cell-like* ICT *pattern $\langle \widetilde{S}_{i,j} \rangle_{i,j}$ such that the center of each $\widetilde{S}_{i,j}$ is c.*

Proof Suppose that in K we have an ICT pattern, with the notation in the paragraph above, where the formulas $\varphi_1(x, \vec{y})$ and $\varphi_2(x, \vec{y})$ are cell-like. We may assume that the parameter sequences $\langle d_{1,j} \rangle_{j \in \mathbb{N}}$ and $\langle d_{2,j} \rangle_{j \in \mathbb{N}}$ are mutually indiscernible. Recall that y_0 is the center of $\varphi_i(x; \vec{y})$. Write $c_{i,j}$ for $(d_{i,j})_0$; that is, $c_{i,j}$ is the center of $S_{i,j}$. Choose k large enough so that it witnesses that both φ_1 and φ_2 are cell-like. We write π as an abbreviation for π_k.

Without loss of generality, we may assume that the sequences $\langle v(c_{1,j} - c_{2,1}) \rangle_j$ and $\langle v(c_{2,j} - c_{1,1}) \rangle_j$ are weakly increasing; that is, $v(c_{1,1} - c_{2,1}) \leq v(c_{1,2} - c_{2,1})$ and $v(c_{2,1} - c_{1,1}) \leq v(c_{2,2} - c_{1,1})$. (If necessary, temporarily replace the parameter sequences with ones of order type $\mathbb{Z}$, then re-index.)

Claim 6.4 *Either $v(c_{1,1} - c_{2,1}) + 2k \leq v(c_{2,2} - c_{2,1})$ or $v(c_{2,1} - c_{1,1}) + 2k \leq v(c_{1,2} - c_{1,1})$.*

Proof We split into two cases: either $v(c_{1,1} - c_{2,1}) = v(c_{1,2} - c_{2,1})$ or not.

Case 1 Assume $v(c_{1,1} - c_{2,1}) = v(c_{1,2} - c_{2,1})$. By the indiscernibility of $\langle c_{1,j} \rangle_j$ over $\{c_{2,j}\}_j$, we have $v(c_{1,j} - c_{2,1}) = v(c_{1,1} - c_{2,1})$ for all j. Since the residue field is finite, a pigeonhole argument shows that there are $j < j'$ such that $\pi_{2k}(c_{1,j} - c_{2,1}) = \pi_{2k}(c_{1,j'} - c_{2,1})$. In fact, by indiscernibility, we have $\pi_{2k}(c_{1,1} - c_{2,1}) = \pi_{2k}(c_{1,2} - c_{2,1})$. In other words,

$$v(c_{1,2} - c_{1,1}) \geq v(c_{2,1} - c_{1,1}) + 2k,$$

as desired.

Case 2 Now assume $v(c_{1,1} - c_{2,1}) \neq v(c_{1,2} - c_{2,1})$. Using the assumption before the statement of the claim, we have $v(c_{1,1} - c_{2,1}) < v(c_{1,2} - c_{2,1})$. By indiscernibility, $\langle v(c_{1,j} - c_{2,1})\rangle_j$ is an increasing sequence in Γ. Thus there is $j > 1$ such that $v(c_{1,1} - c_{2,1}) + 2k < v(c_{1,j} - c_{2,1})$. In fact, by indiscernibility, we have

$$v(c_{1,1} - c_{2,1}) + 2k < v(c_{1,2} - c_{2,1}). \tag{1}$$

We now show that $v(c_{2,2} - c_{2,1}) \geq v(c_{1,2} - c_{2,1})$. Suppose otherwise. By the ultrametric inequality, $v(c_{1,2} - c_{2,2}) = v(c_{2,2} - c_{2,1})$. From the indiscernibility of $\langle c_{1,j}\rangle_j$ over $\{c_{2,j}\}_j$, we have

$$v(c_{1,1} - c_{2,2}) = v(c_{2,2} - c_{2,1}) = v(c_{1,2} - c_{2,2}).$$

Finally, from the indiscernibility of $\langle c_{2,j}\rangle_j$ over $\{c_{1,j}\}_j$, we get

$$v(c_{1,1} - c_{2,1}) = v(c_{1,2} - c_{2,1}),$$

which contradicts (1). Thus, we have established

$$v(c_{2,2} - c_{2,1}) \geq v(c_{1,2} - c_{2,1}).$$

Combining this inequality with (1), we have $v(c_{2,2} - c_{2,1}) \geq v(c_{1,1} - c_{2,1}) + 2k$, which completes the proof of the claim. □

By Claim 6.4, we may assume $v(c_{2,2} - c_{2,1}) \geq v(c_{1,1} - c_{2,1}) + 2k$. (Switch the rows of the ICT pattern if necessary.) Using the indiscernibility of $\langle c_{1,j}\rangle_j$ over $\{c_{2,j}\}_j$, we get

$$v(c_{2,2} - c_{2,1}) \geq v(c_{1,j} - c_{2,1}) + 2k \tag{2}$$

for all $j \in \mathbb{N}$.

Claim 6.5 *If $a \in (K \setminus S_{1,1}) \cap S_{1,2}$, then $v(a - c_{2,1}) < v(c_{1,2} - c_{2,1}) + k$.*

Proof Otherwise, $v(a - c_{2,1}) \geq v(c_{1,2} - c_{2,1}) + k \geq v(c_{1,1} - c_{2,1}) + k$, so $\pi(a - c_{1,2}) = \pi(c_{2,1} - c_{1,2})$ and $\pi(a - c_{1,1}) = \pi(c_{2,1} - c_{1,1})$. Thus, because φ_1 is cell-like, we have $c_{2,1} \in S_{1,2}$ and $c_{2,1} \notin S_{1,1}$. However, this contradicts the indiscernibility of $\langle d_{1,j}\rangle_j$ over $c_{2,1}$. □

Claim 6.6 *If $a \in S_{2,2} \cap \bigcap_{j>2}(K \setminus S_{2,j})$, then $v(a - c_{2,1})$ is not in the interval $(v(c_{1,2} - c_{2,1}) - k, v(c_{1,2} - c_{2,1}) + k)$.*

Proof In order to prove the claim, we prove the following:

$$\text{there is some } \ell > 1 \text{ such that } a \in S_{2,\ell} \cap \bigcap_{j>\ell}(K \setminus S_{2,j}) \text{ implies}$$
$$v(a - c_{2,1}) \text{ is not in the interval } (v(c_{1,2} - c_{2,1}) - k, v(c_{1,2} - c_{2,1}) + k). \tag{3}$$

Once we have established (3), we are done: by saturation, some finite part of the intersection suffices in (3), and thus, by indiscernibility, we can replace ℓ by 2.

Suppose (3) is false. For each $\ell > 1$, choose $a_\ell \in S_{2,\ell} \cap \bigcap_{j>\ell}(K \setminus S_{2,j})$ such that $v(a_\ell - c_{2,1})$ is in the specified interval. The interval is finite; because the residue field is finite, the set $\{\pi(a_\ell - c_{2,1}) : \ell > 1\}$ is finite. Choose $\ell_1 > \ell_2 > 1$ such that

$$\pi(a_{\ell_1} - c_{2,1}) = \pi(a_{\ell_2} - c_{2,1}).$$

For each $\ell > 1$, the choice of a_ℓ and the inequality (2) yield

$$v(a_\ell - c_{2,1}) < v(c_{1,2} - c_{2,1}) + k \le v(c_{2,2} - c_{2,1}) - k.$$

By the indiscernibility of $\langle c_{2,j} \rangle_j$ over $c_{1,2}$, we get

$$v(a_\ell - c_{2,1}) < v(c_{1,2} - c_{2,1}) + k \le v(c_{2,\ell_1} - c_{2,1}) - k;$$

thus, $\pi(a_\ell - c_{2,\ell_1}) = \pi(a_\ell - c_{2,1})$. By the choice of ℓ_1 and ℓ_2, we get

$$\pi(a_{\ell_1} - c_{2,\ell_1}) = \pi(a_{\ell_1} - c_{2,1}) = \pi(a_{\ell_2} - c_{2,1}) = \pi(a_{\ell_2} - c_{2,\ell_1}).$$

Because φ_2 is cell-like, the previous equation contradicts our choices $a_{\ell_1} \in S_{2,\ell_1}$ and $a_{\ell_2} \notin S_{2,\ell_1}$. This contradiction establishes (3) and completes the proof of the claim. □

For $i \in \{1, 2\}$, build a new parameter sequence $\langle \widehat{d}_{i,j} \rangle_{j>1}$, where $\widehat{d}_{i,j} = c_{2,1}{}^\frown(d_{i,j})_{>0}$. Note that these new sequences are mutually indiscernible. Let $\widehat{S}_{i,j}$ be the corresponding definable sets.

Claim 6.7 *The partial type*

$$\widehat{S}_{1,2} \cap (K \setminus \widehat{S}_{1,3}) \cap \widehat{S}_{2,2} \cap (K \setminus \widehat{S}_{2,3}) \tag{4}$$

is consistent.

Proof Choose $a \in (K \setminus S_{1,1}) \cap S_{1,2} \cap (K \setminus S_{1,3}) \cap S_{2,2} \cap \bigcap_{j>2}(K \setminus S_{2,j})$. We show that a satisfies (4). By Claim 6.5, we know $v(a - c_{2,1}) < v(c_{1,2} - c_{2,1}) + k$. By Claim 6.6, $v(a - c_{2,1})$ is not in the interval $(v(c_{1,2} - c_{2,1}) - k, v(c_{1,2} - c_{2,1}) + k)$, so we have $v(a - c_{2,1}) \le v(c_{1,2} - c_{2,1}) - k$. Consequently, $\pi(a - c_{1,2}) = \pi(a - c_{2,1})$. Also, $v(c_{1,2} - c_{2,1}) \le v(c_{1,3} - c_{2,1})$ (by the weakly-increasing assumption), so $v(a - c_{2,1}) \le v(c_{1,3} - c_{2,1}) - k$; that is, $\pi(a - c_{1,3}) = \pi(a - c_{2,1})$. Since φ_1 is cell-like and $a \in S_{1,2}$ (i.e., $\varphi_1(a, c_{1,2}(d_{1,2})_{>0})$ is true), we know $\varphi_1(a, c_{2,1}(d_{1,2})_{>0})$ is true, so $a \in \widehat{S}_{1,2}$. Similarly, since $a \notin S_{1,3}$ (i.e., $\varphi_1(a, c_{1,3}(d_{1,3})_{>0})$ is false), we have $\varphi_1(a, c_{2,1}(d_{1,3})_{>0})$ is false, so $a \notin \widehat{S}_{1,3}$.

We established above that $v(a - c_{2,1}) + k \le v(c_{1,2} - c_{2,1})$. Since $v(c_{1,2} - c_{2,1}) \le v(c_{2,2} - c_{2,1})$ (by (2)) and $v(c_{1,2} - c_{2,1}) \le v(c_{2,3} - c_{2,1})$ (by indiscernibility of $\langle c_{2,j} \rangle_j$ over $c_{1,2}$), we have $\pi(a - c_{2,2}) = \pi(a - c_{2,1})$ and $\pi(a - c_{2,3}) = \pi(a - c_{2,1})$. Since φ_2 is cell-like and $a \in S_{2,2}$, we know $a \in \widehat{S}_{2,2}$. Similarly, since $a \notin S_{2,3}$, we know $a \notin \widehat{S}_{2,3}$. □

Examining the proof of Fact 2.8 we obtain the following result specific to the current setting.

Fact 6.8 There are mutually indiscernible parameter sequences $\langle d'_{1,j} \rangle_{j \in \mathbb{N}}$ and $\langle d'_{2,j} \rangle_{j \in \mathbb{N}}$ such that $\mathrm{tp}(d'_{1,1} d'_{1,2} d'_{2,1} d'_{2,2}) = \mathrm{tp}(\widehat{d}_{1,1} \widehat{d}_{1,2} \widehat{d}_{2,1} \widehat{d}_{2,2})$ and for $i \in \{1, 2\}$, the formulas $\varphi_i(x, \vec{y}_1) \leftrightarrow \neg\varphi_i(x, \vec{y}_2)$ together with the parameter sequences $\langle d'_{i,2j}{}^\frown d'_{i,2j+1} \rangle_{j \in \mathbb{N}}$ form an ICT pattern.

In particular, since $\mathrm{tp}(d'_{1,1}d'_{1,2}d'_{2,1}d'_{2,2}) = \mathrm{tp}(\widehat{d}_{1,1}\widehat{d}_{1,2}\widehat{d}_{2,1}\widehat{d}_{2,2})$, we know that there is some element c such that each $d'_{i,j}$ is $c^\frown(d'_{i,j})_{>0}$. Let $\widetilde{\varphi}_i$ be the formula $\varphi_i(x, y_0, (\vec{y}_1)_{>0}) \leftrightarrow \neg\varphi_i(x, y_0, (\vec{y}_2)_{>0})$. Then, we still get an ICT pattern from the formulas $\widetilde{\varphi}_i$ and the parameters $\widetilde{e}_{i,j} = c^\frown(d_{i,2j})_{>0}{}^\frown(d_{i,2j+1})_{>0}$. Note that each $\widetilde{\varphi}_i$ is cell-like, because both φ_1 and φ_2 are. Therefore, we have found a cell-like ICT pattern in which all sets have the same center. This completes the proof of Proposition 6.3. □

Below, we use Proposition 6.3 to establish dp-minimality of the p-adics. But first, we state a more general "transfer theorem" for ICT patterns that may be useful in other valued fields.

Theorem 6.9 *Let K be as above, and assume K has finite residue field. If there is a cell-like* ICT *pattern in K, then there is an* ICT *pattern in $RV_k(K)$, for some $k \in \mathbb{N}$.*

Proof Suppose there is some cell-like ICT pattern in K. Let $\langle S_{i,j}\rangle_{i\in\{1,2\},j\in\mathbb{N}}$ be the ICT pattern guaranteed by Proposition 6.3; that is, there is $c \in K$ such that each $S_{i,j}$ is cell-like with center c. We may assume the parameter sequences are mutually indiscernible. Choose k to be large enough to witness that $\langle S_{i,j}\rangle_{i,j}$ is cell-like, and let π be an abbreviation for π_k. Let $T_{i,j} \subseteq RV_k(K)$ be the image of $S_{i,j}$ under the map $x \mapsto \pi(x - c)$. By Fact 2.8, it suffices to show that

$$T_{1,1} \cap (RV_k \setminus T_{1,2}) \cap T_{2,1} \cap (RV_k \setminus T_{2,2})$$

is consistent. Take

$$a \in S_{1,1} \cap (K \setminus S_{1,2}) \cap S_{2,1} \cap (K \setminus S_{2,2}).$$

We claim that

$$\pi(a - c) \in T_{1,1} \cap (RV_k \setminus T_{1,2}) \cap T_{2,1} \cap (RV_k \setminus T_{2,2}).$$

Only $\pi(a - c) \notin T_{1,2}$ and $\pi(a - c) \notin T_{2,2}$ require an argument. Suppose for a contradiction that $\pi(a - c) \in T_{1,2}$. Thus, we can choose $b \in S_{1,2}$ such that $\pi(b - c) = \pi(a - c)$. But $S_{1,2}$ is cell-like with center c, so $a \in S_{1,2}$, contradicting our choice of a. An analogous argument shows $\pi(a - c) \notin T_{2,2}$. □

We now introduce background information on cells in a p-minimal field, which we use to show that p-minimality implies dp-minimality. Let K be a p-minimal field (for example, K_0 is either $\mathbb{Q}_p$ or $\mathbb{Q}_p^{\mathrm{an}}$).

Definition 6.10 ([8])

1. An *annulus* in K is a set of the form

$$\mathrm{Ann}(c, \gamma, \delta) = \{x \in K : \gamma \geq v(x - c) \geq \delta\},$$

where $\gamma \in \Gamma \cup \{\infty\}$, $\delta \in \Gamma \cup \{-\infty\}$ and $c \in K$.

2. Let P_n be set of nth powers in K. A *power coset* in K is a set of the form

$$\mathrm{Pow}_{n,\lambda}(c) = \{x \in K : x - c \in \lambda P_n\},$$

where, $n \in \mathbb{N}$, $\lambda \in \mathbb{N} \cup \{0\}$, and $c \in K$.

3. A *cell* in K is a nonempty set of the form

$$\mathrm{Cell}_{n,\lambda}(c, \gamma, \delta) = \mathrm{Ann}(c, \gamma, \delta) \cap \mathrm{Pow}_{n,\lambda}(c).$$

We call c the *center* of $\mathrm{Cell}_{n,\lambda}(c, \gamma, \delta)$.

Remark 6.11 For each n there are only finitely many cosets of P_n in K^*, each represented by some $\lambda \in \mathbb{N}$. Thus we consider λ to be a term in the language rather than a parameter.

Proposition 6.12 *A cell is cell-like.*

Proof Since Definition 6.2 is first-order, it suffices to work in K_0. It is clear that $\mathrm{Ann}(c, \gamma, \delta)$ is cell-like with center c (as witnessed by any $k \in \mathbb{N}$). Fix $n, \lambda \in \mathbb{N}$. It suffices to show that $\mathrm{Pow}_{n,\lambda}(c)$ is also cell-like with center c. We verify that $k = 2v(n) + 1$ is a witness for Definition 6.2.

Take $x, x', c, c' \in \mathbb{Q}_p$ such that $x - c \in \lambda P_n$ and $\pi_k(x - c) = \pi_k(x' - c')$. We want to show that $x' - c' \in \lambda P_n$. The case that $x = c$ is immediate, so we assume $x \neq c$. Set $y = \lambda^{-1}(x - c)$, $y' = \lambda^{-1}(x' - c')$. We know $y \in P_n$, so $v(y)$ is divisible by n. Also, because π_k is multiplicative, we have $\pi_k(y) = \pi_k(y')$. In particular, we know $v(y) = v(y')$. Let $y_0 = p^{-v(y)}y$ and $y_0' = p^{-v(y)}y'$; thus, $v(y_0) = v(y_0') = 0$. Furthermore, $\pi_k(y_0) = \pi_k(y_0')$. By Proposition 6.1, $v(y_0 - y_0') \geq v(y_0) + k = k$. (Here we are using the fact that both y_0 and y_0' are nonzero; this follows from the assumption that $x \neq c$.) Let $f(X) = X^n - y_0'$. Let $z_0 \in R$ be an nth root of y_0. Then,

$$v(f(z_0)) = v(y_0 - y_0') \geq k = 2v(n) + 1 > 2v(n) = 2v(f'(z_0)).$$

Therefore, by the Hensel-Rychlik Theorem (see [10]), we know there is $z_0' \in R$ such that $f(z_0') = 0$. Let $z' = p^{\frac{v(y)}{n}} z_0'$; hence, $(z')^n = y'$, so $y' \in P_n$. Finally, we see that $x' - c' \in \lambda P_n$, as desired. □

Theorem 6.13 *If K is a p-minimal field, then* $\mathrm{Th}(K)$ *is dp-minimal.*

Proof For a contradiction, suppose that there is an ICT pattern in K. Because K is p-minimal and $\mathbb{Q}_p$ has cell-decomposition (see [9]), we know that in K every formula with one free variable is equivalent to a finite disjunction of cells. By Fact 2.13, we get an ICT pattern with formulas $\mathrm{Cell}_{n_1,\lambda_1}(x, y_0, y_1, y_2)$ and $\mathrm{Cell}_{n_2,\lambda_2}(x, y_0, y_1, y_2)$. By Proposition 6.12, this ICT pattern is cell-like. Thus, by Proposition 6.3 (and its proof), we know that there is an ICT pattern comprised of the formulas $\varphi_i(x, \vec{y}) = (\mathrm{Cell}_{n_i,\lambda_i}(x, y_0, y_1, y_2) \leftrightarrow \neg\,\mathrm{Cell}_{n_i,\lambda_i}(x, y_0, y_3, y_4))$ and parameter sequences $\langle d_{i,j}\rangle_{j\in\mathbb{N}}$, where $d_{i,j} = c^\frown e_{i,j}{}^\frown f_{i,j}$. (Here, $i \in \{1, 2\}$ and each $e_{i,j}$ and $f_{i,j}$ is a tuple of length 2.) The power coset in $\mathrm{Cell}_{n_i,\lambda_i}(x, d_{i,j})$ is $\mathrm{Pow}_{n_i,\lambda_i}(x, c)$; thus, for fixed i, the power coset is constant as j varies.

Claim 6.14 *The single formula*

$$\psi(x, \vec{y}) = (\mathrm{Ann}(x, y_0, y_1, y_2) \leftrightarrow \neg\,\mathrm{Ann}(x, y_0, y_3, y_4))$$

and the parameter sequences $\langle d_{1,j}\rangle_j$ and $\langle d_{2,j}\rangle_j$ form an ICT *pattern.*

Proof Fix $j_1, j_2 \in \mathbb{N}$ and let a be a realization of $\varphi_1(x, d_{1,j_1}) \wedge \bigwedge_{j\neq j_1} \neg\varphi_1(x, d_{1,j}) \wedge \varphi_2(x, d_{2,j_2}) \wedge \bigwedge_{j\neq j_2} \neg\varphi_2(x, d_{2,j})$. We show that a satisfies

$$\psi(x, d_{1,j_1}) \wedge \bigwedge_{j\neq j_1} \neg\psi(x, d_{1,j}) \wedge \psi(x, d_{2,j_2}) \wedge \bigwedge_{j\neq j_2} \neg\psi(x, d_{2,j}).$$

Since a satisfies $\varphi_i(x, d_{i,j_i}) = (\mathrm{Cell}_{n_i,\lambda_i}(x, c, e_{i,j_i}) \leftrightarrow \neg\,\mathrm{Cell}_{n_i,\lambda_i}(x, c, f_{i,j_i}))$, we know that the formulas $\mathrm{Cell}_{n_i,\lambda_i}(x, c, e_{i,j_i})$ and $\mathrm{Cell}_{n_i,\lambda_i}(x, c, f_{i,j_i})$ disagree on a. It follows that a satisfies $\mathrm{Pow}_{n_i,\lambda_i}(x, c)$ (for otherwise, both $\mathrm{Cell}_{n_i,\lambda_i}(x, c, e_{i,j_i})$ and

$\mathrm{Cell}_{n_i,\lambda_i}(x, c, f_{i,j_i})$ are false of a). Hence, $\mathrm{Ann}(x, c, e_{i,j_i})$ and $\mathrm{Ann}(x, c, f_{i,j_i})$ disagree on a, so a satisfies $\psi(x, d_{i,j_i})$. Moreover, for $j \neq j_i$, the formulas $\mathrm{Cell}_{n_i,\lambda_i}(x, c, e_{i,j})$ and $\mathrm{Cell}_{n_i,\lambda_i}(x, c, f_{i,j})$ agree on a. Since a satisfies $\mathrm{Pow}_{n_i,\lambda_i}(x, c)$, we conclude that $\mathrm{Ann}(x, c, e_{i,j})$ and $\mathrm{Ann}(x, c, f_{i,j})$ agree on a. Hence, a does not satisfy $\psi(x, d_{i,j})$. This completes the proof of the claim. □

Let $S_{i,j}$ be the set defined by $\psi(x, d_{i,j})$. Let $\widetilde{S}_{ij}$ be the image of $S_{i,j}$ under the map $x \mapsto v(x - c)$. Then, $\langle \widetilde{S}_{i,j} \rangle_{i,j}$ is a (quantifier-free definable) ICT pattern in $(\Gamma, <) \equiv (\mathbb{Z}, <)$, which is clearly impossible. From this contradiction we conclude that there is no ICT pattern in K, so K is dp-minimal. □

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