Notre Dame Journal of Formal Logic
Volume 52, Number 3, 2011

The Field of LE-Series with a Nonstandard Analytic Structure

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Abstract In this paper we prove that the field of Logarithmic-Exponential power series endowed with the exponential function and a class of analytic functions containing both the overconvergent functions in the t-adic norm and the usual strictly convergent power series is o-minimal.

1 Introduction

The real field with restricted analytic functions and exponentiation is the structure

$$\mathbb{R}_{an,\exp} := (\mathbb{R}, <, 0, 1, +, -, \cdot, (\tilde{f})_{f \in \mathbb{R}\{\xi,m\}, m \in \mathbb{N}}, \exp)$$

where $\mathbb{R}\{\xi, m\} := \mathbb{R}\{\xi_1, \dots, \xi_m\}$ is the ring of all power series in $\xi_1, \dots, \xi_m$ that converge in a neighborhood of $[-1, 1]^m$ and where $\tilde{f}$ equals f on the box $[-1, 1]^m$ and is zero outside.

In [2] and [5] it was shown that $\mathbb{R}_{an,\exp}$ admits quantifier elimination and is o-minimal. On the other hand, the field of *Logarithmic-exponential series* $\mathbb{R}((t))^{\mathrm{LE}}$ is a generalized power series field which turns out to be a nonstandard model of the theory of the real field with restricted analytic functions and exponentiation.

In this paper we consider the field $\mathbb{R}((t))^{\mathrm{LE}}$ together with a nonstandard analytic structure in the sense of [1] and the exponential function. We show that the resulting structure is o-minimal in the language $\mathcal{L}^+(\exp)$ of rings together with symbols for restricted analytic functions, restricted t-adically overconvergent functions and the exponential function.

Received July 9, 2010; accepted May 17, 2011; printed July 25, 2011
2010 Mathematics Subject Classification: Primary, 03C64
Keywords: o-minimality, exponential, analytic structure
 10.1215/00294527-1435447

2 The Setting

Let Γ be an Abelian ordered group. We define the field of power series $k((t^{\Gamma}))$ over a field k as the set of power series of the form

$$f = \sum_{\gamma \in I} a_{\gamma} t^{\gamma}$$

with $I \subset \Gamma$ is well ordered and $a_{\gamma} \in k$. We may also write

$$f = \sum_{\gamma \in \Gamma} a_{\gamma} t^{\gamma}$$

such that the set $\operatorname{supp}(f) := \{\gamma \in \Gamma : a_{\gamma} \neq 0\}$ is well ordered. We have the following standard lemma (see [9]).

Lemma 2.1 (Neumann) *Let G be an ordered Abelian group and let $S \subset G$ be well ordered with $s > 0$ for all $s \in S$. Then for all $g \in G$, the set $S_g = \{(h_1, \ldots, h_n) : n \in \mathbb{N}, h_i \in S, \sum h_i = g\}$ is finite and $\{g : S_g \neq \varnothing\}$ is well ordered.*

Addition on $k((t^{\Gamma}))$ is defined componentwise; that is, for $f = \sum_{\gamma \in I} a_{\gamma} t^{\gamma}$ and $g = \sum_{\gamma \in J} b_{\gamma} t^{\gamma}$ with $I, J \subset G$ well ordered we write

$$f + g = \sum_{\gamma \in I \cup J} (a_{\gamma} + b_{\gamma}) t^{\gamma},$$

where we put $a_{\gamma} = 0$ (respectively, $b_{\gamma} = 0$) for $\gamma \in J \setminus I$ (respectively, $\gamma \in I \setminus J$). The product $f \cdot g$ is defined by

$$f \cdot g = \sum_{(\delta,\sigma) \in I \times J} a_{\delta} b_{\sigma} t^{\delta+\sigma} = \sum_{\gamma \in \Gamma} c_{\gamma} t^{\gamma}$$

where $c_{\gamma} = \sum_{(\delta,\sigma) \in K_{\gamma}} a_{\delta} b_{\sigma}$, $K_{\gamma} = \{(\delta, \sigma) \in I \times J : \delta + \sigma = \gamma\}$. It is easily seen by Lemma 2.1 that K_{γ} is finite for each $\gamma \in \Gamma$ and that the set $\{\gamma \in \Gamma : c_{\gamma} \neq 0\}$ is well ordered. Hence, $f \cdot g$ belongs to $k((t^{\Gamma}))$.

The operations $+$ and $\cdot$ defined above make $k((t^{\Gamma}))$ into a field. If Γ is divisible and k is real closed, then $k((t^{\Gamma}))$ is real closed. The usual order on $\mathbb{R}$ may be extended to $\mathbb{R}((t^{\Gamma}))$ by setting $f = \sum_{\gamma \in \Gamma} a_{\gamma} t^{\gamma} > 0$ if and only if $f \neq 0$ and $a_{\delta} > 0$ where $\delta = \min(\operatorname{supp} f)$. Naturally, $\mathbb{R}((t^{\Gamma}))$ comes equipped with a valuation $\operatorname{ord}_t : \mathbb{R}((t^{\Gamma}))^{\times} \to \Gamma$ given by $\operatorname{ord}_t f = \min(\operatorname{supp} f)$.

3 Analytic Structure

Let Γ be a divisible Abelian ordered group. Let $A_{n,\alpha}$ (α is a positive real) be the set

$$A_{n,\alpha} = \{f \in \mathbb{R}[[\xi_1, \ldots, \xi_n]] : \text{radius of convergence of } f > \alpha\}.$$

We have the following definition [1].

Definition 3.1

$$\begin{aligned}
\mathcal{R}_{n,\alpha}(\Gamma) &:= A_{n,\alpha} \otimes_{\mathbb{R}} \mathbb{R}((t^{\Gamma}))\\
&:= \left\{ \sum_{\gamma \in I} f_\gamma t^\gamma \,:\, f_\gamma \in A_{n,\alpha} \text{ and } I \subset \Gamma \text{ is well ordered} \right\}\\
\mathcal{R}_n(\Gamma) &:= \bigcup_{\alpha>1} \mathcal{R}_{n,\alpha}(\Gamma)\\
\mathcal{R}(\Gamma) &:= \bigcup_{n\in\mathbb{N}} \bigcup_{\alpha>1} \mathcal{R}_{n,\alpha}(\Gamma).
\end{aligned}$$

As we shall see below, the elements of $\mathcal{R}_n(\Gamma)$ define functions from $[-1,1]^n \subset \mathbb{R}((t^{\Gamma}))^n$ to $\mathbb{R}((t^{\Gamma}))$. We define the language $\mathcal{L}_\Gamma$ as

$$\mathcal{L}_\Gamma := \langle\, \cdot\,, +, ^{-1}, 0, 1, <, \mathcal{R}(\Gamma)\rangle.$$

Following [2], $\mathbb{R}((t^{\Gamma}))$ may be naturally equipped with an $\mathcal{L}_\Gamma$-structure. Let $\xi = (\xi_1, \ldots, \xi_n)$ be variables and let g be a formal power series in $\xi_1, \ldots \xi_n$, $g \in \mathbb{R}[[\xi]]$,

$$g(\xi) = \sum_{\nu\in\mathbb{N}^n} a_\nu \xi^\nu.$$

Let $\mathfrak{M}^n = \{x \in \mathbb{R}((t^{\Gamma}))^n \,:\, \mathrm{ord}_t(x_i) > 0, \text{ for } i = 1, \ldots, n\}$, and define $\widehat{g} : \mathfrak{M}^n \to \mathbb{R}((t^{\Gamma}))$ by

$$\widehat{g}(x) = \sum_{\nu\in\mathbb{N}^n} a_\nu x^\nu.$$

Using Lemma 2.1, the last sum is a well-defined element of $\mathbb{R}((t^{\Gamma}))$.

Now let $f \in \mathbb{R}\{\xi_1, \ldots, \xi_n\}$. In other words, $f \in A_{n,\alpha}$ for some $\alpha > 1$. Then $\tilde{f}$ is interpreted on $\mathbb{R}((t^{\Gamma}))^n$ as follows: Let $z \in \mathbb{R}((t^{\Gamma}))^n$ such that $-1 \le z_i \le 1$ for $i = 1, \ldots, n$. There exists some $a \in [-1,1]^n \cap \mathbb{R}^n$ such that $\mathrm{ord}_t(z_i - a_i) > 0$ for $i = 1, \ldots, n$. Also, there exists a convergent power series $g_a \in \mathbb{R}[[\xi_1, \ldots, \xi_n]]$ such that for some $\varepsilon > 0$, if $\|x - a\| < \varepsilon$, then $f(x) = g_a(x - a)$. Then define

$$\tilde{f}(z) = \widehat{g}_a(z - a).$$

Having interpreted $\tilde{f}$ for $f \in A_{n,\alpha}$, for $\alpha > 1$, the interpretation of $f \in \mathcal{R}_{n,\alpha}(\Gamma)$ follows clearly.

Recall the following results (due to Denef and van den Dries as pointed by the referee).

Theorem 3.2 (see [1] or [2]) *Let Γ be a divisible ordered Abelian group. Then $\mathbb{R}((t^{\Gamma}))$ admits quantifier elimination and is o-minimal in $\mathcal{L}_\Gamma$.*

Corollary 3.3 (see [1] or [2]) *Let Γ_0 and Γ_1 be divisible ordered Abelian groups with $\Gamma_0 \subset \Gamma_1$. Consider $\mathbb{R}((t^{\Gamma_i}))$ with its induced $\mathcal{L}_{\Gamma_0}$-structure. Then $\mathbb{R}((t^{\Gamma_0})) \preceq \mathbb{R}((t^{\Gamma_1}))$ in $\mathcal{L}_{\Gamma_0}$.*

4 Axioms for the Exponential Function

There exists a set of universal axioms for the theory $T_{an,\exp}$ (see [2]).

Theorem 4.1 *The theory $T_{an,\exp}$ is o-minimal and is axiomatized by T_{an} and the following axioms:*

E1 $\exp(x+y)=\exp(x)\exp(y)$;
E2 $x<y\to\exp(x)<\exp(y)$;
E3 $x>0\to\exists y\ \exp(y)=x$;
E4$_n$ $x>n^2\to\exp(x)>x^n$, *for each natural number* $n>0$;
E5 $-1\le x\le 1\to\exp(x)=e(x)$,
where e *is the function symbol in* $\mathcal{L}_{an}$ *representing the restricted analytic function which is equal to the power series* $\sum\frac{X^n}{n!}\in\mathbb{R}\{X\}$ *on* $[-1,1]$.

5 Analytic Structure over the Field of Logarithmic-Exponential Series

We shall reconstruct the field of logarithmic-exponential power series while replacing all $\mathcal{L}_{an}$-embeddings by $\mathcal{L}^*_{an}$-embeddings.

5.1 Field of exponential series The construction of the field of exponential power series $\mathbb{R}((t))^{\mathrm{E}}$ is parallel to the classical construction in, for example, [3].

Recall that the construction procedure consists of building a chain $\Gamma_0\subset\Gamma_1\subset\dots$ of DAOG (divisible Abelian ordered groups) and a corresponding chain $K_0\subset K_1\subset\dots$ of ordered fields such that for each n we have $K_n=\mathbb{R}((t^{\Gamma_n}))$, and K_n is equipped with a map $E_n:K_n\to K_{n+1}$ with E_{n+1} extending E_n; that is, $E_{n+1}|_{K_n}=E_n$. Now we equip each K_n with an analytic structure as in Section 3. More precisely, we will obtain an $\mathcal{L}_{\Gamma_n}$-structure,

$$\langle K_n,\cdot,+,^{-1},0,1,\mathcal{R}(\Gamma_n)\rangle,$$

for each $n\in\mathbb{N}$.

Consider the language $\mathcal{L}^*_{an}=\bigcup_{n\in\mathbb{N}}\mathcal{L}_{\Gamma_n}$. Now we equip $\mathbb{R}((t))^{\mathrm{E}}$ with an $\mathcal{R}^*_{an}$-analytic structure where $\mathcal{R}^*_{an}=\bigcup_n\mathcal{R}(\Gamma_n)$ so that $\mathbb{R}((t))^{\mathrm{E}}$ becomes an $\mathcal{L}^*_{an}$-structure. Each function symbol in $\mathcal{L}^*_{an}$ belongs to $\mathcal{L}_{\Gamma_n}$ for some $n\in\mathbb{N}$. The interpretation of f on $\mathbb{R}((t))^{\mathrm{E}}$ is clear from its interpretation on $\mathbb{R}((t^{\Gamma_n}))$ (as in Section 3). The $\mathcal{L}^*_{an}$-structure $\mathbb{R}((t))^{\mathrm{E}}$ now satisfies the axioms (E1), (E2), (E4), and (E5). Now consider the $\mathcal{L}^*_{an}$-theory $T^*_{an}:=\bigcup_n T_{\Gamma_n}$ where $T_{\Gamma_n}=\mathrm{Th}(\mathbb{R}((t^{\Gamma_n})))$. Then we have the following lemma.

Lemma 5.1 $\mathbb{R}((t))^{\mathrm{E}}\models T^*_{an}$.

Proof For $n\in\mathbb{N}$ and all $m>n$ we have $\mathbb{R}((t^{\Gamma_n}))\prec\mathbb{R}((t^{\Gamma_m}))$ in $\mathcal{L}_{\Gamma_n}$ by Corollary 3.3. By standard model theory, it follows that $\mathbb{R}((t^{\Gamma_n}))\prec\mathbb{R}((t))^{\mathrm{E}}$ in $\mathcal{L}_{\Gamma_n}$, and hence $\mathbb{R}((t))^{\mathrm{E}}\models T_{\Gamma_n}$ for all $n\in\mathbb{N}$, and consequently $\mathbb{R}((t))^{\mathrm{E}}\models T^*_{an}$ as required. □

Lemma 5.2 $\mathbb{R}((t))^{\mathrm{E}}$ *admits quantifier elimination and is o-minimal in* $\mathcal{L}^*_{an}$.

Proof Consider a set $M\subset K^n$ defined by some $\mathcal{L}^*_{an}$-formula φ. By the definition of $\mathcal{L}^*_{an}$ there exists some $n\in\mathbb{N}$ such that $\varphi\in\mathcal{L}_{\Gamma_n}$. Now observe that K_n is *o*-minimal in $\mathcal{L}_{\Gamma_n}$, and $K_n\prec K_m$ in $\mathcal{L}_{\Gamma_n}$ for all $m>n$. It follows that $K_n\prec K=\bigcup_i K_i$ in $\mathcal{L}_{\Gamma_n}$; hence K is *o*-minimal in $\mathcal{L}_{\Gamma_n}$. So M is a finite union of points and intervals. It follows that $\mathbb{R}((t))^{\mathrm{E}}$ is *o*-minimal in $\mathcal{L}^*_{an}$ as desired. A similar reasoning shows that $\mathbb{R}((t))^{\mathrm{E}}$ admits QE in $\mathcal{L}^*_{an}$. □

Corollary 5.3 *The theory* T^*_{an} *admits* QE *and is o-minimal.*

5.1.1 $\mathbb{R}((t))^{\mathrm{LE}}$ as a power series field Recall that for the construction of $\mathbb{R}((t))^{\mathrm{LE}}$ we built a chain $L_0 \subset L_1 \subset \ldots$ and a set of isomorphisms $\eta_i : L_i \to \mathbb{R}((t))^{\mathrm{E}}$ such that $\eta_{i+1}(z) = \Phi(\eta_i(z))$ for all $z \in L_i$.

By [4] we have the following.

Lemma 5.4 *There exists an ordered field inclusion $\mathbb{R}((t))^{\mathrm{LE}} \subset \mathbb{R}((t^{\Gamma^+}))$ where Γ^+ is a* DAOG.

Proof Let $\Gamma^{E,n} = \eta_n^{-1}(\Gamma)$, where η_n is the isomorphism $\eta_n : L_n \to \mathbb{R}((t))^E$, and $\Gamma^{E,n}$ is a subgroup of the additive group of L_n. By (2.8) of [4] there exists an isomorphism

$$\sum_{\mu \in \Gamma} a_\mu t^\mu \mapsto \sum_{\mu \in \Gamma^{E,n}} a_\mu \eta_n^{-1}(t^\mu) : \mathbb{R}((t^\Gamma)) \to \mathbb{R}((t^{\Gamma^{E,n}}))$$

of ordered fields. Consider the increasing sequence

$$\Gamma^E = \Gamma^{E,0} \subset \Gamma^{E,1} \subset \Gamma^{E,2} \subset \ldots$$

of subgroups of the additive group of $\mathbb{R}((t))^{\mathrm{LE}}$, and let $\Gamma^+ = \bigcup_n \Gamma^{E,n}$. Finally, let $\mathbb{R}((t^{\Gamma^+}))$ be the maximal ordered field with residue field $\mathbb{R}$ and value group Γ^+. Then we obtain an ordered field inclusion $\mathbb{R}((t))^{\mathrm{LE}} \subset \mathbb{R}((t^{\Gamma^+}))$ as required. □

Now similarly to the case of $\mathbb{R}((t))^{\mathrm{E}}$, we consider languages

$$\mathcal{L}_n = \langle +, \ , \cdot, {}^{-1}, 0, 1, \mathcal{R}(\Gamma^{E,n}) \rangle.$$

Let $\mathcal{L}^+ = \bigcup_n \mathcal{L}_n$ and $T^+ = \bigcup_n T_n$ where T_n is the $\mathcal{L}_n$-theory of $\mathbb{R}((t^{\Gamma^{E,n}}))$. By Corollary 3.3, $L_n \preceq L_m$ in $\mathcal{L}_n$ for $m \geq n$; hence $L_n \prec \mathbb{R}((t))^{\mathrm{LE}}$ for all $n \in \mathbb{N}$ so $\mathbb{R}((t))^{\mathrm{LE}} \models T_n$ for all $n \in \mathbb{N}$. Hence we obtain similarly to Lemma 5.1 the following.

Lemma 5.5 $\mathbb{R}((t))^{\mathrm{LE}} \models T^+$.

The next theorem also follows.

Theorem 5.6 $\mathbb{R}((t))^{\mathrm{LE}} \models T^+(\exp)$.

Proof It is easy to see that exp satisfies axioms (E1)–(E5) of Theorem 4.1. The theorem then follows by the above lemma. □

Theorem 5.7 $\mathbb{R}((t))^{\mathrm{LE}}$ *is o-minimal and admits quantifier elimination in the language $\mathcal{L}^+$.*

Proof Easily follows from Corollary 5.3. □

By analogy to Theorem 2.14 of [2] we have the following theorem.

Theorem 5.8 *The theory T^+ admits a universal axiomatization.*

Proof We consider the following axioms in the language $\mathcal{L}^+({}^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots})$:

(1) axioms for ordered fields,
(2) an axiom for each power series identity, and
(3) for each n the axiom saying that each positive element has an nth-root,

$$(x > 0 \to ((\sqrt[n]{x})^n = x \wedge \sqrt[n]{x} > 0)) \wedge (x \leq 0 \to \sqrt[n]{x} = 0),$$

together with the defining axiom for ${}^{-1}$. These axioms are universal and they axiomatize the theory T^+ as required.

□

We shall denote $\mathbb{R}((t))^{\mathrm{LE}}$ by $\mathbb{K}$. Using quantifier elimination we obtain, as in [2], Corollary 2.15.

Corollary 5.9 *For each function $f : \mathbb{K}^n \to \mathbb{K}$ there are $\mathcal{L}^+({}^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots})$-terms $\tau_1(x_1, \ldots, x_n), \ldots, \tau_k(x_1, \ldots, x_n)$ such that f is piecewise given by the terms $\tau_1, \ldots, \tau_k$.*

Proof Cf. [2], Corollary 2.15. □

6 Quantifier Elimination

In this section we prove that $\mathbb{R}((t))^{\mathrm{LE}}$ admits quantifier elimination in $\mathcal{L}^+(\exp, \log)$. We use the following test of quantifier elimination [10]:

(*) A theory T in a language $\mathcal{L}$ has quantifier elimination if and only if T has the Shoenfield property.

Shoenfield property A theory T has the Shoenfield property if and only if, for any two models, $M_1, M_2 \models T$, such that M_2 is $\|M_1\|^+$-saturated and any $\mathcal{L}$-embedding $\sigma : N \to M_2$ with $N \subset M_1$, there is an $\mathcal{L}$-embedding $\sigma^* : M_1 \to M_2$ extending f.

Let $M_1, M_2 \models T^+(\exp)$ such that M_2 is $\|M_1\|^+$-saturated as in $(*)$ above, and let $\sigma : N \to M_2$ be an $\mathcal{L}^+(\exp, \log)$ embedding. An $\mathcal{L}^+$-embedding $\sigma : M \to N$ is called *log-preserving* if $\log(\sigma(x)) = \sigma(\log x)$ for all $x \in M$. For $M \models T^+$, $L \subset^+ M$ (L is an $\mathcal{L}^+$-substructure of M) and $y \in M \setminus L$, we denote by $L\langle y \rangle$ the $\mathcal{L}^+$-definable closure of $L \cup \{y\}$ in M.

Theorem 6.1 *Let K be a model of $T^+(\exp)$ and F_0 an $\mathcal{L}^+$-substructure of K such that F_0 is log-closed and $F_0 \models T^+$. Let L be a $|K|^+$-saturated model of $T^+(\exp)$ and $\sigma_0 : F_0 \to L$ a log-preserving embedding. Then σ_0 can be extended to a log-preserving embedding of K into L.*

Proof Similar to the proof of Theorem 4.1 of [2]. □

Corollary 6.2 $T^+(\exp)$ *admits quantifier elimination in* $\mathcal{L}^+(\exp, \log)$.

Proof Using the Shoenfield test of quantifier elimination and the previous theorem, we proceed similarly to (4.5) of [2]. □

7 Model Completeness of $T^+(\exp)$ in $\mathcal{L}^+(\exp)$

Quantifier elimination as stated above is of no direct use to us, since it would be extremely complicated to replicate the proofs of [2], [5] or others in our context where the base field is a non-Archimedean real closed field. However, there is some progress in the mathematics literature of our understanding of Noetherian varieties in definably complete structures; that is, the base field is assumed definably complete.

To be able to exploit finiteness results in this context toward the proof of o-minimality one has to use model completeness of the theory. However, finiteness results are only known for algebraic-exponential varieties over a definably complete field. Hence, we must show that $T^+(\exp)$ is model complete in the language $\mathcal{L}^+(\exp)$ (i.e., there is no symbol for log).

Proposition 7.1 *Let $\varphi(x)$ be an $\mathcal{L}^+(\exp, \log)$-formula. Then $\varphi(x)$ is equivalent to an existential $\mathcal{L}^+(\exp)$-formula.*

Proof By quantifier elimination in $\mathcal{L}^+(\exp, \log)$ we may assume that $\varphi(x)$ is quantifier free. Assume that $\varphi(x)$ is purely logarithmic (no occurrences of exponential in a subformula of $\varphi(x)$). So $\varphi(x)$ is a Boolean combination of atomic formulas of the form $\tau_i(x)\Box_i 0$, where $\tau_i(x) = t_i(f_1(x), \dots, f_k(x), \log g_1(x), \dots, \log g_l(x))$, and where $f_j(x), g_j(x)$ are simpler $\mathcal{L}^+(\exp, \log)$-terms, and $\Box_i \in \{=, <\}$. We may assume that all the atomic subformulas occurring in such Boolean combination are positive (no atomic subformula is preceded by a negation). For instance, the subformula $\neg(\tau_i(x) = 0)$ is to be replaced by $\tau_i(x) < 0 \vee 0 < \tau_i(x)$.

So the atomic formula $\tau_i(x)\Box_i 0$ is equivalent to

$$t_i(f_1(x), \dots, f_k(x), y_1, \dots, y_l) = 0 \wedge \bigwedge_{j=1}^{l} y_j = \log g_j(x).$$

Replacing the occurrence of the logarithm by an existential subformula we obtain $\tau_i(x) = 0$ is equivalent to

$$t_i(f_1(x), \dots, f_k(x))\Box_i 0 \wedge \bigwedge_{j=1}^{l} \exists \xi_j g_j(x) = \xi_j.$$

In the same way we can unravel each term (getting rid of the logarithm at the expense of adding more existential quantifiers). Since there are no occurrences of negative subformulas in the formulas above, the existential quantifiers can be pulled back and we obtain an existential $\mathcal{L}^+(\exp)$ as required.

Now if $\varphi(x)$ is not purely logarithmic the same reasoning applies. We leave the details to the reader. □

This proposition easily yields the following theorem.

Theorem 7.2 *The theory $T^+(\exp)$ is model complete in the language $\mathcal{L}^+(\exp)$.*

Proof Since every $\mathcal{L}^+(\exp)$-formula is equivalent to an $\mathcal{L}^+(\exp, \log)$-formula and thus to an existential $\mathcal{L}^+(\exp)$-formula it follows that $T^+(\exp)$ is model complete. □

8 o-Minimality

In this section we prove the o-minimality of the field of LE-series in the language $\mathcal{L}^+(\exp)$. We first recall the definition of definable completeness.

Definition 8.1 Consider a language $\mathcal{L} = (+, -, \cdot, <, 0, \dots)$ expansion of the language of ordered rings and let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $\mathcal{M}$ is definably complete if every definable subset $X \subset M$ of the domain M of $\mathcal{M}$ which is bounded above has a least upper bound.

We have the following result [8].

Proposition 8.2 *The following statements are equivalent:*

(i) *$\mathcal{M}$ is definably complete.*
(ii) *$\mathcal{M}$ has the intermediate value property.*
(iii) *$\mathcal{M}$ is definably connected.*

(iv) *Intervals in M are definably connected.*
(v) *If $f : A \to M^n$ is a definable continuous function and A is a closed and bounded subset of M^m, then the set $f(A)$ is closed and bounded.*
(vi) *If $F : A \to M$ is a definable continuous function and A is a closed and bounded subset of M^n, then f attains a maximum and a minimum in A.*

Proof See [7] and [8]. □

Using model completeness proven in Section 7 we have the following equivalence.

Proposition 8.3 *Let $\mathcal{L}$ be an expansion of the language of ordered rings and let $\mathcal{M}$ be an $\mathcal{L}$-structure such that $\mathcal{M}$ is model complete and definably complete. The structure $\mathcal{M}$ is o-minimal in $\mathcal{L}$ if and only if every definable variety can be written as a union of finitely many definably connected components.*

By a "variety" we mean a Boolean combination of sets of the form $V(f)$ for one or finitely many definable functions $f : M^n \to M$ (where M is the underlying universe of $\mathcal{M}$).

Proof The "if only" implication is obvious. Assume that every variety of M^n is a union of finitely many connected components. By model completeness each definable set $X \subset M$ is the projection of a definable variety $V \subset M^n$ for some n. Since V is the union of finitely many definably connected components it easily follows that its projection along any of the coordinate axes is also composed of finitely many definably connected components (not necessarily varieties). Hence it follows that X is a finite union of intervals and points as required. □

Now we let $\mathbb{K}$ be the field of LE-series $\mathbb{R}((t))^{\mathrm{LE}}$ and $\mathcal{M}$ be the structure $\mathbb{K}$ with language $\mathcal{L}^+(\exp)$. We show that $\mathbb{K}$ is definably complete in the language $\mathcal{L}^+(\exp)$.

Lemma 8.4 *The structure $\mathbb{K}$ is definably complete in $\mathcal{L}^+(\exp)$.*

Proof By Proposition 8.2 it suffices to show property (v); that is, for any definable continuous function $f : A \subset \mathbb{K}^m \to \mathbb{K}^n$, if A is definable, closed and bounded (i.e. definably compact) then so is $f(A)$.

Write $f(x_1, \ldots, x_m) = (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m))$. By Theorem 5.8 there are $\mathcal{L}^+(^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots}, \exp, \log)$-terms $\tau_1(x), \ldots, \tau_k(x)$ such that each f_i is given piecewise by terms $\tau_1(x_1, \ldots, x_m), \ldots, \tau_k(x_1, \ldots, x_m)$. Observe that each $\mathcal{L}^+(^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots}, \exp, \log)$-term $\tau(x)$ can be constructed inductively using

$$\mathcal{L}^+(^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots})\text{-terms } \tau'(x) \text{ and}$$
$$\mathcal{L}(\exp, \log) \text{ terms } \tau''(x).$$

It follows that we may restrict to the case

$$f(x_1, \ldots, x_m) = g(h_1(x_1, \ldots, x_m), \ldots, h_l(x_1, \ldots, x_m))$$

where either

(i) g is an $\mathcal{L}^+(^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots})$-definable function and $h_1, \ldots, h_l$ are $\mathcal{L}(\exp, \log)$-definable functions or
(ii) g is an $\mathcal{L}(\exp, \log)$-definable function and $h_1, \ldots, h_l$ are $\mathcal{L}^+(^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots})$-definable functions.

Clearly, by o-minimality of $\mathbb{K}$ in either $\mathcal{L}^+$ or $\mathcal{L}(\exp)$, $\mathbb{K}$ is definably complete in both languages. Consider the case (i) for definiteness.

Then there exists, by the above argument, a partition of A into $\mathcal{L}(\exp)$-definable pieces $A_1, \dots, A_k$ such that on each A_i for $i = 1, \dots, k$ the functions $h_1, \dots, h_l$ are given by terms (this can be done by partitioning A into definable pieces on each of which h_1 is given by an $\mathcal{L}(\exp, \log)$-term and then refining the obtained partition to obtain a partition of A into pieces on each of which h_1 and h_2 are given by terms, and so on). The pieces must be bounded (since A is) but they are not necessarily closed. Consider a piece A_i for some $i \in \{1, \dots, k\}$, and let $\mathrm{Fr}(A_i) := \bar{A}_i \setminus A_i^\circ$ be the frontier of A_i, where $\bar{A}_i$ and A_i° are the closure and the interior of A_i, respectively. If $\mathrm{Fr}(A_i) \subset A_i$, then A_i is closed and bounded and its image B by $(h_1, \dots, h_l)$ is closed and bounded; hence $g(B)$ is closed and bounded as required.

Otherwise, assume $\mathrm{Fr}(A_i) \not\subseteq A_i$. Then $\mathrm{Fr}(A_i)$ is also finitely partitioned into $\mathcal{L}(\exp)$-definable pieces. Let us assume that $f(\mathrm{Fr}(A_i))$ is closed and bounded. Even though A_i may not be closed, the fact that f is continuous forces $f(A_i)$ to be bounded. Then also $f(\bar{A}_i)$ must be closed by continuity of f and the fact that $f(\mathrm{Fr}(A_i))$ is closed and bounded.

More precisely, assume that $f(A_i)$ is a nonclosed set. Let $(x_n)_n$ be a sequence in A_i such that $f(x_n) \to b$, where $b \notin f(A_i)$ is a limit point of $f(A_i)$. Let $\bar{C}$ be the closure of $C = \{x_n | n \in \mathbb{N}\}$. If $\bar{C} \subset A_i$ then necessarily $b \in f(A_i)$, contrary to hypothesis. Then there exists a limit point a of C which lies in $\mathrm{Fr}(A_i)$ and hence, by the continuity of f, $b = f(a) \in f(\mathrm{Fr}(A_i)) \subset f(\bar{A}_i)$. Thus $f(\bar{A}_i)$ is closed and bounded as required.

To show that $f(\mathrm{Fr}(A_i))$ is closed and bounded we consider the partition of $\mathrm{Fr}(A_i)$ into pieces induced by the partition $A_1, \dots, A_k$. Then we apply to each piece $(A_j \cap \mathrm{Fr}(A_i))$ the same reasoning as above, assuming this time that $f(\mathrm{Fr}(A_j \cap \mathrm{Fr}(A_i)))$ is closed and bounded. Hence, by induction, we are led to consider the case of a piece which is equal to its frontier, hence closed and bounded. Finally set $A = \bar{A}_1 \cup \dots \cup \bar{A}_k$ (since A is closed) and then the required result follows. The case (ii) is treated similarly. Hence $\mathbb{K}$ is definably complete in $\mathcal{L}^+(\exp)$. □

Hereafter we borrow some results and definitions from [6]. In the following "definable" means $\mathcal{L}^+$, $\mathcal{L}_{an,\exp}$, or $\mathcal{L}^+(\exp)$-definable. Let $X \subset Y \subset \mathbb{K}^n$ with Y definable.

Definition 8.5 X is nowhere dense (in Y) if $\mathrm{int}_Y(\mathrm{cl}_Y X) = \varnothing$. X is definably meager (in Y) if there exists a definable increasing family $(A(t))_{t \in \mathbb{K}}$ of nowhere dense subsets of Y such that $X \subset \bigcup_{t \in \mathbb{K}} A(t)$.

Definition 8.6 Y is definably Baire if every nonempty open definable subset of Y is not definably meager (in Y).

For more results and discussions about the notion of definably Baire structure we refer the reader to [6]. Here we state the following result needed for the proof of o-minimality.

Proposition 8.7 *Let $Y \subset \mathbb{K}^m$ be definable. The following are equivalent:*

1. *Y is Baire;*
2. *for all $X \subset Y$, if X is meager then* $\mathrm{int} X = \varnothing$*;*
3. *every $x \in Y$ has a definable neighborhood which is Baire;*
4. *every residual subset of Y is dense;*

5. *every open definable nonempty subset of* Y *is not meager in itself;*
6. *every meager closed definable subset of* Y *has empty interior.*

Proof Cf. [6], Lemma 2.5. □

Lemma 8.8 *Let* $V \subset \mathbb{K}^n$ *be an* $\mathcal{L}^+(\exp)$*-definably connected variety. Then there exist* $N \in \mathbb{N}$, $N \geq n$, *and* $\mathcal{L}^+$*- and* $\mathcal{L}(\exp)$*-definably connected varieties* $V_1, V_2, \ldots, V_k \subset \mathbb{K}^N$ *and* $U \subset \mathbb{K}^N$, *respectively, such that* V *is the image of* $(V_1 \cup \cdots \cup V_k) \cap U$ *by the projection* $\mathbb{K}^N \to \mathbb{K}^n$.

Proof Let $\tau_1(x_1, \ldots, x_n), \ldots, \tau_s(x_1, \ldots, x_n)$ be the $\mathcal{L}^+(\exp)$ terms used in the description of V. We separate the occurrences of the exponential and the other $\mathcal{L}^+$ terms in the standard way; for instance, if

$$\tau(x_1, \ldots, x_n) = t_1(\sigma_1(x), \ldots, \sigma_l(x), \exp(\sigma_{l+1}(x)), \ldots, \exp(\sigma_{l+k}(x)))$$

where t_1 is an $\mathcal{L}^+$-term and $\sigma_1, \ldots, \sigma_{l+k}$ are $\mathcal{L}^+(\exp)$-terms of lower complexity (i.e., they have less exp occurrences) then the set $\{x \in \mathbb{K}^n | \tau_1(x) = 0\}$ is the projection on $\mathbb{K}^n$ of the intersection of the sets $Y_1 = \{(x, y, z) \in \mathbb{K}^{n+2k} | t_1(x) = 0, y_1 - \sigma_{l+1}(x) = 0, \ldots, y_k - \sigma_{l+k}(x) = 0\}$ and $Y_2 = \{(x, y, z) \in \mathbb{K}^{n+2k} | z_1 = \exp(y_1), \ldots, z_k = \exp(y_k)\}$. Y_1 is a finite union of $\mathcal{L}^+$-definably connected sets and Y_2 is $\mathcal{L}(\exp)$-definably connected as required. The general case is treated similarly. □

Definition 8.9 A definably complete structure $\mathbb{K}$ is a Baire structure if $\mathbb{K}$ is definably Baire as a definable subset of $\mathbb{K}$ itself.

Lemma 8.10 $\mathbb{K} := \mathbb{R}((t))^{\mathrm{LE}}$ *is an* $\mathcal{L}^+(\exp)$*-definably complete Baire structure.*

Proof Suppose not; then there exists an $\mathcal{L}^+(\exp)$-definable increasing family $(A(t))_{t \in \mathbb{K}}$ of nowhere dense subsets of $\mathbb{K}^n$ for some $n > 0$ such that $\bigcup_t A(t) = \mathbb{K}^n$. Then, by Lemma 8.8, $A(t)$ is the projection of the $\mathcal{L}^+$-definable and $\mathcal{L}(\exp)$-definable subsets $B(t) \subset \mathbb{K}^m$ and $C(t) \subset \mathbb{K}^m$, respectively. There exists a ball $\mathcal{B}(c, r)$ of center $c \in \mathbb{K}^m$ and radius $r > 0$ such that $B(t) \cap \mathcal{B}(c, r)$ or $C(t) \cap \mathcal{B}(c, r)$ is nowhere dense. Otherwise, by o-minimality of $\mathbb{K}$ in $\mathcal{L}^+$, $B(t)$ contains a ball $\mathcal{B}(c_1, r_1)$ of center $c_1 \in \mathbb{K}^m$ and radius $r_1 > 0$. Similarly, by o-minimality of $\mathbb{K}$ in $\mathcal{L}(\exp)$, $C(t)$ contains a ball $\mathcal{B}(c_2, r_2)$ of center $c_2 \in \mathbb{K}^m$ and radius $r_2 > 0$. By our hypothesis, we can choose $\mathcal{B}(c_2, r_2) \subset \mathcal{B}(c_1, r_1)$ or $\mathcal{B}(c_1, r_1) \subset \mathcal{B}(c_2, r_2)$. So there exists a ball $\mathcal{B}(c, r) = \mathcal{B}(c_1, r_1) \cap \mathcal{B}(c_2, r_2)$ for some c and r as above such that, for $t > t_0$, $\mathcal{B}(c, r) \subset B(t) \cap C(t)$ so there exists a ball $\mathcal{B}'$ which is the image of $\mathcal{B}(c, r)$ by the projection $\mathbb{K}^m \to \mathbb{K}^n$ such that $\mathcal{B}' \subset A(t)$, contradicting the hypothesis. Hence $B(t) \cap \mathcal{B}(c, r)$ or $C(t) \cap \mathcal{B}(c, r)$ is nowhere dense for all t and for some c, r as above. Let us say $B(t) \cap \mathcal{B}(c, r)$ is nowhere dense for all $t \in \mathbb{K}$. Hence $\bigcup_t B(t) \cap \mathcal{B}(c, r) = \mathcal{B}(c, r)$, contradicting the fact that $\mathbb{K}$ is definably Baire in $\mathcal{L}^+$. If $C(t) \cap \mathcal{B}(c, r)$ is nowhere dense for all $t \in \mathbb{K}$, a similar reasoning also leads to a contradiction. □

Theorem 8.11 *The structure* $\mathbb{R}((t))^{\mathrm{LE}}$ *is o-minimal in the language* $\mathcal{L}^+(\exp)$.

Proof By Theorem 9.9 of [6] the theory $T^+(\exp)$ is o-minimal since it is definably complete and Baire by Lemma 8.10 (obviously $T^+(\exp)$ is consistent since it has a model). □

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Acknowledgments

I warmly thank François Loeser for proposing to me this fascinating subject. Also I would like to thank Raf Cluckers, Leonard Lipshitz, and Kobi Peterzil for fruitful discussions.

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