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A Nonstandard Counterpart of WWKL

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Abstract In this paper, we introduce a system of nonstandard second-order arithmetic $\mathsf{ns\text{-}WWKL_0}$ which consists of $\mathsf{ns\text{-}BASIC}$ plus Loeb measure property. Then we show that $\mathsf{ns\text{-}WWKL_0}$ is a conservative extension of $\mathsf{WWKL_0}$ and we do Reverse Mathematics for this system.

1 Introduction

In [4] Keisler characterized the "big five" subsystems of second-order arithmetic $\mathsf{RCA_0}$, $\mathsf{WKL_0}$, $\mathsf{ACA_0}$, $\mathsf{ATR_0}$, and Π^1_1-$\mathsf{CA_0}$ in terms of systems of nonstandard arithmetic (for the details of these systems, see [6]). In [11] we introduced systems of nonstandard second-order arithmetic corresponding to $\mathsf{RCA_0}$, $\mathsf{WKL_0}$, and $\mathsf{ACA_0}$ within which we can do nonstandard analysis. In this paper we introduce a nonstandard counterpart of the system $\mathsf{WWKL_0}$. $\mathsf{WWKL_0}$ is an appropriate system for some measure theory. It consists of $\mathsf{RCA_0}$ plus "weak weak König's lemma" (see [1; 12; 13] and [6, Section X.1]). We use some properties of Loeb measure to give a nonstandard characterization of $\mathsf{WWKL_0}$.

2 Systems of Nonstandard Second-Order Arithmetic

We first introduce the language of nonstandard second-order arithmetic.

Definition 2.1 The language of nonstandard second-order arithmetic $\mathcal{L}^*_2$ is defined by the following:

standard number variables:	$x^{\mathrm{s}}, y^{\mathrm{s}}, \ldots,$
nonstandard number variables:	$x^*, y^*, \ldots,$
standard set variables:	$X^{\mathrm{s}}, Y^{\mathrm{s}}, \ldots,$
nonstandard set variables:	$X^*, Y^*, \ldots,$
function and relation symbols:	$0^{\mathrm{s}}, 1^{\mathrm{s}}, =^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}},$
	$0^*, 1^*, =^*, +^*, \cdot^*, <^*, \in^*, \surd.$

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Here, $0^{\mathrm{s}}, 1^{\mathrm{s}}, =^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}}$ denote "the standard structure" of second-order arithmetic; $0^*, 1^*, =^*, +^*, \cdot^*, <^*, \in^*$ denote "the nonstandard structure" of second-order arithmetic; and $\surd$ denotes an embedding from the standard structure to the nonstandard structure.

The terms and formulas of the language of nonstandard second-order arithmetic are as follows. *Standard numerical terms* are built up from standard number variables and the constant symbols 0^{s} and 1^{s} by means of $+^{\mathrm{s}}$ and $\cdot^{\mathrm{s}}$. *Nonstandard numerical terms* are built up from nonstandard number variables, the constant symbols 0^* and 1^* and $\surd(t^{\mathrm{s}})$ by means of $+^*$ and $\cdot^*$, where t^{s} is a numerical term. *Standard set terms* are standard set variables and *nonstandard set terms* are nonstandard set variables and $\surd(X^{\mathrm{s}})$ whenever X^{s} is a standard set term. *Atomic formulas* are $t_1^{\mathrm{s}} =^{\mathrm{s}} t_2^{\mathrm{s}}$, $t_1^{\mathrm{s}} <^{\mathrm{s}} t_2^{\mathrm{s}}$, $t_1^{\mathrm{s}} \in^{\mathrm{s}} X^{\mathrm{s}}$, $t_1^* =^* t_2^*$, $t_1^* <^* t_2^*$, and $t_1^* \in^* X^*$ where $t_1^{\mathrm{s}}, t_2^{\mathrm{s}}$ are standard numerical terms, t_1^*, t_2^* are nonstandard numerical terms, X^{s} is a standard set term, and X^* is a nonstandard set term. *Formulas* are built up from atomic formulas by means of propositional connectives and quantifiers. A *sentence* is a formula without free variables.

Let φ be an $\mathcal{L}_2$-formula. We write φ^{s} for the $\mathcal{L}_2^*$-formula constructed by adding $^{\mathrm{s}}$ to all occurrences of bound variables, relations, and operations of φ. Similarly, we write φ^* for the $\mathcal{L}_2^*$ formula constructed by adding *. We sometimes omit $^{\mathrm{s}}$ and * for relations and operations. We write $t^{\mathrm{s}\surd}$ for $\surd(t^{\mathrm{s}})$ and $X^{\mathrm{s}\surd}$ for $\surd(X^{\mathrm{s}})$. We sometimes write $\vec{x}$ (respectively, $\vec{X}$) for a finite sequence of variables $x_1, \dots, x_k$ (respectively, $X_1, \dots, X_k$).

A *model* for $\mathcal{L}_2^*$ is a triple $\mathcal{M} = (V_{\mathcal{M}}^{\mathrm{s}}, V_{\mathcal{M}}^*, \surd_{\mathcal{M}})$ such that

(s) $V_{\mathcal{M}}^{\mathrm{s}} = (M_{\mathcal{M}}^{\mathrm{s}}, S_{\mathcal{M}}^{\mathrm{s}}, =_{\mathcal{M}}^{\mathrm{s}}, +_{\mathcal{M}}^{\mathrm{s}}, \cdot_{\mathcal{M}}^{\mathrm{s}}, 0_{\mathcal{M}}^{\mathrm{s}}, 1_{\mathcal{M}}^{\mathrm{s}}, <_{\mathcal{M}}^{\mathrm{s}}, \in_{\mathcal{M}}^{\mathrm{s}})$ is a model for

$$\{=^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, 0^{\mathrm{s}}, 1^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}}\},$$

and

(∗) $V_{\mathcal{M}}^* = (M_{\mathcal{M}}^*, S_{\mathcal{M}}^*, =_{\mathcal{M}}^*, +_{\mathcal{M}}^*, \cdot_{\mathcal{M}}^*, 0_{\mathcal{M}}^*, 1_{\mathcal{M}}^*, <_{\mathcal{M}}^*, \in_{\mathcal{M}}^*)$ is a model for

$$\{=^*, +^*, \cdot^*, 0^*, 1^*, <^*, \in^*\};$$

that is, $V_{\mathcal{M}}^{\mathrm{s}}$ and $V_{\mathcal{M}}^*$ are models for $\mathcal{L}_2$, and $\surd_{\mathcal{M}}$ is a function from $M_{\mathcal{M}}^{\mathrm{s}} \cup S_{\mathcal{M}}^{\mathrm{s}}$ to $M_{\mathcal{M}}^* \cup S_{\mathcal{M}}^*$. We usually omit the subscript $_{\mathcal{M}}$.

In Section 4 we will do nonstandard analysis within nonstandard second-order arithmetic without fixing models. When we do nonstandard analysis, we need to mention the standard universe and the nonstandard universe. In such a context, we use V^{s} to denote the standard universe and V^* to denote the nonstandard universe, without mentioning models. We sometimes say that "φ holds in V^{s}" (abbreviated $V^{\mathrm{s}} \models \varphi$) if φ^{s} holds, and we say that "φ holds in V^*" (abbreviated $V^* \models \varphi$) if φ^* holds. Here V^{s} and V^* do not refer to fixed models. We use this notation only in order to make our nonstandard arguments more accessible.

We next introduce some typical axioms of nonstandard second-order arithmetic.

Definition 2.2

embedding principle (EMB): $\forall \vec{x}^{\mathrm{s}} \forall \vec{X}^{\mathrm{s}} (\varphi(\vec{x}^{\mathrm{s}}, \vec{X}^{\mathrm{s}})^{\mathrm{s}} \leftrightarrow \varphi(\vec{x}^{\mathrm{s}\surd}, \vec{X}^{\mathrm{s}\surd})^*)$

where $\varphi(\vec{x}, \vec{X})$ is any atomic $\mathcal{L}_2$-formula with exactly the displayed free variables.

end extension principle (E): $\forall x^* \forall y^{\mathrm{s}} (x^* < y^{\mathrm{s}\surd} \rightarrow \exists z^{\mathrm{s}} (x^* = z^{\mathrm{s}\surd}))$.

Σ^i_j *overspill principle* (Σ^i_j-OS):

$$\forall \vec{x}^* \forall \vec{X}^* (\forall y^s \exists z^s (z^s \geq y^s \wedge \varphi(z^{s\surd}, \vec{x}^*, \vec{X}^*)^*) \\ \rightarrow \exists y^* (\forall w^s (y^* > w^{s\surd}) \wedge \varphi(y^*, \vec{x}^*, \vec{X}^*)^*))$$

where $\varphi(y, \vec{x}, \vec{X})$ is any Σ^i_j $\mathcal{L}_2$-formula with exactly the displayed free variables.

Σ^i_j *equivalence principle* (Σ^i_j-EQ): $(\varphi^s \leftrightarrow \varphi^*)$ where φ is any Σ^i_j $\mathcal{L}_2$-sentence.

Σ^i_j *transfer principle* (Σ^i_j-TP): $\forall \vec{x}^s \forall \vec{X}^s (\varphi(\vec{x}^s, \vec{X}^s)^s \leftrightarrow \varphi(\vec{x^{s\surd}}, \vec{X^{s\surd}})^*)$

where $\varphi(\vec{x}, \vec{X})$ is any Σ^i_j $\mathcal{L}_2$-formula with exactly the displayed free variables.

Now we define the base system of nonstandard second-order arithmetic.

Definition 2.3 (The system ns-BASIC) The axioms of ns-BASIC are the following:

standard structure:	$(\mathsf{RCA}_0)^s$
basic axioms:	EMB, E
nonstandard axioms:	Σ^0_1-OS, Σ^1_2-EQ, Σ^0_0-TP.

Theorem 2.4 (Conservativity) ns-BASIC *is a conservative extension of* RCA_0*,; that is,* ns-BASIC $\vdash \psi^s$ *implies* $\mathsf{RCA}_0 \vdash \psi$ *for any* $\mathcal{L}_2$*-sentence.*

Proof This is a straightforward consequence of Tanaka's self-embedding theorem [9] and Harrington's theorem [6, Theorem IX.2.1]. □

Within ns-BASIC, a standard set A^s is said to be the *standard part* of a nonstandard set B^* (abbreviated $B^* \upharpoonright M^s = A^s$) if $\forall x^s (x^s \in A^s \leftrightarrow x^{s\surd} \in B^*)$. By Σ^0_0-TP, we can show $\forall X^s (X^{s\surd} \upharpoonright M^s = X^s)$. The existence of the standard part of any nonstandard set provides a nonstandard counterpart of WKL_0.

Definition 2.5 (The system ns-WKL_0) ns-WKL_0 consists of ns-BASIC plus standard part principle (ST) which asserts

$$\forall X^* \exists Y^s \forall x^s (x^s \in Y^s \leftrightarrow x^{s\surd} \in X^*).$$

Theorem 2.6 (Conservativity) ns-WKL_0 *is a conservative extension of* WKL_0*; that is,* ns-$\mathsf{WKL}_0 \vdash \psi^s$ *implies* $\mathsf{WKL}_0 \vdash \psi$ *for any* $\mathcal{L}_2$*-sentence..*

Proof This is a straightforward consequence of Tanaka's self-embedding theorem [9]. See also [11]. □

3 ns-WWKL$_0$

In this section we define Loeb measure for trees and introduce another nonstandard axiom LMP (Loeb measure property) and a new system ns-WWKL_0. Then we show that LMP is a nonstandard counterpart of weak weak König's lemma.

Within RCA_0, we define the measure μ for binary trees as

$$\mu(T) = \lim_{i \to \infty} \frac{|\{\sigma \in T \mid \mathrm{lh}(\sigma) = i\}|}{2^i}.$$

Similarly, we define

$$\mu(T) \geq a \Leftrightarrow \forall i \left(\frac{|\{\sigma \in T \mid \mathrm{lh}(\sigma) = i\}|}{2^i} \geq a \right),$$

and $\mu(T) > a \Leftrightarrow \exists b > a\ \mu(T) \geq b$. Note that we have defined the relation $\mu(T) \geq a$ or $\mu(T) > a$ for $a \in \mathbb{R}$ even if $\mu(T)$ does not exist. $\mathsf{WWKL_0}$ consists of $\mathsf{RCA_0}$ plus weak weak König's lemma (WWKL) which asserts that a binary tree T has a path if $\mu(T) > 0$.

Within ns-BASIC, both V^{s} and V^* satisfy $\mathsf{RCA_0}$ by $\mathsf{RCA_0}^{\mathrm{s}} + \Sigma^1_2$-EQ. Thus, we can develop basic parts of mathematics in both V^{s} and V^* just as in $\mathsf{RCA_0}$. We can define real numbers, open sets, continuous functions, complete separable metric spaces, and so on in both V^{s} and V^*. For example, $\mathbb{N}^{\mathrm{s}} = \{x^{\mathrm{s}} \mid x^{\mathrm{s}} = x^{\mathrm{s}}\}$ is a set of (standard) natural numbers in V^{s}, $\mathbb{N}^* = \{x^* \mid x^* = x^*\}$ is a set of (nonstandard) natural numbers in V^*, $\alpha^{\mathrm{s}} = \langle a^{\mathrm{s}}(i^{\mathrm{s}}) \mid i^{\mathrm{s}} \in \mathbb{N}^{\mathrm{s}}\rangle$ is said to be a (standard) real number in V^{s} (abbreviated $\alpha^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}$) if $|a^{\mathrm{s}}(i^{\mathrm{s}}) - a^{\mathrm{s}}(i^{\mathrm{s}} + k^{\mathrm{s}})| < 2^{-i^{\mathrm{s}}}$ for any $i^{\mathrm{s}}, k^{\mathrm{s}} \in \mathbb{N}^{\mathrm{s}}$, $\alpha^* = \langle a^*(i^*) \mid i^* \in \mathbb{N}^*\rangle$ is said to be a (nonstandard) real number in V^* (abbreviated $\alpha^* \in \mathbb{R}^*$) if $|a^*(i^*) - a^*(i^* + k^*)| < 2^{-i^*}$ for any $i^*, k^* \in \mathbb{N}^*$, and so on. Then we can do nonstandard analysis in this system. From now on, we identify a standard natural number $x^{\mathrm{s}} \in \mathbb{N}^{\mathrm{s}}$ with a nonstandard natural number $x^{\mathrm{s}\surd} \in \mathbb{N}^*$ and consider $\mathbb{N}^{\mathrm{s}} \subseteq \mathbb{N}^*$. By this identification, we usually omit superscripts $^{\mathrm{s}}$ and * for natural numbers. Similarly, we consider a set of standard rational numbers $\mathbb{Q}^{\mathrm{s}}$ (defined in V^{s}) as a subset of a set of nonstandard rational numbers $\mathbb{Q}^*$ (defined in V^*) and omit superscripts for rational numbers since rational numbers are coded by natural numbers.

We now define the standard part of a real number.

Definition 3.1 (Standard part) The following definition is made in ns-BASIC. Let $\alpha^* = \langle a^*(i) \mid i \in \mathbb{N}^*\rangle \in \mathbb{R}^*$ in V^* and $\beta^{\mathrm{s}} = \langle b^{\mathrm{s}}(i) \mid i \in \mathbb{N}^{\mathrm{s}}\rangle \in \mathbb{R}^{\mathrm{s}}$ in V^{s}. Then β^{s} is said to be the standard part of α^* (abbreviated $\mathrm{st}(\alpha^*) = \beta^{\mathrm{s}}$) if

$$\forall i \in \mathbb{N}^{\mathrm{s}}\ |a^*(i) - b^{\mathrm{s}}(i)| \leq 2^{-i} \text{ (in } V^*).$$

We sometimes write $\mathrm{st}(\alpha^*) \in \mathbb{R}^{\mathrm{s}}$ if $\exists \gamma^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}\ \mathrm{st}(\alpha^*) = \gamma^{\mathrm{s}}$.

Similarly to the definition of standard parts, we write $\mathrm{st}(\alpha^*) \leq \beta^{\mathrm{s}}$ if

$$\forall i \in \mathbb{N}^{\mathrm{s}}\ a^*(i) \leq b^{\mathrm{s}}(i) + 2^{-i} \text{ (in } V^*).$$

Note that we have defined $\mathrm{st}(\alpha^*) \leq \beta^{\mathrm{s}}$ even if the standard part of α^* does not exist in $\mathbb{R}^{\mathrm{s}}$. We write $\alpha_1^* \approx \alpha_2^*$ if $\mathrm{st}(\alpha_1^* - \alpha_2^*) = 0$. Note that the existence of standard parts of real numbers is equivalent to ns-$\mathsf{WKL_0}$ over ns-BASIC (see [3]).

Now we introduce a nonstandard axiom for Loeb measure and a new system ns-$\mathsf{WWKL_0}$. Let T^* be a tree in V^*, and let $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$. Then we define the Loeb measure L_ω as

$$L_\omega(T^*) = \mathrm{st}(|\{\sigma \in T^* \mid \mathrm{lh}(\sigma) = \omega\}|/2^\omega),$$

and we define the relation $L_\omega(T^*) > \alpha^{\mathrm{s}}$ for $\alpha^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}$ as

$$L_\omega(T^*) > a \Leftrightarrow \mathrm{st}(|\{\sigma \in T^* \mid \mathrm{lh}(\sigma) = \omega\}|/2^\omega) \not\leq a.$$

Note that we have defined the relation $L_\omega(T^*) > \alpha^{\mathrm{s}}$ even if $L_\omega(T^*)$ does not exist.

Definition 3.2 (Loeb measure property) Loeb measure property (LMP) is the following statement: if a tree T^* has positive Loeb measure, that is, there exists $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ such that $L_\omega(T^*) > 0$, then there exists a function $f^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \to 2$ such that $f^{\mathrm{s}}[n] \in T^*$ for any $n \in \mathbb{N}^{\mathrm{s}}$.

LMP asserts that if T^* has positive Loeb measure, then $T^* \restriction M^{\mathrm{s}}$ has a path (even if $T^* \restriction M^{\mathrm{s}}$ does not exist in V^{s}).

Definition 3.3 (The system ns-WWKL$_0$) ns-WWKL$_0$ consists of ns-BASIC plus LMP.

The next two theorems show that LMP is a nonstandard counterpart of weak weak König's lemma.

Theorem 3.4 ns-WWKL$_0$ *implies* (WWKL)$^{\mathrm{s}}$.

Proof We reason within ns-WWKL$_0$. Let T^{s} be a tree such that $\mu(T^{\mathrm{s}}) > 0$. Then there exist $m \in \mathbb{N}^{\mathrm{s}}$ such that

$$\forall i \in \mathbb{N}^{\mathrm{s}} \frac{|\{\sigma \in T^{\mathrm{s}} \mid \mathrm{lh}(\sigma) = i\}|}{2^i} > 2^{-m}.$$

By Σ^0_1-OS there exists $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ such that

$$\frac{|\{\sigma \in T^{\mathrm{s}\surd} \mid \mathrm{lh}(\sigma) = \omega\}|}{2^\omega} > 2^{-m}.$$

Thus, $L_\omega(T^{\mathrm{s}\surd}) > 0$. Hence, by LMP, $T^{\mathrm{s}} = T^{\mathrm{s}\surd} \restriction M^{\mathrm{s}}$ has a path in V^{s}, and this completes the proof. □

Theorem 3.5 (Conservativity) ns-WWKL$_0$ *is a conservative extension of* WWKL$_0$*; that is,* ns-WWKL$_0 \vdash \psi^{\mathrm{s}}$ *implies* WWKL$_0 \vdash \psi$ *for any sentence* ψ *in* $\mathcal{L}_2$.

In order to prove this theorem, we first prove the following lemma concerning models of WWKL$_0$ and WKL$_0$.

Lemma 3.6 *Let* $(M, S) \models$ WWKL$_0$ *be a countable model. Then there exists an* ω*-extension* $\bar{S} \supseteq S$ *such that* $(M, \bar{S}) \models$ WKL$_0$ *and the pair* $\bar{S}$ *and* S *satisfy the following:*

(†) *for any binary tree* $T \in \bar{S}$,

$$\mu(T) > 0 \to \exists f \in S\ (f \text{ is a path through } T).$$

Note that this lemma is a generalization of the following proposition which was independently obtained by Downey, Hirschfeldt, Miller, and Nies [2, Proposition 7.4] and Reimann and Slaman [5, Theorem 4.5].

Proposition 3.7 *If* $X \in 2^\omega$ *is Martin-Löf random, then for any nonempty* Π^0_1*-class* $P \subseteq 2^\omega$ *there exists* $A \in P$ *such that* X *is Martin-Löf random relative to* A.

Proof This follows easily from the special case of Lemma 3.6 where (M, S) is an ω-model. □

Proof of Lemma 3.6 Within RCA$_0$, we can define the notion of Turing reducibility "$A \leq_T B$" and the notion of Martin-Löf randomness "A is B-random" by using a universal Π^0_1 formula. Within WKL$_0$, we can show the following.

1. Let $\varphi(X, Y)$ be a generalized Π^0_1 formula with exactly the displayed free variables; that is, φ is of the form $\varphi(X, Y) \equiv \exists Z \psi(X, Y, Z)$ such that ψ is a Π^0_1 formula, and let $A \subseteq \mathbb{N}$. Then there exists a binary tree $T_{\varphi(\cdot,A)} \leq_T A$ such that B is a path through $T_{\varphi(\cdot,A)}$ if and only if $\varphi(B, A)$. (Here we identify a set with its characteristic function.)

2. There is a Π^0_1 formula $\Theta(X, Y)$ such that for any $A, B \subseteq \mathbb{N}$, $\Theta(B, A)$ implies B is A-random and $\mu(T_{\Theta(\cdot,A)}) > 0$.
3. If B is A-random and $T \leq_T A$ is a binary tree with $\mu(T) > 0$, then there exists $k \in \mathbb{N}$ such that $B^{(k)} = \{n \mid n + k \in B\}$ is a path through T.

(1) is a well-known fact within WKL_0 (see [6, Lemma VIII.2.4]). Formalizing the usual arguments for randomness, we can show (2) and (3). For (2), see [7, Theorem 3.2], and for (3), see [7, Lemma 4.12].

Let (M, S) be a countable model of WWKL_0. For any countable ω-extension $S' \supseteq S$ such that $(M, S') \models \mathsf{RCA}_0$, we write $S' \supseteq_r S$ if for any $A \in S'$, there exists $B \in S$ such that B is A-random. Clearly, $S \supseteq_r S$ by (2). For a binary tree T, we write $X \in [T]$ if (the characteristic function of) X is a path through T.

Claim Let $S' \supseteq S$ be a countable ω-extension such that $(M, S') \models \mathsf{RCA}_0$ and $S' \supseteq_r S$. Let $T \in S'$ be an infinite binary tree. Then there exists a countable ω-extension $S'' \supseteq S'$ such that there exists a path through T in S'', $(M, S'') \models \mathsf{RCA}_0$ and $S'' \supseteq_r S$.

We show this claim by a forcing argument. By Harrington's theorem, there exists an extension $\tilde{S} \supseteq S'$ such that $(M, \tilde{S})$ is a model of WKL_0 (see, e.g., [6, Theorem IX.2.1]). We will argue in this model. Let $S' = \{A_i\}_{i<\omega}$ and let $\{\langle U^i_m \subseteq 2^{<\mathbb{N}} \mid m < d_i\rangle \in S'\}_{i<\omega}$ be an enumeration of all finite sequences of binary trees belonging to S'. We construct a descending sequence of infinite binary trees $T = T_0 \supseteq T_1 \supseteq T_2 \supseteq \dots$ such that $T_i \in S'$ and $\{T_i\}_{i<\omega}$ satisfies the following:

(i) $\forall m < d_i([T_{i+1}] \cap [U^i_m] = \varnothing \vee T_{i+1} \subseteq U^i_m)$ (this condition is for Harrington's forcing argument);

(ii) there exists $B_i \in S$ such that $X \in [T_{i+1}] \to \Theta(B_i, X \oplus A_0 \oplus \cdots \oplus A_i)$ where Θ is defined in (2) (this means that T_{i+1} forces 'B_i is $X \oplus A_0 \oplus \cdots \oplus A_i$-random if $X \in [T_i]$').

For given T_i, we construct $T_{i+1} \subseteq T_i$ as follows. By (1), there exists a tree $\hat{T} \leq_T T_i \oplus A_0 \oplus \cdots \oplus A_i$ such that $Z \in [\hat{T}] \leftrightarrow \exists X(X \in [T_i] \wedge \Theta(Z, X \oplus A_0 \oplus \cdots \oplus A_i))$. Since $\hat{T} \in S'$, there exists $B \in S$ such that B is $\hat{T}$-random. By (2), $\mu(\hat{T}) > 0$. Thus, there exists $k \in M$ such that $B^{(k)} \in [\hat{T}]$ by (3). Define B_i as $B_i = B^{(k)}$. By (1), take a tree $T'_i \leq_T B_i \oplus T_i \oplus A_0 \oplus \cdots \oplus A_i$ such that $X \in [T'_i] \leftrightarrow \Theta(B_i, X) \wedge X \in [T_i]$. Then T'_i is infinite since $\exists X \Theta(B_i, X) \wedge X \in [T_i]$, and $T'_i \in S'$. Take $C \in \tilde{S}$ such that $C \in [T'_i]$ and define $k \in M$ as $k = \min\{k' \mid \forall m < d_i(C[k'] \in U^i_m \to C \in [U^i_m]\}$. Define $\tau = C[k]$ and define $T_{i+1} \in S'$ as

$$T_{i+1} = \{\sigma \in T'_i \mid \sigma \subseteq \tau \vee (\tau \subseteq \sigma \wedge \forall m < d_i(\tau \in U^i_m \to \sigma \in U^i_m))\}.$$

Then T_{i+1} is infinite since $C \in [T_{i+1}]$, and T_{i+1} satisfies the desired conditions.

Now we construct an ω-extension S''. Define a set $G \subseteq M$ as $G = \bigcap_i [T_i]$; that is, $a \in G$ if and only if there exists $\sigma \in T$ such that $\sigma(a) = 1$ and $\sigma \in T_i$ for any $i < \omega$. For given Σ^0_1 formula $\varphi(m, X)$ and $d \in M$, there exists $i < \omega$ such that $\forall m < d\ \forall X(\varphi(m, X) \leftrightarrow X \notin [U^i_m])$. Thus, for any $m < d$, $\varphi(m, G)$ is equivalent to $T_{i+1} \subseteq U^i_m$, and this means that $(M, S' \cup \{G\})$ satisfies bounded Σ^0_1 comprehension, which is equivalent to $\mathrm{I}\Sigma^0_1$. Define S'' as $S'' = \Delta^0_1\text{-Def}(M, S' \cup \{G\})$. Then $(M, S'') \models \mathsf{RCA}_0 \wedge G \in [T]$. To show $S'' \supseteq_r S$, we only need to show that B_i is $G \oplus A_0 \oplus \cdots \oplus A_i$-random for any $i < \omega$. By the normal form theorem [6, Theorem II.2.7], there exists a Σ^0_0 formula

$\theta(n, X, Y)$ such that $\mathsf{RCA}_0 \vdash \Theta(X, Y) \leftrightarrow \forall n \theta(n, X[n], Y[n])$. We show that $(M, S'') \models \forall n \theta(n, B_i[n], G \oplus A_0 \oplus \cdots \oplus A_i[n])$. For any $n \in M$, there exists $C_n \in \tilde{S}$ such that $C_n \in [T_{i+1}]$ and $C_n \oplus A_0 \oplus \cdots \oplus A_i[n] = G \oplus A_0 \oplus \cdots \oplus A_i[n]$. Since $(M, \tilde{S}) \models \forall X(X \in [T_{i+1}] \rightarrow \Theta(B_i, X \oplus A_0 \oplus \cdots \oplus A_i))$, we have $(M, \tilde{S}) \models \theta(B_i[n], C_n \oplus A_0 \oplus \cdots \oplus A_i[n])$. Thus, $(M, S'') \models \forall n \theta(n, B_i[n], G \oplus A_0 \oplus \cdots \oplus A_i[n])$. Hence $(M, S'') \models \Theta(B_i, G \oplus A_0 \oplus \cdots \oplus A_i)$; thus B_i is $G \oplus A_0 \oplus \cdots \oplus A_i$-random. This completes the proof of the claim.

Using the claim repeatedly, we can construct a sequence of ω-extensions $S = S_0 \subseteq S_1 \subseteq \ldots$ such that $S_i \supseteq_r S$ and for any infinite binary tree $T \in S_i$, there exists a path $A \in S_{i+1}$ through T. Define $\bar{S}$ as $\bar{S} = \bigcup_i S_i$. Then $(M, \bar{S}) \models \mathsf{WKL}_0$ and for any binary tree $T \in \bar{S}$, $\mu(T) > 0 \rightarrow \exists f \in S$ f is a path through T by (3). □

Proof of Theorem 3.5 We show that $\mathsf{WWKL}_0 \nvdash \psi$ implies $\mathsf{ns\text{-}WWKL}_0 \nvdash \psi^{\mathrm{s}}$. Let $(M, S) \models \mathsf{WWKL}_0$ be a countable model such that $(M, S) \models \neg\psi$ and $M \not\cong \omega$. By Lemma 3.6, there exists $\bar{S} \supseteq S$ such that $(M, \bar{S}) \models \mathsf{WKL}_0$ and $\bar{S}$ and S satisfy the condition (†). We prepare some notation for a self-embedding of a model of second-order arithmetic. Let $M^{<c} = \{a \in M \mid a < c\}$, $\bar{S}^{<c} = \{A \cap M^{<c} \mid A \in \bar{S}\}$, and $\bar{S} \upharpoonright \surd(M) = \{A \cap \surd(M) \mid A \in \bar{S}\}$. Note that $\bar{S}^{<c} \subseteq S$. Then, by Tanaka's self-embedding theorem for WKL_0 (see [9]), there exist $c \in M$ and a homomorphism $\surd : M \cup \bar{S} \rightarrow M^{<c} \cup \bar{S}^{<c}$ such that $\surd(M)$ is a semi-regular cut of M and $(\surd(M), \bar{S} \upharpoonright \surd(M)) = (\surd(M), \surd(\bar{S}) \upharpoonright \surd(M)) \cong (M, \bar{S})$. Define a model $\mathcal{M}$ as $V^{\mathrm{s}} = (M^{\mathrm{s}}, S^{\mathrm{s}}) = (M, S)$, $V^* = (M^*, S^*) = (M, S)$, and $\mathcal{M} = (V^{\mathrm{s}}, V^*, \surd \upharpoonright M \cup S)$, and define a model $\bar{\mathcal{M}}$ as $\bar{V}^{\mathrm{s}} = (M^{\mathrm{s}}, \bar{S}^{\mathrm{s}}) = (M, \bar{S})$, $\bar{V}^* = (M^*, \bar{S}^*) = (M, \bar{S})$, and $\bar{\mathcal{M}} = (\bar{V}^{\mathrm{s}}, \bar{V}^*, \surd)$. Then we can easily check that $\mathcal{M} \models \mathsf{ns\text{-}BASIC}$, $\bar{\mathcal{M}} \models \mathsf{ns\text{-}WKL}_0$, and $\mathcal{M} \models \neg\psi^{\mathrm{s}}$. We show that $\mathcal{M} \models \mathrm{LMP}$. Let $H \in M^* \setminus M^{\mathrm{s}}$ and let $T^* \in S^*$ be a binary tree such that $L_H(T^*) > 0$. By STP in $\bar{\mathcal{M}}$, $T^* \upharpoonright M^{\mathrm{s}} \in \bar{S}^{\mathrm{s}}$. Then $\mu(T^* \upharpoonright M^{\mathrm{s}}) \geq L_H(T^*) > 0$. By the condition (†), there exists a path $A^{\mathrm{s}} \in S^{\mathrm{s}}$ through $T^* \upharpoonright M^{\mathrm{s}}$. Thus, $\mathcal{M} \models \mathrm{LMP}$, and $\mathcal{M} \models \mathsf{ns\text{-}WWKL}_0$ by Theorem 3.4. Therefore, $\mathsf{ns\text{-}WWKL}_0 \nvdash \psi^{\mathrm{s}}$, and this completes the proof. □

Remark 3.8 LMP is a purely nonstandard axiom; that is, for any true $\mathcal{L}_2$-sentence φ, $\mathsf{ns\text{-}BASIC} + \varphi^{\mathrm{s}}$ does not imply LMP. To show this, let (M, S) be a nonstandard countable model of $\mathsf{WKL}_0 + \varphi$. Then we can construct $S', S'' \subseteq \mathcal{P}(M)$ such that $S \subseteq S' \subseteq S''$, (M, S'') is a model of WKL_0, and $S' = \Delta^0_1\text{-Def}(M; \{X\})$ for some $X \in S'$ (for this construction, see, e.g., [8]). Note that $S' \not\subseteq_r S''$; thus, $S \not\subseteq_r S''$. By Tanaka's self-embedding theorem, there exist $c \in M$ and a homomorphism $\surd : M \cup S'' \rightarrow M^{<c} \cup S''^{<c}$ such that

$$(\surd(M), S'' \upharpoonright \surd(M)) = (\surd(M), \surd(S'') \upharpoonright \surd(M)) \cong (M, S'').$$

Then $\mathcal{M} = ((M, S), (M, S), \surd \upharpoonright M \cup S)$ is a model of $\mathsf{ns\text{-}BASIC} + (\mathsf{WKL}_0)^{\mathrm{s}} + \varphi^{\mathrm{s}}$. Since $S \not\subseteq_r S'' = S'' \upharpoonright \surd(M) = S \upharpoonright \surd(M)$, $\mathcal{M}$ is not a model of LMP.

4 Reverse Mathematics for Nonstandard Measure Theory

4.1 Nonstandard Reverse Mathematics In this section we do Reverse Mathematics for some basic propositions of nonstandard measure theory and nonstandard integral theory. We reason within $\mathsf{ns\text{-}BASIC}$. For $m \in \mathbb{N}^*$ we define $\Omega_m \subseteq [0, 1]$ in V^* as $\Omega_m = \{i/2^m \in \mathbb{Q}^* \mid 0 \leq i < 2^m\}$. We mainly consider the case $m \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$.

We define the image and the inverse image of st for nonstandard finite sets $A^* \subseteq \mathbf{\Omega}_m$ and standard open sets $A^{\mathrm{s}} \subseteq [0,1]$ as

$$\begin{aligned} \mathrm{st}(A^*) &:= \{x \in \mathbb{R}^{\mathrm{s}} \mid \exists x^* \in A^*\ x = \mathrm{st}(x^*)\},\\ \mathrm{st}_m^{-1}(A^{\mathrm{s}}) &:= \{x^* \in \mathbf{\Omega}_m \mid \exists x^{\mathrm{s}} \in A^{\mathrm{s}}\ x^{\mathrm{s}} = \mathrm{st}(x^*)\},\\ (A^{\mathrm{s}})^{\mathbf{\Omega}_m} &:= \{x^* \in \mathbf{\Omega}_m \mid \exists a^{\mathrm{s}}, b^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}\ a^{\mathrm{s}} \leq \mathrm{st}(x^*) \leq b^{\mathrm{s}} \wedge [a^{\mathrm{s}}, b^{\mathrm{s}}] \subseteq A^{\mathrm{s}}\}. \end{aligned}$$

Here, $[a^{\mathrm{s}}, b^{\mathrm{s}}] \subseteq A^{\mathrm{s}}$ means that $\forall x^{\mathrm{s}} \in [a^{\mathrm{s}}, b^{\mathrm{s}}]\ x^{\mathrm{s}} \in A^{\mathrm{s}}$. Note that $\mathrm{st}_m^{-1}(A^{\mathrm{s}})$ and $(A^{\mathrm{s}})^{\mathbf{\Omega}_m}$ are "external sets"; that is, they are not in V^*.

Definition 4.1 (The Loeb measure) The following definition is made in ns-BASIC. Let $m \in \mathbb{N}^*$. We define the Loeb measure $L_m(A^*)$ for a (nonstandard) finite set $A^* \subseteq \mathbf{\Omega}_m$ as

$$L_m(A^*) = \mathrm{st}(\mathrm{card}(A^*)2^{-m}).$$

We also define Loeb measure for external sets $\mathrm{st}_m^{-1}(A^{\mathrm{s}})$ or $(A^{\mathrm{s}})^{\mathbf{\Omega}_m}$ as follows:

$$\begin{aligned} L_m(\mathrm{st}_m^{-1}(A^{\mathrm{s}})) &= \sup\{L_m(B^*) \mid B^* \subseteq_{\mathrm{al}} \mathrm{st}_m^{-1}(A^{\mathrm{s}})\},\\ L_m((A^{\mathrm{s}})^{\mathbf{\Omega}_m}) &= \sup\{L_m(B^*) \mid B^* \subseteq_{\mathrm{al}} (A^{\mathrm{s}})^{\mathbf{\Omega}_m}\}, \end{aligned}$$

where B^* ranges over nonstandard finite subsets of $\mathbf{\Omega}_m$ in V^*, and $C \subseteq_{\mathrm{al}} D$ means that $L(X^*) = 0$ for any nonstandard finite set $X^* \subseteq C \setminus D$ (C and D may be external). In fact, we cannot prove the existence of $L_m(\mathrm{st}_m^{-1}(A^{\mathrm{s}}))$ or $L_m((A^{\mathrm{s}})^{\mathbf{\Omega}_m})$ within ns-BASIC, but the relations $L_m(\mathrm{st}_m^{-1}(A^{\mathrm{s}})) \leq \alpha^{\mathrm{s}}$ and $L_m((A^{\mathrm{s}})^{\mathbf{\Omega}_m}) \leq \alpha^{\mathrm{s}}$ can be expressed by $\mathcal{L}_2^*$-formulas, as for the definition of the measure μ.

Lemma 4.2 *The following are equivalent over* ns-BASIC.

1. ns-WWKL_0.
2. *There exists* $m \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ *such that*

$$\forall A^* \subseteq \mathbf{\Omega}_m\ (L_m(A^*) > 0 \to \exists x^* \in A^*\ \exists x^{\mathrm{s}} \in [0,1]\ \mathrm{st}(x^*) = x^{\mathrm{s}}).$$

3. *For any* $m \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$*, we have*

$$\forall A^* \subseteq \mathbf{\Omega}_m\ (L_m(A^*) > 0 \to \exists x^* \in A^*\ \exists x^{\mathrm{s}} \in [0,1]\ \mathrm{st}(x^*) = x^{\mathrm{s}}).$$

Proof Define $\lambda_m : 2^{=m} \to \mathbf{\Omega}_m$ as $\lambda(\sigma) = \sum_{i<m} 2^{-i-1}\sigma(i)$ where $2^{=m} = \{\sigma \in 2^{<\mathbb{N}^*} \mid \mathrm{lh}(\sigma) = m\}$. Then λ_m is a natural isomorphism from $2^{=m}$ to $\mathbf{\Omega}_m$. By this isomorphism, (3) $\leftrightarrow$ (1) is trivial. (3) $\to$ (2) is also trivial, so we show (2) $\to$ (3). Let $n \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$, $B^* \subseteq \mathbf{\Omega}_n$, and $L_n(B^*) > 0$. By (2), take $m \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ such that $\forall A^* \subseteq \mathbf{\Omega}_m\ L_m(A^*) > 0 \to \exists x^* \in A^*\ \exists x^{\mathrm{s}} \in [0,1]\ \mathrm{st}(x^*) = x^{\mathrm{s}}$. If $n \leq m$, define $B_0^* = \{x^* \in \mathbf{\Omega}_m \mid \exists y^* \in B^* y^* \leq x^* < y^* + 2^{-n}\}$, and if $n > m$, define $B_0^* = \{x^* \in \mathbf{\Omega}_m \mid \exists y^* \in B^*\ x^* \leq y^* < x^* + 2^{-m}\}$. In either case, $L_m(B_0^*) > 0$; hence there exist $x^* \in B_0^*$ and $x^{\mathrm{s}} \in [0,1]$ such that $\mathrm{st}(x^*) = x^{\mathrm{s}}$. Thus, there exists $y^* \in B^*$ such that $x^* \approx y^*$; that is, $\mathrm{st}(y^*) = x^{\mathrm{s}}$. □

Using the notion of Loeb measure, we do Reverse Mathematics for nonstandard measure and integral theory. Recall that we can define the Riemann integral for a continuous function within RCA_0. For a continuous function f on $[0,1]$ and a splitting $\Delta = \{0 = a_0 \leq \xi_0 \leq a_1 \leq \cdots \leq \xi_{k-1} \leq a_k = 1\}$ of $[0,1]$, we define $S(f;\Delta) = \sum_{i<k} f(\xi_i)(a_{i+1} - a_i)$. We define a splitting Δ_m as $a_i = i2^{-m}$ for any

$i \leq 2^m$ and $\xi_i = i2^{-m}$ for any $i < 2^m$. We write $\Delta \in \Omega_m$ if $a_i, \xi_i \in \Omega_m$ for all $i \leq k$. Then the Riemann integral of f is defined as

$$\int_0^1 f(x)dx = \lim_{|\Delta|\to 0} S(f; \Delta)$$

where $|\Delta| = \max\{a_{i+1} - a_i \mid i < k\}$. Moreover, we can define the internal measure for open sets within RCA_0. Let $\mathbb{Q}[x]$ be a set of rational polynomials. For an open set $U \subseteq [0, 1]$, we define the internal measure $\mu(U)$ as

$$\mu(U) = \sup\left\{\int_0^1 f(x)dx \,\middle|\, f \in \mathbb{Q}[x], \forall x \in [0,1] \setminus U\ f(x) \leq 0, \forall x \in [0,1]\ f(x) \leq 1\right\}.$$

Note that we can integrate polynomial functions within RCA_0. In fact, we have only defined the relation $\mu(U) \leq \alpha$ and $\mu(U) \geq \alpha$ here, as for the definition of the measure for trees.

The following theorem shows that we can get a nonstandard approximation for an open set or for an L^1-function within ns-BASIC. Since a rational polynomial can be coded by a natural number, we consider $\mathbb{Q}^s[x]$ as a subset of $\mathbb{Q}^*[x]$. Let $\Omega_{\mathbb{N}^s} = \bigcup_{m\in\mathbb{N}^s} \Omega_m$.

Theorem 4.3 *Let $\omega \in \mathbb{N}^* \setminus \mathbb{N}^s$, and let $\Omega = \Omega_\omega$, $L = L_\omega$, and $\mathrm{st}^{-1} = \mathrm{st}_\omega^{-1}$. The following are provable within* ns-BASIC.

1. *For any standard open set $A^s \subseteq [0, 1]$ and $l \in \mathbb{N}^s$, there exist nonstandard finite sets $A^*_+, A^*_- \subseteq \Omega$ such that $A^*_- \subseteq (A^s)^\Omega \subseteq A^*_+$ and $\mu(A^s) - 2^{-l} \leq L(A^*_-) \leq \mu(A^s) \leq L(A^*_+) \leq \mu(A^s) + 2^{-l}$, where μ is the internal measure for open sets. Particularly, $L((A^s)^\Omega) = \mu(A^s)$.*
2. *(Nonstandard L^1-function): An L^1-function can be expressed by one nonstandard polynomial in the following sense. Let $\mathcal{F}^s = \langle f_i \in \mathbb{Q}^s[x] \mid i \in \mathbb{N}^s\rangle$ be a sequence of rational polynomials in V^s such that $\int_0^1 |f_i(x) - f_j(x)|\, dx \leq 2^{-i}$ if $i \leq j \in \mathbb{N}^s$. Then there exist a rational polynomial f^* and a sequence $\langle A^*_n \subseteq \Omega \mid n < H\rangle$ in V^* such that $H \in \mathbb{N}^* \setminus \mathbb{N}^s$, $L(\bigcup_{n\in\mathbb{N}^s} A^*_n) = 1$, and*

$$\forall x \in \bigcup_{n\in\mathbb{N}^s} A^*_n \ \lim_{m\to\mathbb{N}^s} f_m(x) \approx f^*(x),$$

where

$$\lim_{m\to\mathbb{N}^s} \alpha^*_k \approx \beta^* \Leftrightarrow \forall n \in \mathbb{N}^s \exists m \in \mathbb{N}^s \forall k \in \mathbb{N}^s\, k \geq m \to |\alpha^*_k - \beta^*| < 2^{-n}.$$

Proof We first prove (1). Let $A^s \subseteq [0, 1]$ be an open set in V^s and $l \in \mathbb{N}^s$. Take $p, q \in \Omega_{\mathbb{N}^s}$ such that $\mu(A^s) - 2^{-l} \leq p < \mu(A^s) < q \leq \mu(A^s) + 2^{-l}$. By the definition of $\mu(A^s)$, there exists a finite sequence of intervals $\langle [a_i, b_i] \mid i < k\rangle$ in V^s such that $k \in \mathbb{N}^s$, $a_i, b_i \in \Omega_{\mathbb{N}^s}$, $b_i \leq a_j$ if $i < j$, $\bigcup_{i<k}[a_i, b_j] \subseteq A^s$, and $\sum_{i<k}(b_i - a_i) \geq p$. Then $A^*_- := \{x \in \Omega \mid \exists i < k\ a_i \leq x \leq b_i\}$ is the desired finite set in V^*. On the other hand, by the definition of $\mu(A^s)$, there exists a sequence of intervals $\Lambda^s = \langle [c_i, d_i] \mid i \in \mathbb{N}^s\rangle$ in V^s such that $c_i, d_i \in \Omega_{\mathbb{N}^s}$, $(c_i, d_i) \cap (c_j, d_j) = \varnothing$ if $i \neq j$, $\bigcup_{i\in\mathbb{N}^s}[a_i, b_j] \supseteq A^s$, and $\sum_{i\in\mathbb{N}^s}(c_i - d_i) \leq q$. By Σ^0_1-OS, there exists $m \in \mathbb{N}^* \setminus \mathbb{N}^s$ such that $\Lambda^{s\surd} \upharpoonright m = \langle [c_i, d_i] \mid i < m\rangle$ is an extended sequence of Λ^{s} in V^* such that $c_i, d_i \in \Omega_m$ and $\sum_{i<m}(c_i - d_i) \leq q$. Then $A^*_+ := \{x \in \Omega \mid \exists i < m\ c_i \leq x \leq d_i\}$ is the desired finite set in V^*.

Next we prove (2). Let $\mathcal{F}^{\mathrm{s}} = \langle f_i \mid i \in \mathbb{N}^{\mathrm{s}}\rangle$ be a sequence of rational polynomials in V^{s} such that $\int_0^1 |f_i(x) - f_j(x)|dx \leq 2^{-i}$ if $i \leq j \in \mathbb{N}^{\mathrm{s}}$. By Σ^0_1-OS, there exists $H \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ such that $\mathcal{F}^{\mathrm{s}\surd} \restriction_{H+1} = \langle f_i \mid i \leq H\rangle$ is a sequence of polynomials in V^* which satisfies the following:

$$\int_0^1 |f_i(x) - f_j(x)|dx \leq 2^{-i} \text{ if } i \leq j \leq H;$$
$$H \max\{|f_i'(x)| \mid i \leq H, x \in [0,1]\} < \omega.$$

Define a sequence $\langle C^*_n \mid n < H\rangle$ of finite unions of closed intervals in V^* as

$$C^*_n = \left\{ z^* \in [0,1] \;\middle|\; \forall k \leq H \sum_{i=n+2k+3}^{H-1} |f_i(z^*) - f_{i+1}(z^*)| \leq 2^{-k} \right\}.$$

We show that $\mathrm{card}(([0,1] \setminus C^*_n) \cap \Omega) \leq 2^{\omega-n}$. Let $M > \max\{|f_i'(x)| \mid i \leq H, x \in [0,1]\}$, and let $p_{nk} = \mathrm{card}(\{x \in \Omega \mid \sum_{i=n+2k+3}^{H} |f_i(x) - f_{i+1}(x)| > 2^{-k}\})$. Then

$$\left| \sum_{x\in\Omega} |f_i(x) - f_{i+1}(x)| 2^{-\omega} - \int_0^1 |f_i(x) - f_{i+1}(x)|dx \right| \leq \frac{M}{2^\omega}.$$

Thus,

$$\begin{aligned} p_{nk} \cdot 2^{-k-\omega} &\leq \sum_{x\in\Omega} \sum_{i=n+2k+3}^{H} |f_i(x) - f_{i+1}(x)| 2^{-\omega} \\ &\leq \frac{MH}{2^\omega} + \sum_{i=n+2k+3}^{H} \int_0^1 |f_i(x) - f_{i+1}(x)|dx \\ &< 2^{-n-2k-1}. \end{aligned}$$

Then $\mathrm{card}(([0,1] \setminus C_n) \cap \Omega) \leq \sum_{k\leq H} 2^{-n-2k-1} \leq 2^{\omega-n}$.

Define $A^*_n \subseteq \Omega$ as $A^*_n = C^*_n \cap \Omega$ and define f^* as $f^*(x) = f_H(x)$. Then $L(A^*_n) \geq 1 - 2^{-n}$ and $L(\bigcup_{n\in\mathbb{N}^{\mathrm{s}}} A^*_n) = 1$. We can easily check that for any $n, k \in \mathbb{N}^{\mathrm{s}}$, $n \leq m \in \mathbb{N}^{\mathrm{s}}$ and $x \in A^*_n$, $|f_m(x) - f^*(x)| \leq 2^{-k}$; that is, $\lim_{m\to\mathbb{N}^{\mathrm{s}}} f_m(x) \approx f^*(x)$ for any $x \in \bigcup_{n\in\mathbb{N}^{\mathrm{s}}} A^*_n$. □

Remark 4.4 Theorem 4.3(2) showed that "every L^1-convergent sequence of rational polynomials converges almost everywhere in V^*" within ns-BASIC. However, this does not imply "every L^1-convergent sequence of rational polynomials converges almost everywhere in V^{s}" within ns-BASIC, which means that the same statement holds in RCA_0 by conservativity, since we cannot prove that the standard part of measure one set is measure one within ns-BASIC. Compare this with Theorem 2.2 of [12].

On the other hand, we need ns-WWKL_0 for a nonstandard approximation for a measure or an integral. A (nonstandard) continuous function F^* in V^* is said to be s-*bounded* if $\forall x^* \in \mathrm{dom}(F^*)\ \exists K \in \mathbb{N}^{\mathrm{s}}\ |F^*(x^*)| < K$; that is, if each element of the range of F^* is bounded above by a standard integer.

Theorem 4.5 *The following are equivalent over* ns-BASIC.

1. ns-WWKL_0.
2. *Let* $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$, *and let* $\boldsymbol{\Omega} = \boldsymbol{\Omega}_\omega$, $L = L_\omega$, *and* $\mathrm{st}^{-1} = \mathrm{st}_\omega^{-1}$. *Then, for any standard open set* $A^{\mathrm{s}} \subseteq [0,1]$ *and* $l \in \mathbb{N}^{\mathrm{s}}$, *there exist nonstandard finite sets* $A_+^*, A_-^* \subseteq \boldsymbol{\Omega}$ *such that* $A_-^* \subseteq_{\mathrm{al}} \mathrm{st}^{-1}(A^{\mathrm{s}}) \subseteq_{\mathrm{al}} A_+^*$ *and* $\mu(A^{\mathrm{s}}) - 2^{-l} \le L(A_-^*) \le \mu(A^{\mathrm{s}}) \le L(A_+^*) \le \mu(A^{\mathrm{s}}) + 2^{-l}$, *where* μ *is the internal measure for open sets. Particularly,* $L(\mathrm{st}^{-1}(A^{\mathrm{s}})) = \mu(A^{\mathrm{s}})$.
3. *Let* f^{s} *be a continuous function on* $[0,1]$ *in* V^{s} *and* F^* *be an* s*-bounded continuous function on* $[0,1]$ *in* V^* *such that* f^{s} *is a prestandard part of* F^*; *that is,* $\mathrm{st}(F^*(x^*)) = f^{\mathrm{s}}(x^{\mathrm{s}})$ *if* $\mathrm{st}(x^*) = x^{\mathrm{s}}$. *Then* f^{s} *is Riemann integrable on* $[0,1]$, *and for any* $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$,
$$\int_0^1 f^{\mathrm{s}}(x)dx = \mathrm{st}\left(\sum_{i<2^\omega} \frac{F^*(i2^{-\omega})}{2^\omega}\right).$$

Proof We first show (1) → (2). We reason within ns-WWKL_0. Let $l \in \mathbb{N}^{\mathrm{s}}$, and let $A^{\mathrm{s}} \subseteq [0,1]$ be a standard open set. By Theorem 4.3(1), there exist nonstandard finite sets $A_+^*, A_-^* \subseteq \boldsymbol{\Omega}$ such that
$$A_-^* \subseteq (A^{\mathrm{s}})^{\boldsymbol{\Omega}} \subseteq A_+^*$$
and
$$\mu(A^{\mathrm{s}}) - 2^{-l} \le L(A_-^*) \le \mu(A^{\mathrm{s}}) \le L(A_+^*) \le \mu(A^{\mathrm{s}}) + 2^{-l}.$$
Since $\mathrm{st}^{-1}(A^{\mathrm{s}}) \subseteq (A^{\mathrm{s}})^{\boldsymbol{\Omega}}$, $A_+^* \supseteq_{\mathrm{al}} \mathrm{st}^{-1}(A^{\mathrm{s}})$. If $B^* \subseteq A_-^* \setminus \mathrm{st}^{-1}(A^{\mathrm{s}})$, then $\mathrm{st}(B^*) = \varnothing$; thus $L(B^*) = 0$ by LMP. Therefore, $A_-^* \subseteq_{\mathrm{al}} \mathrm{st}^{-1}(A^{\mathrm{s}})$.

Next we show (1) → (3). We reason within ns-WWKL_0. Let f^{s} be a continuous function on $[0,1]$ in V^{s} and F^* be an s-bounded continuous function on $[0,1]$ in V^* such that f^{s} is a prestandard part of F^*. Since F^* is s-bounded, for any $K \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$, $|F^*| < K$ on $[0,1]$. Thus, by Σ_1^0-OS, there exists $K \in \mathbb{N}^{\mathrm{s}}$ such that $|F^*| < K$. For (3), we will show the following by contradiction:

(∗) for any $l \in \mathbb{N}^{\mathrm{s}}$, there exists $n \in \mathbb{N}^{\mathrm{s}}$ such that
$$\forall \Delta \in \boldsymbol{\Omega}_{\mathbb{N}^{\mathrm{s}}}(|\Delta| \le 2^{-n} \to S(F^*; \Delta_\omega) - 2^{-l} \le S(f; \Delta) \le S(F^*; \Delta_\omega) + 2^{-l}).$$

Assume (∗) fails. Then, without loss of generality, we assume that

¬(∗) there exists $l \in \mathbb{N}^{\mathrm{s}}$ such that for all $n \in \mathbb{N}^{\mathrm{s}}$,
$$\exists \Delta \in \boldsymbol{\Omega}_{\mathbb{N}^{\mathrm{s}}}(|\Delta| \le 2^{-n} \wedge S(F^*; \Delta_\omega) + 2^{-l} < S(f^{\mathrm{s}}; \Delta)).$$

Since f^{s} is a prestandard part of F^*, we have

(∗∗) there exists $l \in \mathbb{N}^{\mathrm{s}}$ such that for all $n \in \mathbb{N}^{\mathrm{s}}$,
$$\exists \Delta \in \boldsymbol{\Omega}_\omega(|\Delta| \le 2^{-n} \wedge S(F^*; \Delta_\omega) + 2^{-l} < S(F^*; \Delta)).$$

Then, by Σ_1^0-OS, there exist $\bar{m} \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ and $\bar{\Delta} \in \boldsymbol{\Omega}_\omega$ such that $|\bar{\Delta}| \le 2^{-\bar{m}}$ and $S(F^*; \Delta_\omega) + 2^{-l} < S(F^*; \bar{\Delta})$. For $x \in \boldsymbol{\Omega}_\omega$, we define $i_x = \max\{j \mid a_j \in \bar{\Delta} \wedge a_j \le x\}$ and $\delta_x = |F^*(x) - F^*(\xi_{i_x})|$. Define a nonstandard finite set $A^* \subseteq \boldsymbol{\Omega}_\omega$ as $A^* = \{x \in \boldsymbol{\Omega}_\omega \mid \delta_x > 2^{-l-1}\}$. For any $x \in \boldsymbol{\Omega}_\omega$, if $\exists y^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}\, \mathrm{st}(x) = y^{\mathrm{s}}$, then $\mathrm{st}(F^*(x)) = f^{\mathrm{s}}(y^{\mathrm{s}}) = \mathrm{st}(F^*(\xi_{i_x}))$ since $|x - \xi_{i_x}| \le 2^{-\bar{m}}$. Hence, $\mathrm{st}(A^*) = \varnothing$, and $L_\omega(A^*) = \mathrm{st}(\mathrm{card}(A^*)/2^\omega) = 0$ by LMP. Then

$$\begin{aligned}|S(F^*;\bar{\Delta}) - S(F^*;\Delta_\omega)| &\le \sum_{x\in\Omega_\omega} 2^{-\omega}\delta_x \\ &\le 2^{-l-1} + \sum_{x\in A^*} 2^{-\omega}\delta_x \\ &\le 2^{-l-1} + 2K L_\omega(A^*).\end{aligned}$$

This contradicts $S(F^*;\Delta_\omega) + 2^{-l} < S(F^*;\bar{\Delta})$; thus $(*)$ holds.

Finally, we show $\neg(1) \to \neg(2)$ and $\neg(1) \to \neg(3)$. By $\neg$ns-WWKL$_0$ and Lemma 4.2, for any $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$, there exists $A^* \subseteq \Omega_\omega$ such that $L_\omega(A^*) > 0$ but $\mathrm{st}(A^*) = \varnothing$. Then $\mu([0,1]) = 1$ while $L_\omega(\mathrm{st}^{-1}([0,1])) \le L_\omega(\Omega \setminus A^*) < 1$. Thus, we have $\neg(1) \to \neg(2)$. Also, we can easily construct a continuous function F^* on $[0,1]$ such that $F^*(x) = 1$ if $x \in A^*$ and $F^*(x) = 0$ if $x \in \Omega \setminus A^*$. Then F^* is s-bounded and $f^{\mathrm{s}} = 0$ is a prestandard part of F^*. However,

$$\mathrm{st}\left(\sum_{i<2^\omega} \frac{F^*(i2^{-\omega})}{2^\omega}\right) = L_\omega(A^*) > 0.$$

This means $\neg(1) \to \neg(3)$. □

4.2 Application for standard Reverse Mathematics Using the previous results for nonstandard Reverse Mathematics, we can use some techniques of nonstandard analysis for standard Reverse Mathematics within WWKL$_0$, and we can get some nonstandard proofs for some theorems of WWKL$_0$ by the conservation result in Section 3.

The next lemma shows that a standard continuous function in V^{s} can be expanded to a nonstandard continuous function in V^* within ns-BASIC. For this expansion, we recall the definition of continuous function within RCA$_0$. A (code for a) continuous function f on $[0,1]$ is a set of quintuples $\Phi \subseteq \mathbb{N}\times\mathbb{Q}\times\mathbb{Q}^+\times\mathbb{Q}\times\mathbb{Q}^+$ which satisfies the following three conditions and the domain condition:

(1) if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$, then $|b-b'| \le s+s'$;
(2) if $(a,r)\Phi(b,s)$ and $|a'-a|+r' < r$, then $(a',r')\Phi(b,s)$;
(3) if $(a,r)\Phi(b,s)$ and $|b-b'|+s < s'$, then $(a,r)\Phi(b',s')$;
(dom) for any $x \in [0,1]$ and for any $\varepsilon > 0$ there exists $(m,a,r,b,s) \in \Phi$ such that $|x-a| < r$ and $s < \varepsilon$,

where $(a,r)\Phi(b,s)$ is an abbreviation for $\exists m((m,a,r,b,s) \in \Phi)$. We define the value $f(x)$ to be the unique $y \in \mathbb{R}$ such that $|y-b| < s$ for all $(a,r)\Phi(b,s)$ with $|x-a| < r$.

Lemma 4.6 *The following are provable within* ns-BASIC.

1. *For any continuous function f^{s} on $[0,1]$ in V^{s}, there exists a continuous function F^* on $[0,1]$ in V^* whose prestandard part is f^{s}. Moreover, if f^{s} is bounded, then we can find F^* as an* s-*bounded function.*
2. *For any monotone sequence of bounded continuous functions $\{f_n^{\mathrm{s}}\}_{n\in\mathbb{N}^{\mathrm{s}}}$ on $[0,1]$ which converges to a continuous function f^{s} pointwisely in V^{s}, there exist $m \in \mathbb{N}^*\setminus\mathbb{N}^{\mathrm{s}}$ and a sequence of* s-*bounded continuous functions $\{F_i^*\}_{i\le m}$ such that if $i \in \mathbb{N}^{\mathrm{s}}$ then the prestandard part of F_i^* is f_i^{s} and if $i \in \mathbb{N}^*\setminus\mathbb{N}^{\mathrm{s}}$ then the prestandard part of F_i^* is f^{s}.*

Proof We reason within ns-BASIC. We first prove (1). Let $\Phi^{s} \subseteq \mathbb{N}^{s} \times \mathbb{Q}^{s} \times \mathbb{Q}^{s+} \times \mathbb{Q}^{s} \times \mathbb{Q}^{s+}$ be a code for a continuous function f^{s} on $[0,1]$. By Σ^0_1-OS, there exists $m_0 \in \mathbb{N}^* \setminus \mathbb{N}^{s}$ such that $\Phi^{s\surd} \upharpoonright m_0$ is a code for a continuous function in V^*; that is, $\Phi^{s\surd} \upharpoonright m_0$ satisfies three conditions for a code for a continuous function. Since Φ^{s} satisfies (dom) in V^{s} and $\Phi^{s} \subseteq \Phi^{s\surd} \upharpoonright m_0$, for any $k \in \mathbb{N}^{s}$, there exists $\sigma = \langle (n_x, a_x, r_x, b_x, s_x) \mid x \in \Omega_k \rangle \subseteq \Phi^{s\surd} \upharpoonright m_0$ such that $r_x, s_x < 2^{-k}$ and $x \in (a_x - r_x, a_x + r_x)$ for any $x \in \Omega_k$. Applying Σ^0_1-OS again, there exists $m \in \mathbb{N}^* \setminus \mathbb{N}^{s}$ such that there exists $\bar{\sigma} = \langle (n_x, a_x, r_x, b_x, s_x) \mid x \in \Omega_m \rangle \subseteq \Phi^{s\surd} \upharpoonright m_0$ such that $r_x, s_x < 2^{-m}$ and $x \in (a_x - r_x, a_x + r_x)$ for any $x \in \Omega_m$. In V^*, we define a (piecewise linear) continuous function F^* as $F^*(i2^{-m} + t2^{-m}) = b_{i2^{-m}} + t(b_{(i+1)2^{-m}} - b_{i2^{-m}})$ for each $i < 2^m$ and $t \in [0,1]$. For any $x \in \Omega_m$, if $\mathrm{st}(x) \in (a - r, a + r) \wedge (n, a, r, b, s) \in \Phi^{s}$, then $|\,\mathrm{st}(b_x - b)| \leq s$. Thus, f^{s} is a prestandard part of F^*. Moreover, if f^{s} is bounded by $K \in \mathbb{N}^{s}$, then $|b_x| \leq K + 1$, thus F^* is s-bounded.

We can prove (2) similarly. Let $\{f^{s}_i\}_{i \in \mathbb{N}^{s}}$ be a monotone sequence of bounded continuous functions on $[0,1]$ which converges to a continuous function f^{s} pointwise in V^{s}. Then, similar to the previous construction, we can construct a sequence of s-bounded continuous functions $\{F^*_i\}_{i \leq m}$ ($m \in \mathbb{N}^* \setminus \mathbb{N}^{s}$) and an s-bounded continuous function F^* such that

(i) f^{s} is a prestandard part of F^* and f^{s}_n is a prestandard part of F^*_n for any $n \in \mathbb{N}^{s}$,

(ii) $F^*_i(x) + 2^{-m} \geq F^*_j(x) \geq F^*(x)$ if $i \leq j$.

Let $x \in \Omega_m$, $\mathrm{st}(x) = y^{s}$, and $i \in \mathbb{N}^* \setminus \mathbb{N}^{s}$. Then, for any $l \in \mathbb{N}^{s}$, there exists $j \in \mathbb{N}^{s}$ such that $f^{s}(y^{s}) \leq f^{s}_j(y^{s}) + 2^{-l} \leq f^{s}(y^{s})$; hence $\mathrm{st}(F^*(x)) \leq \mathrm{st}(F^*_i(x)) \leq \mathrm{st}(F^*_j(x)) \leq \mathrm{st}(F^*(x)) + 2^{-l}$. This means that $\mathrm{st}(F^*_i(x)) = f^{s}(\mathrm{st}(x))$. □

The following are nonstandard proofs for theorems of measure or integral theory within ns-WWKL_0.

Lemma 4.7 *The following assertions (for V^{s}) are provable within* ns-WWKL_0.

1. *Countable subadditivity of Lebesgue measure: for any open sets $\langle A^{s}_n \subseteq [0,1] \mid n \in \mathbb{N}^{s} \rangle$ and $B^{s} \subseteq [0,1]$, $\sum_{n \in \mathbb{N}^{s}} \mu(A^{s}_n) \geq \mu(B^{s})$ if $\bigcup_{n \in \mathbb{N}^{s}} A^{s}_n \supseteq B^{s}$.*
2. *Existence of Riemann integral: every bounded continuous function on $[0,1]$ is Riemann integrable.*
3. *Ascoli's lemma: if a monotone sequence of bounded continuous functions $\{f^{s}_n\}_{n \in \mathbb{N}}$ on $[0,1]$ converges to a continuous function f^{s} pointwise, then, each of f^{s}_n and f^{s} is integrable and*

$$\lim_{n \to \infty} \int_0^1 f^{s}_n(x)\, dx = \int_0^1 f^{s}(x)\, dx.$$

Proof We reason within ns-WWKL_0. We first show (1) by way of contradiction. Let $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{s}$, and let $\Omega = \Omega_\omega$, $L = L_\omega$, and $\mathrm{st}^{-1} = \mathrm{st}^{-1}_\omega$. Assume that $\sum_{n \in \mathbb{N}^{s}} \mu(A^{s}_n) < q < q' < \mu(B^{s})$ for some $q, q' \in \Omega_{\mathbb{N}^{s}}$ and $\bigcup_{n \in \mathbb{N}^{s}} A^{s}_n \supseteq B^{s}$. Without loss of generality, we may assume that $\mathcal{A}^{s} = \langle A^{s}_n \mid n \in \mathbb{N}^{s} \rangle = \langle (a_n, b_n) \mid n \in \mathbb{N}^{s} \rangle$ with $a_n, b_n \in \Omega_{\mathbb{N}^{s}}$. Then, by Σ^0_1-OS, there exists $m \in \mathbb{N}^* \setminus \mathbb{N}^{s}$ such that $\mathcal{A}^{s\surd} \upharpoonright m = \langle (a_n, b_n) \mid n < m \rangle$ satisfies $a_n, b_n \in \Omega$ for any $n < m$ and $\sum_{n<m} |a_n - b_n| < q$. Define $(\mathcal{A}^{s\surd} \upharpoonright m)^{\Omega} = \{x \in \Omega \mid \exists n < m\ a_n < x < b_n\}$ (note that $(\mathcal{A}^{s\surd} \upharpoonright m)^{\Omega}$ is internal, that is, a nonstandard finite set in V^*). Then

$(\bigcup_{n\in\mathbb{N}^s} A_n^s)^\Omega \subseteq (\mathcal{A}^{s\surd} \restriction m)^\Omega$ and $\mathrm{card}((\mathcal{A}^{s\surd} \restriction m)^\Omega) < q2^\omega$. By Theorem 4.5(2), take $B_-^* \subseteq_{\mathrm{al}} \mathrm{st}(B^s)$ such that $L(B_-^*) > q'$. Since $\mathrm{st}(B_-^*) \subseteq (\mathcal{A}^{s\surd} \restriction m)^\Omega$, $L(\mathrm{st}(B_-^*) \setminus (\mathcal{A}^{s\surd} \restriction m)^\Omega) = 0$. This contradicts $\mathrm{card}(\mathrm{st}(B_-^*) \setminus (\mathcal{A}^{s\surd} \restriction m)^\Omega) > q'2^\omega - q2^\omega$.

(2) is a straightforward direction from Theorem 4.5(3) and Lemma 4.6(1).

To prove (3), we only need to show $\lim_{n\to\infty} \int_0^1 f_n^s(x)\,dx = \int_0^1 f^s(x)\,dx$. Let $\{F_n^*\}_{n\le m}$ be a sequence of s-bounded continuous functions taken by Lemma 4.6(2), and let $\omega \in \mathbb{N}^* \setminus \mathbb{N}^s$. Then, for any $n \in \mathbb{N}^* \setminus \mathbb{N}^s$, $\mathrm{st}(|S(F_n^*;\Delta_\omega) - S(F_m^*;\Delta_\omega)|) = 0$ by Theorem 4.5(3). Thus, by Σ_1^0-OS,

$$\forall k \in \mathbb{N}^s \exists l \in \mathbb{N}^s \forall n \ge l\ |S(F_n^*;\Delta_\omega) - S(F_m^*;\Delta_\omega)| \le 2^{-k}.$$

Again by Theorem 4.5(3), we have $\lim_{n\to\infty} \int_0^1 f_n^s(x)\,dx = \int_0^1 f^s(x)\,dx$. □

Theorem 4.8 *The following assertions are provable within* $\mathsf{WWKL_0}$.

1. *Countable subadditivity of Lebesgue measure: for any open sets* $\langle A_n \subseteq [0,1] \mid n \in \mathbb{N}\rangle$ *and* $B \subseteq [0,1]$, $\sum_{n\in\mathbb{N}} \mu(A_n) \ge \mu(B)$ *if* $\bigcup_{n\in\mathbb{N}} A_n \supseteq B$.
2. *Existence of Riemann integral: every bounded continuous function on* $[0,1]$ *is Riemann integrable.*
3. *Ascoli's lemma: if a uniformly bounded monotone sequence of continuous functions* $\{f_n\}_{n\in\mathbb{N}}$ *on* $[0,1]$ *converges to a continuous function* f *pointwise, then each of* f_n *and* f *is integrable and*

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

Proof Straightforward application of the previous lemma and Theorem 3.5. □

Note that the assertions of the previous theorem are known theorems of $\mathsf{WWKL_0}$. Actually they are equivalent to $\mathsf{WWKL_0}$ ((1) is due to Yu and Simpson [13] and (2) and (3) are due to Yokoyama [10]).

References

[1] Brown, D. K., M. Giusto, and S. G. Simpson, "Vitali's theorem and WWKL," *Archive for Mathematical Logic*, vol. 41 (2002), pp. 191–206. Zbl 1030.03044. MR 1890192. 229

[2] Downey, R., D. R. Hirschfeldt, J. S. Miller, and A. Nies, "Relativizing Chaitin's halting probability," *Journal of Mathematical Logic*, vol. 5 (2005), pp. 167–92. Zbl 1093.03025. MR 2188515. 233

[3] Horihata, Y., and K. Yokoyama, "Nonstandard second-order arithmetic and Riemann's mapping theorem," in preparation. 232

[4] Keisler, H. J., "Nonstandard arithmetic and reverse mathematics," *Bulletin of Symbolic Logic*, vol. 12 (2006), pp. 100–125. Zbl 1101.03040. MR 2209331. 229

[5] Reimann, J., and T. A. Slaman, "Measures and their random reals," Available online at arxiv.org/abs/0802.2705, 2008. 233

[6] Simpson, S. G., *Subsystems of Second Order Arithmetic*, Perspectives in Mathematical Logic. Springer-Verlag, 1999. Second Edition, Perspectives in Logic, Association for Symbolic Logic, Cambridge University Press, 2009. Zbl 0909.03048. MR 1723993. 229, 231, 234

[7] Simpson, S. G., "Almost everywhere domination and superhighness," *Mathematical Logic Quarterly*, vol. 53 (2007), pp. 462–82. Zbl 1123.03040. MR 2351944. 234

[8] Simpson, S. G., K. Tanaka, and T. Yamazaki, "Some conservation results on weak König's lemma," *Annals of Pure and Applied Logic*, vol. 118 (2002), pp. 87–114. Zbl 1016.03064. MR 1933397. 235

[9] Tanaka, K., "The self-embedding theorem of WKL_0 and a non-standard method. Fifth Asian Logic Conference (Singapore, 1993)," *Annals of Pure and Applied Logic*, vol. 84 (1997), pp. 41–49. Zbl 0871.03044. MR 1440462. 231, 235

[10] Yokoyama, K., *Standard and Non-standard Analysis in Second Order Arithmetic*, Ph.D. thesis, Tohoku University, 2007. Available as *Tohoku Mathematical Publications* 34, 2009. 242

[11] Yokoyama, K., "Formalizing non-standard arguments in second-order arithmetic," *The Journal of Symbolic Logic*, vol. 75 (2010), pp. 1199–1210. Zbl 1214.03047. MR 2767964. 229, 231

[12] Yu, X., "Lebesgue convergence theorems and reverse mathematics," *Mathematical Logic Quarterly*, vol. 40 (1994), pp. 1–13. Zbl 0804.03047. MR 1284435. 229, 238

[13] Yu, X., and S. G. Simpson, "Measure theory and weak König's lemma," *Archive for Mathematical Logic*, vol. 30 (1990), pp. 171–80. Zbl 0718.03043. MR 1080236. 229, 242

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