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Euclidean Functions of Computable Euclidean Domains

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Abstract We study the complexity of (finitely-valued and transfinitely-valued) Euclidean functions for computable Euclidean domains. We examine both the complexity of the minimal Euclidean function and any Euclidean function. Additionally, we draw some conclusions about the proof-theoretical strength of minimal Euclidean functions in terms of reverse mathematics.

1 Introduction

One of the first algorithms discussed in almost any elementary algebra course is Euclid's algorithm for computing the greatest common divisor of two integers. Later, this algorithm is extended to other principal ideal domains like $\mathbb{Q}[X]$. In a first course in abstract algebra, this idea is explained by describing both $\mathbb{Z}$ and $\mathbb{Q}[X]$ as Euclidean domains. We recall the definition of a Euclidean domain.

Definition 1.1 A commutative ring $\mathcal{R}$ is a *Euclidean domain* if it is an integral domain (i.e., there are no zero divisors) and there is a function $\phi : R \setminus \{0\} \to \omega$ satisfying

$$(\forall a, d \in R)(\exists q \in R)\big[d = 0 \text{ or } a + qd = 0 \text{ or } \phi(a + qd) < \phi(d)\big].$$

If there is such a function $\phi : R \setminus \{0\} \to ON$ (where ON is the class of ordinals), then $\mathcal{R}$ is a *transfinite Euclidean domain*.

In the former case, we say the function ϕ is a *(finitely-valued) Euclidean function* for $\mathcal{R}$; in the latter case, we say the function ϕ is a *transfinitely-valued Euclidean function* for $\mathcal{R}$.

The reader should note that it is equivalent to demand $\phi(a - qd) < \phi(d)$; indeed, this inequality better aligns with the intuition that if $a = 18$ and $d = 7$, then the quotient is $q = 2$ (rather than $q = -2$ as above). The reader may also note that

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texts often restrict attention to Euclidean domains rather than transfinite Euclidean domains, though the greatest common divisor algorithm works provided the range of ϕ is well-founded. Remarkably, it is a forty-year-old open question (implicitly a sixty-year-old open question) whether there exists a transfinite Euclidean domain having no finitely-valued Euclidean function.

If the commutative ring does not need to be an integral domain, then $\mathbb{Z} \oplus \mathbb{Z}$ (the direct product of two copies of $\mathbb{Z}$) serves as an example of a ring having a transfinitely-valued Euclidean function (with range $\omega^2 + \omega^2$) but no finitely-valued Euclidean function (see [7]). Some integral domains are known to have both finitely-valued and transfinitely-valued Euclidean functions. For example, the functions $\phi_1(z) = |z|$, $\phi_2(z) = \lfloor \log_2 |z| \rfloor$, and $\phi_3(z) = \omega \cdot i + j + 1$ (where $z = \pm 2^i(2j + 1)$) are all (transfinitely-valued) Euclidean functions for $\mathcal{R} = \mathbb{Z}$ (see [8]). Of course, these functions demonstrate the well-known fact that Euclidean functions are not unique for a ring $\mathcal{R}$.

A fact that is not presented in many texts on (transfinite) Euclidean domains is that they can be defined *without* recourse to the existence of a (transfinitely-valued) Euclidean function. The idea is to define a hierarchy of sets with the property that it exhausts the set $R \setminus \{0\}$ of nonzero elements if and only if $\mathcal{R}$ is a (transfinite) Euclidean domain. At the bottom level R_0 of this hierarchy, we have the units. At the next level R_1, we have all those elements which either exactly divide all elements or give remainder a unit upon division. More generally, at level R_α, we have all those elements which either exactly divide all elements or give remainder in $R_{<\alpha}$ upon division.

Definition 1.2 (Motzkin [5], Samuel [7]) If $\mathcal{R}$ is an integral domain, define a sequence of sets $\{R_\alpha\}_{\alpha \in ON}$ via recursion by

$$R_\alpha = \left\{d \in R \setminus \{0\} : (\forall a \in R \setminus \{0\})(\exists q \in R)\left[a + dq = 0 \text{ or } \phi(a + dq) \in R_{<\alpha}\right]\right\}.$$

Theorem 1.3 (Motzkin [5], Samuel [7]) *If $\mathcal{R}$ is an integral domain, then $\mathcal{R}$ is a (transfinite) Euclidean domain if and only if*

$$R \setminus \{0\} = \bigcup\nolimits_{\alpha \in \omega} R_\alpha \quad (R \setminus \{0\} = \bigcup\nolimits_{\alpha \in ON} R_\alpha).$$

In the case that $R \setminus \{0\} = \cup_{\alpha \in ON} R_\alpha$, the function $\phi_{\mathcal{R}}$ mapping x to the least ordinal α with $x \in R_\alpha$ satisfies $\phi_{\mathcal{R}}(x) \leq \phi(x)$ for any transfinitely-valued Euclidean function.

If $\mathcal{R}$ is a transfinite Euclidean domain, the second part of Theorem 1.3 says there is always a *least* transfinitely-valued Euclidean function. As a consequence, it is also possible to define $\phi_{\mathcal{R}}$ as the infimum (minimum) of all transfinitely-valued Euclidean functions; that is,

$$\phi_{\mathcal{R}}(x) = \inf\{\phi(x) : \phi \text{ is a transfinitely-valued Euclidean function for } \mathcal{R}\}$$

(see [7]). Naturally, we seek to understand the complexity of this least function $\phi_{\mathcal{R}}$ and of any transfinitely-valued Euclidean function ϕ for $\mathcal{R}$.

The goal of the current paper is to add to our understanding of the complexity of the possible transfinitely-valued Euclidean functions ϕ on effectively-presented Euclidean domains. Thus, we are studying computable commutative algebra in a tradition going back to Herrmann (see [3]) and van der Waerden (see [11]), and in its modern incarnation certainly going back to Fröhlich and Shepherdson (see [2]), Mal′cev (see [4]), and Rabin (see [6]). We refer the reader to the survey article [10] for background in effective commutative ring theory.

In this paper, we will be extending earlier work of Schrieber (see [8]), solving the questions posed in that paper. There, Schrieber showed that there is a computable Euclidean domain with no computable finitely-valued Euclidean function, that there is a computable Euclidean domain with a computable Euclidean function but whose units are noncomputable, and that there is a computable Euclidean domain with neither computable units nor a computable Euclidean function.

A coarse analysis based on quantifiers in the definition of the R_α reveals some upper bounds. As the set R_0 is Σ^0_1, being the collection of units, the set R_n is Π^0_{2n} for $0 < n < \omega$. Thus in a computable Euclidean domain $\mathcal{R}$, if $\phi_{\mathcal{R}}$ is finitely-valued, then $\phi_{\mathcal{R}}$ is $\varnothing^{(\omega)}$-computable.

Any Euclidean function ϕ for $\mathcal{R}$, where $\mathcal{R}$ is Schrieber's computable Euclidean domain with no finitely-valued computable Euclidean function, computes $\varnothing'$. Schrieber asked if it was possible to remove the restriction of being finitely-valued. We show that it is.

Theorem 1.4 *There is a computable Euclidean domain $\mathcal{R}$ having no transfinitely-valued computable Euclidean function ϕ. Moreover, every transfinitely-valued Euclidean function ϕ for $\mathcal{R}$ computes $\varnothing'$.*

Schrieber's computable Euclidean domain for which the set of units R_0 is noncomputable has the property that R_0 is Σ^0_1-complete in any computable presentation. As we utilize this ring when later discussing relevant reverse mathematics, we sketch his proof (see [8]). First, we recall the fact that if $\mathcal{R}$ is a Euclidean domain with Euclidean function ϕ and S is a multiplicatively closed set in $\mathcal{R}$ containing the multiplicative identity but not the additive identity, then $S^{-1}\mathcal{R} := \{s^{-1}r \mid s \in S \text{ and } r \in \mathcal{R}\}$ is also a Euclidean domain (see [7]). Let the Halting Problem be represented by a set of primes P and apply the above with S as the multiplicative closure of $P \cup \{\pm 1\}$ in $\mathbb{Z}$. Then the Euclidean domain $S^{-1}\mathbb{Z}$ has Σ^0_1-complete units.

This result shows that it is possible to have a computable Euclidean domain for which R_0 is as complicated as possible and for which the least Euclidean function is as complex as $\varnothing'$ (since it can compute the units). We strengthen this by exhibiting a computable Euclidean domain for which R_1 is as complicated as possible, namely, Π^0_2-complete, and thus for which the least Euclidean function is as complex as $\varnothing''$ (since it can compute R_1).

Theorem 1.5 *There is a computable Euclidean domain $\mathcal{R}$ for which the set R_1 is Π^0_2-complete.*

We do not know whether this result can be extended and will make some remarks about R_j for $j \geq 2$ in Section 4. We also show that Schrieber's result can be extended to any $\varnothing'$-computable Euclidean function.

Theorem 1.6 *There is a computable Euclidean domain $\mathcal{R}$ for which there is no finitely-valued $\varnothing'$-computable Euclidean function ϕ.*

We note Schrieber's computable Euclidean domain with no computable finitely-valued Euclidean function does have a computable transfinitely-valued Euclidean function.

Theorem 1.7 *There is a computable Euclidean domain $\mathcal{R}$ having no computable finitely-valued Euclidean function but having a computable transfinitely-valued Euclidean function.*

It is well known that results in effective algebra (which seeks to understand algebra via computability theory) often go hand in hand with results in reverse mathematics (which seeks to understand the logical strength of theorems of mathematics via their proof-theoretical strength in second-order arithmetic). To conclude the paper, we discuss the implications of these results to the proof-theoretic strength (within the framework of reverse mathematics) of the theorem asserting the existence of a minimal Euclidean function. Though we offer some background in Section 3, in a short paper such as this, we do not include all the necessary background.

Theorem 1.8 (RCA_0) *The statement*

MEF: *every Euclidean domain has a minimal Euclidean function*

proves ACA_0.

2 Proofs of Results

It is really quite difficult to construct complicated Euclidean domains. Our results will use extensions of methods of Schrieber. Thus, as preparation for proving the theorems, we recall some notation and results introduced by Samuel and Schrieber.

Definition 2.1 (Schrieber [8]) If K is a field and $\{X_i\}_{i\in\omega}$ is a set of variables, denote by $K\langle X_i\rangle_{i\in\omega}$ the commutative ring of reduced fractions p/q with $p, q \in K[X_i]_{i\in\omega}$ and q not divisible by X_i for any i.

Thus, every element x of the commutative ring $K\langle X_i\rangle_{i\in\omega}$ is the product of a monomial m and a unit u.

Theorem 2.2 (Schrieber [8]) *The function $\phi(x) = \phi(mu) := \deg(m)$, where m is a monomial and u is a unit, is the least Euclidean function for $K\langle X_i\rangle_{i\in\omega}$. In particular, $K\langle X_i\rangle_{i\in\omega}$ is a Euclidean domain.*

All the Euclidean domains we construct will be of the form $K\langle X_i\rangle_{i\in\omega}$, where the field K is either $\mathbb{Q}$ or $\mathbb{Q}(Z_j)_{j\in\omega}$, for some sets of formal variables $\{X_i\}_{i\in\omega}$ and $\{Z_j\}_{j\in\omega}$.

Proposition 2.3 (Samuel [7]) *If $\mathcal{R}$ is an integral domain and $B, T \in R$ are nonzero, then $\phi_{\mathcal{R}}(T) \leq \phi_{\mathcal{R}}(BT)$.*

Proof We consider the function $\phi(T) := \min_{0\neq B\in R} \phi_{\mathcal{R}}(BT)$. We note ϕ is a Euclidean function for $\mathcal{R}$ as given a nonzero $D \in R$, there is a nonzero $B \in R$ with $\phi(D) = \phi_{\mathcal{R}}(BD)$. Thus for any $A \in R$, there is a $Q \in R$ such that either $A + QBD = 0$ or $\phi_{\mathcal{R}}(A + QBD) < \phi_{\mathcal{R}}(BD)$. Considering BQ, we have either $A+(BQ)D = 0$ or $\phi(A+(BQ)D) \leq \phi_{\mathcal{R}}(A+(BQ)D) < \phi_{\mathcal{R}}(BD) = \phi(D)$. This verifies ϕ is a Euclidean function for $\mathcal{R}$.

Moreover, we have $\phi = \phi_{\mathcal{R}}$ since

$$\phi_{\mathcal{R}}(T) \leq \phi(T) \leq \phi_{\mathcal{R}}(1T) = \phi_{\mathcal{R}}(T)$$

as a consequence of the minimality of $\phi_{\mathcal{R}}$ and taking 1 for B. It is clear that ϕ, and thus $\phi_{\mathcal{R}}$, has the desired property. □

Proposition 2.4 (Folklore) *If $\mathcal{R}$ is an integral domain, $B, T \in R$ are nonzero, and B is a nonunit, then $\phi_{\mathcal{R}}(T) < \phi_{\mathcal{R}}(BT)$.*

Proof Since B is a nonunit, it follows BT does not divide T. Thus

$$\min_{Q \in R}\{\phi_{\mathcal{R}}(T + QBT)\} < \phi_{\mathcal{R}}(BT)$$

by virtue of the definition of R_α. By Proposition 2.3 (as $1 + QB \neq 0$ for all $Q \in R$), we have $\min_{Q \in R}\{\phi_{\mathcal{R}}(T + QBT)\} = \min_{Q \in R}\{\phi_{\mathcal{R}}(T(1 + QB))\} \geq \phi_{\mathcal{R}}(T)$. □

We are now prepared to demonstrate the theorems.

Proof of Theorem 1.4 It would seem difficult to diagonalize against all computable functions from elements of the ring to ordinal notations, but we realize that any such function would simply map the elements of the ring to some computable subordering of the rational numbers (as a dense linear ordering) with various extra constraints. Thus, rather than construct $\mathcal{R}$ to diagonalize against transfinitely-valued Euclidean functions ϕ, we diagonalize against computable relations

$$E_\phi(x, y) := \{(x, y) \in R \times R : \phi(x) \leq \phi(y)\}.$$

This is justified because E_ϕ is computable if ϕ is a computable transfinitely-valued Euclidean function.

Therefore, fix an enumeration $\{E_i\}_{i\in\omega}$ of computable binary relations. The idea is to determine whether $E_i(X_i, Y_i)$ or $E_i(Y_i, X_i)$ (if either computation converges) and assure this cannot be the case by making either X_i a power of Y_i or Y_i a positive power of X_i.

Construction At stage s, we introduce terms X_s and Y_s. For each $i \leq s$, we check whether $E_i(X_i, Y_i)\downarrow = 1$ or $E_i(Y_i, X_i)\downarrow = 1$. If either has newly converged, we put $X_i = Y_i^s$ if $E(X_i, Y_i)\downarrow = 1$ and $Y_i = X_i^s$ otherwise.

Finally, at each stage s, we continue the enumeration of the ring, working toward $\mathbb{Q}\langle X_i, Y_i\rangle_{i\in\omega}$ (with a slight abuse of notation).

Verification It is clear that we construct a computable ring. By Theorem 2.2, it is a Euclidean domain. Moreover, it cannot have a computable transfinitely-valued Euclidean function ϕ. For if it did, the binary relation E_ϕ would be total computable. Fixing an index i for which $E_\phi(x, y) = E_i(x, y)$, the relationship between the terms X_i and Y_i contradicts E_i by Proposition 2.4. □

The idea for Theorem 1.5 and Theorem 1.6 is to construct a computable ring $\mathcal{R}$ classically isomorphic to $\mathbb{Q}(X_i)_{i\in\omega}\langle Y_j\rangle_{j\in\omega}$, where $\{X_i\}_{i\in\omega}$ and $\{Y_j\}_{j\in\omega}$ are some set of formal variables. However, the ring $\mathcal{R}$ we construct will not be computably isomorphic as it will be difficult to determine whether a formal variable $Z \in R$ is invertible.

Proof of Theorem 1.5 Fix a Π^0_2-complete set S and a computable predicate $P(i, s)$ so that $i \in S$ if and only if $\exists^\infty s\,[P(i, s)]$. The idea is to start with the rationals $\mathbb{Q}$ and expressions $\{Z_i\}_{i\in\omega}$. As the construction proceeds, each expression Z_i will be declared equal to a product of two variables $Z_i = X_{i,j}Y_{i,j}$ (starting with $j = 0$). Every time i appears in a fixed Π^0_2 set, we make $X_{i,j}$ a unit and declare Z_i also equal to the product $X_{i,j+1}Y_{i,j+1}$. The point is that if $\exists^{<\infty} s\,[P(i, s)]$, then Z_i will have rank two (being a product of two variables); if $\exists^\infty s\,[P(i, s)]$, then Z_i will have rank one (being a product of only a variable and a unit).

Construction At stage s, we introduce two new terms $X_{s,s}$ and $Y_{s,s}$ and denote their product by Z_s. For each $i \leq s$, we test whether $P(i,s)$ holds. If it does, we enumerate $X_{i,s'}^{-1}$ into the ring, where s' is the greatest $t < s$ where $P(i,t)$ held and $s' = i$ if no such t exists; introduce two new terms $X_{i,s}$ and $Y_{i,s}$ into the ring; and equate Z_i with the product $X_{i,s}Y_{i,s}$. If it does not, we take no action.

Finally, at each stage s, we continue the enumeration of the ring, working toward the ring $\mathbb{Q}(A)\langle B\rangle$, where

$$A := \{X_i : X_i^{-1} \text{ exists}\} \text{ and } B := \{X_i : X_i^{-1} \text{ does not exist}\} \cup \{Y_i : i \in \omega\}.$$

Verification It is clear that we construct a computable integral domain. Moreover, if $\mathcal{R}$ is a Euclidean ring, then $Z_s \in R_1$ if and only if $\exists^\infty s\, P(i,s)$ (as noted earlier). Thus, it suffices to show that $\mathcal{R}$ is classically a Euclidean ring.

We show that $\mathcal{R}$ is a Euclidean ring by showing $\mathcal{R} \cong \mathbb{Q}\left(A_i\right)_{i\in\omega}\left\langle B_j\right\rangle_{j\in\omega}$ for appropriate sets of variables $\{A_i\}_{i\in\omega}$ and $\{B_j\}_{j\in\omega}$. Indeed, any bijection between $\{X_{i,s} : X_{i,s}^{-1} \text{ exist}\}$ and $\{A_i\}_{i\in\omega}$ and $\{X_{i,s} : X_{i,s}^{-1} \text{ does not exist}\} \cup \{Y_{i,s}\}$ and $\{B_i\}_{i\in\omega}$ induces a bijection between $\mathcal{R}$ and $\mathbb{Q}\left(A_i\right)_{i\in\omega}\left\langle B_j\right\rangle_{j\in\omega}$. □

Utilizing larger products of variables allows diagonalizing against finitely-valued $\varnothing'$-computable Euclidean functions.

Proof of Theorem 1.6 Fix an effective enumeration $\{\psi_e(x)\}_{e\in\omega}$ of the partial $\varnothing'$-computable functions and an effective enumeration $\{\phi_e(x,s)\}_{e\in\omega}$ of total computable functions with the property $\psi_e(x) = \lim_s \phi_e(x,s)$ if $\psi_e(x)\downarrow$. Additionally, we assume that if $0 \neq \phi_e(x,s)$ and $\phi_e(x,s) \neq \phi_e(x,s+1)$, then $\phi_e(x,s+1) = 0$, that is, that the value zero is taken for at least one stage if the approximation changes value. We construct a computable Euclidean domain $\mathcal{R}$ for which $\phi_{\mathcal{R}} \not\leq \psi_e$ for any e.

Construction At stage s, we introduce a fresh term X_s into $\mathcal{R}$ and compute the value of $\phi_s(X_s,s)$. We then introduce $\phi_s(X_s,s)+1$ many new terms $X_{s,s,0}$, $X_{s,s,1}$, $\ldots$, $X_{s,s,\phi_s(X_s,s)}$ and declare their product equal to X_s.

Then, for each $e < s$, we compare the values of $\phi_e(X_e,s)$ and $\phi_e(X_e,s-1)$. If $\phi_e(X_e,s) \neq \phi_e(X_e,s-1) = 0$, we introduce $\phi_e(X_e,s)+1$ many new terms $X_{e,s,0}, X_{e,s,1}, \ldots, X_{e,s,\phi_e(X_e,s)}$ to the ring R and declare their product equal to X_e. In the case that $\phi_e(X_e,s) \neq \phi_e(X_e,s-1)$ and $\phi_e(X_e,s) = 0$, we enumerate $X_{e,s',j}^{-1}$ into the ring for $1 \leq j \leq \phi_e(X_e,s')$ (where s' is the last stage at which the approximation changed), making $X_e = q \cdot X_{e,s',0}$ for some unit $q \in R$.

Finally, at each stage s, we continue the enumeration of the ring, closing under addition, multiplication, and additive inverse.

Verification It is clear that we construct a computable integral domain. Moreover, if $\mathcal{R}$ is a Euclidean ring, it cannot have a finitely valued $\varnothing'$-computable Euclidean function. This is because if $\lim_s \phi_e(X_e,s)$ fails to exist, then $\psi_e(x) = \lim_s \phi_e(x,s)$ is not a total function, and if $\psi_e(X_e) = \lim_s \phi_e(X_e,s)$ exists, then $\phi_{\mathcal{R}}(X_e) = \psi_e(X_e) + 1 > \psi_e(X_e)$, contradicting the minimality of $\phi_{\mathcal{R}}$. Thus, it suffices to show that $\mathcal{R}$ is classically a Euclidean ring.

We show $\mathcal{R}$ is a Euclidean ring by showing $\mathcal{R} \cong \mathbb{Q}(A_i)_{i\in\omega}\left\langle B_j\right\rangle_{j\in\omega}$ for appropriate sets of variables $\{A_i\}_{i\in\omega}$ and $\{B_j\}_{j\in\omega}$. Indeed, any bijection between

$\{X_{i,s,j} : X_{i,s,j}^{-1} \text{ exists}\}$ and $\{A_i\}_{i\in\omega}$, and $\{X_{i,s,j} : X_{i,s,j}^{-1} \text{ does not exist}\}$ and $\{B_i\}_{i\in\omega}$ induces a bijection between $\mathcal{R}$ and $\mathbb{Q}\,(A_i)_{i\in\omega}\,\langle B_j\rangle_{j\in\omega}$. □

We continue by sketching Schrieber's construction of a computable Euclidean domain with no computable finitely-valued Euclidean function and noting it has a computable transfinitely-valued Euclidean function.

Proof of Theorem 1.7 Fix an effective enumeration $\{\phi_e(x)\}_{e\in\omega}$ of the partial computable functions.

Construction At each stage s, we create a term X_s. For each $i \leq s$ for which $\phi_i(X_i)$ newly converges, we create a new variable Y_i and set $X_i = Y_i^{\phi_i(X_i)+1}$. Finally, at each stage s, we continue the enumeration of the ring, working toward $\mathbb{Q}\,\langle X_i\rangle_{i\in\omega}\,\langle Y_i\rangle_{Y_i \text{ exists}}$ (with a slight abuse of notation).

Verification The ring $\mathcal{R}$ is a Euclidean domain with no computable finitely-valued Euclidean function (see [8]). On the other hand, the computable transfinitely-valued function induced by mapping Y_i to 1 and X_i to ω is a transfinite Euclidean function for $\mathcal{R}$. More precisely, the function ϕ assigns the rank $\omega\cdot\sum_{t=1}^{t=m} k_t + \sum_{t=1}^{t=n} \ell_t$ to the monomial $X_{i_1}^{k_1}\dots X_{i_m}^{k_m} Y_{j_1}^{\ell_1}\dots Y_{j_n}^{\ell_n}$, where $\ell_i \leq \phi_i(X_i)$ if $\phi_i(X_i)\downarrow$. This suffices as every monomial is assigned a rank and $\phi(x) < \phi(y)$ whenever $x \mid y$ and $y \nmid x$. □

Remark 2.5 It is not difficult to show that the ring $\mathcal{R}$ of Theorem 1.7 has no computable α-valued Euclidean function for any $\alpha < \omega^2$. Indeed, if there were a computable α-valued Euclidean function ϕ, then for each i we could find $a, c \in \omega$ such that $\phi(X_i^{a+1}) = \phi(X_i^a) + c$. As a consequence of Proposition 2.4, this would imply $\phi_i(X_i)+1 \leq \phi_i(Y_i^{\phi_i(X_i)+1}) = \phi_i(X_i) < c$ if $\phi_i(X_i)\downarrow$, an impossibility. On the other hand, the function given shows there is a computable ω^2-valued Euclidean function.

A generalization of the construction (noted by the anonymous referee) yields, for any integer k, a ring $\mathcal{R}$ having no computable α-valued Euclidean function for any $\alpha < \omega^k$, but having a computable ω^k-valued Euclidean function. For example, with $k = 3$, it is possible to ensure $\phi(Y_i) \geq \omega$ (if Y_i exists) for any computable Euclidean function ϕ by creating a new variable Z_i when (if) we see $\phi_i(Y_i)$ converge and setting $Y_i = Z_i^{\phi_i(Y_i)+1}$. In turn, this ensures $\phi(X_i) \geq \omega^2$ for any computable Euclidean function.

By taking an appropriate "union" of such rings, it is possible to construct a computable Euclidean domain having no computable ω^k-valued Euclidean function for any $k \in \omega$, but having a computable ω^ω-valued Euclidean function. Unfortunately, it does not seem as if this technique can be extended to ordinals higher than this.

3 Connections with Reverse Mathematics

Reverse mathematics is the subfield of mathematics that attempts to calibrate the proof-theoretic strength of theorems within the framework of second-order arithmetic. This is done by considering a theorem T of classical mathematics and asking what set existence axioms A are necessary to prove T over a base set of axioms B. If T is provable from A and A is provable from T (the *reversal*), both over B, then A and T have the same proof-theoretic strength. This program was introduced by Friedman (see [1]). We refer the reader to other sources (see [9], for example) for further discussion of reverse mathematics and a formal definition of the axiom systems within this paper.

Often, the axiom system RCA_0 is chosen as the base set of axioms. Roughly speaking, the axiom system RCA_0 only requires the model to contain the computable sets and be a Turing ideal. A theorem T is provable from RCA_0 (over RCA_0) only if the computable sets witness the conclusion of T.

The axiom system ACA_0 is strictly stronger than RCA_0, requiring the model also be closed under Turing jump. A theorem T is provable from ACA_0 (over RCA_0) only if the arithmetic sets witness the conclusion of T. The system ACA_0 is equivalent to many natural theorems of classical mathematics (see [9]).

Before doing so, we need to formalize terminology within the framework of second-order arithmetic.

Definition 3.1 (RCA_0) If $\mathcal{M}$ is a model, then a commutative ring $\mathcal{R}$ is a *(transfinite) Euclidean domain* [in $\mathcal{M}$] if it is an integral domain and there is a (transfinitely-valued) Euclidean function ϕ for $\mathcal{R}$ in $\mathcal{M}$.

A (transfinitely-valued) Euclidean function ϕ [in $\mathcal{M}$] for $\mathcal{R}$ is *minimal* if $\phi \leq \phi'$ for all (transfinitely-valued) Euclidean functions ϕ' [in $\mathcal{M}$] for $\mathcal{R}$.

A priori, there is no reason for the minimal (transfinitely-valued) Euclidean function ϕ [in $\mathcal{M}$] to satisfy $\phi = \phi_{\mathcal{R}}$. The following observation, however, is the key step in showing that any classically nonminimal (transfinitely-valued) Euclidean function has classically a strictly smaller (transfinitely-valued) Euclidean function of the same Turing degree. This will enable us to conclude that if $\mathcal{R}$ has minimal (transfinitely-valued) Euclidean function [in $\mathcal{M}$], then it is $\phi_{\mathcal{R}}$.

Lemma 3.2 (RCA_0) *Fix a Euclidean domain $\mathcal{R}$ and a nonminimal finitely-valued Euclidean function ϕ for $\mathcal{R}$. Let α be the least ordinal for which there is a $T \in R$ with $\alpha = \phi_{\mathcal{R}}(T) < \phi(T)$. Then (fixing such a T)*

$$\hat{\phi}(z) = \begin{cases} \phi(z) & \text{if } z \neq T \\ \phi_{\mathcal{R}}(T) & \text{if } z = T \end{cases}$$

is a finitely-valued Euclidean function for $\mathcal{R}$ and satisfies $\phi \not\leq \hat{\phi}$.

Proof Since $\hat{\phi}(T) = \phi_{\mathcal{R}}(T) < \phi(T)$, it is immediate that $\phi \not\leq \hat{\phi}$. As α was chosen minimal, for any $A \in R$, there exists a $Q \in R$ with $\phi(A + QT) < \phi(T)$ as $\phi(A + QT) = \phi_{\mathcal{R}}(A + QT) < \phi_{\mathcal{R}}(T) = \alpha$. □

Proof of Theorem 1.8 Fixing a set X in the model, we show X' exists. We consider the X-computable ring whose units are $\Sigma^0_1(X)$-complete constructed by relativizing Schrieber's construction of a computable subring of the rationals whose units are intrinsically Σ^0_1-complete. As noted in the introduction, the (relativized) ring $\mathcal{R}$ has a computable Euclidean function, namely, $\phi(a/b) = a$. Thus, it is a Euclidean domain; that is, it has a Euclidean function in the model.

Consequently, by MEF, we may fix a minimal Euclidean function ϕ for $\mathcal{R}$ (so that $\phi \leq \phi'$ for all ϕ' in the model). We argue that $\phi = \phi_{\mathcal{R}}$. If not, the function $\hat{\phi}$ of Lemma 3.2 is in the model as $\hat{\phi} \equiv_T \phi \equiv_T \varnothing$ and is a Euclidean function for $\mathcal{R}$. But then we would have $\hat{\phi} < \phi$, contradicting the minimality of ϕ. Thus it must be the case that $\phi = \phi_{\mathcal{R}}$. As $\phi_{\mathcal{R}}$ computes X', the model must be closed under the Turing Jump. □

4 Questions and Comments

We close with some questions that remain open. We begin with the question explicitly stated in Samuel's classic paper.

Question 4.1 (Samuel [7]) *Is there (classically) a transfinite Euclidean domain that is not a Euclidean domain?*

We note Theorem 1.7 demonstrates that the effective analogue of this question has a positive answer. However, the computable ring of Theorem 1.7 does have computable presentations with computable finitely-valued Euclidean functions. We would like to know how much the complexity of (minimal) Euclidean functions can vary across presentations. Indeed, is there always a computable presentation having a computable finitely-valued Euclidean function?

Question 4.2 *Is there a computable Euclidean domain $\mathcal{R}$ such that no isomorphic presentation $\mathcal{R}'$ has a computable finitely-valued Euclidean function?*

For finitely-valued Euclidean functions, we would like to know if the upper bound of the complexity on the Euclidean function can be achieved.

Question 4.3 *Is there a computable Euclidean domain $\mathcal{R}$ for which any Euclidean function for $\mathcal{R}$ computes $\varnothing^{(\omega)}$ (or even $\varnothing^{(3)}$)?*

The Euclidean domains both Schrieber and we use have the property that the least Euclidean function is determined by the rank one elements. As a consequence, it is impossible to have R_2 be more complex than $\varnothing''$ using this approach. Thus to answer Question 4.3 in the positive direction, it is necessary to construct a Euclidean domain where the rank two elements are somehow more independent than the rank one elements.

It is interesting to note that all classical constructions (at least those of which we are aware) seem to have the property that the rank two elements are determined by the rank one elements, *or* are somehow easily definable from the rank one elements. It is conceivable that Question 4.3 has a negative answer as a consequence of some rather deep algebra. If the answer is positive, likely new algebra will be needed too!

Finally, determining the exact proof-theoretical strength of the results above in terms of reverse mathematics would seem interesting. For example, Theorem 1.8 shows MEF proves ACA_0 over RCA_0. Is it strictly stronger?

Conjecture 4.4 (RCA_0) *The theorem* MEF *is equivalent to* ACA_0^+.

Also interesting would be a similar analysis for the more general case of Euclidean rings.

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