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APPLICATION AND SIMPLIFIED PROOF OF A SHARP L^2 EXTENSION THEOREM

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To Professor Masatake Kuranishi on his 90th birthday

Abstract. As an application of a sharp L^2 extension theorem for holomorphic functions in Guan and Zhou, a stability theorem for the boundary asymptotics of the Bergman kernel is proved. An alternate proof of the extension theorem is given, too. It is a simplified proof in the sense that it is free from ordinary differential equations.

§0. Introduction

In [19] it was proved that there exists a positive number $C \leq 1620\pi$ such that, for any pseudoconvex domain D contained in $\{z = (z', z_n) \in \mathbb{C}^n; |z_n| < 1\}$, for any plurisubharmonic function $\varphi(z)$ on D, and for any holomorphic function $f(z')$ on $D' = \{z' \in \mathbb{C}^{n-1}; (z', 0) \in D\}$ satisfying

$$\int_{D'} e^{-\varphi(z',0)} \left| f(z') \right|^2 d\lambda_{n-1} < \infty,$$

where $d\lambda_{n-1}$ denotes the Lebesgue measure on $\mathbb{C}^n$, there exists a holomorphic function $\tilde{f}(z)$ on D such that $\tilde{f}(z', 0) = f(z')$ $(z' \in D')$ and

$$\int_D e^{-\varphi(z)} \left| \tilde{f}(z) \right|^2 d\lambda_n \leq C \int_D e^{-\varphi(z',0)} \left| f(z') \right|^2 d\lambda_{n-1}$$

hold.

A motivation of [19] was to develop an analytic aspect of the harmonic integrals due to Hodge [9] and Kodaira [11], in view of the works of Hörmander [10], Andreotti and Vesentini [1], and Skoda [20]. More practically, we wanted to improve the preceding work [16] on the boundary behavior of

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the Bergman kernel of weakly pseudoconvex domains in $\mathbb{C}^n$. As a result, it turned out that the Bergman kernel function $K_D(z,w)$ grows at least as fast as $\delta(z)^{-2}$ when $z=w$, and z tends to the boundary ∂D of D, if D is a bounded pseudoconvex domain with C^1-smooth boundary. Here $\delta(z)$ denotes the Euclidean distance from z to ∂D.

In [17], where Theorem 0.1 below was established under a stronger assumption and with a nonoptimal constant, the author was content to know that, if D is a bounded pseudoconvex domain with C^2-smooth boundary, application of the result implies that $K_D(z,z)$ grows at least as fast as $\delta(z)^{-2-\nu}$ as z tends to a point $z_0 \in \partial D$ nontangentially, where ν denotes the rank of the Levi form of ∂D at z_0.

Recently, by virtue of the works of Zhu, Guan, and Zhou [22] and Chen [4], the proof in [19], as well as the bound of the above-mentioned constant C, was considerably improved. Exploiting a variant of the methods of [22] and [4], Błocki [2] improved the results further and settled a long-standing conjecture of Suita (see [2] and [21]), by proving that π can be taken as C. Obviously π is optimal as a universal constant. After this breakthrough, Guan and Zhou [6] showed sharp L^2 extension theorems in this sense, including the following refinement of [17] as a special case.

THEOREM 0.1 (see [6, Corollary 3.15]; see also [3], [7], [8]). *Let X be a Stein manifold of dimension n, let φ and ψ be plurisubharmonic functions on X, and let w be a holomorphic function on X such that* $\sup(\psi + 2\log|w|) \leq 0$ *and dw is not identically zero on every irreducible component of $w^{-1}(0) = H$. Then, for any holomorphic $(n-1)$-form f on $H_0 = H - \operatorname{Sing} H$ satisfying*

$$\Big|\int_{H_0} e^{-\varphi} f \wedge \bar{f}\Big| < \infty,$$

there exists a holomorphic n-form F on X such that $F = dw \wedge f$ at any point of H_0 and

$$\Big|\int_X e^{-\varphi+\psi} F \wedge \bar{F}\Big| \leq 2\pi \Big|\int_{H_0} e^{-\varphi} f \wedge \bar{f}\Big|, \tag{0.1}$$

where $\operatorname{Sing} H$ *denotes the set of singular points of H.*

Note that Theorem 0.1 implies the validity of the extension theorem of [19] for $C = \pi$ by letting $\psi = 0$, because $|dz_n|^2 = 2$ with respect to the Euclidean metric.

The methods of [2], [3], and [6] are essentially the same: they separate the smaller side of the basic L^2 inequality, a modification of Hörmander's or Kodaira and Nakano's methods, into two parts, say, the principal and the secondary terms, and choose a twist function and an auxiliary weight function to make the secondary term zero. The idea is that the optimal one among such choices would lead to the optimal constant. To realize this, they solve an ordinary differential equation (ODE) problem with two unknowns. The use of such auxiliary weight had been introduced in [13].

The purpose of the present note is to give remarks on Theorem 0.1 and its proof. First we shall show that Theorem 0.1 implies a stability result on the asymptotic behavior of the ratio of the Bergman kernel of D and a weighted Bergman kernel of D'.

THEOREM 0.2. *Let D be a bounded pseudoconvex domain with C^2-smooth boundary in $\mathbb{C}^n$, and let $\phi(z)$ be a real-valued C^2 function on a neighborhood of $\bar{D}$ such that $\{z; \phi(z) > 0\}$ is pseudoconvex and $D = \{z; \phi(z) > |z_n|^2\}$. Assume that $\phi|_{\bar{D}'} = v\delta^t$ $(t > 0)$ for some positive C^2 function v, that $-\log\phi$ is plurisubharmonic on D, and that*

$$\phi(z) = \phi(z', 0) + o\big(\phi(z', 0) + |z_n|^2\big) \tag{0.2}$$

as z tends to $\partial D \cap \{z_n = 0\}$. Then, the reproducing kernel $K_{D',\phi}(z', w')$ of the space of L^2 holomorphic functions on $D' = \{z' \in \mathbb{C}^{n-1}; (z', 0) \in D\}$ with respect to the measure $\phi(z', 0)d\lambda_{n-1}$ satisfies

$$\frac{K_{D',\phi}(z', z')}{K_D((z', 0), (z', 0))} \longrightarrow 1$$

as $(z', 0)$ tends to $\partial D \cap \{z_n = 0\}$.

It is known from a formula of Forelli and Rudin that $K_{D',\phi}(z', z') = K_D((z', 0), (z', 0))$ if $\phi(z) = \phi(z', 0)$ (see [12]), so Theorem 0.2 asserts that the formula is stable under the deformation of D of the above type. For the proof of Theorem 0.2, a lemma on the L^2 norm of $K_D(z, w)/\sqrt{K_D(z, z)}$ is needed besides Theorem 0.1. Since it is only implicitly contained in [5], the proof will be given in Section 1 for the convenience of the reader. Another remark on Theorem 0.1 is that one can avoid solving an ODE by choosing a family of cutoff functions producing the $\bar{\partial}$-data in such a way that they are naturally related to the Poincaré metric of the punctured disk. This change causes a difference in adjusting the twist and the weight and simplifies the situation very much. Since the new cutoff functions are of the same nature as those appearing in the proof of Theorem 0.2, they seem to deserve special attention in the future research.

§1. A lemma on maximizing functions

Let the notation be as in the Introduction, and let ϕ be as in Theorem 0.2. Since ∂D is C^2-smooth, one can find a neighborhood U of $\partial D \cap \{z=0\}$ and $M>0$ such that the function $\Psi = -\log\phi + M\|z\|^2$ satisfies an inequality $\partial\bar\partial\Psi \geq M^2\partial\Psi\bar\partial\Psi$ on $U\cap D$. (For the argument verifying this assertion, see [18, Theorem 1.1].) Once and for all we shall fix such Ψ and put

$$D_c = \{z\in D; \Psi(z) < c\}.$$

Lemma 1. *For any $c\in\mathbb{R}$ and for any sequence $\{a_\mu\}_\mu^\infty \subset D$ such that $\lim_{\mu\to\infty}\Psi(a_\mu)=\infty$,*

$$\lim_{\mu\to\infty}\int_{D_c}\left|\frac{K_D(z,a_\mu)}{\sqrt{K_D(a_\mu,a_\mu)}}\right|^2 d\lambda_n = 0. \tag{1.1}$$

Proof. If the assertion were false, there would exist $c\in\mathbb{R}$, $a\in D$ such that $\lim_{\mu\to\infty}\Psi(a_\mu)=\infty$, and $b>0$ such that

$$\int_{D_c}\left|\frac{K_D(z,a_\mu)}{\sqrt{K_D(a_\mu,a_\mu)}}\right| d\lambda_n \geq b \tag{1.2}$$

holds for all μ. Without loosing the generality, one may assume that the diameter of D is less than 1. Let $R>1$, and let $\varphi_\mu : D\setminus\{a_\mu\}\to(-\infty,0]$ be C^∞ functions satisfying the following properties:

(i) $\varphi_\mu(z) - \log\|z-a_\mu\|$ is bounded;

(ii) $\varphi_\mu(z) - \log\Psi - \log\big(-\log\|z-a_\mu\|\big)$ is plurisubharmonic; and

(iii) $\operatorname{supp}\varphi_\mu \subset \{z\in D; \Psi(a_\mu)/R < \Psi(z) < R\Psi(a_\mu)\}$.

It is known that such functions exist for sufficiently large R (see [5, proof of Proposition 2.1]). Therefore, a standard application of the L^2 method of solving the $\bar\partial$-equation (see [5] for an instance of how (ii) works for that), one has C^∞ functions u on D such that

$$\bar\partial u_\mu = \bar\partial\Big(\chi\big(R\Psi(z)/\Psi(a_\mu)\big)\cdot\frac{K(z,a_\mu)}{\sqrt{K(a_\mu,a_\mu)}}\Big) \tag{1.3}$$

and

$$\lim_{\mu\to\infty}\int_D e^{-2n\varphi_\mu}|u_\mu|^2\,d\lambda_n = 0, \tag{1.4}$$

where χ denotes a C^∞ real-valued function on $\mathbb{R}$ such that $\chi(t)=1$ if $t<1/2$ and $\chi(t)=0$ if $t>1$. Since $u_\mu(a_\mu)=0$ by (i), the sequence of L^2 holomorphic functions

$$f_\mu(z) = \big(1-\chi\big(R\Psi(z)/\Psi(a_\mu)\big)\big)\cdot\frac{K_D(z,a_\mu)}{\sqrt{K_D(a_\mu,a_\mu)}} + u_\mu(z) \tag{1.5}$$

satisfies

$$\lim_{\mu\to\infty}\int_D \big|f_\mu(z)\big|^2\,d\lambda_n \le 1-b<1 \tag{1.6}$$

and

$$\lim_{\mu\to\infty}\frac{|f_\mu(a_\mu)|^2}{K_D(a_\mu,a_\mu)} = 1, \tag{1.7}$$

which is a contradiction because

$$K_D(z,z) = \sup\Big\{\big|f(z)\big|^2;\ \int_D \big|f(z)\big|^2\,d\lambda_n = 1\Big\}. \tag{1.8}$$

□

§2. Proof of Theorem 0.2

Let D and ϕ be as in Theorem 0.2. We will compare the values of $K_D(z,z)$ and $K_{D',\phi}(z',z')$ when $z=(z',0)$ and z is sufficiently close to ∂D. Note that this makes sense because $z'\in D'$ whenever a point (z',z_n) is in D and sufficiently close to $\partial D\cap\{z_n=0\}$, which is obvious from assumption (0.2). Since $D=\{z;\log\phi(z)-\log|z_n|^2>0\}$, Theorem 0.1 implies that

$$K_{D',\phi}(z',z') \le K_D\big((z',0),(z',0)\big)$$

holds for any $z'\in D'$. Therefore, it suffices to show that

$$\liminf_{z'\to\partial D'}\frac{K_{D',\phi}(z',z')}{K_D((z',0),(z',0))} \ge 1$$

but, in view of condition (0.2), this is obvious from Cauchy's estimate, Fubini's theorem, and Lemma 1. □

§3. The $\bar\partial$-problem and a basic estimate

In view of the standard limiting procedure, for the proof of Theorem 0.1 it suffices to show that, given any bounded pseudoconvex domain Ω in $X \setminus \{dw = 0\}$, there exists a holomorphic n-form F_Ω on Ω such that

$$F_\Omega = dw \wedge f \quad \text{on } H \cap \Omega \tag{3.1}$$

and

$$\Big| \int_\Omega e^{\varphi+\psi} F_\Omega \wedge \bar{F_\Omega} \Big| \le 2\pi \Big| \int_{H_0} e^{-\varphi} f \wedge \bar{f} \Big|. \tag{3.2}$$

Moreover, we may assume that φ and ψ are C^∞ functions. Then the problem to be solved is a set of $\bar\partial$-equations

$$\bar\partial u_\epsilon = \tilde{f} \bar\partial \chi_\epsilon\big(-\psi/2 - \log|w|\big) \wedge dw \quad \text{for } 0 < \epsilon < 1, \tag{3.3}$$

where $\tilde{f}$ is a holomorphic extension of f to X and where

$$\chi_\epsilon(t) = \begin{cases} 0 & \text{if } t < -\log\epsilon, \\ \log t - \log(-\log\epsilon) & \text{if } -\log\epsilon \le t \le -e\log\epsilon, \\ 1 & \text{if } t > -e\log\epsilon. \end{cases}$$

Clearly, if one can find solutions u_ϵ such that u_ϵ are extendible to Ω continuously to be zero along $\Omega \cap H$ and

$$\liminf_{\epsilon\to 0} \Big| \int_\Omega e^{-\varphi+\psi} u_\epsilon \wedge \bar{u_\epsilon} \Big| \le 2\pi \Big| \int_{H_0} e^{-\varphi} f \wedge \bar{f} \Big|,$$

a subsequence of $\tilde{f}\chi_\epsilon(-\psi/2 - \log|w|) \wedge dw - u_\epsilon$ will converge to a desired extension F_Ω. As in [6], our proof is based on a variant of Nakano's identity on Kähler manifolds. We recall this general formula below. Let (M, g) be a Kähler manifold of dimension n, and let ϕ be a C^∞ function on M. Then, with respect to the weighted L^2 norm

$$\|u\|_\phi = \Big(\int_M e^{-\phi} |u|^2 \, dv \Big)^{1/2}$$

and the associated inner product $(\ ,\ )_\phi$, the following holds for any compactly supported C^∞ $(n,1)$-form on M:

$$(i\partial\bar\partial\phi \Lambda u, u)_\phi \le \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2, \tag{3.4}$$

where Λ denotes the adjoint of the exterior multiplication by the fundamental form of g, and $\bar\partial^*$ $(=\bar\partial^*(\phi))$ is the adjoint of $\bar\partial$ (see [14] or [15]). Since (3.4) is a consequence of a formula for the difference between the complex Laplacian $\bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and its conjugate, by inserting the multiplication of a positive C^∞ function, say, η, on M between $\bar\partial$ and $\bar\partial^*$, we obtain

$$(3.5)\quad \big(i(\eta\partial\bar\partial\phi - \partial\bar\partial\eta - c\partial\eta\wedge\bar\partial\eta)\Lambda u, u\big)_\phi \le \|\sqrt{\eta}\bar\partial u\|^2_\phi + \|\sqrt{\eta + c^{-1}}\bar\partial^* u\|^2_\phi$$

for any positive C^∞ function c on M (see [19]).

§4. Proof of Theorem 0.1

To simply display the main ingredient of the method, let us prove Theorem 0.1 first for the special case where $\phi = 0$ and $\psi = 0$. For that, we put $M = \Omega \setminus H$, and we fix a Kähler metric g on M. We may assume that $|w| < 1$ on M. Then we put

$$\eta_0 = \begin{cases} -\log|w| \\ \quad \text{if } \epsilon < |w| < 1, \\ (-\log|w|)\log(-\log|w|) - \log|w| + \log(-\log\epsilon)(-\log|w|) \\ \quad \text{if } \epsilon^e \le |w| \le \epsilon, \\ (-e\log\epsilon)\log(-e\log\epsilon) - e\log\epsilon + \log(-\log\epsilon)(-e\log\epsilon) \\ \quad \text{if } |w| < \epsilon^e, \end{cases}$$

$\eta = \eta_0 + (1 - |w|^2)/4$, and $\phi = \log|w|^2$. Note that $\eta < |w|^{-2}$,

$$\partial\eta_0 = \begin{cases} \partial\log|w| & \text{if } \epsilon < |w| < 1, \\ -(\log(-\log|w|) + \log(-\log\epsilon))\partial\log|w| & \text{if } \epsilon^e \le |w| \le \epsilon, \\ 0 & \text{if } |w| < \epsilon^e, \end{cases}$$

$-\eta$ is plurisubharmonic,

$$-\partial\bar\partial\eta = |w|^2\partial\log|w|\wedge\bar\partial\log|w| \quad \text{if } \epsilon < |w| < 1,$$

and

$$-\partial\bar\partial\eta = \partial\log|w|\wedge\bar\partial\log|w|/\big(-\log|w|\big) + |w|^2\partial\log|w|\wedge\bar\partial\log|w| \\ \quad \text{if } \epsilon^e \le |w| \le \epsilon.$$

Therefore, one has positive continuous functions $c = c(\epsilon)$, $b = b(\epsilon)$, and $d = d(\epsilon)$ on M with $\lim_{\epsilon\to 0} b(\epsilon) = 0$, $\lim_{\epsilon\to 0} d(\epsilon) = 0$, and $\eta + c^{-1} = |w|^{-2}$, such that

$$i(\eta\partial\bar\partial\phi - \partial\bar\partial\eta - c\partial\eta\bar\partial\eta) \geq 0 \tag{4.1}$$

and

$$i(\eta\partial\bar\partial\phi - \partial\bar\partial\eta - c\partial\eta\bar\partial\eta) \geq -i\big(1 - d(\epsilon)\big)\partial\bar\partial\eta \quad \text{on } \epsilon^e < |w| < \epsilon. \tag{4.2}$$

Therefore, a standard application of (3.5) yields solutions u_ϵ to (3.3) on M which are extendible holomorphically along $\Omega \cap H$ in such a way that their restrictions to $\Omega \cap H$ are zero and

$$\big\|(\sqrt{\eta + c^{-1}})^{-1} u_\epsilon\big\|^2 \leq \big(1 + a(\epsilon)\big) 2\pi \int_{H_0} f \wedge \bar f. \tag{4.3}$$

Here $a(\epsilon) \to 0$ as $\epsilon \to 0$. For these u_ϵ it is clear that $\liminf_{\epsilon\to 0} \|u_\epsilon\|_0^2 \leq 2\pi \int_{H_0} f \wedge \bar f$. In general, one has only to replace ϕ and $\log|w|$ by $\varphi + \phi$ and $\psi/2 + \log|w|$, respectively, and proceed similarly as above. ∎

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