

A GLOBAL ESTIMATE FOR THE DIEDERICH–FORNAESS INDEX OF WEAKLY PSEUDOCONVEX DOMAINS

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Abstract. A uniform upper bound for the Diederich–Fornaess index is given for weakly pseudoconvex domains whose Levi form of the boundary vanishes in ℓ -directions everywhere.

§1. Introduction

The aim of this article is to reveal a relation between the Diederich–Fornaess index of weakly pseudoconvex domains and the rank of the Levi form of their boundaries.

Let us first recall the definition of the Diederich–Fornaess index. Consider a complex manifold X and a relatively compact domain $\Omega \Subset X$ with \mathcal{C}^2 -smooth boundary. A defining function of Ω is a \mathcal{C}^2 -smooth function $\rho : \overline{\Omega} \rightarrow \mathbb{R}$ satisfying $\Omega = \{\rho < 0\}$ and whose gradient does not vanish on $\partial\Omega$. To avoid using too many minus signs, we will associate to a fixed defining function ρ the nonnegative function $\hat{\delta} = \hat{\delta}_\rho = -\rho$, which can be thought of as a boundary distance function of Ω with respect to a certain Hermitian metric on X (depending on ρ).

The *Diederich–Fornaess exponent* $\eta_{\hat{\delta}}$ of a defining function $-\hat{\delta}$ is the supremum of $\eta \in (0, 1)$ such that $-\hat{\delta}^\eta$ is a bounded, strictly plurisubharmonic exhaustion function of Ω . If there is no such η , we let $\eta_{\hat{\delta}} := 0$. The *Diederich–Fornaess index* $\eta(\Omega)$ of Ω is the supremum of the Diederich–Fornaess exponents of defining functions of Ω .

The Diederich–Fornaess index is a numerical index on the strength of a certain pseudoconvexity, more precisely that of hyperconvexity. If $\partial\Omega$ is strictly pseudoconvex, we know that $\partial\Omega$ admits a strictly plurisubharmonic defining function; hence, $\eta(\Omega) = 1$. For Ω to have positive $\eta(\Omega)$, Ω must be

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Stein, and in fact we need more. A theorem of Ohsawa and Sibony [12, Theorem 1.1]; (see also [11]) tells us that $\eta_\delta > 0$ if and only if $i\partial\bar{\partial}(-\log \hat{\delta}) \geq \omega_0$ in Ω for some Hermitian metric ω_0 of X . The domains Ω with positive $\eta(\Omega)$ should carry such a special exhaustion as if they are proper pseudoconvex domains in $X = \mathbb{C}\mathbb{P}^n$, where Takeuchi's theorem guarantees this kind of exhaustion. Many techniques using such exhaustions have been developed for solving the $\bar{\partial}$ -equation on weakly pseudoconvex domains (see, e.g., [2]–[5], [11]).

Let us give several examples to illustrate the situation we are considering. In a celebrated series of works, Diederich and Fornaess (see [7], [8]) showed that, if X is Stein, $\eta(\Omega) > 0$ for any domain $\Omega \Subset X$ with \mathcal{C}^2 -smooth pseudoconvex boundary. Note that in this situation $\partial\Omega$ must have a strictly pseudoconvex point, for we can find a level set of a strictly plurisubharmonic exhaustion of X touching $\partial\Omega$ at some points and bounding Ω . They also showed that, for any $\varepsilon > 0$, there is $\Omega \Subset X = \mathbb{C}^2$ with $0 < \eta(\Omega) < \varepsilon$ by using the worm domains, where a Levi-flat portion sits on $\partial\Omega$. Adachi in [1] proved that certain holomorphic disk bundles Ω over compact Riemann surfaces in their associated flat ruled surfaces X satisfy $\eta(\Omega) > 0$ even though $\partial\Omega$ is totally Levi-flat.

A natural question, therefore, is to ask to what extent the Diederich–Fornaess exponent gets smaller when $\partial\Omega$ is nearly Levi-flat everywhere. Our answer is the following.

MAIN THEOREM. *Let X be a complex manifold of dimension $n \geq 2$, and let $\Omega \Subset X$ be a relatively compact domain with \mathcal{C}^3 -smooth boundary. Assume that the Levi form of the boundary $\partial\Omega$ has at least ℓ zero eigenvalues everywhere on $\partial\Omega$ where $0 \leq \ell \leq n - 1$. Then $\eta(\Omega) \leq (n - \ell)/n$.*

In particular, we obtain the following.

COROLLARY 1.1. *If $\eta(\Omega) > 1/n$, then $\partial\Omega$ is not Levi-flat.*

COROLLARY 1.2. *If $\eta(\Omega) > (n - 1)/n$, then $\partial\Omega$ has a strictly pseudoconvex point.*

Let us explain the idea of our proof of the **Main Theorem**. When X is Stein, we found a strictly pseudoconvex point on $\partial\Omega$ by approximating $\partial\Omega$ by strictly pseudoconvex real hypersurfaces from outside. Since no such approximation exists in general, we use the following method inside. We assume by contradiction that $\eta(\Omega) > (n - \ell)/n$. Then we show in Theorem 4.1, using weighted L^2 -estimates, that any smooth, top-degree form

with compact support in Ω is $\bar{\partial}$ -exact in the sense of currents on X . This is impossible essentially because the top-degree cohomology with compact support does not vanish.

For the proof of Theorem 4.1, we use an estimate of Donnelly–Fefferman type (see [9]) to pass from an L^2 vanishing result in $L^2_{n,n}(\Omega, \hat{\delta}^\eta)$ to an L^2 vanishing result in $L^2_{n,n}(\Omega, \hat{\delta}^{-\eta})$. We also modify this argument by using a special Kähler metric $\omega := i\partial\bar{\partial}(-\hat{\delta}^\eta)$ in Ω for some $\eta \in (0, \eta_{\hat{\delta}})$. This metric respects the degeneracy of the Levi form of $\partial\Omega$ in a certain manner and permits the proof that the trivial extension of this solution is in fact a solution on all of X .

§2. Preliminaries on L^2 -estimate

In this section we introduce some notation that we use throughout this article. Also, for the convenience of the reader, we recall some of the basic facts concerning a priori estimates and solvability results for the $\bar{\partial}$ operator.

Let X be a complex manifold equipped with a Hermitian metric ω_0 , and let $\Omega \subset X$ be a domain with \mathcal{C}^2 -smooth boundary. We let $-\hat{\delta} : \bar{\Omega} \rightarrow \mathbb{R}$ be a defining function.

We denote by $L^2_{p,q}(\Omega, \hat{\delta}^s)$ the Hilbert space of (p, q) -forms u which satisfy

$$\|u\|_{\hat{\delta}^s}^2 := \int_{\Omega} |u|_{\omega_0}^2 \hat{\delta}^s dV_{\omega_0} < +\infty.$$

Here dV_{ω_0} is the canonical volume element associated with the metric ω_0 , and $|\cdot|_{\omega_0}$ is the norm of (p, q) -forms induced by ω_0 . For $s = 0$ the L^2 -spaces just defined coincide with the usual L^2 -spaces on Ω ; in this case, we will omit the index $\hat{\delta}^0$.

In our proofs it is sometimes necessary to replace the base metric ω_0 with a different metric ω . The corresponding Hilbert spaces (resp., norms) will then be denoted by $L^2_{p,q}(\Omega, \hat{\delta}^s, \omega)$ (resp., $\|\cdot\|_{\hat{\delta}^s, \omega}$).

For later use, we recall the well-known Bochner–Kodaira–Nakano inequality for Kähler metrics for the special case of the trivial line bundle \mathbb{C} on Ω equipped with a weight function $\varphi \in \mathcal{C}^2(\Omega)$, which is the key point when establishing L^2 existence theorems for the $\bar{\partial}$ operator (see [6]), as follows.

Let ω be a Kähler metric on Ω . Then for every $u \in \mathcal{D}^{p,q}(\Omega)$ we have

$$(2.1) \quad \|\bar{\partial}u\|_{e^{-\varphi}}^2 + \|\bar{\partial}^*_{e^{-\varphi}}u\|_{e^{-\varphi}}^2 \geq \langle [i\partial\bar{\partial}\varphi, \Lambda]u, u \rangle_{e^{-\varphi}}.$$

Here Λ is the adjoint of multiplication by ω .

A standard computation for the curvature term yields that

$$(2.2) \quad \langle [i\partial\bar{\partial}\varphi, \Lambda]u, u \rangle \geq \left(\lambda_1 + \cdots + \lambda_q - \sum_{j=1}^n \lambda_j \right) |u|^2$$

for any form $u \in \Lambda^{0,q} T^* \Omega$. Here $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of $i\partial\bar{\partial}\varphi$ with respect to ω .

§3. A special metric

When Ω has a defining function $-\hat{\delta}$ with positive Diederich–Fornaess exponent $\eta_{\hat{\delta}}$, taking $0 < \eta < \eta_{\hat{\delta}}$, we will equip the domain Ω with another Kähler metric $\omega := i\partial\bar{\partial}(-\hat{\delta}^\eta)$ different from ω_0 .

Let us study the behavior of the metric ω near $\partial\Omega$ for later use.

LEMMA 3.1. *Suppose that $\partial\Omega$ is C^3 -smooth and that the Levi form of $\partial\Omega$ has at least ℓ zero eigenvalues everywhere. Then, we have*

$$(3.1) \quad dV_\omega \lesssim \hat{\delta}^{n\eta-2-(n-\ell-1)} dV_{\omega_0}$$

near $\partial\Omega$.

Proof. First fix a finite covering of $\partial\Omega$ by holomorphic charts $\{(U; z_U)\}$ equipped with the Euclidean metrics ω_U associated with their coordinates z_U . We can fix the covering so that

- $|d\hat{\delta}|_{\omega_U} > 1$ on each chart U ;
- ω_U are uniformly comparable to ω_0 ; and
- a C^k -norm for functions defined on a neighborhood of $\bar{\Omega}$, say, $\|\cdot\|_{C^k(\bar{\Omega})}$, bounds the C^k -norm associated with the coordinate z_U from above for functions compactly supported in U .

Let $p \in \partial\Omega$, and take one of the holomorphic charts that contains p , say, $(U; z_U = (z_1, z_2, \dots, z_n))$. For small $\varepsilon > 0$, consider a nontangential cone $\Gamma_{p,\varepsilon} := \{z \in U \cap \Omega \mid |z - p| < 2\hat{\delta}(z), |z - p| < \varepsilon\}$ with vertex at p . Note that $\Gamma_{p,\varepsilon}$ is nonempty as $\bar{\Gamma}_{p,\varepsilon}$ contains a segment starting from p normal to $\ker d\hat{\delta}_p$. It suffices to find a positive constant C independent of the choice of p so that

$$D_U := \frac{dV_\omega}{dV_{\omega_U}} \leq C \hat{\delta}^{n\eta-2-(n-\ell-1)}$$

holds on $\Gamma_{p,\varepsilon}$ for some $\varepsilon = \varepsilon(p) > 0$. That is because $\bigcup_{p \in \partial\Omega} \Gamma_{p,\varepsilon(p)} = W \cap \Omega$ for some neighborhood W of $\partial\Omega$ and ω_0 is comparable to every ω_U with a uniform constant; we can prove the desired inequality on $W \cap \Omega$.

To compute dV_ω/dV_{ω_U} , we will select an orthonormal frame of $T^{1,0}U$. By a unitary transformation, we can suppose that $\ker d\hat{\delta}_p = \mathbb{C}^{n-1} \times \mathbb{R}$ and that $\mathbb{C}^\ell \times \{0\}$ is contained in the kernel of the Levi form of $\partial\Omega$ at p . Define a \mathcal{C}^2 -smooth frame $\mathcal{Y} = (Y_1, Y_2, \dots, Y_n)$ of $T^{1,0}U$ by

$$Y_j := \frac{\partial}{\partial z_j} - \frac{\partial \hat{\delta} / \partial z_j}{\partial \hat{\delta} / \partial z_n} \frac{\partial}{\partial z_n} \quad (j = 1, 2, \dots, n-1), \quad Y_n := \frac{\partial}{\partial z_n}.$$

Note that $\{Y_1, Y_2, \dots, Y_{n-1}\}$ spans $\ker \partial \hat{\delta}$ on U . We apply the Gram-Schmidt procedure to \mathcal{Y} and obtain an orthonormal frame $\mathcal{X} = (X_1, X_2, \dots, X_n)$ with respect to ω_U . Denote by $A(z) = (a_{jk}(z))$ the change-of-base matrices at each point: $X_k = \sum_{j=1}^n Y_j a_{jk}$ on U .

We would like to estimate each $\lambda_{j\bar{k}} := \omega(X_j, \overline{X_k})$ on $\Gamma_{p,\varepsilon}$. To achieve it, we combine two estimates: one is about $\mu_{j\bar{k}} := \omega(Y_j, \overline{Y_k})$, and the other is about the change-of-base matrices $A(z)$.

First consider the behavior of $\mu_{j\bar{k}}$ on $\Gamma_{p,\varepsilon}$. The equality

$$(3.2) \quad \omega = i\eta \hat{\delta}^\eta \left\{ \frac{\partial \bar{\partial}(-\hat{\delta})}{\hat{\delta}} + (1-\eta) \frac{\partial \hat{\delta} \wedge \bar{\partial} \hat{\delta}}{\hat{\delta}^2} \right\}$$

yields that, if $j = k = n$,

$$\lim_{z \rightarrow p, z \in U \cap \Omega} \frac{\mu_{n\bar{n}}(z)}{\hat{\delta}(z)^{\eta-2}} = \eta(1-\eta) |\partial \hat{\delta}(Y_n(p))|^2 \leq \|\hat{\delta}\|_{\mathcal{C}^1(\bar{\Omega})}^2;$$

otherwise,

$$\lim_{z \rightarrow p, z \in U \cap \Omega} \frac{|\mu_{j\bar{k}}(z)|}{\hat{\delta}(z)^{\eta-1}} = \eta |\partial \bar{\partial}(-\hat{\delta})(Y_j(p), \overline{Y_k(p)})| \leq \|\hat{\delta}\|_{\mathcal{C}^2(\bar{\Omega})}.$$

We can say more for directions in which the Levi form vanishes. If $1 \leq j \leq \ell$, $1 \leq k \leq n-1$ or $1 \leq j \leq n-1$, $1 \leq k \leq \ell$,

$$\begin{aligned} & \limsup_{z \rightarrow p, z \in \Gamma_{p,\varepsilon}} \frac{|\mu_{j\bar{k}}(z)|}{\hat{\delta}(z)^\eta} \\ &= \limsup_{z \rightarrow p, z \in \Gamma_{p,\varepsilon}} \eta \left| \frac{\partial \bar{\partial}(-\hat{\delta})(Y_j(z), \overline{Y_k(z)})}{\hat{\delta}(z)} \right| \\ &= \limsup_{z \rightarrow p, z \in \Gamma_{p,\varepsilon}} \eta \frac{|z-p|}{\hat{\delta}(z)} \left| \frac{\partial \bar{\partial}(-\hat{\delta})(Y_j(z), \overline{Y_k(z)}) - 0}{|z-p|} \right| \end{aligned}$$

$$\begin{aligned}
&\leq 2|d(\partial\bar{\partial}(-\hat{\delta})(Y_j, \bar{Y}_k))(p)|_{\omega_U} \\
&\leq 2(\|\hat{\delta}\|_{C^3(\bar{\Omega})} + 2\|\hat{\delta}\|_{C^2(\bar{\Omega})}^2).
\end{aligned}$$

Next we proceed to estimate the change-of-base matrices $A(z)$. We identify an n -tuple of $(1, 0)$ -vectors with an $n \times n$ matrix by using our coordinate z_U . Then, we have $\mathcal{X}(p) = \mathcal{Y}(p) = I_n$ and $A(z) = \mathcal{Y}^{-1}(z) \cdot \mathcal{X}(z)$, where I_n denotes the identity matrix. As a matrix-valued 1-form, we have

$$dA(p) = \mathcal{Y}^{-1}(p) \cdot d\mathcal{X}(p) + d\mathcal{Y}^{-1}(p) \cdot X(p) = d\mathcal{X}(p) + d\mathcal{Y}^{-1}(p).$$

Since $I_n = \mathcal{Y}^{-1}(z) \cdot \mathcal{Y}(z)$, we also have

$$0 = d(\mathcal{Y}^{-1} \cdot \mathcal{Y})(p) = d\mathcal{Y}^{-1}(p) + d\mathcal{Y}(p).$$

Now let $\text{GS} : \text{GL}(n, \mathbb{C}) \rightarrow U(n)$ be the map determined by the Gram–Schmidt procedure. Its differential at I_n defines $d\text{GS}_{I_n} : \mathfrak{gl}(n, \mathbb{C}) \rightarrow \mathfrak{u}(n)$. We linearly extend this map on matrix-valued, that is, $\mathfrak{gl}(n, \mathbb{C})$ -valued 1-forms, and we also write $d\text{GS}_{I_n}$ for the extended linear map by abuse of notation. Then, $d\text{GS}_{I_n}(d\mathcal{Y}(p)) = d\mathcal{X}(p)$ follows from $\text{GS}(\mathcal{Y}(z)) = \mathcal{X}(z)$. Combining these equalities, we therefore have

$$dA(p) = d\text{GS}_{I_n}(d\mathcal{Y}(p)) - d\mathcal{Y}(p).$$

We use the norm $|A| = \max_{j,k} |a_{jk}|$ for matrices, and we consider the induced norm for linear maps between spaces of matrices. Since a straightforward computation yields $|d\mathcal{Y}(p)|_{\omega_U} \leq \|\hat{\delta}\|_{C^2(\bar{\Omega})}$, we have

$$\begin{aligned}
\limsup_{z \rightarrow p, z \in \Gamma_{p,\varepsilon}} \frac{|A(z) - I_n|}{\hat{\delta}(z)} &= \limsup_{z \rightarrow p, z \in \Gamma_{p,\varepsilon}} \frac{|z - p| |A(z) - I_n|}{\hat{\delta}(z) |z - p|} \\
&\leq 2|dA(p)|_{\omega_U} \\
&\leq 2(|d\text{GS}_{I_n}| + 1)|d\mathcal{Y}(p)|_{\omega_U} \\
&\leq 2(|d\text{GS}_{I_n}| + 1)\|\hat{\delta}\|_{C^2(\bar{\Omega})}.
\end{aligned}$$

Note that $|d\text{GS}_{I_n}|$ is independent of p and depends only on n .

By combining the estimates on $\mu_{j\bar{k}}$ and $A(z)$ above, we can find a positive constant C depending only on $n = \dim X$ and $\|\hat{\delta}\|_{C^3(\bar{\Omega})}$ so that

$$(3.3) \quad |\lambda_{j\bar{k}}(z)| = \left| \sum_{l,m} \mu_{l\bar{m}}(z) a_{jl}(z) \overline{a_{km}(z)} \right| \leq \begin{cases} C\hat{\delta}^{\eta-2} & (\text{for } j = k = n) \\ C\hat{\delta}^{\eta} & (\text{for } 1 \leq j \leq \ell, 1 \leq k \leq n-1) \\ C\hat{\delta}^{\eta} & (\text{for } 1 \leq j \leq n-1, 1 \leq k \leq \ell) \\ C\hat{\delta}^{\eta-1} & (\text{otherwise}) \end{cases}$$

holds on $\Gamma_{p,\varepsilon}$ for $0 < \varepsilon \ll 1$. It follows that

$$\begin{aligned} D_U &= \det(\lambda_{j\bar{k}})_{j,k=1}^n \\ &\leq n! C^n \hat{\delta}^{\ell\eta + (n-\ell-1)(\eta-1) + (\eta-2)} \\ &= n! C^n \hat{\delta}^{n\eta-2-(n-\ell-1)} \end{aligned}$$

on $\Gamma_{p,\varepsilon}$, which completes the proof. \square

LEMMA 3.2. *Suppose that $\partial\Omega$ is C^3 -smooth and that the Levi form of $\partial\Omega$ has at least ℓ zero eigenvalues everywhere. Then, for any $(n, n-1)$ -form u on Ω ,*

$$|u|_{\omega_0}^2 dV_{\omega_0} \lesssim |u|_{\omega}^2 \hat{\delta}^{(n-1)\eta-2-(n-\ell-1)} dV_{\omega}$$

near $\partial\Omega$ with positive constant independent of u .

Proof. It suffices to prove the inequality on $\Gamma_{p,\varepsilon}$ with ω_U instead of ω_0 , where we work in the same local situation as in the proof of Lemma 3.1. Consider the induced frame of $\wedge^n T^{1,0}U \otimes \wedge^{n-1} T^{0,1}U$ from $\{X_1, X_2, \dots, X_n\}$ over U . It follows from (3.3) that

$$\begin{aligned} &|X_1 \wedge X_2 \wedge \cdots \wedge X_n \otimes \bar{X}_1 \wedge \bar{X}_2 \wedge \cdots \wedge \widehat{\bar{X}}_k \wedge \cdots \wedge \bar{X}_n|_{\omega}^2 \\ &= D_U |\bar{X}_1 \wedge \bar{X}_2 \wedge \cdots \wedge \widehat{\bar{X}}_k \wedge \cdots \wedge \bar{X}_n|_{\omega}^2 \\ &\leq D_U (n-1)! C^{n-1} \begin{cases} \hat{\delta}^{(\ell-1)\eta + (n-\ell-1)(\eta-1) + (\eta-2)} & (\text{for } 1 \leq k \leq \ell) \\ \hat{\delta}^{\ell\eta + (n-\ell-2)(\eta-1) + (\eta-2)} & (\text{for } \ell+1 \leq k \leq n-1) \\ \hat{\delta}^{\ell\eta + (n-\ell-1)(\eta-1)} & (\text{for } k = n) \end{cases} \\ &\leq D_U (n-1)! C^{n-1} \hat{\delta}^{(n-1)\eta-2-(n-\ell-1)}. \end{aligned}$$

Hence, we can estimate $|u|_\omega^2$ as

$$\begin{aligned} |u|_\omega^2 &\geq \max_{1 \leq k \leq n} \frac{|u(X_1, X_2, \dots, X_n, \bar{X}_1, \bar{X}_2, \dots, \widehat{X}_k, \dots, \bar{X}_n)|^2}{|X_1 \wedge X_2 \wedge \dots \wedge X_n \otimes \bar{X}_1 \wedge \bar{X}_2 \wedge \dots \wedge \widehat{X}_k \wedge \dots \wedge \bar{X}_n|_\omega^2} \\ &\geq \frac{\max_{1 \leq k \leq n} |u(X_1, X_2, \dots, X_n, \bar{X}_1, \bar{X}_2, \dots, \widehat{X}_k, \dots, \bar{X}_n)|^2}{(n-1)! C^{n-1} D_U \widehat{\delta}^{(n-1)\eta-2-(n-\ell-1)}} \\ &\geq C' \frac{|u|_{\omega_U}^2}{D_U \widehat{\delta}^{(n-1)\eta-2-(n-\ell-1)}}, \end{aligned}$$

with constant $C' > 0$ independent of u . We therefore have the desired inequality

$$\begin{aligned} |u|_\omega^2 dV_\omega &\geq C' \frac{1}{D_U} |u|_{\omega_U}^2 \widehat{\delta}^{-(n-1)\eta+2+(n-\ell-1)} D_U dV_{\omega_U} \\ &= C' |u|_{\omega_U}^2 \widehat{\delta}^{-(n-1)\eta+2+(n-\ell-1)} dV_{\omega_U}. \quad \square \end{aligned}$$

§4. The $\bar{\partial}$ equation in top degree

In this section we will study a version of an L^2 $\bar{\partial}$ -Cauchy problem in top degree on a smoothly bounded domain with weakly pseudoconvex boundary, which, by duality, implies a restriction on the rank of the Levi form of $\partial\Omega$.

THEOREM 4.1. *Let X be a complex manifold of dimension $n \geq 2$, and let $\Omega \Subset X$ be a relatively compact domain with C^3 -smooth boundary. Suppose that the Levi form of $\partial\Omega$ has at least ℓ zero eigenvalues everywhere on $\partial\Omega$ for some $0 \leq \ell \leq n-1$. If $\eta(\Omega) > (n-\ell)/n$, then for any $f \in L_{n,n}^2(X)$ which is compactly supported in Ω there exists a current $T \in \mathcal{D}'_{0,1}(X)$ supported in $\bar{\Omega}$ such that $\bar{\partial}T = f$ in the distribution sense on X .*

Theorem 4.1 is based on the following estimate of Donnelly–Fefferman type.

THEOREM 4.2. *Let X be a complex manifold of dimension $n \geq 2$, and let $\Omega \Subset X$ be a relatively compact domain with C^2 -smooth boundary. Let $-\widehat{\delta}$ be a defining function of Ω with Diederich–Fornaess exponent $\eta_\delta > 0$. For an arbitrary but fixed $\eta \in (0, \eta_\delta)$ we define $\omega := i\partial\bar{\partial}(-\widehat{\delta}^\eta)$. Then, for any $f \in L_{n,n}^2(\Omega, \widehat{\delta}^{-\eta}, \omega)$, there exists $u \in L_{n,n-1}^2(\Omega, \widehat{\delta}^{-\eta}, \omega)$ satisfying $\bar{\partial}u = f$ in the distribution sense in Ω .*

Proof. Let us first see that the conclusion follows in a standard manner from the following a priori estimate.

CLAIM. *There exists a constant $C > 0$ such that*

$$(4.1) \quad \|v\|_{\hat{\delta}^{-\eta}, \omega}^2 \leq C \|\bar{\partial}^* v\|_{\hat{\delta}^{-\eta}, \omega}^2$$

for any $v \in \mathcal{D}^{n,n}(\Omega)$. Here $\bar{\partial}^* = \bar{\partial}_{\hat{\delta}^{-\eta}, \omega}^*$ is the adjoint of $\bar{\partial}$ with respect to the scalar product induced by $\|\cdot\|_{\hat{\delta}^{-\eta}, \omega}$.

Note that in the top degree we can work with noncomplete metrics, since there is no compatibility condition. Indeed, let us take $f \in L_{n,n}^2(\Omega, \hat{\delta}^{-\eta}, \omega)$ and define a linear functional ϕ on $\bar{\partial}^*(\mathcal{D}^{n,n}(\Omega)) \subset L_{n,n-1}^2(\Omega, \hat{\delta}^{-\eta}, \omega)$ by $\phi(\bar{\partial}^* v) = \langle\langle v, f \rangle\rangle_{\hat{\delta}^{-\eta}, \omega}$, which is well defined and bounded from (4.1). The Hahn–Banach theorem allows us to extend ϕ to a bounded linear functional on $L_{n,n-1}^2(\Omega, \hat{\delta}^{-\eta}, \omega)$, and the Riesz representation theorem yields $u \in L_{n,n-1}^2(\Omega, \hat{\delta}^{-\eta}, \omega)$ satisfying

$$\langle\langle \bar{\partial}^* v, u \rangle\rangle_{\hat{\delta}^{-\eta}, \omega} = \langle\langle v, f \rangle\rangle_{\hat{\delta}^{-\eta}, \omega}$$

for all $v \in \mathcal{D}^{n,n}(\Omega)$; that is, $\bar{\partial}u = f$ in the distribution sense in Ω .

Let us proceed to prove (4.1). For a direct proof of it, we would have to work with different adjoint operators. Therefore, it is somewhat more convenient to actually prove the dual a priori estimate

$$(4.2) \quad \|v\|_{\hat{\delta}^\eta, \omega} \leq C \|\bar{\partial}v\|_{\hat{\delta}^\eta, \omega}$$

for any $v \in \mathcal{D}^{0,0}(\Omega)$. Equation (4.1) then follows from (4.2) using a weighted Hodge star operator.

So let us proceed to prove (4.2). Since $\eta < \eta_{\hat{\delta}}$, there exists some small $\varepsilon > 0$ such that $\eta + \varepsilon < \eta_{\hat{\delta}}$, which means that

$$i\partial\bar{\partial}(-\hat{\delta}^{\eta+\varepsilon}) \geq 0 \quad \text{in } \Omega.$$

But then

$$i\partial\bar{\partial} \log \hat{\delta}^{\eta+\varepsilon} = \frac{i\partial\bar{\partial}\hat{\delta}^{\eta+\varepsilon}}{\hat{\delta}^{\eta+\varepsilon}} - i\partial \log \hat{\delta}^{\eta+\varepsilon} \wedge \bar{\partial} \log \hat{\delta}^{\eta+\varepsilon} \leq -i\partial \log \hat{\delta}^{\eta+\varepsilon} \wedge \bar{\partial} \log \hat{\delta}^{\eta+\varepsilon}.$$

Hence, we get

$$\text{Trace}_\omega(i\partial\bar{\partial} \log \hat{\delta}^{\eta+\varepsilon}) \leq -|\bar{\partial} \log \hat{\delta}^{\eta+\varepsilon}|_\omega^2 \quad \text{in } \Omega.$$

Putting $\psi = \hat{\delta}^\eta$, we have $i\partial\bar{\partial}\psi = -\omega$ by definition of ω ; thus, $\text{Trace}_\omega(i\partial\bar{\partial}\psi) = -n$. Hence, we get

$$(4.3) \quad \text{Trace}_\omega(i\partial\bar{\partial}\psi + i\partial\bar{\partial}\log \hat{\delta}^{\eta+\varepsilon}) \leq -n - |\partial \log \hat{\delta}^{\eta+\varepsilon}|_\omega^2 \quad \text{on } \Omega.$$

On Ω , we consider the weight function $e^{-\psi}$. Since $e^{-\psi}$ is bounded from below and from above by positive constants on Ω , we can replace the norm $\|\cdot\|$ by $\|\cdot\|_{e^{-\psi}}$ for forms on Ω .

Multiplying the metric of the trivial bundle \mathbb{C} further by $\hat{\delta}^{-(\eta+\varepsilon)} = e^{-\log \hat{\delta}^{\eta+\varepsilon}}$ on Ω , it then follows from (2.1) and (2.2) that for $u \in \mathcal{D}^{0,0}(\Omega)$ one has

$$\langle\langle -\text{Trace}_\omega(i\partial\bar{\partial}\psi + i\partial\bar{\partial}\log \hat{\delta}^{\eta+\varepsilon})u, u \rangle\rangle_{e^{-\psi}\hat{\delta}^{-(\eta+\varepsilon)}, \omega} \leq \|\bar{\partial}u\|_{e^{-\psi}\hat{\delta}^{-(\eta+\varepsilon)}, \omega}^2.$$

Using (4.3) we obtain

$$\langle\langle (n + |\bar{\partial}\log \hat{\delta}^{\eta+\varepsilon}|_\omega^2)u, u \rangle\rangle_{e^{-\psi}\hat{\delta}^{-(\eta+\varepsilon)}, \omega} \leq \|\bar{\partial}u\|_{e^{-\psi}\hat{\delta}^{-(\eta+\varepsilon)}, \omega}^2$$

for $u \in \mathcal{D}^{0,0}(\Omega)$. Observing that $\partial \log \hat{\delta}^{\eta+\varepsilon} = (\eta + \varepsilon)\partial \log \hat{\delta}$ and setting $u = v\hat{\delta}^{\eta+\varepsilon/2}$, we obtain

$$(4.4) \quad \begin{aligned} & \langle\langle (n + (\eta + \varepsilon)^2 |\bar{\partial}\log \hat{\delta}|_\omega^2)v, v \rangle\rangle_{e^{-\psi}\hat{\delta}^{\eta, \omega}} \\ & \leq \left\| \bar{\partial}v + \left(\eta + \frac{\varepsilon}{2}\right)v\bar{\partial}\log \hat{\delta} \right\|_{e^{-\psi}\hat{\delta}^{\eta, \omega}}^2 \\ & \leq \left(1 + \frac{1}{a}\right) \|\bar{\partial}v\|_{e^{-\psi}\hat{\delta}^{\eta, \omega}}^2 + (1+a) \left(\eta + \frac{\varepsilon}{2}\right)^2 \|v\bar{\partial}\log \hat{\delta}\|_{e^{-\psi}\hat{\delta}^{\eta, \omega}}^2. \end{aligned}$$

Choosing a so small that $(1+a)(\eta + \varepsilon/2)^2 \leq (\eta + \varepsilon)^2$, we can thus absorb the last term in (4.4) in the left-hand side, which immediately gives the a priori estimate (4.2). \square

Now let us give the proof of Theorem 4.1.

Proof of Theorem 4.1. By the assumption on Ω , we can find a defining function $-\hat{\delta}$ with $\eta_{\hat{\delta}} > (n - \ell)/n$. We fix some real η such that $(n - \ell)/n < \eta < \eta_{\hat{\delta}}$, and we apply Theorem 4.2 with this choice of η .

Now let $f \in L_{n,n}^2(X)$ be compactly supported in Ω , which implies that $f \in L_{n,n}^2(\Omega, \hat{\delta}^{-\eta}, \omega)$. Hence, it follows from Theorem 4.2 that there exists $u \in L_{n,n-1}^2(\Omega, \hat{\delta}^{-\eta}, \omega)$ satisfying $\bar{\partial}u = f$ in Ω .

We first claim that, if we extend u by zero outside Ω , then it defines a current $T = T_u \in \mathcal{D}'_{0,1}(X)$. Indeed, we see from Lemma 3.2 that

$$\int_{\Omega} |u|_{\omega_0}^2 \hat{\delta}^{1-\nu} dV_{\omega_0} \lesssim \int_{\Omega} |u|_{\omega}^2 \hat{\delta}^{1-\nu} \hat{\delta}^{(n-1)\eta-2-(n-\ell-1)} dV_{\omega}.$$

Now a straightforward computation shows that the last integral can be estimated by $\int_{\Omega} |u|_{\omega}^2 \hat{\delta}^{-\eta} dV_{\omega} < +\infty$ if $\nu \leq n\eta - n + \ell$. But by assumption on η we have $n\eta - n + \ell > 0$; hence, we may deduce that for some small $\nu > 0$ we have $u \in L_{n,n-1}^2(\Omega, \hat{\delta}^{1-\nu})$.

But then for any $v \in \mathcal{C}_{0,1}^{\infty}(X)$ we have

$$(4.5) \quad \begin{aligned} \left| \int_{\Omega} u \wedge v \right|^2 &\leq \left(\int_{\Omega} |u|_{\omega_0}^2 \hat{\delta}^{1-\nu} dV_{\omega_0} \right) \cdot \left(\int_{\Omega} |v|_{\omega_0}^2 \hat{\delta}^{-1+\nu} dV_{\omega_0} \right) \\ &\leq \|u\|_{\hat{\delta}^{1-\nu}}^2 \cdot \left(\int_{\Omega} \hat{\delta}^{-1+\nu} dV_{\omega_0} \right) \sup_{\Omega} |v|_{\omega_0}^2. \end{aligned}$$

Since $\nu > 0$, we have $\int_{\Omega} \hat{\delta}^{-1+\nu} dV_{\omega_0} < +\infty$. Therefore, u defines a current $T \in \mathcal{D}'_{0,1}(X)$.

It remains to see that $T = T_u$ satisfies $\bar{\partial}T = f$ in the sense of distributions on X . Let $\alpha \in \mathcal{C}_{0,0}^{\infty}(X)$. We must show that

$$(4.6) \quad \int_{\Omega} u \wedge \bar{\partial}\alpha = \int_{\Omega} f \wedge \alpha.$$

Let $\chi \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R})$ be a function such that $\chi(t) = 0$ for $t \leq 1/2$ and $\chi(t) = 1$ for $t \geq 1$. Set $\chi_j = \chi(j\hat{\delta}) \in \mathcal{D}^{0,0}(\Omega)$. Then $\chi_j \alpha \in \mathcal{D}^{0,0}(\Omega)$, and since $\bar{\partial}u = f$ in Ω , we therefore have

$$\int_{\Omega} f \wedge \chi_j \alpha = \int_{\Omega} u \wedge \bar{\partial}(\chi_j \alpha) = \int_{\Omega} u \wedge (\alpha \bar{\partial}\chi_j + \chi_j \wedge \bar{\partial}\alpha).$$

As f has L^2 coefficients on Ω , the integral of $f \wedge \chi_j \alpha$ converges to the integral of $f \wedge \alpha$ as j tends to infinity. The convergence of the integral of $u \wedge \chi_j \bar{\partial}\alpha$ to the integral of $u \wedge \bar{\partial}\alpha$ follows from $u \in L_{n,n-1}^2(\Omega, \hat{\delta}^{1-\nu})$ (use the estimate (4.5)).

The remaining term can be estimated as follows. Using the Cauchy–Schwarz inequality we have

$$\begin{aligned}
\left| \int_{\Omega} u \wedge \alpha \bar{\partial} \chi_j \right|^2 &= \left| \int_{\{\frac{1}{2j} \leq \hat{\delta} \leq \frac{1}{j}\}} \langle u \hat{\delta}^{-\eta/2}, \overline{\star_{\omega} \alpha \bar{\partial} \chi_j \hat{\delta}^{\eta/2}} \rangle_{\omega} dV_{\omega} \right|^2 \\
&\leq \int_{\{\frac{1}{2j} \leq \hat{\delta} \leq \frac{1}{j}\}} |u \hat{\delta}^{-\eta/2}|_{\omega}^2 dV_{\omega} \cdot \int_{\{\frac{1}{2j} \leq \hat{\delta} \leq \frac{1}{j}\}} |\star_{\omega} \alpha \bar{\partial} \chi_j \hat{\delta}^{\eta/2}|_{\omega}^2 dV_{\omega} \\
&\leq \sup_{\Omega} |\alpha|^2 \int_{\{\hat{\delta} \leq \frac{1}{j}\}} |u|_{\omega}^2 \hat{\delta}^{-\eta} dV_{\omega} \cdot \int_{\Omega} |\bar{\partial} \chi_j|_{\omega}^2 \hat{\delta}^{\eta} dV_{\omega},
\end{aligned}$$

where \star_{ω} denotes the Hodge star operator with respect to ω in Ω . Since $u \in L^2_{n, n-1}(\Omega, \hat{\delta}^{-\eta}, \omega)$, the integral $\int_{\{\hat{\delta} \leq 1/j\}} |u|_{\omega}^2 \hat{\delta}^{-\eta} dV_{\omega}$ converges to 0 when j tends to infinity.

To estimate the second integral, we look at the behavior of its integrand $|\bar{\partial} \chi_j|_{\omega}^2$ near $\partial\Omega$. From $\bar{\partial} \chi_j = j \chi' \bar{\partial} \hat{\delta}$,

$$\begin{aligned}
\hat{\delta}(z)^{\eta-2} |\bar{\partial} \chi_j|_{\omega}^2(z) &\leq j^2 \|\chi'\|_{C^1(\mathbb{R})}^2 |\bar{\partial} \hat{\delta}|_{\hat{\delta}^{2-\eta} \omega}^2(z) \\
&= j^2 \|\chi'\|_{C^1(\mathbb{R})}^2 \max_{0 \neq v \in T_z^{1,0} X} \frac{|\partial \hat{\delta}(v)|^2}{\eta(\hat{\delta}(z) i \partial \bar{\partial}(-\hat{\delta})(v, v) + |\partial \hat{\delta}(v)|^2)} \\
&\rightarrow j^2 \|\chi'\|_{C^1(\mathbb{R})}^2 \frac{1}{\eta} \quad \text{as } z \rightarrow \partial\Omega.
\end{aligned}$$

Therefore, $|\bar{\partial} \chi_j|_{\omega}^2 \lesssim j^2 \hat{\delta}^{2-\eta}$ near $\partial\Omega$. Since the Levi form of $\partial\Omega$ has ℓ zero eigenvalues, we can estimate it with Lemma 3.1 as

$$\begin{aligned}
\int_{\Omega} |\bar{\partial} \chi_j|_{\omega}^2 \hat{\delta}^{\eta} dV_{\omega} &\lesssim \int_{\{\hat{\delta} \leq \frac{1}{j}\}} j^2 \hat{\delta}^{2-\eta} \hat{\delta}^{\eta} \hat{\delta}^{n\eta-2-(n-\ell-1)} dV_{\omega_0} \\
&= \int_{\{\hat{\delta} \leq \frac{1}{j}\}} j^2 \hat{\delta}^{1+n\eta-(n-\ell)} dV_{\omega_0} \\
&\lesssim j^{2-(2+n\eta-(n-\ell))} \\
&= j^{-n\eta+n-\ell} \rightarrow 0
\end{aligned}$$

as $j \rightarrow \infty$ since $-n\eta + n - \ell < 0$ by the assumption that $\eta > (n - \ell)/n$.

Therefore, $\int_{\Omega} u \wedge \alpha \bar{\partial} \chi_j$ converges to 0 when j tends to infinity. Equation (4.6) follows. \square

§5. Proof of the **Main Theorem**

The proof of the **Main Theorem** easily follows from Theorem 4.1 using a duality argument.

*Proof of the **Main Theorem**.* Assume by contradiction that the Levi form of the boundary $\partial\Omega$ has ℓ zero eigenvalues, and assume that $\eta(\Omega) > (n - \ell)/n$. Let $f \in \mathcal{D}^{n,n}(\Omega)$ be a smooth form of top degree with compact support in Ω satisfying $\int_{\Omega} f = 1$. Applying Theorem 4.1, we can find a current $T \in \mathcal{D}'_{0,1}(X)$ satisfying $\bar{\partial}T = f$ in the current sense. Let χ be a compactly supported smooth function on X which is equal to one on $\bar{\Omega}$. But then

$$1 = \int_{\Omega} f = \langle f, \chi \rangle = \langle T, \bar{\partial}\chi \rangle = 0.$$

This contradiction proves that $\eta(\Omega) \leq (n - \ell)/n$. □

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