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TORSION IN TENSOR POWERS OF MODULES

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Abstract. Tensor products usually have nonzero torsion. This is a central theme of Auslander's 1961 paper; the theme continues in the work of Huneke and Wiegand in the 1990s. The main focus in this article is on tensor powers of a finitely generated module over a local ring. Also, we study torsion-free modules N with the property that $M \otimes_R N$ has nonzero torsion unless M is very special. An important example of such a module N is the Frobenius power ${}^{p^e}R$ over a complete intersection domain R of characteristic $p > 0$.

§1. Introduction

In a 1961 paper, Auslander [1] studied torsion in tensor products of nonzero finitely generated modules M and N over unramified regular local rings R. Under the assumption that $M \otimes_R N$ is torsion-free, he proved that

(1) M and N must be torsion-free, and
(2) M and N are Tor independent; that is, $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

The two conclusions are cleverly intertwined in his proof, which we revisit in Section 3 of the present paper. We show, over a reduced complete intersection ring R of positive characteristic p, that $M \otimes_R {}^{\varphi^e}R$ is torsion-free if and only if M is torsion-free and of finite projective dimension, in which case $\mathrm{Tor}_i^R(M, {}^{\varphi^e}R) = 0$ for all $i \geq 1$. (Here $\varphi : R \to R$ is the Frobenius endomorphism and ${}^{\varphi^e}R$ is the module obtained from R by restriction of scalars along φ^e.) When R is F-finite, we obtain a criterion for regularity: R is regular if and only if $({}^{\varphi^e}M) \otimes_R {}^{\varphi^e}R$ is torsion-free for some (equivalently, every) nonzero finitely generated R-module M.

Our main results are in Section 2, where we study torsion in tensor powers. We obtain detailed information on annihilators of elements in $\otimes_R^n M$ and draw several conclusions. Suppose, for example, that $\underline{r} = r_1, \ldots, r_d$ is a

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regular sequence in R and that M is the cokernel of the $d \times 1$ matrix $[\underline{r}]^{\mathrm{t}}$. We show in Theorem 2.5 that $\otimes_R^t M$ is torsion-free if and only $t \leq d$. This result should be compared with Auslander's observation in [1, p. 638] that the same holds when M is the $(d-1)$st syzygy of a module of projective dimension d over a d-dimensional regular local ring (see also [7, Proposition 3.1]). If R is local, the "only if" direction holds much more generally: we show in Theorem 2.7 that, if we write M as the cokernel of an $m \times n$ matrix θ with entries in the maximal ideal of R, and if some entry of θ is a nonzerodivisor, then $\otimes_R^t M$ has nonzero torsion for every $t \geq m$.

Throughout this article, R is a commutative, Noetherian ring.

§2. Torsion in tensor powers

In this section we establish results on annihilators of elements in tensor powers of modules.

NOTATION 2.1. Given elements $\underline{m} := m_1, \ldots, m_d$ in an R-module M, we consider the element in $\otimes_R^d M$ defined by

$$\tau(\underline{m}) := \sum_{\sigma \in S_d} \operatorname{sign}(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(d)}.$$

PROPOSITION 2.2. *Let M be an R-module. If elements $m_1, \ldots, m_d$ in M and $r_1, \ldots, r_d$ in R satisfy*

$$r_1 m_1 + \cdots + r_d m_d = 0, \tag{2.2.1}$$

then $(r_1, \ldots, r_d) \cdot \tau(\underline{m}) = 0$ in $\otimes_R^d M$.

Proof. The *twisted shuffle product* gives the graded R-algebra $\bigoplus_{n \geq 0} \otimes_R^n M$ a strictly skew-commutative structure (see [11, Chapter X, (12.4)]. Strictly skew-commutative means that for any $a \in \otimes_R^i M$ and $b \in \otimes_R^j M$ there are equalities

$$a \star b = (-1)^{ij} b \star a, \qquad \text{and} \qquad a \star a = 0 \quad \text{when } i \text{ is odd.}$$

By definition of the shuffle product, $\tau(\underline{m}) = m_1 \star \cdots \star m_d$. Thus, for each j we have

$$\begin{aligned} r_j \cdot \tau(\underline{m}) &= m_1 \star \cdots \star m_{j-1} \star r_j m_j \star m_{j+1} \star \cdots \star m_n \\ &= -\sum_{i \neq j} r_i (m_1 \star \cdots \star m_{j-1} \star m_i \star m_{j+1} \star \cdots \star m_n) = -\sum_{i \neq j} r_i 0 = 0. \end{aligned}$$

∎

There is a "universal" source for the element $\tau(\underline{m})$ in the following sense.

REMARK 2.3. Consider the polynomial ring $\mathbb{Z}[\underline{x}]$ on indeterminates $\underline{x} := x_1, \dots, x_d$, and let U be the $\mathbb{Z}[\underline{x}]$-module with presentation

$$0 \to \mathbb{Z}[\underline{x}] \xrightarrow{[x_1,\dots,x_d]^{\mathrm{t}}} \mathbb{Z}[\underline{x}]^d \to U \to 0.$$

Let $u_1, \dots, u_d$ be the generators of U corresponding to the standard basis for $\mathbb{Z}[\underline{x}]^d$, so that $x_1u_1 + \cdots + x_du_d = 0$; that is, $\underline{x}$ and $\underline{u}$ satisfy (2.2.1). Then $\operatorname{ann}_{\mathbb{Z}[\underline{x}]} \tau(\underline{u}) \supseteq (\underline{x})$ by Proposition 2.2; we will see, in Theorem 2.5 below, that in fact $\operatorname{ann}_{\mathbb{Z}[\underline{x}]} \tau(\underline{u}) = (\underline{x})$.

Given any R-module M with a syzygy relation (2.2.1), consider the ring homomorphism $\mathbb{Z}[\underline{x}] \to R$ taking x_i to r_i, for each i, and extending the structure homomorphism $\mathbb{Z} \to R$. The hypothesis on M implies that there is a homomorphism of $\mathbb{Z}[\underline{x}]$-modules

$$f \colon U \to M \quad \text{with } f(u_i) = m_i \text{ for } i = 1, \dots, d.$$

Under the induced map $\otimes^d f\colon \otimes^d_{\mathbb{Z}[\underline{x}]} U \to \otimes^d_R M$, the element $\tau(\underline{u})$ maps to $\tau(\underline{m})$.

This remark prompts the discussion below, culminating in Theorem 2.5. First we review some notions regarding depth. For details, see [4, Chapter 1].

2.1. Depth

Let M be a finitely generated R-module, and let I be an ideal of R satisfying $IM \neq M$. The I-*depth* of M is the number

$$\operatorname{depth}_R(I, M) = \inf\{n \geq 0 \mid \operatorname{Ext}^n_R(R/I, M) \neq 0\}.$$

The I-depth of M is always finite and is equal to the length of every maximal M-regular sequence in I.

If $\underline{x} := x_1, \dots, x_d$ is a sequence of elements in R, and if K is the Koszul complex on $\underline{x}$, then the $(\underline{x})$-depth of M may be computed from its Koszul homology:

$$\operatorname{depth}_R\big((\underline{x}), M\big) = d - \sup\{i \geq 0 \mid \mathrm{H}_i(K \otimes_R M) \neq 0\}.$$

This is the *depth sensitivity* of the Koszul complex.

Suppose now that $\underline{x}$ is R-regular. Then K is a free resolution of $R/(\underline{x})$, and hence $\mathrm{H}_*(K \otimes_R M) \cong \operatorname{Tor}^R_*(R/(\underline{x}), M)$. In this case, we have

$$\operatorname{depth}_R\big((\underline{x}), M\big) = d - \sup\{i \geq 0 \mid \operatorname{Tor}^R_i\big(R/(\underline{x}), M\big) \neq 0\}. \tag{2.3.1}$$

If R is local with maximal ideal $\mathfrak{m}$, we write $\operatorname{depth}_R M$ for the $\mathfrak{m}$-depth of M and call it the *depth* of M.

2.2. A Koszul syzygy module

Let R be a Noetherian ring, and let $\underline{r} := r_1, \dots, r_d$ be a regular sequence in R with $(\underline{r}) \neq R$. Consider the complex

$$F := 0 \to R \xrightarrow{[r_1,\dots,r_d]^{\mathrm{t}}} R^d \to 0$$

concentrated in degrees 0 and 1. Set $M = \mathrm{H}_0(F)$; as r_1 is a nonzerodivisor, F is a free resolution of M.

LEMMA 2.4. *Let M, d, and F be as in Section 2.2. For each $n = 1, \dots, d$, the following statements hold:*

(1) *M and $\otimes_R^{n-1} M$ are* Tor *independent; and*

(2) *$\otimes_R^n F$ is a free resolution of $\otimes_R^n M$, and $\mathrm{pd}_R(\otimes_R^n M) = n$.*

Proof. The base case is $n = 1$, and then (1) and (2) are clear. Fix an integer n with $2 \leq n \leq d$, and assume that these statements hold for all integers $\leq n-1$. Set $I = (\underline{r})$. Since $\otimes_R^{n-1} F$ is a free resolution of $\otimes_R^{n-1} M$, we have

$$\mathrm{Tor}_*^R(R/I, \otimes_R^{n-1} M) = \mathrm{H}_*\big((R/I) \otimes_R (\otimes_R^{n-1} F)\big) \cong \big(\otimes_R^{n-1}\big((R/I) \otimes_R F\big)\big)_*,$$

where the last isomorphism holds because the complex in question has zero differential. In particular, $\mathrm{Tor}_{n-1}^R(R/I, \otimes_R^{n-1} M) \cong R/I \neq 0$, so that

$$\sup\{i \geq 0 \mid \mathrm{Tor}_i^R\big(R/(\underline{r}), \otimes_R^{n-1} M\big) \neq 0\} = n-1. \tag{2.4.1}$$

We can now complete the induction step.

(1) The induction hypothesis implies that $\otimes_R^{n-1} F$ is a free resolution of $\otimes_R^{n-1} M$, so (2.4.1) and (2.3.1) show that

$$\mathrm{depth}_R(I, \otimes_R^{n-1} M) = d - (n-1) \geq 1. \tag{2.4.2}$$

Moreover, $\mathrm{Tor}_*^R(M, \otimes_R^{n-1} M)$ is the homology of the complex

$$F \otimes_R (\otimes_R^{n-1} M) : 0 \to \otimes_R^{n-1} M \xrightarrow{[\underline{r}]^{\mathrm{t}}} (\otimes_R^{n-1} M)^d \to 0$$

(concentrated in degrees 0 and 1). By (2.4.2), some r_i is a nonzerodivisor on $\otimes_R^{n-1} M$, and it follows that M and $\otimes_R^{n-1} M$ are Tor independent.

(2) By hypothesis, F and $\otimes_R^{n-1} F$ are free resolutions of M and $\otimes_R^{n-1} M$, respectively. We have already proved, in (1), that these modules are Tor independent, so the complex $F \otimes_R (\otimes_R^{n-1} F)$, that is, $\otimes_R^n F$, is a free resolution of $\otimes_R^n M$. In particular, $\mathrm{pd}_R(\otimes_R^n M) \leq n$; that the equality holds follows from (2.4.1). ∎

2.3. Torsion submodule

Let $\mathrm{Q}(R)$ be the total quotient ring of R. The *torsion submodule* $\top_R M$ of M is the kernel of the natural homomorphism $M \to \mathrm{Q}(R) \otimes_R M$. The inclusion $\top_R M \subseteq M$ gives rise to an exact sequence

$$0 \to \top_R M \to M \to \bot_R M \to 0. \tag{2.4.3}$$

The module M is *torsion* if $\top_R M = M$ (i.e., $M_\mathfrak{p} = 0$ for each $\mathfrak{p} \in \mathrm{Ass}(R)$), and M is *torsion-free* if $\top_R M = 0$. Thus, M is torsion-free if and only if $\bigcup \mathrm{Ass}\, M \subseteq \bigcup \mathrm{Ass}\, R$. The stronger condition, that $\mathrm{Ass}\, M \subseteq \mathrm{Ass}\, R$, is therefore a sufficient condition for M to be torsion-free. We will invoke this criterion twice in the proof of the next theorem.

Part (1) of the next result is reminiscent of Auslander's discussion in [1, p. 638] (see also [7, Proposition 3.1]).

THEOREM 2.5. *Let M and $\underline{r}$ be as in Section 2.2. The following statements hold:*

(1) *$\otimes_R^n M$ is torsion-free if and only if $n \le d-1$;*
(2) *the element $\tau(\underline{m})$ in $\otimes_R^d M$ satisfies $\mathrm{ann}_R\, \tau(\underline{m}) = (\underline{r})$; and*
(3) *the map $R/(\underline{r}) \to \otimes_R^d M$ of R-modules with $1 \mapsto \tau(\underline{m})$ induces a splitting*

$$\otimes_R^d M \cong \big(R/(\underline{r})\big) \oplus W,$$

where W is torsion-free; in particular, we have

$$\mathrm{Hom}_R\big(R/(\underline{r}), \otimes_R^d M\big) = R\tau(\underline{m}) \neq 0.$$

Proof. Set $I = (\underline{r})$, let $n \le d-1$, and fix a prime $\mathfrak{p} \in \mathrm{Ass}(\otimes_R^n M)$. If $I \subseteq \mathfrak{p}$, it follows from Lemma 2.4 that $(\otimes_R^n F)_\mathfrak{p}$ is a minimal free resolution of $(\otimes_R^n M)_\mathfrak{p}$; therefore,

$$\mathrm{depth}_{R_\mathfrak{p}} (\otimes_R^n M)_\mathfrak{p} = \mathrm{depth}\, R_\mathfrak{p} - n \ge d - n \ge 1,$$

which is a contradiction. Thus, $I \not\subseteq \mathfrak{p}$, and then the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is a nonzero free module; hence, so is $(\otimes_R^n M)_\mathfrak{p}$. Therefore, $\mathrm{depth}\, R_\mathfrak{p} = \mathrm{depth}_{R_\mathfrak{p}} (\otimes_R^n M)_\mathfrak{p}$. We have shown that $\mathrm{Ass}(\otimes_R^n M) \subseteq \mathrm{Ass}\, R$, and hence that $\otimes_R^n M$ is torsion-free. The "only if" direction of (1) will follow from (3).

For (2) and (3), by construction $r_1 m_1 + \cdots + r_d m_d = 0$, so Proposition 2.2 gives an inclusion $I \subseteq \mathrm{ann}_R\, \tau(\underline{m})$. The reverse inclusion will follow, once we ascertain that the map in (3) splits. Consider the homomorphisms of R-modules

$$\begin{aligned}\otimes_R^d(F_0) \twoheadrightarrow \otimes_R^d M \twoheadrightarrow (\otimes_R^d M)\otimes_R R/I &\cong \mathrm{H}_0\big((\otimes_R^d F)\otimes_R R/I\big)\\ &= \otimes_R^d(F_0\otimes_R R/I),\end{aligned}$$

where the surjections are the natural ones; the isomorphism holds because $\otimes_R^d F$ is a free resolution of $\otimes_R^d M$, and the equality holds because the differential on F has its image in IF. Let $\underline{e}=e_1,\dots,e_d$ be the standard basis for $F_0=R^d$, in Section 2.2, and let $\underline{e}'$ be the induced basis of the free R/I-module $F_0\otimes_R R/I$. Under the composite map, the element $\tau(\underline{e})$ maps to $\tau(\underline{e}')$, and $\{\tau(\underline{e}')\}$ extends to a basis of the R/I-module $\otimes_R^d(F_0\otimes_R(R/I))$. Since $\tau(\underline{e})$ maps to $\tau(\underline{m})$ in $\otimes_R^d M$, the map in (2) splits and gives a decomposition

$$\otimes_R^d M\cong (R/I)\oplus W.$$

It remains to verify that W is torsion-free; given the decomposition above, the other parts of (3) are a consequence of this fact.

For $\mathfrak{p}\in\operatorname{Spec}R$ with $I\nsubseteq\mathfrak{p}$, the $R_\mathfrak{p}$-module is $M_\mathfrak{p}$ free, and hence so is $W_\mathfrak{p}$. Assume now that $I\subseteq\mathfrak{p}$. The Koszul complex on $\underline{r}$, viewed as elements in $R_\mathfrak{p}$, is a minimal resolution of $(R/I)_\mathfrak{p}$, and so it is a direct summand of $(\otimes_R^n F)_\mathfrak{p}$, the minimal free resolution of $(\otimes_R^n M)_\mathfrak{p}$. The ranks of the free modules in the top degree, d, of these complexes coincide (and equal 1), whence $\operatorname{pd}_{R_\mathfrak{p}}W_\mathfrak{p}\le d-1$ and

$$\operatorname{depth}_{R_\mathfrak{p}}W_\mathfrak{p}=\operatorname{depth}R_\mathfrak{p}-\operatorname{pd}_{R_\mathfrak{p}}W_\mathfrak{p}\ge 1.$$

These observations show that $\operatorname{Ass}W\subseteq\operatorname{Ass}R$, so W is torsion-free as claimed. ∎

2.4. Local rings

Next we focus on local rings, where the preceding results can be strengthened to some extent.

LEMMA 2.6. *Let M be a finitely generated module over a local ring $(R,\mathfrak{m})$, and let $m_1,\dots,m_d\in M$. If the images of $\{m_1,\dots,m_d\}$ in $M/\mathfrak{m}M$ are linearly independent, then $\tau(\underline{m})$ is not in $\mathfrak{m}(\otimes_R^d M)$.*

Proof. Let m_i' be the image of m_i in the k-vector space $M/\mathfrak{m}M$. Since $\{m_1',\dots,m_d'\}$ is linearly independent, $\tau(\underline{m}')\neq 0$. Hence, $\tau(\underline{m})\notin\mathfrak{m}(\otimes_R^d M)$. ∎

Given an R-module M, we write $I(M)$ for the ideal (r_{ij}) defined by the entries in a matrix in some minimal presentation

$$R^{\mu} \xrightarrow{[r_{ij}]} R^{\nu} \to M \to 0 \quad \text{where } \nu = \nu_R(M).$$

This ideal is independent of the presentation. Moreover, $I(M)$ contains a nonzerodivisor if and only if, over $\mathrm{Q}(R)$, the total quotient ring of R, the module $\mathrm{Q}(R) \otimes_R M$ can be generated by fewer than ν elements. To see this we note that, since $\mathrm{Q}(R)$ is semilocal, the module $\mathrm{Q}(R) \otimes_R M$ needs ν generators if and only if $\nu_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \nu$ for some $\mathfrak{p} \in \operatorname{Ass} R$; moreover, $\nu_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \nu$ if and only if the presentation remains minimal when localized at $\mathfrak{p}$, that is, if and only if $I(M) \subseteq \mathfrak{p}$. Thus, $\mathrm{Q}(R) \otimes_R M$ needs ν generators if and only if $I(M) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} R$, that is to say, if and only if $I(M)$ consists of zerodivisors.

Recall that M is said to have *rank* r if $\mathrm{Q}(R) \otimes_R M$ is free over $\mathrm{Q}(R)$ of rank r (see [4, Proposition 1.4.3] for different characterizations of this property).

Theorem 2.7. *Let R be a local ring, and let M be a nonzero finitely generated R-module satisfying one of the following conditions:*

(1) *$I(M)$ contains a nonzerodivisor; in this case, set $b = \nu_R(M)$; or*

(2) *M has rank; in this case, set $b = \operatorname{rank}_R(M) + 1$.*

If M is not free, then for each nonzero finitely generated R-module N one has

$$\top_R\big((\otimes_R^n M) \otimes_R N\big) \neq 0 \quad \textit{for each } n \geq b.$$

Proof. It suffices to prove the statement for $n = b$, since

$$(\otimes_R^n M) \otimes_R N \cong (\otimes_R^b M) \otimes_R \big((\otimes_R^{n-b} M) \otimes_R N\big),$$

and $N \neq 0$ implies that $(\otimes_R^i M) \otimes_R N \neq 0$ for each $i \geq 0$, by Nakayama's lemma.

(1) Let $m_1, \ldots, m_b$ be a minimal generating set for the R-module M. The element $\tau(\underline{m})$ in $\otimes_R^b M$ is annihilated by $I(M)$, by Proposition 2.2, and is not in $\mathfrak{m}(\otimes_R^b M)$, by Lemma 2.6. It follows that, for each x in $N \setminus \mathfrak{m}N$, the element $\tau(\underline{m}) \otimes x$ in $(\otimes_R^b M) \otimes_R N$ is nonzero and is annihilated by $I(M)$ and hence is in the torsion submodule; this is where the hypothesis that $I(M)$ contains a nonzerodivisor is used.

(2) We claim that there exists a syzygy relation (2.2.1) with $\underline{m}$ a minimal generating set for M, $(\underline{r}) \subseteq \mathfrak{m}$, and some r_i a nonzerodivisor.

Indeed, $\nu_R(M) \geq b$ since M is not free. Choose elements $m_1, \dots, m_b$ that form part of a minimal generating set for M and such that $m_1, \dots, m_{b-1}$ form a basis for $\mathrm{Q}(R) \otimes_R M$ over $\mathrm{Q}(R)$. Then there is a syzygy relation as in (2.2.1) in which r_b is a nonzerodivisor.

The element $\tau(\underline{m})$ in $\otimes_R^b M$ is annihilated by $(\underline{r})$, by Proposition 2.2, and is not in $\mathfrak{m}(\otimes_R^b M)$, by Lemma 2.6. Since $(\underline{r})$ has a nonzerodivisor, it follows as in (1) that the torsion submodule of $(\otimes_R^b M) \otimes_R N$ is nonzero. □

We learned recently that in 2011, in response to a query on MathOverflow, David Speyer gave a proof in [13] of (1) that is quite similar to ours when R is a domain.

One cannot always expect torsion in tensor powers of nonfree modules, as the following shows.

Example 2.8. Let $R = k[[x,y]]/(xy)$, where k is a field. The torsion-free R-module $M := R/(x)$ is not free; however, $\otimes_R^n M$ is isomorphic to $R/(x)$ for every $n \geq 1$ and hence is torsion-free.

The preceding results bring to the fore the following question:

Question 2.9. Let R be a local domain. Is there an integer b, depending only on R, such that $\otimes_R^n M$ has torsion for every finitely generated nonfree R-module M and every integer $n \geq b$?

The condition that R be a domain is to avoid the situation of Example 2.8. When R is regular, one can take $b = \dim R$, by results of Auslander [1, Theorem 3.2] and Lichtenbaum [10, Corollary 3].

§3. Torsion "carriers"

Some modules, even though they are torsion-free, usually generate torsion in tensor products. For example, over a local ring $(R, \mathfrak{m}, k)$ of positive depth, the maximal ideal $\mathfrak{m}$ is such a module: for any finitely generated nonfree R-module M, the tensor product $\mathfrak{m} \otimes_R M$ has torsion. To see this, observe that the short exact sequence

$$0 \to \mathfrak{m} \to R \to k \to 0$$

yields an injection from the torsion module $\mathrm{Tor}_1^R(k, M)$ into $\mathfrak{m} \otimes_R M$; moreover, $\mathrm{Tor}_1^R(k, M) \neq 0$ because M is not free.

We give two more examples of torsion carriers: the integral closure $\overline{R}$ of a 1-dimensional analytically unramified ring R, and the Frobenius powers

${}^{\varphi^e}R$ of a complete intersection R of characteristic p. Recall that a local ring is *analytically unramified* provided that its completion is reduced. If R is 1-dimensional, an equivalent condition is that R be Cohen–Macaulay with finitely generated integral closure $\overline{R}$ (see [9, Theorem 4.6]).

THEOREM 3.1. *Let R be a 1-dimensional analytically unramified local ring, and let $\overline{R}$ be the integral closure of R in its total quotient ring. If M is a finitely generated R-module for which $\overline{R} \otimes_R M$ is torsion-free, then M is free.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of R, and for each i let r_i be the dimension of the $R_{\mathfrak{p}_i}$-vector space $M_{\mathfrak{p}_i}$. Put $n = \nu_R M$, the minimal number of generators of the R-module M, and choose an exact sequence

$$0 \to K \to R^{(n)} \to M \to 0.$$

If we can show that $r_i = n$ for each i, we will know that K is torsion and hence zero, and we will be done.

Put $D_i = \overline{R/\mathfrak{p}_i}$, the integral closure of the domain $R/\mathfrak{p}_i$. Since R is reduced, we have inclusions

$$R \hookrightarrow \prod_{i=1}^{s} R/\mathfrak{p}_i \hookrightarrow \prod_{i=1}^{s} D_i \hookrightarrow \prod_{i=1}^{s} \mathrm{Q}(R/\mathfrak{p}_i) = \mathrm{Q}(R).$$

We see that $\overline{R} = \prod_{i=1}^{s} D_i$; moreover, each D_i is a semilocal Dedekind domain and therefore a principal ideal domain. Since $\overline{R} \otimes_R M$ is torsion-free, it is projective, in fact free of rank r_i on the component D_i. Therefore, setting $e_i = \nu_R D_i$, we have the equations

$$r_1 e_1 + \cdots + r_s e_s = \nu_R(\overline{R} \otimes_R M) = (\nu_R \overline{R}) \cdot (\nu_R M) = (e_1 + \cdots + e_s)n.$$

Since $r_i \leq n$ for each i, it follows from these equations that $r_i = n$ for each i. ∎

Let R be a Noetherian ring of positive characteristic p, and let $\varphi\colon R \to R$ be the Frobenius endomorphism $r \mapsto r^p$. Given an R-module M and a positive integer e, we write ${}^{\varphi^e}M$ for the R-module obtained from M by restriction of scalars along φ^e; thus, $r \cdot m = r^{p^e} m$ for $r \in R$ and $m \in M$. Observe that M is torsion-free if and only if ${}^{\varphi^e}M$ is torsion-free for some (equivalently, all) $e \geq 1$. Following [12], we write $F^e(M)$ for the tensor product $M \otimes_R {}^{\varphi^e}R$. One views $F^e(M)$ as a right R-module: the action of R on $F^e(M)$

comes from the right (ordinary) action of R on ${}^{\varphi^e}\!R$. Thus, $F^e(R) \cong R$ as R-modules, and it follows that $F^e(M)$ is finitely generated if M is finitely generated.

THEOREM 3.2 ([12, Corollaire 1.10]). *Let R be a local ring of characteristic p, and let M be a finitely generated R-module. If M has finite projective dimension, then* $\operatorname{Tor}_i^R(M, {}^{\varphi^e}\!R) = 0$ *for all $e \geq 1$ and all $i \geq 1$.*

The converse of Theorem 3.2 is true and was proved by Herzog [5, Theorem 3.1]. For complete intersections, the following strong converse was proved by Avramov and Miller.

THEOREM 3.3 ([3, Main Theorem]). *Let $(R, \mathfrak{m})$ be a complete intersection of characteristic p, and let M be a finitely generated R-module. If* $\operatorname{Tor}_i^R(M, {}^{\varphi^e}\!R) = 0$ *for some $e \geq 1$ and some $i \geq 1$, then M has finite projective dimension.*

The proof that (1) $\Longrightarrow$ (2) in the next theorem follows many of the same steps Auslander used in [1, proof of Lemma 3.1]. The main differences are that we have to allow for the possibility that ${}^{\varphi^e}\!R$ is not finitely generated and that we appeal to Theorems 3.2 and 3.3 for a replacement of rigidity of Tor over regular local rings. Recall that a module M is generically free provided that $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for each $\mathfrak{p} \in \operatorname{Ass} R$.

THEOREM 3.4. *Let $(R, \mathfrak{m})$ be a complete intersection of characteristic p, and let M be a finitely generated, generically free R-module. Fix a positive integer e. The following conditions are equivalent:*

(1) *$F^e(M)$ is torsion-free, and*
(2) *M is torsion-free and of finite projective dimension.*

Proof. Suppose that (1) holds, and apply $- \otimes_R {}^{\varphi^e}\!R$ to the short exact sequence (2.4.3), getting an exact sequence

$$F^e(\top_R M) \xrightarrow{\alpha} F^e(M) \xrightarrow{\beta} F^e(\perp_R M) \to 0.$$

Since $F^e(\top_R M)$ is torsion and $F^e(M)$ is torsion-free, we see that $\alpha = 0$, whence β is an isomorphism. In particular, $F^e(\perp_R M)$ is torsion-free. Next, consider the universal pushforward (see [6, Section 1]):

$$0 \to \perp_R M \to R^{(m)} \to N \to 0. \tag{3.4.1}$$

Applying $-\otimes_R {}^{\varphi^e}R$ to this sequence, we obtain an injection

$$\operatorname{Tor}_1^R(N, {}^{\varphi^e}R) \hookrightarrow F^e(\perp_R M).$$

Now $\perp_R M$ is clearly generically free, and from the construction of the universal pushforward in [6, Section 1], one checks that N is generically free as well. It follows that $\operatorname{Tor}_1^R(N, {}^{\varphi^e}R)$ is torsion. Since $F^e(\perp_M)$ is torsion-free, we have $\operatorname{Tor}_1^R(N, {}^{\varphi^e}R) = 0$. Now we invoke Theorems 3.2 and 3.3 to see that

$$\operatorname{Tor}_i^R(N, {}^{\varphi^e}R) = 0 \quad \text{for all } i \geq 1$$

and, moreover, that N has finite projective dimension. From (3.4.1) it follows that $\operatorname{Tor}_i^R(\perp_R M, {}^{\varphi^e}R) = 0$ for all $i \geq 1$ and that $\perp_R M$ has finite projective dimension. Therefore, we will have (2) once we show that $\top_R M = 0$. For this, we apply $-\otimes_R {}^{\varphi^e}R$ once again to (2.4.3), to get an injection

$$F^e(\top_R M) \hookrightarrow F^e(M).$$

Since $F^e(\top_R M)$ is torsion and $F^e(M)$ is torsion-free, we have $F^e(\top_R M) = 0$. If $\top_R M$ were nonzero, there would be a surjection $\top_R M \twoheadrightarrow R/\mathfrak{m}$. But then $F^e(R/\mathfrak{m}) = 0$, that is, $\mathfrak{m}{}^{\varphi^e}R = {}^{\varphi^e}R$, an obvious contradiction, since $\mathfrak{m}{}^{\varphi^e}R \subseteq \mathfrak{m}$. Thus $\top_R M = 0$, and the proof that (1) $\implies$ (2) is complete.

Now assume that (2) holds. Since M is torsion-free, we can build the universal pushforward (see [6, Section 1]):

$$0 \to M \to R^{(\nu)} \to N \to 0,$$

where $\nu = \nu_R M^*$. Then N has finite projective dimension. Now Theorem 3.2 implies that $\operatorname{Tor}_i^R(N, {}^{\varphi^e}R) = 0$ for all $i \geq 1$. Therefore, $\operatorname{Tor}_1^R(M, {}^{\varphi^e}R) = 0$, and we get an injection $F^e(M) \hookrightarrow ({}^{\varphi^e}R)^{(\nu)}$, whence $F^e(M)$ is torsion-free. ∎

From Theorem 3.2 (alternatively, from the proof of Theorem 3.4), we get Tor independence (item (2) in the Introduction), as follows.

Corollary 3.5. *If R and M satisfy the equivalent conditions of Theorem 3.4, then* $\operatorname{Tor}_i^R(M, {}^{\varphi^{e'}}R) = 0$ *for every* $i \geq 1$ *and every* $e' \geq 1$.

Of course, if M is torsion-free, the converse of Corollary 3.5 holds, by Theorem 3.3. In fact, it suffices to check that $\operatorname{Tor}_i^R(M, {}^{\varphi^{e'}}R) = 0$ for a single e' and a single i.

Recall that R is F-finite provided that φ is a finite map, that is, that R is module-finite over $\varphi(R)$. In this case, φ^e is a finite map for each $e \geq 1$. Note that the action of R on the module $({}^{\varphi^e}M)$ in items (1) and (2) below is the Frobenius action $m \cdot r = mr^{p^e}$.

COROLLARY 3.6. *Assume that* $(R, \mathfrak{m})$ *is a reduced local ring, is* F*-finite, and is a complete intersection. The following conditions are equivalent:*

(1) $F^e({}^{\varphi^{e'}}M)$ *is torsion-free for every torsion-free* R*-module* M *and every pair* e, e' *of positive integers;*
(2) $F^e({}^{\varphi^{e'}}M)$ *is torsion-free for some nonzero finitely generated* R*-module* M *and some pair* e, e' *of positive integers; and*
(3) R *is regular.*

Proof. Obviously (1) $\Longrightarrow$ (2), and the implication (3) $\Longrightarrow$ (1) holds by Kunz's theorem [8, Theorem 2.1] that the R-module ${}^{\varphi^e}R$ is flat when R is regular.

To prove that (2) $\Longrightarrow$ (3), we note that ${}^{\varphi^{e'}}M$ is a finitely generated R-module, by F-finiteness. Also, ${}^{\varphi^{e'}}M$ is generically free because R is reduced. By Theorem 3.4, ${}^{\varphi^{e'}}M$ has finite projective dimension, and now [2, Theorem 1.1] implies that R is regular. ∎

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