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SPHERICAL FUNCTORS ON THE KUMMER SURFACE

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Abstract. We find two natural spherical functors associated to the Kummer surface and analyze how their induced twists fit with Bridgeland's conjecture on the derived autoequivalence group of a complex algebraic K3 surface.

§1. Introduction

Let $\mathcal{D}(X)$ be the bounded derived category of coherent sheaves on a smooth complex projective variety X, and let $\operatorname{Aut}(\mathcal{D}(X))$ denote the set of isomorphism classes of exact $\mathbb{C}$-linear autoequivalences of $\mathcal{D}(X)$. Then we always have a subgroup $\operatorname{Aut}_{\mathrm{st}}(\mathcal{D}(X)) \subset \operatorname{Aut}(\mathcal{D}(X))$ of *standard* autoequivalences which is generated by pushforwards along automorphisms, twists by line bundles, and shifts. The complement of this subgroup, if nonempty, is usually very interesting and mysterious; its elements will be called *nonstandard* autoequivalences.

The most successful way to construct nonstandard autoequivalences was discovered in the groundbreaking work of Seidel and Thomas [14] on spherical objects. This was extended by Huybrechts and Thomas [8] to a notion of $\mathbb{P}$-objects and further still to a theory of spherical and $\mathbb{P}$-functors (see [13], [3], [1]).

The first example of a series of $\mathbb{P}$-functors was constructed by Addington [1, Theorem 2] for the Hilbert scheme $X^{[n]}$ of n points on a K3 surface X. In particular, he showed that the natural functor $F : \mathcal{D}(X) \to \mathcal{D}(X^{[n]})$ induced by the universal ideal sheaf on $X \times X^{[n]}$ is a $\mathbb{P}^{n-1}$-functor in the sense of [1, Section 3] and thus gives rise to a nonstandard autoequivalence of $\mathcal{D}(X^{[n]})$ for each $n \geq 2$. Notice that when $n = 1$ this F is Mukai's reflection functor (see [10, p. 362]), which coincides (up to a shift) with the spherical twist around the structure sheaf $\mathcal{O}_X$.

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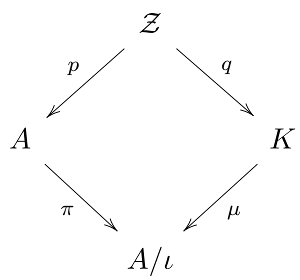
Inspired by this example, the second author [9, Theorem 4.1] provided the analogous result for the generalized Kummer variety $K_n \subset A^{[n+1]}$ associated to an abelian surface A. More precisely, he proved that the natural Fourier–Mukai functor $F_K : \mathcal{D}(A) \to \mathcal{D}(K_n)$ induced by the universal ideal sheaf on $A \times K_n$ is again a $\mathbb{P}^{n-1}$-functor yielding a new nonstandard autoequivalence of $\mathcal{D}(K_n)$ for each $n \geq 2$.

This short note completes this theorem to the case $n = 1$ where the generalized Kummer variety is the classical Kummer surface. The motivation to understand this particular case comes from Bridgeland's conjecture in [5, Conjecture 1.2] on the derived autoequivalence group of a complex algebraic K3 surface; roughly speaking, it says that $\operatorname{Aut}(\mathcal{D}(X))$ should be generated by standard autoequivalences and twists around spherical objects.

Summary of main results

Every abelian surface A has a natural K3 surface associated to it, namely, the *Kummer surface* $K := K_1$. It can either be defined as the blowup of the quotient A/ι along the sixteen ordinary double points, where ι denotes the involution $a \mapsto -a$ or, equivalently, as the fiber of the Albanese map $m : A^{[2]} \to A$ over zero. That is, we can identify K with the subvariety of the Hilbert scheme $A^{[2]}$ consisting of those points representing length 2 subschemes of A whose weighted support sums to zero. In other words, there is a universal family $\mathcal{Z} \subset A \times K$ giving rise to the commutative diagram



Recall that a Fourier–Mukai functor $F : \mathcal{D}(Y) \to \mathcal{D}(X)$ with left adjoint L and right adjoint R is said to be *spherical* if the cotwist $C_F := \operatorname{cone}(\mathrm{id} \xrightarrow{\eta} RF)$ is an autoequivalence of $\mathcal{D}(Y)$ and we have a functorial isomorphism $R \simeq CL$. In particular, if F is spherical, then the *twist* $T_F := \operatorname{cone}(FR \xrightarrow{\epsilon} \mathrm{id})$ is an autoequivalence of $\mathcal{D}(X)$. A spherical object $\mathcal{E} \in \mathcal{D}(X)$ corresponds to the case $F := (\_) \otimes \mathcal{E} : \mathcal{D}(\mathrm{pt}) \to \mathcal{D}(X)$.

In this article, we focus on the exact triangle $F \to F' \to F''$ of Fourier–Mukai functors $\Phi_{\mathcal{E}} : \mathcal{D}(A) \to \mathcal{D}(K)$ induced by the structure sequence of $\mathcal{Z}$:

$$F := \Phi_{\mathcal{I}_{\mathcal{Z}}}, \qquad F' := \Phi_{\mathcal{O}_{A\times K}} = \mathrm{H}^*(\_) \otimes \mathcal{O}_K, \qquad F'' := \Phi_{\mathcal{O}_{\mathcal{Z}}} = q_* p^*.$$

Our main result is the following.

Theorem 1 (Proposition 2.1 and Corollary 2.4). *Both F and F'' are spherical functors with cotwists $C_F \simeq C_{F''} \simeq \iota^*$.*

In light of [5, Conjecture 1.2], this immediately raises the question of whether the twists $T_F, T_{F''} \in \mathrm{Aut}(\mathcal{D}(K))$ associated to these functors F, F'' can be decomposed into twists $T_{\mathcal{E}}$ around spherical objects $\mathcal{E} \in \mathcal{D}(K)$. We answer this question with the following.

Theorem 2 (Proposition 2.1 and Corollary 2.4). *The induced twists T_F, $T_{F''} \in \mathrm{Aut}(\mathcal{D}(K))$ decompose in the following way:*

$$T_{F''} \simeq \prod_i T^{-1}_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)}[1] \simeq \prod_i T_{\mathcal{O}_{E_i}} \circ M_{\mathcal{O}_K(-E/2)}[1]$$

and

$$F[1] \simeq T_{\mathcal{O}_K} \circ F'' \quad \Longrightarrow \quad T_F \simeq T_{\mathcal{O}_K} \circ T_{F''} \circ T^{-1}_{\mathcal{O}_K},$$

where $E = \bigcup_i E_i$ for the exceptional curves E_i of the Hilbert–Chow morphism μ and $M_{\mathcal{O}_K(E/2)} := (\_) \otimes \mathcal{O}_K(E/2)$.

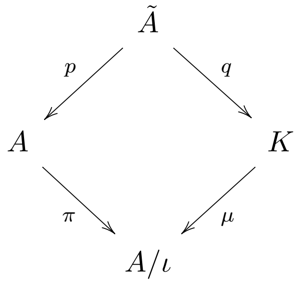
It is easy to see that the squares $T_F^2, T_{F''}^2$ of our twists act trivially on the cohomology of K (see [1, Section 1.4]). In fact, Corollary 2.5 shows that $T_F^2 \simeq T_{F''}^2 \simeq [2]$.

In this paper we give a different proof of Corollary 2.4 from that which could have been obtained from adapting the arguments in [9]. The advantage of our approach is that it immediately provides us with the decompositions of T_F and $T_{F''}$ as stated above.

§2. Natural functors on the Kummer surface

Another way of describing K is by first blowing up the fixed points $\tilde{A} \to A$. Since the fixed points are ι-invariant, the involution ι lifts to an involution

$\tilde{\iota}$ of $\tilde{A}$:



The quotient $\tilde{A} \to K$ is a double cover ramified over 16 exceptional curves E_i. Moreover, the canonical bundle formula for the blowup yields $\omega_{\tilde{A}} \simeq \mathcal{O}(\sum \tilde{E}_i)$, where the $\tilde{E}_i$ are the exceptional divisors in $\tilde{A}$. Their images E_i in K satisfy $q^*\mathcal{O}(E_i) \simeq \mathcal{O}(2\tilde{E}_i)$ and $q_*\mathcal{O}_{\tilde{A}} \simeq \mathcal{O}_K \oplus \mathcal{O}(-\frac{1}{2}\sum E_i)$. (See [7, Chapter 1.1] for more details.) We set $E := \bigcup_i E_i$ and $\tilde{E} := \bigcup_i \tilde{E}_i$ from now on.

PROPOSITION 2.1. *We have that* $F'' : \mathcal{D}(A) \to \mathcal{D}(K)$ *is a spherical functor with cotwist* $C_{F''} \simeq \iota^*$ *and twist*

$$T_{F''} \simeq \prod_i T^{-1}_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)}[1].$$

Proof. The pushforward along the double cover $q_* : \mathcal{D}(\tilde{A}) \to \mathcal{D}(K)$ is a spherical functor with cotwist $C_{q_*} \simeq M_{\mathcal{O}_{\tilde{A}}(\tilde{E})} \circ \tilde{\iota}^* \simeq S_{\tilde{A}} \circ \tilde{\iota}^*[-2]$ and twist $T_{q_*} \simeq M_{\mathcal{O}_K(E/2)}[1]$ (see [1, Section 1.2, Examples 5 and 6]).

By [11, Theorem 4.3], we have a semiorthogonal decomposition

$$\mathcal{D}(\tilde{A}) \simeq \big\langle \mathcal{O}_{\tilde{E}_1}(-1), \dots, \mathcal{O}_{\tilde{E}_{16}}(-1), p^*\mathcal{D}(A) \big\rangle.$$

We set $\mathcal{A} := \langle \mathcal{O}_{\tilde{E}_1}(-1), \dots, \mathcal{O}_{\tilde{E}_{16}}(-1) \rangle$ and $\mathcal{B} := p^*\mathcal{D}(A)$ so that $\mathcal{D}(\tilde{A}) \simeq \langle \mathcal{A}, \mathcal{B} \rangle$. Since $\mathcal{D}(\tilde{A}) \simeq \langle S_{\tilde{A}}\mathcal{B}, \mathcal{A} \rangle$ by [4, Proposition 3.6] and $C_{q_*}\mathcal{B} \simeq S_{\tilde{A}}\mathcal{B}$, we have $\mathcal{D}(\tilde{A}) \simeq \langle C_{q_*}\mathcal{B}, \mathcal{A} \rangle$. Thus, by [6, Theorem 4.13], the restrictions $q_*|_{\mathcal{A}} : \mathcal{D}(A[2]) \to \mathcal{D}(K)$ (to the set $A[2] \subset A$ of 2-torsion points) and $q_*|_{\mathcal{B}} \simeq q_*p^* =: F'' : \mathcal{D}(A) \to \mathcal{D}(K)$ are spherical functors with $T_{q_*} \simeq T_{q_*|_{\mathcal{A}}} \circ T_{q_*|_{\mathcal{B}}}$. Since $q_*\mathcal{O}_{\tilde{E}_i}(-1) \simeq \mathcal{O}_{E_i}(-1)$, we see that $T_{q_*|_{\mathcal{A}}} \simeq \prod_i T_{\mathcal{O}_{E_i}(-1)}$; hence,

$$T_{F''} \simeq T^{-1}_{q_*|_{\mathcal{A}}} \circ T_{q_*} \simeq \prod_i T^{-1}_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)}[1].$$

Notice that the cotwist of $F'' \simeq q_*|_{\mathcal{B}}$ is given by $S_A \circ \iota^*[-2] \simeq \iota^*$. ∎

REMARK 2.2. We can use (1) below to rewrite this decomposition as

$$T_{F''} \simeq \prod_i T_{\mathcal{O}_{E_i}} \circ M_{\mathcal{O}_K(-E/2)}[1].$$

LEMMA 2.3. *We have the following isomorphism of functors*

$$F[1] \simeq T_{\mathcal{O}_K} \circ F''.$$

Proof. Consider the following exact triangles of functors:

$$\mathrm{Hom}^*(\mathcal{O}_K, F'') \otimes \mathcal{O}_K \to F'' \to T_{\mathcal{O}_K} \circ F'' \qquad \text{and} \qquad F' \to F'' \to F[1].$$

Then it is enough to show that $\mathrm{Hom}^*(\mathcal{O}_K, F'') \otimes \mathcal{O}_K \simeq F' \simeq \mathrm{H}^*(A, \_) \otimes \mathcal{O}_K$. In other words, it is sufficient to show that $\mathrm{H}^*(K, F''(\_)) \simeq \mathrm{H}^*(A, \_)$, but this follows from the fact that p is a blowup. Indeed, we have

$$\begin{aligned} \mathrm{H}^*\big(K, F''(\_)\big) &\simeq \mathrm{H}^*\big(K, q_*p^*(\_)\big) \simeq \mathrm{H}^*\big(\tilde{A}, p^*(\_)\big) \\ &\simeq \mathrm{H}^*\big(A, p_*p^*(\_)\big) \simeq \mathrm{H}^*(A, \_). \end{aligned}$$

∎

COROLLARY 2.4. *We have that $F : \mathcal{D}(A) \to \mathcal{D}(K)$ is a spherical functor with cotwist $C_F \simeq \iota^*$ and twist*

$$T_F \simeq T_{\mathcal{O}_K} \circ T_{F''} \circ T_{\mathcal{O}_K}^{-1}.$$

Proof. Recall that if $F : \mathcal{D}(Z) \to \mathcal{D}(Y)$ is a spherical functor and $\Phi : \mathcal{D}(Y) \xrightarrow{\sim} \mathcal{D}(X)$ is an equivalence of categories, then $\Phi \circ F : \mathcal{D}(Z) \to \mathcal{D}(X)$ is also a spherical functor with the same cotwist and $T_{\Phi \circ F} \simeq \Phi \circ T_F \circ \Phi^{-1}$. In particular, we see immediately from Lemma 2.3 that F is a spherical functor with cotwist $C_F \simeq \iota^*$ and twist

$$T_F \simeq T_{F[1]} \simeq T_{\mathcal{O}_K} \circ T_{F''} \circ T_{\mathcal{O}_K}^{-1}.$$

∎

COROLLARY 2.5. *The squares of the spherical twists are given by*

$$T_F^2 \simeq T_{F''}^2 \simeq [2].$$

In particular, $T_F^2, T_{F''}^2$ act trivially on cohomology.

Proof. Let $j : E \to K$ denote the inclusion of the exceptional divisor. Since E is smooth, we can apply [1, Section 1.2, Example 5] to see that $j_* : \mathcal{D}(E) \to \mathcal{D}(K)$ is spherical with cotwist $C_{j_*} \simeq M_{\mathcal{O}_E(E)}[-1] \simeq S_E[-2]$ and twist $T_{j_*} \simeq M_{\mathcal{O}_K(E)}$.

Set $\mathcal{A}_1 := \langle \mathcal{O}_{E_1}(-1), \dots, \mathcal{O}_{E_{16}}(-1)\rangle$ and $\mathcal{A}_2 := \mathcal{A}_1 \otimes \mathcal{O}_E(1)$ to be subcategories of $\mathcal{D}(E)$. Then, by [11, Theorem 2.6], we have a semiorthogonal decomposition

$$\mathcal{D}(E) \simeq \langle \mathcal{A}_1, \mathcal{A}_2\rangle.$$

Thus, using Kuznetsov's trick in [2, Theorem 11] (which is a special case of [6, Theorem 4.13]), we see that the restriction $j_\ell := j_*|_{\mathcal{A}_\ell} : \mathcal{D}(A[2]) \to \mathcal{D}(K)$ is spherical for each $\ell = 1, 2$, and the twists satisfy $T_{j_1} \circ T_{j_2} \simeq T_{j_*}$. That is,

$$\prod_i T_{\mathcal{O}_{E_i}(-1)} \circ \prod_i T_{\mathcal{O}_{E_i}} \simeq M_{\mathcal{O}_K(E)}. \tag{1}$$

Furthermore, we have $j_1 \simeq M_{\mathcal{O}_K(E/2)} \circ j_2$ since $\mathcal{O}_{E_i}(E/2) \simeq \mathcal{O}_{E_i}(-1)$, and so

$$T_{j_1} \simeq T_{M_{\mathcal{O}_K(E/2)} \circ j_2} \simeq M_{\mathcal{O}_K(E/2)} \circ T_{j_2} \circ M_{\mathcal{O}_K(-E/2)},$$

which, after taking inverses, equates to

$$\prod_i T^{-1}_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)} \simeq M_{\mathcal{O}_K(E/2)} \circ \prod_i T^{-1}_{\mathcal{O}_{E_i}}. \tag{2}$$

This expression allows us to reduce the formula for $T^2_{F''}$ in the following way:

$$\begin{aligned} T^2_{F''} &\simeq \prod_i T^{-1}_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)} \circ \prod_i T^{-1}_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)}[2] \\ &\simeq M_{\mathcal{O}_K(E/2)} \circ \prod_i T^{-1}_{\mathcal{O}_{E_i}} \circ \prod_i T^{-1}_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)}[2] \\ &\simeq M_{\mathcal{O}_K(E/2)} \circ M_{\mathcal{O}_K(-E)} \circ M_{\mathcal{O}_K(E/2)}[2] \\ &\simeq [2], \end{aligned}$$

where the second and third lines follow from (2) and (1), respectively.

The fact that $T^2_F \simeq [2]$ now follows immediately from Corollary 2.4. ∎

Corollary 2.6. *The collections* $\operatorname{im} F$ *and* $\operatorname{im} F''$ *are spanning classes for* $\mathcal{D}(K)$.

Proof. For any spherical functor $F : \mathcal{D}(Y) \to \mathcal{D}(X)$, we have a natural spanning class for $\mathcal{D}(X)$ given by $\operatorname{im} F \cup (\operatorname{im} F)^\perp \simeq \operatorname{im} F \cup \ker R$ (see [1, Section 1.4]). However, in our case we have $\ker R = 0$. Indeed, let $\mathcal{E} \in \ker R$. Then the defining triangle for the twist $FR(\mathcal{E}) \to \mathcal{E} \to T_F(\mathcal{E})$ shows that $T_F(\mathcal{E}) \simeq \mathcal{E}$. But by Corollary 2.5 we have $\mathcal{E} \simeq T^2_F(\mathcal{E}) \simeq \mathcal{E}[2]$, which implies that $\mathcal{E} \simeq 0$; a similar argument works for F''. ∎

REMARK 2.7. This should be contrasted to the object case where every spherical object $\mathcal{E}$ is expected to have a nonempty perpendicular $\mathcal{E}^\perp$ (see [12, Question 1.25]).

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