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L-FUNCTIONS OF p-ADIC CHARACTERS

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Abstract. We define a p-adic character to be a continuous homomorphism from $1 + t\mathbb{F}_q[[t]]$ to $\mathbb{Z}_p^*$. For $p > 2$, we use the ring of big Witt vectors over $\mathbb{F}_q$ to exhibit a bijection between p-adic characters and sequences $(c_i)_{(i,p)=1}$ of elements in $\mathbb{Z}_q$, indexed by natural numbers relatively prime to p, and for which $\lim_{i\to\infty} c_i = 0$. To such a p-adic character we associate an L-function, and we prove that this L-function is p-adic meromorphic if the corresponding sequence (c_i) is overconvergent. If more generally the sequence is C log-convergent, we show that the associated L-function is meromorphic in the open disk of radius q^C. Finally, we exhibit examples of C log-convergent sequences with associated L-functions which are not meromorphic in the disk of radius $q^{C+\epsilon}$ for any $\epsilon > 0$.

§1. Introduction

Let $\mathbb{F}_q$ be a finite field of q elements with characteristic p. Let $K = \mathbb{F}_q(t)$ be the rational function field, which is the function field of the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$. Let G_K denote the absolute Galois group of K, namely, the Galois group of a fixed separable closure of K. Given a continuous p-adic representation

$$\rho : G_K \longrightarrow \mathrm{GL}_n(\mathbb{Z}_p),$$

unramified on $U = \mathbb{P}^1 - S$ with S being a finite set of closed points of $\mathbb{P}^1$, the L-function of the representation ρ is defined by

$$L(\rho, s) := L(\rho/U, s) = \prod_{x\in|U|} \frac{1}{\det(I - \rho(\mathrm{Frob}_x)s^{\deg(x)})} \in 1 + s\mathbb{Z}_p[[s]],$$

where $|U|$ denotes the set of closed points of U and Frob_x denotes the geometric Frobenius conjugacy class at x. It is clear that the power series $L(\rho, s)$ is convergent (or analytic) in the open unit disk $|s|_p < 1$.

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A basic object of study in number theory is the L-function $L(\rho, s)$. The first question about $L(\rho, s)$ is its possible analytic or meromorphic continuation. This question has been studied extensively in the literature; however, it remains quite mysterious, even in the abelian case when $n = 1$. We now briefly review the limited known results. If ρ is of finite order, then $L(\rho, s)$ is a rational function in s according to Brauer, and it satisfies a Riemann hypothesis according to Weil. If, in addition, ρ is irreducible and nontrivial, then $L(\rho, s)$ is a polynomial in s (Artin's conjecture for function fields), which follows from Grothendieck's trace formula.

If ρ is a p-adic representation of infinite order, the situation is much more complicated, even in the abelian case $n = 1$. First, it is easy to construct examples such that $L(\rho, s)$ is not rational in s. For an arbitrary p-adic representation ρ, the L-function $L(\rho, s)$ is known to be meromorphic on the closed unit disk $|s|_p \leq 1$, and its unit root part (a rational function in s) is given by the Frobenius action on the p-adic étale cohomology of ρ. This was conjectured by Katz in [10], and proved by Crew in [4] in the rank 1 case, and proved more generally by Emerton and Kisin in [7]. A stronger conjecture of Katz [10, Conjecture 6.1.1] stated that $L(\rho, s)$ is meromorphic in $|s|_p < \infty$. This turned out to be false in general, even in the case $n = 1$ (see [17]). It suggests that the L-function $L(\rho, s)$ is much more complicated than previously thought.

Motivated by his pioneering work on p-adic variation of zeta functions, Dwork ([5], [6]) conjectured that if ρ is geometric (arising from the relative p-adic étale cohomology of a family of varieties), then the L-function $L(\rho, s)$ is p-adic meromorphic in $|s|_p < \infty$. This was proved by the second author in ([18], [19], [20]). It suggests that the class of geometric p-adic representations behaves reasonably well from the L-function point of view. We note that even in the geometric rank 1 case, although the L-function $L(\rho, s)$ is p-adic meromorphic in $|s|_p < \infty$, it is not expected to be rational in s, nor should one expect that it is p-adic entire. (Namely, the Artin entireness conjecture fails for nontrivial rank 1 geometric p-adic representations of G_K.) One such example follows from Coleman's work [2] in the elliptic modular case.

The aim of this article is to reexamine this L-function from a new point of view via Witt vectors in the hope that it will provide new insight into this mysterious meromorphic continuation problem. We will focus on the abelian case $n = 1$. Then the representation ρ factors through the maximal abelian quotient G_K^{ab}:

$$\rho : G_K^{\mathrm{ab}} \longrightarrow \mathrm{GL}_1(\mathbb{Z}_p) = \mathbb{Z}_p^*.$$

That is, ρ is a p-adic character. By class field theory, G_K^{ab} is isomorphic to the profinite completion of the idèle class group of K. Precisely, in our case of the rational function field $K = \mathbb{F}_q(t)$, we have

$$G_K^{\mathrm{ab}} \equiv \widehat{\langle t \rangle} \times \left(1 + \frac{1}{t}\mathbb{F}_q\left[\left[\frac{1}{t}\right]\right]\right) \times \prod_{x \in |\mathbb{A}^1|} \mathbb{F}_q[t]_x^*,$$

where $\mathbb{F}_q[t]_x$ denotes the completion of $\mathbb{F}_q[t]$ at the prime x and $\widehat{\langle t \rangle}$ denotes the profinite completion of the infinite cyclic multiplicative group generated by t. Since the character ρ is unramified on $U = \mathbb{P}^1 - S$, the restriction of ρ to the x-factor $\mathbb{F}_q[t]_x^*$ is trivial for all $x \in U$. To further simplify the situation, we will assume that S is the one-point set consisting of the origin corresponding to the prime t in $\mathbb{F}_q[t]$. In this case, $\mathbb{F}_q[t]_t = \mathbb{F}_q[[t]]$. Twisting by a harmless finite character, we may further assume that ρ factors through the character

$$\chi : \mathbb{F}_q[[t]]^* / \mathbb{F}_q^* = 1 + t\mathbb{F}_q[[t]] \longrightarrow \mathbb{Z}_p^*.$$

If $f(t) \in 1 + t\mathbb{F}_q[t]$ is an irreducible polynomial, then one checks that

$$\rho(\mathrm{Frob}_{f(t)}) = \chi\big(f(t)\big).$$

Thus, the L-function $L(\rho, s)$ reduces to the following L-function of the p-adic character χ

$$L(\chi, s) = \prod_f \frac{1}{1 - \chi(f) s^{\deg(f)}},$$

where f now runs over all monic irreducible polynomials of $\mathbb{F}_q[t]$ different from t. Expanding the product, the L-function of χ is also the series

$$L(\chi, s) = \sum_g \chi(g) s^{\deg(g)} \in 1 + s\mathbb{Z}_p[[s]],$$

where g runs over all monic polynomials in $\mathbb{F}_q[t]$ different from t. (Alternatively, one defines $\chi(t) = 0$.) A related function, which is of great interest to us, is the characteristic series of χ defined by

$$C(\chi, s) = \prod_{k=0}^{\infty} L(\chi, q^k s) \in 1 + s\mathbb{Z}_p[[s]].$$

Equivalently,

$$L(\chi,s)=\frac{C(\chi,s)}{C(\chi,qs)}.$$

Thus, the L-function and the characteristic series determine each other.

In summary, we reduce to studying the following basic question.

QUESTION 1.1. Given a continuous p-adic character

$$\chi:1+t\mathbb{F}_q[[t]]\longrightarrow\mathbb{Z}_p^*,$$

when is its L-function $L(\chi,s)$ as defined above p-adic meromorphic in s?

To give an idea of what we prove, we state our result in this introduction only in the simpler special case that $q=p$. For the more general case, as well as the proof, see Theorem 4.7.

THEOREM 1.2. *Fix $p>2$. There is a one-to-one correspondence between continuous p-adic characters $\chi : 1+t\mathbb{F}_p[[t]]\longrightarrow\mathbb{Z}_p^*$ and sequences $\pi=(\pi_i)_{(i,p)=1}$, where $\pi_i\in p\mathbb{Z}_p$ and $\lim_{i\to\infty}\pi_i=0$. Denote the associated character by χ_π. For an irreducible polynomial $f(t)\in 1+t\mathbb{F}_p[t]$ with degree d, let $\bar{\lambda}$ denote a reciprocal root of $f(t)$, and let λ denote the Teichmüller lifting of $\bar{\lambda}$. Then the character χ_π is given by*

$$\chi_\pi\big(f(t)\big)=\prod_{(i,p)=1}(1+\pi_i)^{\mathrm{Tr}(\lambda^i)},$$

where Tr *denotes the trace map from $\mathbb{Z}_{p^d}$ to $\mathbb{Z}_p$. Assume that the sequence π satisfies the $\infty\log$-condition*

$$\liminf_{i\to\infty}\frac{v_p(\pi_i)}{\log_p i}=\infty.$$

Then the characteristic power series

$$C(\chi,s)=\prod_{k=0}^{\infty}L(\chi,p^k s)$$

is entire in $|s|_p<\infty$, and thus the L-function

$$L(\chi_\pi,s)=\frac{C(\chi,s)}{C(\chi,ps)}$$

is p-adic meromorphic in $|s|_p<\infty$.

We show further in Section 5 that Theorem 1.2 is optimal in the sense that the ∞ log-condition cannot be weakened to a C log-condition for any finite C (see Example 5.7). The key tool used by this example is Theorem 5.6, which states that one can lift from power series to sequences in a way that (a) preserves C log-convergence and (b) does not change the associated L-series. We then produce the desired example by using [17], which describes an analogous example in terms of power series.

REMARK 1.3. The result here cannot be used to provide a new proof of the rank 1 case of Dwork's conjecture. The reason is that characters coming from geometry do not in general yield ∞ log-convergent sequences. Instead, they are only guaranteed to yield C log-convergent sequences for $C \geq 1$. On the other hand, most C log-convergent sequences for $C \geq 1$ do not come from geometry. So, while related, the condition of "coming from geometry" is rather different from the condition studied here (see [18, p. 893] for more details).

To prove Theorem 1.2, we link the character χ_π via the binomial power series to a power series in λ with a good convergence condition and then apply the results from [17]. Thus, our proof ultimately depends on Dwork's trace formula. It would be very interesting to find a self-contained proof of Theorem 1.2 without using Dwork's trace formula, as this would pioneer an entirely new (and likely motivic) approach.

In this article we treat only L-functions of the simplest nontrivial p-adic characters, that is, p-adic characters with values in $\mathbb{Z}_p^*$ ramified only at the origin. There are several interesting ways to extend the present work. One can consider p-adic characters ramified at several closed points (not just the origin t). One can replace the projective line $\mathbb{P}^1$ by a higher genus curve or even a higher-dimensional variety. One can consider higher-rank p-adic representations instead of considering only p-adic characters.

One can also consider p-adic characters with values in the unit group of other p-adic rings such as $\mathcal{O}_{\mathbb{C}_p}$ or the 2-dimensional local ring $\mathbb{Z}_p[[T]]$. The latter will be very useful in studying the variation of the L-function when the character χ moves in a p-adic analytic family; here, T is the analytic parameter. Two related examples which have been studied in depth are the eigencurves (see [3]) and T-adic L-functions (see [14]). We conclude this introduction by briefly discussing one case in more detail.

Fix a character

$$\chi : 1 + t\mathbb{F}_q[[t]] \to \mathbb{Z}_p^*.$$

Let $\mathcal{W} = \mathrm{Hom}_{\mathrm{gp}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ denote the weight space of $\mathbb{C}_p$-valued p-adic characters. This can be viewed as a rigid analytic space, and we let Λ denote its ring of rigid analytic functions. We compose χ with the universal character $\mathbb{Z}_p^* \to \Lambda^*$:

$$1 + t\mathbb{F}_q[[t]] \to \mathbb{Z}_p^* \to \Lambda^*.$$

Now, for any $x \in \mathcal{W}$, we have a natural "evaluation" map $\Lambda^* \to \mathbb{C}_p^*$. Composing all these maps, we get, for fixed $x \in \mathcal{W}$, a character

$$\chi_x : 1 + t\mathbb{F}_q[[t]] \to \mathbb{C}_p^*, \tag{1.3.1}$$

of a type which is only slightly more general than those considered in the present paper. If $x_0 \in \mathcal{W} = \mathrm{Hom}_{\mathrm{gp}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ corresponds to the natural inclusion $\mathbb{Z}_p^* \hookrightarrow \mathbb{C}_p^*$, then χ_{x_0} in (1.3.1) is simply the character χ we chose initially. In future work, we will use the techniques of the present paper to consider the following questions.

- If χ has the ∞ log-condition of Theorem 1.2, does the same hold true for the deformed characters χ_x?
- Assume that we have an affirmative answer to the previous question for (certain) characters χ_x. We then have p-adic entire power series $C(\chi, s)$ and $C(\chi_x, s)$. How do the slopes of the Newton polygons vary, as we move x through (a suitable part of) weight space?

The second question is similar in spirit to what was studied by the second author in [21]. The above questions were posed to the authors by Liang Xiao, in connection to his work with Kedlaya and Pottharst [11]. In fact, it was discussions with Xiao during a visit he made to the University of California at Irvine that led to the present paper. We will return to these and other questions in future work.

Notation and conventions

Let $q = p^a$ denote a power of p. Beginning in Section 3, we require $p > 2$. For a ring R (always assumed to be commutative and with unity), we denote by $W(R)$ the p-typical Witt vectors with coefficients in R, and we denote by $\mathbb{W}(R)$ the big Witt vectors with coefficients in R (for an explanation of these Witt vectors, see Section 2.2). We write $\mathbb{Z}_p$ for the ring of p-adic integers, $\mathbb{Q}_p$ for the field of p-adic numbers, $\mathbb{Q}_q$ for the unramified degree a extension of $\mathbb{Q}_p$, $\mathbb{Z}_q$ for the ring of integers in $\mathbb{Q}_q$, $\widehat{\mathbb{Q}_p^{\mathrm{nr}}}$ for the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$, and $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ for the ring of integers in $\widehat{\mathbb{Q}_p^{\mathrm{nr}}}$. When we have a fixed unramified extension $\mathbb{Q}_q/\mathbb{Q}_p$, we write σ for the Frobenius map, the unique automorphism which induces the Frobenius in characteristic p. It is a generator of the cyclic group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$. We let $f_\lambda(t)$ denote a general irreducible polynomial in $1+t\mathbb{F}_q[t]$. We call its degree d; we write $\bar{\lambda}$ for one of its reciprocal roots, and we write λ for the corresponding Teichmüller lift in $\mathbb{Z}_{q^d}$. In other words, λ will denote the unique root of unity in $\mathbb{Z}_{q^d}$ with reduction modulo p equal to $\bar{\lambda}$. We write $v_p(x)$ to denote the p-adic valuation of x. If R is a topological ring, we let $R\langle t\rangle$ denote convergent power series with coefficients in R. If $g(t) \in \mathbb{Q}_q[[t]]$, then we let $g^\sigma(t) \in \mathbb{Q}_q[[t]]$ denote the power series obtained by applying σ to each coefficient. Unfortunately, we will need both the p-adic logarithm and the classical base-p logarithm. We denote the p-adic logarithm by Log and the classical logarithm by $\log_p$.

§2. Preliminaries

In this section, we introduce the objects we will study (p-adic characters and their associated L-functions) and we introduce one of the key tools we will use to study them (big Witt vectors).

2.1. p-adic characters

We begin by defining p-adic characters and their associated L-functions.

Definition 2.1. By *p-adic character*, we mean a nontrivial continuous homomorphism

$$\chi : \left(1+t\mathbb{F}_q[[t]]\right)^* \to \mathbb{Z}_p^*.$$

Remark 2.2. When we refer to continuity in the preceding definition, we are using the t-adic topology on $(1+t\mathbb{F}_q[[t]])^*$ and the p-adic topology on $\mathbb{Z}_p^*$. We check that if $y \in \mathbb{Z}_p^*$ is in the image of χ, then $y \equiv 1 \mod p$; we call such an element a *1-unit.* Assume that $x := 1+tf(t)$ is some element of $(1+t\mathbb{F}_q[[t]])^*$, and let $y := \chi(x)$. The reduction $y \mod p \in \mathbb{F}_p$ is a unit, so we have $y \equiv \zeta \mod p$, where ζ is some $(p-1)$st root of unity. The sequence $x, x^p, x^{p^2}, \ldots$ clearly converges t-adically to 1. The sequence $y, y^p, y^{p^2}, \ldots$ converges p-adically to ζ. Hence, if the character χ is to be continuous, we must have $\zeta = \chi(1) = 1$. This shows that the image of χ contains only 1-units.

To a p-adic character, we associate an L-function as follows.

DEFINITION 2.3. The L-function associated to a p-adic character χ is the formal power series associated to either and both of the following:

$$L(\chi, s) = \prod_{\substack{f(t) \text{ irred poly,} \\ f(t) \equiv 1 \bmod t}} \frac{1}{1 - \chi(f(t)) s^{\deg f}} \tag{2.3.1}$$

or

$$L(\chi, s) = \exp\Big(\sum_{k=1}^{\infty} \frac{S_k(\chi)}{k} s^k\Big), \tag{2.3.2}$$

where

$$S_k(\chi) = \sum_{\bar{\lambda} \in \mathbb{F}_{q^k}} \chi(f_\lambda)^{k/\deg(\bar{\lambda})}$$

and where f_λ denotes the irreducible polynomial with constant term 1 and having $\bar{\lambda}$ as a reciprocal root.

The most basic question to ask about these functions is the following.

QUESTION 2.4. When is $L(\chi, s)$ a p-adic meromorphic function in the variable s?

To approach this question, we will use Witt vectors to give a new characterization of p-adic characters. Section 2.2 reviews Witt vectors and states the results we will use. Later we apply these results to give a simple description of all p-adic characters.

2.2. Big and p-typical Witt vectors

We now introduce big and p-typical Witt vectors. These are both functors from rings to rings. Witt vectors are used in the proofs of our main theorems, but the statements of the theorems do not require Witt vectors. The reader who is unfamiliar with Witt vectors should focus on the special cases concerning characteristic p, as that is what we will need below. Our references for this section are [9] and [16], but there are many other places to read about Witt vectors (see, e.g., [8, Section 17], [1, Chapter 9, exercises], [13, notes], or [15, notes]).

The big Witt vector functor has an imposing definition, but when it is evaluated on a perfect field of characteristic p, as it will be in our case, it is quite accessible.

DEFINITION 2.5. Let R denote a ring. The ring of *big Witt vectors with coefficients in* R, denoted $\mathbb{W}(R)$, is, as a set, $R^{\mathbb{N}} = \{(r_1, r_2, \ldots) \mid r_i \in R\}$. To uniquely describe $\mathbb{W}(R)$ as a ring, we require the following two properties.

(1) The *ghost map* $w : \mathbb{W}(R) \to R^{\mathbb{N}}$ defined by

$$(r_1, r_2, \ldots, r_i, \ldots) \mapsto \Big(r_1, r_1^2 + 2r_2, \ldots, \sum_{d|i} d r_d^{i/d}, \ldots\Big)$$

is a ring homomorphism, where the ring operations on the target are component-wise.

This uniquely determines $\mathbb{W}(R)$ as a ring in the case that R is $\mathbb{Z}$-torsion-free. To determine the ring operations in general, we need also the following functoriality property.

(2) For any ring homomorphism $f : R \to S$, the map $\mathbb{W}(f) : \mathbb{W}(R) \to \mathbb{W}(S)$ given by $\mathbb{W}(f) : (r_1, r_2, \ldots) \mapsto (f(r_1), f(r_2), \ldots)$ is a ring homomorphism.

REMARK 2.6. Proof is required that such a functor $\mathbb{W}(-)$ exists (see, e.g., [9, Proposition 1.2]).

The more classical version of Witt vectors are the *p-typical Witt vectors*. Again, the general definition may be imposing, but when evaluated on a perfect field of characteristic p, there is a down-to-earth description.

DEFINITION 2.7. Let R denote a ring. The ring of *p-typical Witt vectors with coefficients in* R, denoted $W(R)$, is, as a set, $R^{\mathbb{N}} = \{(r_1, r_p, r_{p^2}, \ldots) \mid r_{p^i} \in R\}$. The ring operations on $W(R)$ are again defined using the ghost map.

(1) The *ghost map* $w : W(R) \to R^{\mathbb{N}}$ defined by

$$(r_1, r_p, \ldots, r_{p^i}, \ldots) \mapsto \Big(r_1, r_1^p + p r_p, \ldots, \sum_{d|p^i} d r_d^{p^i/d}, \ldots\Big)$$

is a ring homomorphism, where the ring operations on the target are component-wise.

This determines the ring operations on $W(R)$ uniquely when R is p-torsion-free. The definition in general follows by making a functoriality requirement as in the big case.

REMARK 2.8. Note that big Witt vectors are written using a outline "$\mathbb{W}$", while p-typical Witt vectors are written using an italic "W". Also, note that, from the definition of ring operations in terms of the ghost map, it is clear that $W(R)$ is a quotient of $\mathbb{W}(R)$, but it is not a subring.

When the ring R is perfect of characteristic p, we have the following classical definition of p-typical Witt vectors, taken from Serre [16, Theorem 2.5.3].

DEFINITION 2.9. If k is a perfect field in characteristic p, then the ring $W(k)$ is the unique (up to canonical isomorphism) p-adically complete discrete valuation ring with maximal ideal (p) and residue field k.

EXAMPLE 2.10. Thus, $W(\mathbb{F}_p) = \mathbb{Z}_p$ and $W(\mathbb{F}_{p^a}) = \mathbb{Z}_{p^a}$ (which by definition is the ring of integers in the unramified extension of $\mathbb{Q}_p$ of degree a). Similarly, if $\overline{\mathbb{F}_p}$ denotes the algebraic closure of $\mathbb{F}_p$, then $W(\overline{\mathbb{F}_p}) = \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$.

We are now ready to give the simpler characterization of big Witt vectors $\mathbb{W}(R)$ when R is perfect of characteristic p or, more generally, a $\mathbb{Z}_{(p)}$-algebra. (The latter is the same as requiring that all integers relatively prime to p are invertible in R.)

PROPOSITION 2.11. *Let R denote a $\mathbb{Z}_{(p)}$-algebra. Then we have an isomorphism of rings*

$$\mathbb{W}(R) \cong \prod_{i \in \mathbb{N}, (i,p)=1} W(R).$$

Proof. For a proof, see [9, Proposition 1.10]. The idea is to prescribe ghost components of many mutually orthogonal idempotents and then to use the fact that all primes $l \neq p$ are invertible in R in order to find Witt vectors with the prescribed ghost components. ∎

REMARK 2.12. Let $U_j \subseteq \mathbb{W}(R)$ denote the ideal $U_j = \{\underline{x} \in \mathbb{W}(R) \mid x_i = 0 \text{ for } i < j\}$. These ideals generate a topology on $\mathbb{W}(R)$ called the *V-adic topology.* Under the isomorphism in Proposition 2.11, these ideals correspond to $U_j' \subseteq \prod_{(i,p)=1} W(R)$, where

$$U_j' = \Big\{ (\underline{x}_i) \in \prod_{(i,p)=1} W(R) \,\Big|\, \text{the } p^k \text{ component of } \underline{x}_i \text{ equals zero for all } ip^k < j \Big\}.$$

EXAMPLE 2.13. It follows immediately from the proposition that as rings

$$\mathbb{W}(\mathbb{F}_q) \cong \prod_{i \in \mathbb{N}, (i,p)=1} \mathbb{Z}_q.$$

We are also interested in the topology on these rings. Viewing an element $\alpha \in \mathbb{Z}_q$ as a Witt vector $\underline{x} \in W(\mathbb{F}_q)$, we have that $p^i \mid \alpha$ if and only if the first i coordinates of $\underline{x}$ are zero: $x_1, \ldots, x_{p^{i-1}} = 0$. (This follows from the fact that multiplication by p corresponds to shifting the Witt vector coordinates to the right and raising each component to the pth power; see [9, Lemma 1.12] or [16, Section II.6].) It is now easy to see that the V-adic topology described above is the same as the product topology on $\prod \mathbb{Z}_q$, where each component is equipped with the p-adic topology.

The following is another useful description of $\mathbb{W}(R)$. (More precisely, it provides a useful description of addition in $\mathbb{W}(R)$.) It is not as simple as the above description, but it hints at how we will use Witt vectors in our work on p-adic characters.

Definition–Theorem 2.14. For any ring R, let $\Lambda(R) := 1 + tR[[t]]$ consist of power series with coefficients in R with constant term 1. View $\Lambda(R)$ as a group under multiplication. View $\mathbb{W}(R)$ as a group under addition. Then the map $E : \mathbb{W}(R) \to \Lambda(R)$ defined by

$$E : (r_1, r_2, \ldots) \mapsto \prod_{i=1}^{\infty} (1 - r_i t^i)$$

is a group isomorphism. If we view $\mathbb{W}(R)$ as a topological group using the V-adic topology described above and if we view $\Lambda(R)$ as a topological group using the t-adic topology, then the isomorphism is a homeomorphism.

Remark 2.15. There are four different reasonable normalizations for E. These can be obtained by replacing $1 - r_i t^i$ above with $(1 - r_i(\pm t)^i)^{\pm 1}$. We have chosen the normalization which gives us easiest access to reciprocal roots.

Proof of Definition–Theorem 2.14. See [9, Proposition 1.14] for everything except the continuity claims, and these are obvious. The proof given there is for a different normalization, but this does not matter. $\square$

In either p-typical or big Witt vectors, we have a notion of Teichmüller lift.

Definition 2.16. If $r \in R$ is any element, we let $[r] \in W(R)$ or $\mathbb{W}(R)$ denote the Witt vector with components $(r, 0, 0, \ldots)$. This Witt vector is called the *Teichmüller lift of* r. We have $[rs] = [r][s]$, but of course $[\cdot] : R \to$

$W(R)$ is not a ring homomorphism (since traditionally R is characteristic p and $W(R)$ is characteristic 0).

REMARK 2.17. For a nonzero element $x \in \mathbb{F}_q$, the Teichmüller lift $[x] \in \mathbb{Z}_q$ is the unique $(q-1)$st root of unity which is a lift of x. This follows from the equalities $[x]^{q-1} = [x^{q-1}] = [1] = 1$.

REMARK 2.18. In general it is not so easy to describe explicitly the map $\mathbb{W}(R) \to \prod_{(i,p)=1} W(R)$ from Proposition 2.11. However, for the case of Teichmüller lifts, we can describe it explicitly. In that case, we have $[\lambda] \mapsto ([\lambda^i])_{(i,p)=1}$. To prove this, work in terms of ghost components, using the explicit description from [9, Proposition 1.10].

EXAMPLE 2.19. We have natural inclusions $1 + t\mathbb{F}_q[[t]] \subseteq 1 + t\overline{\mathbb{F}_q}[[t]]$ and $\mathbb{W}(\mathbb{F}_q) \subseteq \mathbb{W}(\overline{\mathbb{F}_q})$. The latter corresponds to the inclusion

$$\prod_{(i,p)=1} \mathbb{Z}_q \subseteq \prod_{(i,p)=1} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}.$$

This yields a commutative diagram

$$\begin{array}{ccccc} 1 + t\mathbb{F}_q[[t]] & \longrightarrow & 1 + t\overline{\mathbb{F}_q}[[t]] & & (1 - \bar{\lambda} t) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{(i,p)=1} \mathbb{Z}_q & \longrightarrow & \prod_{(i,p)=1} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} & & (\lambda, \lambda^2, \ldots) \end{array}$$

where we write λ for the Teichmüller lift of $\bar{\lambda}$.

Consider now an irreducible degree d polynomial $f(t) \in 1 + t\mathbb{F}_q[t]$ with reciprocal root $\bar{\lambda}$. This polynomial can be factored as $(1 - \bar{\lambda} t)(1 - \bar{\lambda}^q t) \cdots (1 - \bar{\lambda}^{q^{d-1}} t)$. The conjugates of the Teichmüller lift λ over $\mathbb{Z}_q$ are $\lambda, \lambda^q, \ldots, \lambda^{q^{d-1}}$. (Proof: They are roots of unity and have the correct reductions modulo p.) The above commutative diagram then shows that the polynomial $f(t)$ corresponds to the element $(\mathrm{Tr}(\lambda^i))_{(i,p)=1} \in \prod_{(i,p)=1} \mathbb{Z}_q$, where Tr denotes the trace from $\mathbb{Z}_{q^d}$ to $\mathbb{Z}_q$.

§3. Characters in terms of Witt vectors

We begin this section with a result which gives us a new description of p-adic characters. It follows directly from combining Definition–Theorem 2.14 with Proposition 2.11.

COROLLARY 3.1. *Giving a continuous homomorphism* $\chi : (1+t\mathbb{F}_q[[t]])^* \to \mathbb{Z}_p^*$ *is the same as giving a continuous homomorphism*

$$\prod_{i\in\mathbb{N},(i,p)=1} \mathbb{Z}_q \to \mathbb{Z}_p^*.$$

A first step toward understanding such continuous homomorphisms is to understand the component homomorphisms $\mathbb{Z}_q \to \mathbb{Z}_p^*$. Before describing these, we describe the homomorphisms $\mathbb{Z}_q \to \mathbb{Z}_p$.

LEMMA 3.2. *For any continuous group homomorphism* $\phi : \mathbb{Z}_q \to \mathbb{Z}_p$*, there exists a unique element* $c \in \mathbb{Z}_q$ *such that* $\phi : \alpha \mapsto \mathrm{Tr}(c\alpha)$*, where* Tr *is the trace map from* $\mathbb{Z}_q$ *to* $\mathbb{Z}_p$*.*

Proof. We know from [16, Section II.5] that as topological groups $\mathbb{Z}_q \cong \bigoplus_{i=1}^{a} \mathbb{Z}_p$. Fix a basis $e_1, \dots, e_a$ of $\mathbb{Z}_q$ as a $\mathbb{Z}_p$-module. Write $\phi(e_i) = r_i$. We want to find $c \in \mathbb{Z}_q$ such that for any $b \in \mathbb{Z}_q$, we have $\phi(b) = \mathrm{Tr}(cb)$. Write $b = b_1e_1 + \cdots + b_ae_a$, where each $b_i \in \mathbb{Z}_p$. Writing $c = c_1e_1 + \cdots + c_ae_a$, our goal now becomes to find the elements $c_i \in \mathbb{Z}_p$. We want

$$\begin{aligned} \phi(b) &= \mathrm{Tr}(cb), \\ \phi(b_1e_1 + \cdots + b_ae_a) &= \mathrm{Tr}(cb), \\ b_1r_1 + \cdots + b_ar_a &= \sum c_ib_j \,\mathrm{Tr}(e_ie_j). \end{aligned}$$

Considering the case $b_j = 1$ and $b_i = 0$ for $i \neq j$, we see that we must find c_i so that

$$r_j = \sum_{i=1}^{a} c_i \,\mathrm{Tr}(e_ie_j). \tag{3.2.1}$$

In fact, if we find such c_i, then we are done. (Simply compare the coefficients of b_i above.) We now show that c_i satisfying (3.2.1) exist and are uniquely determined.

Reducing everything modulo p, we know that $\overline{e_1}, \dots, \overline{e_a}$ is a basis for $\mathbb{F}_q/\mathbb{F}_p$. This is a separable extension, and so the matrix with (i,j)-entry $\mathrm{Tr}(\overline{e_ie_j})$ is invertible (see, e.g., [16, p. 50].) Hence, the determinant of the matrix with (i,j)-entry $\mathrm{Tr}(e_ie_j)$ is nonzero modulo p; hence, it is invertible in $\mathbb{Z}_p$. Hence, we can find $c_i \in \mathbb{Z}_p$ which satisfy (3.2.1) for all j, and these c_i are uniquely determined. ∎

We will study maps $\mathbb{Z}_q \mapsto \mathbb{Z}_p^*$ by factoring them as $\mathbb{Z}_q \to \mathbb{Z}_p \to \mathbb{Z}_p^*$. Lemma 3.2 concerned the first map in this composition; Lemma 3.3 concerns the second map.

LEMMA 3.3. *Assume that* $p > 2$. *Let* $\pi \in p^n\mathbb{Z}_p$ *for* $n \geq 1$. *There exists a unique element* $r \in p^{n-1}\mathbb{Z}_p$ *such that* $1+\pi = (1+p)^r$.

Proof. It suffices to show that, for any $c \in \{0,1,\ldots,p-1\}$, we can find unique $r \in p^{n-1}\mathbb{Z}_p/p^n\mathbb{Z}_p$ such that $(1+p)^r \equiv 1 + cp^n \mod p^{n+1}$. Using the fact that $p \neq 2$, we see from the binomial expansion that only $r = cp^{n-1} \mod p^n\mathbb{Z}_p$ works. ∎

We now realize our first goal of characterizing all continuous group homomorphisms from $\mathbb{Z}_q$ to $\mathbb{Z}_p^*$. We will then be able to join these together to describe all continuous group homomorphisms from $\prod_{(i,p)=1} \mathbb{Z}_q$ to $\mathbb{Z}_p^*$. The following proposition will relate this to the case $q = p$.

PROPOSITION 3.4. *Assume that* $p > 2$. *Any continuous group homomorphism* $\chi : \mathbb{Z}_q \to \mathbb{Z}_p^*$ *can be factored as*

$$\chi' \circ \mathrm{Tr} \circ c : \mathbb{Z}_q \xrightarrow{\cdot c} \mathbb{Z}_q \xrightarrow{\mathrm{Tr}} \mathbb{Z}_p \xrightarrow{\chi'} \mathbb{Z}_p^*,$$

where c *denotes multiplication by some element* $c \in \mathbb{Z}_q$ *and where* $\chi' : \mathbb{Z}_p \to \mathbb{Z}_p^*$. *Conversely, any such factorization yields a continuous group homomorphism.*

Moreover, we will see that χ' *can be taken to be the map* $\alpha \mapsto (1+p)^\alpha$. *With this restriction, then the corresponding element* c *is unique.*

Proof. That any such composition yields a continuous homomorphism is clear because all maps in the composition are continuous homomorphisms. (For instance, trace is a sum of continuous homomorphisms.)

Now consider any continuous homomorphism $\chi : \mathbb{Z}_q \to \mathbb{Z}_p^*$. By Lemma 3.3, there exists a unique map $\phi : \mathbb{Z}_q \to \mathbb{Z}_p$ such that $\chi(\alpha) = (1+p)^{\phi(\alpha)}$ for any $\alpha \in \mathbb{Z}_q$. By our assumptions on χ, it is easy to see that ϕ is a (continuous group) homomorphism. Hence, by Lemma 3.2, we are done. ∎

PROPOSITION 3.5. *Let* $q = p^a$, *where* p *is an* odd *prime. Giving a* p*-adic character*

$$\chi : \left(1 + t\mathbb{F}_q[[t]]\right)^* \to \mathbb{Z}_p^*$$

is equivalent to giving a sequence of elements $(c_i)_{(i,p)=1}$, *where each* $c_i \in \mathbb{Z}_q$, *subject to the constraint that* $\lim_{i\to\infty} c_i = 0$. *More explicitly, given such a sequence* (c_i), *the associated character* χ *sends an irreducible degree* d

polynomial $f(t)$ with root $\bar\lambda$ to

$$\prod_{(i,p)=1} (1+p)^{\mathrm{Tr}_{\mathbb{Z}_{q^d}/\mathbb{Z}_p}(c_i\lambda^i)},$$

where λ is the Teichmüller lift of $\bar\lambda$.

Proof. We can realize our p-adic character as a continuous homomorphism

$$\prod_{(i,p)=1} \mathbb{Z}_q \to \mathbb{Z}_p^*.$$

By Proposition 3.4, and temporarily ignoring continuity, giving such a homomorphism is equivalent to giving a sequence of elements $(c_i)_{(i,p)=1}$. The sequence of elements in $\prod_{(i,p)=1}\mathbb{Z}_q$ given by $(1,0,0,\ldots),(0,1,0,\ldots),\ldots$ converges to 0 in the product topology. Hence, the images of the maps $\mathbb{Z}_q \to \mathbb{Z}_p$ given by $\alpha \mapsto (1+p)^{\mathrm{Tr}(c_i\alpha)}$ should be converging p-adically to 1 as i increases. We claim that such an image is contained in $1+p^{j+1}\mathbb{Z}_p$ if and only if $c_i \in p^j\mathbb{Z}_q$. The "if" direction is obvious. We now prove the "only if" direction. Using Lemma 3.3, it suffices to show that if $c_i \notin p^j\mathbb{Z}_q$, then $\mathrm{Tr}(c_i\alpha) \notin p^j\mathbb{Z}_p$ for some α. It suffices to prove the claim for $j=0$.

Write $c_i = c_{i1}e_1+\cdots+c_{ia}e_a$, where $e_1,\ldots,e_a$ is a basis for $\mathbb{Z}_q$ over $\mathbb{Z}_p$ and where $c_{ij}\in\mathbb{Z}_p$. We claim that if one of the c_{ij} is nonzero mod p, then $\mathrm{Tr}(c_ie_j)$ is nonzero mod p. Writing $\mathrm{Tr}(c_ie_j) = c_{i1}\,\mathrm{Tr}(e_1e_j)+\cdots+c_{ia}\,\mathrm{Tr}(e_ae_j)$, the claim follows as above from the fact that the matrix $\mathrm{Tr}(e_ie_j)$ is nonsingular mod p.

It remains only to show that χ has the explicit description in terms of the sequence c_i given in the statement of the proposition. Let $f(t)\in 1+t\mathbb{F}_q[t]$ denote an irreducible polynomial of degree d with reciprocal root $\bar\lambda$, and let λ denote the Teichmüller lift of $\bar\lambda$. Then $\lambda\in\mathbb{Z}_{q^d}$. For any i, the conjugates of λ^i over $\mathbb{Z}_q$ are $\lambda^i,\lambda^{ip^a},\lambda^{ip^{2a}},\ldots,\lambda^{ip^{(d-1)a}}$. As shown in Example 2.19, the polynomial $f_\lambda(t)$ corresponds to $(\mathrm{Tr}_{\mathbb{Z}_{q^d}/\mathbb{Z}_q}(\lambda^i))_{(i,p)=1}\in\prod_{(i,p)=1}\mathbb{Z}_q$. Then by Proposition 3.4 there are unique elements $c_i\in\mathbb{Z}_q$ such that

$$\begin{aligned}\chi\bigl(f_\lambda(t)\bigr) &= \prod_{(i,p)=1}(1+p)^{\mathrm{Tr}_{\mathbb{Z}_q/\mathbb{Z}_p}(c_i\,\mathrm{Tr}_{\mathbb{Z}_{q^d}/\mathbb{Z}_q}(\lambda^i))}\\ &= \prod_{(i,p)=1}(1+p)^{\mathrm{Tr}_{\mathbb{Z}_q/\mathbb{Z}_p}\circ\mathrm{Tr}_{\mathbb{Z}_{q^d}/\mathbb{Z}_q}(c_i\lambda^i)} = \prod_{(i,p)=1}(1+p)^{\mathrm{Tr}_{\mathbb{Z}_{q^d}/\mathbb{Z}_p}(c_i\lambda^i)}.\end{aligned}$$

∎

PROPOSITION 3.6. *The correspondence described in Proposition 3.5, between characters and sequences, is a bijection.*

Proof. We must show that there is a bijection between continuous homomorphisms $(1+t\mathbb{F}_q[[t]])^* \to \mathbb{Z}_p^*$ and sequences $(c_i)_{(i,p)=1}$ of elements in $\mathbb{Z}_q$ converging to 0. It suffices to show that there is a bijection between continuous homomorphisms

$$\prod_{(i,p)=1} \mathbb{Z}_q \to \mathbb{Z}_p^*$$

and convergent sequences $(c_i)_{(i,p)=1}$. Therefore, it suffices to show that there is a bijection between continuous homomorphisms $\mathbb{Z}_q \to \mathbb{Z}_p^*$ and elements $c \in \mathbb{Z}_q$. This was shown above. □

§4. Meromorphic continuation

For the rest of this article, we assume that $p > 2$.

In this section we use our earlier descriptions of characters and their associated L-functions to address the question of when the L-functions are p-adic meromorphic. Our main strategy is to consider our L-functions as being associated to certain convergent power series and then to use results from [17] to study meromorphic continuation.

4.1. The result for a special character χ

To introduce our techniques, we first consider the simplest nontrivial example in detail. Let $q = p$. We are going to fix a p-adic character $\chi : (1+t\mathbb{F}_p[[t]])^* \to \mathbb{Z}_p^*$, which by Proposition 3.5 is the same as fixing a sequence of elements $(c_i)_{(i,p)=1}$, where each $c_i \in \mathbb{Z}_p$ and where $\lim_{i\to\infty} c_i = 0$. For our simple introductory case, we further assume that $c_i = 0$ for $i > 1$. Write $1 + \pi_1 := (1+p)^{c_1}$.

Fix an irreducible degree d polynomial $f(t) \in 1 + t\mathbb{F}_p[t] \subseteq (1+t\mathbb{F}_p[[t]])^*$. Let $\bar{\lambda}$ denote a reciprocal root of $f(t)$, and let λ denote the Teichmüller lift of $\bar{\lambda}$ to $\mathbb{Z}_{p^d}$. Then for the specific character χ chosen above, by Proposition 3.5 we have

$$\chi : f(t) \mapsto (1+\pi_1)^{\mathrm{Tr}(\lambda)} = (1+\pi_1)^{\lambda}(1+\pi_1)^{\lambda^p} \cdots (1+\pi_1)^{\lambda^{p^{d-1}}}.$$

To this character χ, we can associate an L-function as in Definition 2.3, and we wish to consider the meromorphic continuation of that L-function. For our techniques, it is most convenient to consider this L-function as also being associated to a certain power series in the variable λ, which we now describe.

For each of the above factors, we can associate its binomial power series expansion

$$B_{\pi_1}(\lambda^{p^i}) := (1+\pi_1)^{\lambda^{p^i}} = \sum_{j=0}^{\infty} \binom{\lambda^{p^i}}{j} \pi_1^j.$$

We will consider this as a power series for which λ is the variable and π_1 is some fixed constant in $p\mathbb{Z}_p$. We then have

$$B_{\pi_1}(\lambda) = \sum a_k \lambda^k, \quad a_k \in \frac{\pi_1^k}{k!}\mathbb{Z}_p. \tag{4.0.1}$$

Because $v_p(k!) \leq k/(p-1)$ (see, e.g., [12, p. 79]) and because $v_p(\pi_1^k) \geq k$, we have that

$$v_p(a_k) \geq k\Big(\frac{p-2}{p-1}\Big). \tag{4.0.2}$$

The coefficients are clearly in $\mathbb{Q}_p$. Because $p > 2$, the previous inequality guarantees $a_k \in p\mathbb{Z}_p$ for $k > 0$. In terminology to be introduced now, a power series $\sum a_k \lambda^k \in \mathbb{Z}_p[[\lambda]]$ with coefficients satisfying a growth condition as in (4.0.2) is called *overconvergent*.

Definition 4.1. Let $g(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k \in \widehat{\mathbb{Z}_p^{\mathrm{nr}}}[[\lambda]]$. The power series $g(\lambda)$ is called *convergent* if

$$\liminf v_p(a_k) = \infty.$$

The power series $g(\lambda)$ is called *overconvergent* if

$$\liminf \frac{v_p(a_k)}{k} > 0.$$

For a positive constant $0 < C \leq \infty$, the power series $g(\lambda)$ is called *Clog-convergent* if

$$\liminf \frac{v_p(a_k)}{\log_p k} \geq C.$$

Remark 4.2. The terms *convergent* and *overconvergent* are used because a convergent power series converges on the closed unit disk, while an overconvergent power series converges on some strictly larger disk. A crucial example for us is that the above power series $B_{\pi_1}(\lambda)$ is overconvergent.

Given a convergent power series $g(\lambda)$, we now describe how to associate a character and an L-function to it. We momentarily return to the general case $q = p^a$ because we will need the general definition in Section 4.2.

DEFINITION 4.3. For $q = p^a$, the character χ associated to a convergent power series $g(\lambda) \in \mathbb{Z}_q[[\lambda]]$ is defined to be the unique character which sends an irreducible degree d polynomial over $\mathbb{F}_q$ with reciprocal root $\bar{\lambda}$ to

$$g(\lambda)g^{\sigma}(\lambda^p)\cdots g^{\sigma^{ad-1}}(\lambda^{p^{ad-1}}) = h(\lambda)h(\lambda^q)\cdots h(\lambda^{q^{d-1}}),$$

where λ is the Teichmüller lift of $\bar{\lambda}$ and $h(\lambda)$ is the power series

$$h(\lambda) = g(\lambda)g^{\sigma}(\lambda^p)\cdots g^{\sigma^{a-1}}(\lambda^{p^{a-1}}) \in \mathbb{Z}_q[[\lambda]],$$

and where we write σ for the Frobenius automorphism in $\mathrm{Gal}(\mathbb{Q}_{q^d}/\mathbb{Q}_p)$.

DEFINITION 4.4. The L-function of a convergent power series $h(\lambda) \in \mathbb{Z}_q[[\lambda]]$ over $\mathbb{F}_q$ is defined to be

$$L(h/\mathbb{F}_q, s) := \prod_{\bar{\lambda} \in \overline{\mathbb{F}_q}^*} \frac{1}{(1 - h(\lambda)h(\lambda^q)\cdots h(\lambda^{q^{d-1}})s^d)^{1/d}} \in 1 + s\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[[s]],$$

where λ is the Teichmüller lifting of $\bar{\lambda}$ and d denotes the degree of $\bar{\lambda}$ over $\mathbb{F}_q$. The associated characteristic series is defined to be

$$C(h, s) = \prod_{k=0}^{\infty} L(h/\mathbb{F}_q, q^k s).$$

The following results are known about the meromorphic continuation of the L-function $L(h/\mathbb{F}_q, s)$ and its characteristic series (see [17]).

THEOREM 4.5. *Let $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k \in \mathbb{Z}_q[[\lambda]]$. If the power series $h(\lambda)$ is* overconvergent, *then the characteristic series $C(h/\mathbb{F}_q, s)$ is entire in $|s|_p < \infty$, and thus the L-function $L(h/\mathbb{F}_q, s)$ is p-adic meromorphic in $|s|_p < \infty$.*

More generally, if the power series $h(\lambda)$ is C log-convergent *for some constant $0 < C \le \infty$, then the characteristic series $C(h/\mathbb{F}_q, s)$ is entire in $|s|_p < q^C$, and thus the L-function $L(h/\mathbb{F}_q, s)$ is p-adic meromorphic in the open disk $|s|_p < q^C$.*

Before considering the meromorphic continuation of the L-function associated to a general p-adic character, we return to the simple character χ fixed at the beginning of this section. Recall that we are temporarily assuming that $q = p$.

The L-functions we are considering are related as follows:

$$L(\chi, s) = \prod_{\substack{f(t) \text{ irred poly,} \\ f(t) \equiv 1 \bmod t}} \frac{1}{1 - \chi(f(t))s^{\deg f}}$$

$$= \prod_{\substack{f(t) \text{ irred poly,} \\ f(t) \equiv 1 \bmod t}} \frac{1}{1 - (1+\pi_1)^{\lambda + \lambda^p + \cdots + \lambda^{p^{d-1}}} s^d}$$

(where we write λ for the Teichmüller lift of a reciprocal root of f and write d for the degree of f over $\mathbb{F}_p$)

$$= \prod_{\bar{\lambda} \in \overline{\mathbb{F}_p}^*} \frac{1}{(1 - (1+\pi_1)^{\lambda + \lambda^p + \cdots + \lambda^{p^{d-1}}} s^d)^{1/d}}$$

$$= \prod_{\bar{\lambda} \in \overline{\mathbb{F}_p}^*} \frac{1}{(1 - B_{\pi_1}(\lambda) B_{\pi_1}(\lambda^p) \cdots B_{\pi_1}(\lambda^{p^{d-1}}) s^d)^{1/d}}$$

$$= L(B_{\pi_1}/\mathbb{F}_p, s).$$

Combining this equality with the preceding results, to prove that $L(\chi, s)$ is p-adic meromorphic for our easy introductory example, we need only demonstrate overconvergence or ∞ log-convergence of the power series $B_{\pi_1}(\lambda)$. The overconvergence of this power series was already mentioned in Remark 4.2. This completes our treatment of the simple character χ we fixed at the beginning of this section.

4.2. The general case

Continue to assume that p is odd, but we now allow $q = p^a$ with any $a \geq 1$. For two reasons, working with general characters $\chi : (1 + t\mathbb{F}_q[[t]])^* \to \mathbb{Z}_p^*$ is more difficult than the situation in Section 4.1. First of all, in the associated sequence $(c_i)_{(i,p)=1}$, there will typically be infinitely many nonzero terms. Second, if $q = p^a$ with $a > 1$, then the elements c_i are in $\mathbb{Z}_q$, not $\mathbb{Z}_p$. Thus the absolute traces of the elements $c_i\lambda^i$ are more complicated.

To follow the same approach as above, we want to find a power series in λ from which we can recover the character values found in Proposition 3.5:

$$\chi\big(f_\lambda(t)\big) = \prod_{(i,p)=1} (1+p)^{\mathrm{Tr}_{\mathbb{Z}_{q^d}/\mathbb{Z}_p}(c_i \lambda^i)}.$$

(Here and throughout we let $f_\lambda(t)$ denote a general irreducible polynomial in $1+t\mathbb{F}_q[t]$. We call its degree d, and we write λ for the Teichmüller lift of one of its reciprocal roots. Hence, $\lambda \in \mathbb{Z}_{q^d}$.) If we again let σ denote Frobenius in $\mathrm{Gal}(\mathbb{Q}_{q^d}/\mathbb{Q}_p)$, then we can rewrite

$$\begin{aligned}\chi\big(f_\lambda(t)\big) &= \prod_{j=0}^{ad-1} \prod_{(i,p)=1} (1+p)^{\sigma^j(c_i\lambda^i)} \\ &= \prod_{j=0}^{ad-1} \prod_{(i,p)=1} (1+p)^{\sigma^j(c_i)\lambda^{ip^j}}.\end{aligned}$$

Note now that because $c_i \in \mathbb{Z}_q$, we have $\sigma^j(c_i) = c_i$ whenever j is a multiple of a. Let $\pi_{ij} \in \mathbb{Z}_q$ denote the unique element for which $1+\pi_{ij} = (1+p)^{\sigma^j(c_i)}$. We then have

$$\chi\big(f_\lambda(t)\big) = \prod_{j=0}^{a-1} \prod_{(i,p)=1} (1+\pi_{ij})^{\lambda^{ip^j}} (1+\pi_{ij})^{\lambda^{ip^j q}} \cdots (1+\pi_{ij})^{\lambda^{ip^j q^{d-1}}}.$$

Finally, using the same notation as in Section 4.1, we have

$$\chi\big(f_\lambda(t)\big) = \prod_{j=0}^{a-1} \prod_{(i,p)=1} B_{\pi_{ij}}(\lambda^{ip^j}) B_{\pi_{ij}}(\lambda^{ip^j q}) \cdots B_{\pi_{ij}}(\lambda^{ip^j q^{d-1}}).$$

Abbreviating the sequence of elements π_{ij} by π, we define

$$\mathscr{O}_\pi(\lambda) := \prod_{j=0}^{a-1} \prod_{(i,p)=1} B_{\pi_{ij}}(\lambda^{ip^j}). \tag{4.5.1}$$

Note that this series is independent of d; in other words, it does not depend on the degree of λ over $\mathbb{Z}_q$. We now have immediately that

$$L(\chi, s) = L\big(\mathscr{O}_\pi(\lambda)/\mathbb{F}_q, s\big), \qquad C(\chi, s) = C\big(\mathscr{O}_\pi(\lambda)/\mathbb{F}_q, s\big).$$

This enables us to study $L(\chi, s)$ via the series $\mathscr{O}_\pi(\lambda)$.

THEOREM 4.6. *Fix a prime $p > 2$ and a prime power $q = p^a$. Let*

$$\chi : \big(1 + t\mathbb{F}_q[[t]]\big)^* \to \mathbb{Z}_p^*$$

denote a continuous character, and let $(c_i)_{(i,p)=1}$ denote the sequence of elements in $\mathbb{Z}_q$ defined in Proposition 3.5. If the series $\sum c_i x^i$ is overconvergent, then the characteristic series $C(\chi, s)$ is entire in $|s|_p < \infty$ and the associated L-function $L(\chi, s)$ is p-adic meromorphic in $|s|_p < \infty$.

Proof. By Theorem 4.5, it suffices to prove that $\mathscr{O}_\pi(\lambda)$ is overconvergent. We assume that there exists $C > 0$ such that $v_p(c_i) \geq Ci - ((p-2)/(p-1))$. This is certainly true for suitably large i, and after shrinking C we can assume that it is true for all i.

Define $R_C := \{\sum a_k \lambda^k \mid v_p(a_k) \geq Ck\}$. We want to show that the series $\mathscr{O}_\pi(\lambda)$ is overconvergent. Since R_C is a ring, it suffices to show that each factor $B_{\pi_{ij}}(\lambda^{ip^j}) \in R_{C/(p^{a-1})}$. Write

$$B_{\pi_{ij}}(\lambda^{ip^j}) = \sum_{k=0}^{\infty} a_{ijk} \lambda^{kip^j}.$$

We know as in (4.0.2) that

$$\begin{aligned} v_p(a_{ijk}) &\geq v_p(\pi_{ij}^k) - v_p(k!) \\ &= kv_p(\pi_{ij}) - v_p(k!). \end{aligned}$$

Recalling the definition of π_{ij}, and using the fact that valuation is not changed by automorphisms, $v_p(c_i) = v_p(\sigma^j(c_i))$, we have

$$\begin{aligned} v_p(a_{ijk}) &\geq k\Big(Ci - \frac{p-2}{p-1} + 1\Big) - \frac{k}{p-1} \\ &\geq kCi \\ &\geq \frac{C}{p^{a-1}} kip^j, \end{aligned}$$

as required. ∎

The following theorem, which treats log-convergent series, is a generalization of Theorem 4.6, which treated overconvergent series. We prove the two results separately because, on one hand, the proofs are rather different, and on the other hand, the proof of the log-convergent result is simplified by our ability to reference the proof from the overconvergent case.

THEOREM 4.7. *Keep notation as in Theorem 4.6. If the series* $\sum c_i x^i$ *is* C log*-convergent, then the characteristic series* $C(\chi, s)$ *is entire in the disk* $|s|_p < q^C$ *and the L-function* $L(\chi, s)$ *is p-adic meromorphic in the disk* $|s|_p < q^C$.

Proof. It again suffices to show that $\mathscr{O}_\pi(\lambda)$ is C log-convergent. Fix any ϵ such that $(C/2) > \epsilon > 0$. By our assumption on $\sum c_i x^i$, there exists a constant $M(\epsilon)$ such that for all $i > M(\epsilon)$, we have $v_p(c_i) \geq (C-\epsilon)(\log_p(i)+1)$. Consider $S_C \subseteq 1 + \lambda\mathbb{Z}_q[[\lambda]]$ defined by

$$S_C = \left\{ 1 + \sum_{k=1}^{\infty} a_k \lambda^k \,\middle|\, a_k \in \mathbb{Z}_q, v_p(a_k) \geq C\big(\log_p(k)+1\big) \right\}.$$

An easy computation shows that S_C is a ring. Our strategy is to show that for i suitably large, we have $B_{\pi_{ij}}(\lambda^{ip^j}) \in S_{C-2\epsilon}$, and that for all other i, the power series $B_{\pi_{ij}}(\lambda^{ip^j})$ is overconvergent.

First fix $i > M(\epsilon)$. We proceed as in the proof of Theorem 4.6. Write

$$B_{\pi_{ij}}(\lambda^{ip^j}) = 1 + \sum_{k=1}^{\infty} a_{ijk} \lambda^{kip^j}.$$

We have again

$$\begin{aligned} v_p(a_{ijk}) &\geq v_p(\pi_{ij}^k) - v_p(k!) \\ &= k v_p(\pi_{ij}) - v_p(k!). \end{aligned}$$

Now we have our first departure from the proof of Theorem 4.6, because we now have a weaker growth condition on the sequence (c_i), and hence a weaker growth condition on the terms π_{ij}, which have valuation $v_p(c_i)+1$. In our current case $i > M(\epsilon)$, we have

$$\begin{aligned} v_p(a_{ijk}) &\geq k(C-\epsilon)\big(\log_p(i)+1\big) + k - \frac{k}{p-1} \\ &\geq (C-\epsilon)\big(\log_p(i) + \log_p(k)\big) \\ &= (C-\epsilon)\big(\log_p(i) + \log_p(k) + \log_p(p^j)\big) - (C-\epsilon)j \\ &= (C-\epsilon)\log_p(kip^j) - (C-\epsilon)j \\ &\geq (C-2\epsilon)\log_p(kip^j) + \epsilon\log_p(i) - Cj. \end{aligned}$$

Because C and ϵ are fixed, and because j is bounded, we have that, for all but finitely many i,

$$v_p(a_{ijk}) \geq (C-2\epsilon)\big(\log_p(kip^j)+1\big).$$

Hence, for almost all values of i, we have $B_{\pi_{ij}}(\lambda^{ip^j}) \in S_{C-2\epsilon}$.

Now consider the finitely many remaining values of i. We can find $D > 0$ such that $v_p(\pi_{ij}) \geq D(i+1)$ for all remaining i and all j. We can then use the proof of Theorem 4.6 to see that all the corresponding series $B_{\pi_{ij}}$ are overconvergent.

It is clear that the product of a C log-convergent power series and an overconvergent series is again C log-convergent. We thus have that $\mathscr{O}_\pi(\lambda)$ is $(C-2\epsilon)$ log-convergent for every ϵ in the range $C/2 > \epsilon > 0$. It follows that $\mathscr{O}_\pi(\lambda)$ is C log-convergent. ∎

§5. Converting from a power series to a sequence

We have a bijection between characters χ and sequences (c_i) (see Proposition 3.6). On the other hand, many power series can induce the same character; for example, any power series of the form $g(\lambda)/g^\sigma(\lambda^p)$ induces the trivial character. Thus, the function

$$(c_i) \mapsto \prod_{(i,p)=1} (1+p)^{c_i \lambda^i}$$

cannot be a bijection between sequences and power series. Write F to denote this function. Write $g(\lambda)$ for the image of (c_i) under F. Then, in the notation of (4.5.1), we have $\mathscr{O}_\pi(\lambda) = g(\lambda) g^\sigma(\lambda^p) \cdots g^{\sigma^{a-1}}(\lambda)^{p^{a-1}}$. The point of the function F is that the character associated to (c_i) as in Proposition 3.5 is the same as the character associated to $g(\lambda)$ as in Definition 4.3.

Our goal in this section is to describe a one-sided inverse G from convergent power series in $\mathbb{Z}_q\langle\lambda\rangle$ to sequences of elements in $\mathbb{Z}_q$, written $G : g(\lambda) \mapsto (d_i)$, with the following two properties.

(1) The composition $G \circ F$ is the identity.
(2) If $g(\lambda)$ is C log-convergent, then so is $G(g(\lambda))$.

How should we define G? Given a power series $g(\lambda)$, we would like to find a sequence $(d_i)_{(i,p)=1}$ such that

$$g(\lambda) = (1+p)^{\sum d_i \lambda^i}.$$

Let Log denote the p-adic logarithm (see, e.g., [12, Chapter 4]). (Note the difference in notation between this and the base p logarithm $\log_p$ used to define log-convergence in the last section.) Applying Log to both sides of the above equation, we find that

$$\frac{\mathrm{Log}(g(\lambda))}{\mathrm{Log}(1+p)} = \sum d_i \lambda^i.$$

The problem is that the left-hand side probably includes nonzero terms $d_i\lambda^i$ even when $(i,p) \neq 1$. The following definition helps us remove unwanted p-powers.

DEFINITION 5.1. Define a map $\psi_p : \mathbb{Z}_q\langle\lambda\rangle \to \mathbb{Z}_q\langle\lambda\rangle$ by sending

$$\psi_p : \sum_{k=0}^{\infty} a_k\lambda^k \mapsto \sum_{(i,p)=1} b_i\lambda^i, \quad \text{where } b_i = \sum_{j=0}^{\infty} \sigma^{-j}(a_{ip^j}).$$

REMARK 5.2. Note that our definition makes sense only for convergent power series. Also note that ψ_p is additive but not linear.

LEMMA 5.3. *If $g(\lambda)$ is overconvergent (resp., $C\log$-convergent), then $\psi_p(g(\lambda))$ is overconvergent (resp., $C\log$-convergent).*

Proof. This is obvious. Note that for both of these conditions, the requirement on the coefficient of λ^i is stricter when i is larger. Because ψ_p potentially decreases the i exponents, it will preserve overconvergence and $C\log$-convergence. ∎

The following lemma is the reason Definition 5.1 is useful.

LEMMA 5.4. *Let $c(x) \in \mathbb{Z}_q\langle x\rangle$ denote a convergent power series. Let $\bar{\lambda} \in \overline{\mathbb{F}_p}$ denote some nonzero element of degree d over $\mathbb{F}_q$, let λ denote its Teichmüller lift, and let Tr denote the absolute trace from $\mathbb{Z}_{q^d}$ to $\mathbb{Z}_p$. Then we have*

$$\mathrm{Tr}\big(c(\lambda)\big) = \mathrm{Tr}\big(\psi_p(c)(\lambda)\big).$$

Proof. Because trace is additive (and p-adically continuous), it suffices to show that $\mathrm{Tr}(\sigma^{-1}(c_i)\lambda^i) = \mathrm{Tr}(c_i\lambda^{pi})$ for all i, which is obviously true by the definition of absolute trace. ∎

LEMMA 5.5. *Let $\sum c_i\lambda^i \in 1 + p\lambda\mathbb{Z}_q[[\lambda]]$ denote a $C\log$-convergent series. Then $\mathrm{Log}(\sum c_i\lambda^i)$ is also $C\log$-convergent.*

Proof. By our assumption, for every ϵ in the range $C > \epsilon > 0$, there exists a positive integer constant $M(\epsilon) \geq 2$ such that $v_p(c_k) \geq (C-\epsilon)\log_p(k) + 1$ for all $k > M(\epsilon)$. (The reason for the "+1" will become apparent later.) Write

$$\mathrm{Log}\Big(\sum c_k\lambda^k\Big) =: \sum d_k\lambda^k.$$

Our goal is to show that for every ϵ there exists a constant $N(\epsilon)$ such that $v_p(d_k) \geq (C-\epsilon)\log_p(k)$ whenever $k > N(\epsilon)$.

Using the Taylor series expansion for Log, we have

$$\sum d_k \lambda^k = \sum_{k=1}^{\infty} \sum_{k_1+2k_2+\cdots+rk_r=k} \frac{(-1)^{k_1+\cdots+k_r-1}}{k_1+k_2+\cdots+k_r} \binom{k_1+\cdots+k_r}{k_1,k_2,\ldots,k_r} c_1^{k_1}\cdots c_r^{k_r}\lambda^k.$$

Because multinomial coefficients are integers, we have

$$v_p(d_k) \geq k_1 v_p(c_1) + \cdots + k_r v_p(c_r) - v_p(k_1+\cdots+k_r)$$

for some choice of $k_1, k_2, \ldots, k_r$ satisfying $k_1 + 2k_2 + \cdots + rk_r = k$. Note that we always have $v_p(c_i) \geq 1$. Abbreviate the index $M(\epsilon) \geq 2$ chosen above by s. In the case $s \leq r$, we have

$$\begin{aligned}
v_p(d_k) &\geq k_1 + \cdots + k_{s-1} + \big[k_s(C-\epsilon)\log_p(s) + k_s\big] + \cdots \\
&\quad + \big[k_r(C-\epsilon)\log_p(r) + k_r\big] \\
&\quad - v_p(k_1+k_2+\cdots+k_r) \\
&\geq \frac{1}{2}(k_1+\cdots+k_{s-1}) + (C-\epsilon)\sum_{j=s}^{r} k_j \log_p j \\
&\geq \frac{1}{2}(k_1+\cdots+k_{s-1}) + (C-\epsilon)\log_p\Big(\sum_{j=s}^{r} jk_j\Big).
\end{aligned}$$

If $\sum_{j=s}^{r} jk_j \geq k/2$, then

$$v_p(d_k) \geq (C-\epsilon)\log_p\Big(\frac{1}{2}k\Big) = (C-\epsilon)\log_p k - (C-\epsilon)\log_p 2.$$

If $\sum_{j=s}^{r} jk_j < k/2$, then $\sum_{j=1}^{s-1} jk_j \geq k/2$, and thus

$$v_p(d_k) \geq \frac{1}{2}(k_1+\cdots+k_{s-1}) \geq \frac{1}{4(s-1)}k > (C-\epsilon)\log_p k$$

for all sufficiently large k.

The previous step easily adapts to the case $s > r$, because then

$$v_p(d_k) \geq \frac{1}{2}(k_1+\cdots+k_r) \geq \frac{1}{2r}k > \frac{1}{2s}k > (C-\epsilon)\log_p k$$

for all sufficiently large k.

Combining these cases, we see that

$$\liminf_k \frac{v_p(d_k)}{\log_p k} \geq C - \epsilon.$$

The theorem is proved. ∎

When we assemble the above results, we attain the following. It describes the function G promised at the beginning of this section.

THEOREM 5.6. *Let $g(\lambda)$ denote a $C \log$-convergent series. Convert this into the power series $\psi_p \circ \mathrm{Log}(g(\lambda))$, and let d_i denote the coefficient of λ^i in the new power series. Then the sequence $(d_i)_{(i,p)=1}$ is $C \log$-convergent, and its associated L-series is the same as the L-series associated to $g(\lambda)$.*

We close this section with a special example.

EXAMPLE 5.7. Let $q = p$, and let $C > 0$ denote some constant. Power series were constructed in [17, Theorem 1.2] which were $C \log$-convergent and whose associated L-functions failed to have meromorphic continuation to the disk $|s| < p^{C+\epsilon}$ for any ϵ. We briefly mention some implications in our context. Define

$$g_C(\lambda) = 1 + \sum_{i \geq 1} p^{Ci+1} u_i \lambda^{p^i - 1},$$

where $u_i \in \mathbb{Z}$ is such that the reduction modulo p of $\sum_i u_i t^i$ is not in $\mathbb{F}_p(t)$. This series $g_C(\lambda)$ is clearly $C \log$-convergent, so by Theorem 5.6, the associated sequence $(d_i)_{(i,p)=1}$ is $C \log$-convergent. On the other hand, we know by [17] that the associated L-function is meromorphic in the disk $|s|_p < p^C$ but not meromorphic in any larger disk $|s|_p < p^{C+\epsilon}$ for any $\epsilon > 0$. Then by Theorem 4.7, the associated sequence $(d_i)_{(i,p)=1}$ is not $(C+\epsilon)$-convergent.

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REFERENCES

[1] N. Bourbaki, *Éléments de mathématique: Algèbre commutative, chapitres 8 et 9*, reprint of the 1983 original, Springer, Berlin, 2006. MR 2284892.

[2] R. F. Coleman, *p-adic Banach spaces and families of modular forms*, Invent. Math. **127** (1997), 417–479. MR 1431135. DOI 10.1007/s002220050127.

[3] R. Coleman and B. Mazur, “The eigencurve” in *Galois Representations in Arithmetic Algebraic Geometry (Durham, England, 1996)*, London Math. Soc. Lecture Note Ser. **254**, Cambridge University Press, Cambridge, 1998, 1–113. MR 1696469. DOI 10.1017/CBO9780511662010.003.

[4] R. Crew, *L-functions of p-adic characters and geometric Iwasawa theory*, Invent. Math. **88** (1987), 395–403. MR 0880957. DOI 10.1007/BF01388914.

[5] B. Dwork, *Normalized period matrices, I: Plane curves*, Ann. of Math. (2) **94** (1971), 337–388. MR 0396579.

[6] ———, *Normalized period matrices, II*, Ann. of Math. (2) **98** (1973), 1–57. MR 0396580.

[7] M. Emerton and M. Kisin, *Unit L-functions and a conjecture of Katz*, Ann. of Math. (2) **153** (2001), 329–354. MR 1829753. DOI 10.2307/2661344.

[8] M. Hazewinkel, *Formal Groups and Applications*, Pure Appl. Math. **78**, Academic Press, New York, 1978. MR 0506881.

[9] L. Hesselholt, *The big de Rham–Witt complex*, preprint, `http://www.math.nagoya-u.ac.jp/~larsh/papers/028/` (accessed 27 August 2013).

[10] N. Katz, “Travaux de Dwork” in *Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 409*, Lecture Notes in Math. **317**, Springer, Berlin, 1973, 167–200. MR 0498577.

[11] K. S. Kedlaya, J. Pottharst, and L. Xiao, *Cohomology of arithmetic families of* (φ,Γ)*-modules*, preprint, `arXiv:1203.5718` [math.NT].

[12] N. Koblitz, *p-Adic Numbers, p-Adic Analysis, and zeta-Functions*, 2nd ed., Grad. Texts in Math. **58**, Springer, New York, 1984. MR 0754003. DOI 10.1007/978-1-4612-1112-9.

[13] H. Lenstra, *Construction of the ring of Witt vectors*, preprint, `http://www.math.leidenuniv.nl/~hwl` (accessed 27 August 2013).

[14] C. Liu and D. Wan, *T-adic exponential sums over finite fields*, Algebra Number Theory **3** (2009), 489–509. MR 2578886. DOI 10.2140/ant.2009.3.489.

[15] J. Rabinoff, *The theory of Witt vectors*, preprint, `http://www.math.harvard.edu/~rabinoff/misc/witt.pdf` (accessed 27 August 2013).

[16] J.-P. Serre, *Local Fields*, Grad. Texts in Math. **67**, Springer, New York, 1979. MR 0554237.

[17] D. Wan, *Meromorphic continuation of L-functions of p-adic representations*, Ann. of Math. (2) **143** (1996), 469–498. MR 1394966. DOI 10.2307/2118533.

[18] ———, *Dwork’s conjecture on unit root zeta functions*, Ann. of Math. (2) **150** (1999), 867–927. MR 1740990. DOI 10.2307/121058.

[19] ———, *Higher rank case of Dwork’s conjecture*, J. Amer. Math. Soc. **13** (2000), 807–852. MR 1775738. DOI 10.1090/S0894-0347-00-00339-8.

[20] ———, *Rank one case of Dwork’s conjecture*, J. Amer. Math. Soc. **13** (2000), 853–908. MR 1775761. DOI 10.1090/S0894-0347-00-00340-4.

[21] ———, *Variation of p-adic Newton polygons for L-functions of exponential sums*, Asian J. Math. **8** (2004), 427–471. MR 2129244.

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