THE BRAUER–MANIN PAIRING, CLASS FIELD THEORY, AND MOTIVIC HOMOLOGY

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Abstract. For a smooth proper variety over a p-adic field, its Brauer group and abelian fundamental group are related to higher Chow groups by the Brauer–Manin pairing and class field theory. We generalize this relation to smooth (possibly nonproper) varieties, using motivic homology and a variant of Wiesend's ideal class group. Several examples are discussed.

§1. Introduction

1.1. The Brauer–Manin pairing and class field theory

Let p be a prime number, and let k be a finite extension of $\mathbb{Q}_p$. Let X be a smooth variety over k. For $i \in \mathbb{Z}_{\geq 0}$, we write $X_{(i)}$ for the set of all points of X of dimension i. Let $\mathrm{Br}(X)$ be the cohomological Brauer group of X, and let $\pi_1^{\mathrm{ab}}(X)$ be the abelian étale fundamental group of X. For any $x \in X_{(0)}$, local class field theory yields canonical maps

$$\psi_x^* : \mathrm{Br}(x) \cong \mathbb{Q}/\mathbb{Z}, \rho_x : k(x)^* \to \pi_1^{\mathrm{ab}}(x).$$

As for the first map, it is convenient for our purpose to consider its dual. Putting $A^* := \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ for an abelian group A, we define ψ_x to be the composition of the dual of ψ_x^* and the canonical inclusion

$$\psi_x : \mathbb{Z} \hookrightarrow \hat{\mathbb{Z}} \cong \mathrm{Br}(x)^*.$$

Since both of $\mathrm{Br}(-)^*$ and $\pi_1^{\mathrm{ab}}(-)$ are covariant functorial, we get homomorphisms

$$\tilde{\psi}_X : Z_0(X) := \bigoplus_{x \in X_{(0)}} \mathbb{Z} \to \mathrm{Br}(X)^*,$$

(1.1.1)

$$\tilde{\rho}_X : Z_0^1(X) := \bigoplus_{x \in X_{(0)}} k(x)^* \to \pi_1^{\mathrm{ab}}(X)$$

by taking the direct sum of the ψ_x and ρ_x values.

When X is proper over k, Manin [23], Bloch [1], and Saito [29] observed that $\tilde{\psi}_X$ and $\tilde{\rho}_X$ factor, respectively, through

$$\mathrm{CH}_0(X) := \mathrm{coker}\Big[\bigoplus_{y \in X_{(1)}} k(y)^* \to Z_0(X) \Big],$$

$$SK_1(X) := \mathrm{coker}\Big[\bigoplus_{y \in X_{(1)}} K_2 k(y) \to Z_0^1(X) \Big];$$

the induced pairing $\mathrm{CH}_0(X) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$ and the induced map $SK_1(X) \to \pi_1^{\mathrm{ab}}(X)$ are called the *Brauer–Manin pairing* and *reciprocity map* of class field theory, respectively. Both are studied intensively by several authors (for the Brauer–Manin pairing, see [4]–[6], [28], [31], and [46]; for the reciprocity map, see [18], [19], [32], [40], [41], and [47]).

If X is not proper over k, however, $\tilde{\psi}_X$ and $\tilde{\rho}_X$ do not factor through $\mathrm{CH}_0(X)$ or $SK_1(X)$. To this end, we will introduce good quotients of $Z_0(X)$ and $Z_0^1(X)$.

1.2. Wiesend's tame ideal class group

Let V be a variety over a field F. Take $y \in V_{(1)}$. Let $C(y)$ be the closure of $\{y\}$ in V, let $\tilde{C}(y) \twoheadrightarrow C(y)$ be the normalization, let $\tilde{C}(y) \hookrightarrow \bar{C}(y)$ be the smooth completion, and let $C_\infty(y) = \bar{C}(y) \setminus \tilde{C}(y)$. For $x \in C_\infty(y)$, we take a uniformizer $\pi_x \in F(y)^*$ at x. We define for $r \in \mathbb{Z}_{>0}$

$$UK_r^M F(y) := \ker\Big[K_r^M F(y) \to \bigoplus_{x \in C_\infty(y)} \big(K_{r-1}^M F(x) \oplus K_r^M F(x) \big) \Big].$$

Here the x-component of the map is defined by $a \mapsto (\partial_x(a), \partial_x(\{\pi_x\} \cup a))$ for $a \in K_r^M F(y)$, where ∂_x is the tame symbol at x. This group does not depend on the choice of π_x. If V is proper over F, then $UK_r^M F(y) = K_r^M F(y)$. For example, $UK_1^M F(y)$ is the group of rational functions on $\bar{C}(y)$ which takes value 1 at all points of $C_\infty(y)$. The following definition is a natural outcome of an idea of Wiesend [45].

DEFINITION 1.1. Let V be a variety over a field F, and let $r \in \mathbb{Z}_{\geq 0}$. We define *Wiesend's tame ideal class group* of degree r to be

$$C_r(V) := \operatorname{coker}\Big[\bigoplus_{y \in V_{(1)}} U K_{r+1}^M F(y) \hookrightarrow \bigoplus_{y \in V_{(1)}} K_{r+1}^M F(y) \overset{(*)}{\to} \bigoplus_{x \in V_{(0)}} K_r^M F(x) \Big],$$

where $(*)$ is the boundary map of Gersten complex of the Milnor K-sheaf. (In particular, its image is in the direct sum.)

By definition, $C_0(V)$ and $C_1(V)$ are quotients of $Z_0(V)$ and $Z_0^1(V)$, respectively. If V is proper over F, then we have $C_0(V) = \mathrm{CH}_0(V)$ and $C_1(V) = SK_1(V)$.

REMARK 1.2. Suppose that $V = \bar{C} \setminus C_\infty$, where $\bar{C}$ is a smooth projective geometrically irreducible curve over F and where C_∞ is a closed reduced subvariety of $\bar{C}$. Then our definition of $C_0(V)$ coincides with that of the *group of classes of divisors on $\bar{C}$ prime to C_∞ modulo C_∞-equivalence* defined in [36, Chapitre 5, Section 2]. It is proved in [36, Chapitre 5, Théorème 1] that $\ker(\deg : C_0(V) \to \mathbb{Z})$ is represented by a semiabelian variety J, which is called the *generalized Jacobian variety* of $\bar{C}$ with modulus C_∞. (Here the degree map $\deg : C_0(V) \to \mathbb{Z}$ is induced by the usual degree map $Z_0(V) \to \mathbb{Z}$.) For simplicity, we call J the *generalized Jacobian* of V. One can also interpret $C_0(V)$ as the Picard group of the singular curve obtained from $\bar{C}$ by contracting all points of C_∞ to one point (see [36, Chapitre 5, Section 2, and Chapitre 4, Section 4]). On the other hand, $C_0(V)$ is also isomorphic to the relative Picard group $\mathrm{Pic}(\bar{C}, C_\infty)$ (see [38, Theorem 3.1]).

1.3. Motivic homology

Let V be a smooth variety over a perfect field F, and let $i, j \in \mathbb{Z}$. Its motivic homology $H_i^M(V, \mathbb{Z}(j))$ is defined to be the group of homomorphisms $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(F)}(\mathbb{Z}(j)[i], M(V))$ in Voevodsky's category (see [11], [26, (14.17)]). When $j = 0$, the group $H_i^M(V, \mathbb{Z}(0))$ agrees with Suslin's algebraic singular homology $h_i(V)$ (see [38]) and admits a (relatively simple) description in terms of algebraic cycles (see [42, (3.2.7)], [26, (14.18)]). There is a similar but complicated description also for $j \neq 0$ (see [11, (9.4)]). The following theorem, which plays a crucial role in our paper, provides a simpler description in a special case.

THEOREM 1.3. *Let V be a smooth variety over a perfect field F, and let $r \in \mathbb{Z}_{\geq 0}$. Then there is a canonical isomorphism*

$$(1.3.1) \qquad\qquad C_r(V) \cong H_{-r}^M\big(V, \mathbb{Z}(-r)\big).$$

When $r = 0$, using the comparison of $H_0^M(V, \mathbb{Z}(0))$ with $h_0(V)$ recalled above, this follows from a result of Schmidt ([34, Theorem 5.1]). We will prove this theorem by reducing to the case $r = 0$ in Section 2. In what follows, we often identify $C_r(V)$ and $H_{-r}^M(V, \mathbb{Z}(-r))$.

REMARK 1.4. For a smooth projective variety V over a finite field, unramified class field theory (see [21]) relates $\mathrm{CH}_0(V)$ with the abelian fundamental group of V. This has been generalized by Schmidt and Spieß [35] to a smooth (possibly nonproper) variety V, in which $\mathrm{CH}_0(V)$ is replaced by Suslin's algebraic singular homology $h_0(V)$. (Note that $h_0(V) \cong C_0(V)$ by Schmidt's theorem mentioned above. See also recent works of Geisser [12], [13] for a further generalization.) The basic strategy of our paper is to follow their argument over a p-adic base field.

1.4. Open varieties over a p-adic field

We go back to the situation of Section 1.1. Schmidt and Spieß [35] constructed a map connecting motivic homology and étale cohomology with compact support, which we will recall in Section 3. As an application, we deduce the following proposition in Section 4.

PROPOSITION 1.5. *Let X be a smooth variety over a finite extension k of $\mathbb{Q}_p$. The homomorphisms (1.1.1) induce well-defined homomorphisms*

$$(1.4.1) \qquad \psi_X : C_0(X) \to \mathrm{Br}(X)^*, \qquad \rho_X : C_1(X) \to \pi_1^{\mathrm{ab}}(X).$$

In Section 5, we consider the case where X is a curve. In this case, the map ψ_X was already studied by Scheiderer and van Hamel [33] (see Theorem 5.1 below), and the map ρ_X is closely related to work of Hiranouchi [16] (see Theorem 5.4 and Remark 5.5 below). When X is of dimension 2 or higher, the maps ψ_X and ρ_X are not very close to an isomorphism even if X is projective over k. We study several examples of surfaces in Section 6. As a sample, here we mention the following result. (We write $\bar{V} = V \times_k \bar{k}$ for a variety V over a field k with an algebraic closure $\bar{k}$.)

THEOREM 1.6. *Let X be a smooth projective geometrically irreducible surface over a finite extension k of $\mathbb{Q}_p$. Suppose that $\bar{X}$ is rational. Let U be an open subvariety of X.*

(1) *Suppose that the irreducible components of $\bar{X} \setminus \bar{U}$ generate the Néron–Severi group $NS(\bar{X})$ of $\bar{X}$. Then, the kernel of ψ_U is the maximal divisible subgroup of $C_0(U)$. However, there is an example of U such that $\ker(\psi_U \otimes \mathbb{Z}/n\mathbb{Z}) \neq 0$ for all $n \in \mathbb{Z}_{>0}$ divisible by some (fixed) integer N.*
(2) *The kernel of ρ_U is the maximal divisible subgroup of $C_1(U)$, and $\rho_U \otimes \mathbb{Z}/n\mathbb{Z}$ is bijective for all $n \in \mathbb{Z}_{>0}$.*

Note that, if X is a (projective smooth) rational surface, then $\ker(\psi_X) = 0$ and $\ker(\psi_X \otimes \mathbb{Z}/n\mathbb{Z}) = 0$ for all sufficiently divisible n (see Theorem 6.2). Note also that there are examples of (nonrational) projective smooth surfaces X and X' for which $\ker(\psi_X)$ and $\ker(\rho_{X'})$ are not divisible (see [28], [32]).

1.5. Conventions

Let A be an abelian group. For a nonzero integer n, we write $A[n]$ and A/n for the kernel and cokernel of the map $n : A \to A$. This notation is sometimes used when $n = \infty$, in which case we mean that $A[n] = A_{\mathrm{Tor}}$ is the subgroup of torsion elements in A, and $A/n = A \otimes \mathbb{Q}/\mathbb{Z}$. We write $A_{\mathbb{Q}}$ for $A \otimes_{\mathbb{Z}} \mathbb{Q}$. We define $A_{\mathrm{Div}} := \mathrm{Im}[\mathrm{Hom}(\mathbb{Q}, A) \to \mathrm{Hom}(\mathbb{Z}, A) = A]$ to be the maximal divisible subgroup in A, and we define $A_{\mathrm{div}} := \bigcap_{n \in \mathbb{Z}_{>0}} nA$ to be the subgroup of divisible elements in A. Note that we always have $A_{\mathrm{Div}} \subset A_{\mathrm{div}}$ and that $A_{\mathrm{Div}} = A_{\mathrm{div}}$ holds if $A[n]$ is finite for all $n \in \mathbb{Z}_{>0}$.

Let $f : A \to B$ be a homomorphism of abelian groups. We write f/n for $f \otimes \mathbb{Z}/n$ when $n \in \mathbb{Z}_{>0}$, and for $f \otimes \mathbb{Q}/\mathbb{Z}$ when $n = \infty$. Let $m, n \in \mathbb{Z}_{>0}$. The map $\ker(f/n) \to \ker(f/nm)$ induced by the map $m : A \to A$ is designated the canonical map, so that $\{\ker(f/n)\}_n$ becomes an inductive system whose limit is $\ker(f \otimes \mathbb{Q}/\mathbb{Z})$. The map $\ker(f/nm) \to \ker(f/n)$ induced by the identity map on A is designated the canonical map, so that $\{\ker(f/n)\}_n$ becomes an inverse system.

Let F be a field. A separated scheme of finite type over F is called a *variety over F*. Let X be a variety over F. For $i \in \mathbb{Z}_{\geq 0}$, we write $X_{(i)}$ and $X^{(i)}$ for the set of all points on X of dimension i and of codimension i, respectively. We write $\bar{F}$ for an algebraic closure of F, and we write $\bar{X}$ for the base change $X \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \bar{F}$. A closed subset of X is always regarded as a *reduced* subvariety of X.

§2. Wiesend's ideal class group and motivic homology

Let F be a perfect field. Except in the beginning of Section 2.1 and in Section 2.3, we will assume that the characteristic of F is zero.

2.1. Motivic homology and cohomology

We recall some facts from [42] and [26]. Let $\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(F)$ denote the rigid triangulated tensor category of effective motivic complexes (see [42, Section 3.1], [26, Definition 14.1]). There is a functor M from the category of varieties over F to $\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(F)$. For a variety X and $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, we write $M(X, \mathbb{Z}/n\mathbb{Z}) = M(X) \otimes^{\mathbb{L}} \mathbb{Z}/n\mathbb{Z}$ (see Section 1.5). Let $i, j \in \mathbb{Z}$. Motivic cohomology and motivic homology of a variety X are defined by

$$H_M^i\big(X, \mathbb{Z}(j)\big) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(F)}\big(M(X), \mathbb{Z}(j)[i]\big),$$

$$H_i^M\big(X, \mathbb{Z}(j)\big) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(F)}\big(\mathbb{Z}(j)[i], M(X)\big).$$

For $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, their coefficient versions are defined by

$$H_M^i\big(X, \mathbb{Z}/n\mathbb{Z}(j)\big) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(k)}\big(M(X), \mathbb{Z}/n\mathbb{Z}(j)[i]\big),$$

$$H_i^M\big(X, \mathbb{Z}/n\mathbb{Z}(j)\big) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(k)}\big(\mathbb{Z}(j)[i], M(X, \mathbb{Z}/n\mathbb{Z})\big).$$

They fit into (obvious) exact sequences

$$(2.1.1) \qquad 0 \to H_M^i\big(X, \mathbb{Z}(j)\big)/n$$
$$\to H_M^i\big(X, \mathbb{Z}/n\mathbb{Z}(j)\big) \to H_M^{i+1}\big(X, \mathbb{Z}(j)\big)[n] \to 0,$$

$$(2.1.2) \qquad 0 \to H_i^M\big(X, \mathbb{Z}(j)\big)/n$$
$$\to H_i^M\big(X, \mathbb{Z}/n\mathbb{Z}(j)\big) \to H_{i-1}^M\big(X, \mathbb{Z}(j)\big)[n] \to 0.$$

Motivic cohomology $H_M^i(X, \mathbb{Z}(j))$ is contravariantly functorial in X. It also has covariant functionality: if $f : X \to Y$ is a proper flat equidimensional morphism of relative dimension d, then there is an induced map $H_M^i(X, \mathbb{Z}(j)) \to H_M^{i-2d}(Y, \mathbb{Z}(j-d))$. Motivic homology $H_i^M(X, \mathbb{Z}(j))$ is covariantly functorial in X. It also has contravariant functionality: if $f : X \to Y$ is a proper flat equidimensional morphism of relative dimension d, then there is an induced map $H_i^M(Y, \mathbb{Z}(j)) \to H_{i+2d}^M(X, \mathbb{Z}(j+d))$.

The following fact will play an important role in the proof of Theorem 1.3: for any variety X, there is a decomposition

$$(2.1.3) \qquad H_i^M\big(X \times \mathbb{G}_m, \mathbb{Z}(j)\big) \cong H_{i-1}^M\big(X, \mathbb{Z}(j-1)\big) \oplus H_i^M\big(X, \mathbb{Z}(j)\big).$$

This is deduced from the Mayer–Vietoris sequence, the projective bundle formula, and $\mathbb{A}^1$-homotopy invariance.

From now through the end of Section 2.2, we assume that the characteristic of F is zero. If X is a smooth variety, we have a canonical isomorphism

$$(2.1.4) \qquad H_M^i\big(X,\mathbb{Z}(j)\big) \cong \mathrm{CH}^j(X, 2j-i),$$

where the right-hand side is Bloch's higher Chow group. If X is a smooth projective variety of pure dimension d, we also have

$$(2.1.5) \qquad H_i^M\big(X,\mathbb{Z}(j)\big) \cong H_M^{2d-i}\big(X,\mathbb{Z}(d-j)\big) \cong \mathrm{CH}^{d-j}(X, i-2j).$$

In particular, if $X = \mathrm{Spec}\,F$, then we have, for any $r \in \mathbb{Z}_{\geq 0}$,

$$(2.1.6) \qquad \begin{aligned} H_{-r}^M\big(\mathrm{Spec}\,F,\mathbb{Z}(-r)\big) &\cong H_M^r\big(\mathrm{Spec}\,F,\mathbb{Z}(r)\big) \\ &\cong \mathrm{CH}^r(\mathrm{Spec}\,F, r) \cong K_r^M F. \end{aligned}$$

Let Z be a closed subvariety of a smooth variety X. Suppose that Z is smooth and of pure codimension c. Then we have a long exact sequence

$$(2.1.7) \qquad \begin{aligned} \cdots \to H_{i+1}^M\big(X,\mathbb{Z}(j)\big) &\to H_{i+1-2c}^M\big(Z,\mathbb{Z}(j-c)\big) \\ &\to H_i^M\big(X \setminus Z,\mathbb{Z}(j)\big) \to H_i^M\big(X,\mathbb{Z}(j)\big) \to \cdots. \end{aligned}$$

LEMMA 2.1. *Let X be a smooth variety of pure dimension d, and let $i, j \in \mathbb{Z}$.*
(1) *Suppose that $j < 0, i > j+d$, or $i > 2j$. Then we have $H_M^i(X,\mathbb{Z}(j)) = 0$.*
(2) *Suppose that $j > i$. Then we have $H_i^M(X,\mathbb{Z}(j)) = 0$.*
The same holds for the $\mathbb{Z}/n$-coefficient version for any $n \in \mathbb{Z}_{>0} \cup \{\infty\}$.

Proof. For (1), see [26, (3.6), (19.3)]. If X is projective, (2) follows from (1) and (2.1.5). The general case follows by induction on $\dim X$ and (2.1.7). $\quad\square$

By Lemma 2.1 and (2.1.2), we can identify $H_i^M(X,\mathbb{Z}(i))/n = H_i^M(X, \mathbb{Z}/n(i))$ for any $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{>0}$, which will be frequently used without further notice.

REMARK 2.2. If X is a smooth variety of pure dimension d, then motivic homology $H_i^M(X,\mathbb{Z}(j))$ is isomorphic to motivic cohomology with compact support $H_{M,c}^{2d-i}(X,\mathbb{Z}(d-j))$. However, the work of Geisser (see Remark 1.4) suggests that motivic homology would be better for a further generalization, so we opt to use homology theory.

2.2. Motivic complex

We recall some results on the motivic complex. Let X be a smooth variety over F, and let $j \in \mathbb{Z}_{\geq 0}$. There is a complex $\mathbb{Z}(j)_X$ of Zariski sheaves on X (see [26, (3.1)]), which is concentrated in degrees $\leq j$. The hypercohomology of $\mathbb{Z}(j)_X$ agrees with motivic cohomology (see [26, (14.16)]):

$$H^i_{\mathrm{Zar}}\big(X, \mathbb{Z}(j)_X\big) \cong H^i_M\big(X, \mathbb{Z}(j)\big).$$

Thanks to the recent resolution of Bloch–Kato conjecture by Rost and Voevodsky (see [37], [43], [44]), the following important result of Suslin and Voevodsky [39] (see also Geisser and Levine [14]) holds unconditionally.

THEOREM 2.3 ([14], [39]). *Let X be a smooth variety over F, and let $\pi : X_{\text{ét}} \to X_{\mathrm{Zar}}$ be the natural map of sites. Let $j \in \mathbb{Z}_{\geq 0}$, and let $n \in \mathbb{Z}_{>0}$.*

(1) *There is a canonical isomorphism*

$$(2.2.1) \qquad\qquad \pi^* \mathbb{Z}(j)_X \otimes^{\mathbb{L}} \mathbb{Z}/n \cong \mu_n^{\otimes j}.$$

Consequently, we have a canonical map for any $i \in \mathbb{Z}_{\geq 0}$

$$(2.2.2) \qquad\qquad H^i_M\big(X, \mathbb{Z}/n(j)\big) \to H^i_{\text{ét}}(X, \mu_n^{\otimes j}).$$

(2) *The map (2.2.1) induces an isomorphism*

$$\mathbb{Z}(j)_X \otimes^{\mathbb{L}} \mathbb{Z}/n \to \tau_{\leq j} R\pi_* \mu_n^{\otimes j}.$$

Consequently, the map (2.2.2) is an isomorphism if either $i \leq j$ or $j \geq \dim X + \mathrm{cd}\, F$. (Here cd means the cohomological dimension.) It is an injection if $i = j + 1$.

In what follows, we frequently write $H^i_{\text{ét}}(X, \mathbb{Z}/n(j))$ for $H^i_{\text{ét}}(X, \mu_n^{\otimes j})$.

2.3. Wiesend's ideal class group

In this section, we do not assume that $\mathrm{Char}\, F = 0$. We will prove Theorem 1.3 after introducing a simple lemma.

LEMMA 2.4. *Let $r \in \mathbb{Z}_{\geq 0}$. Let $f : X \to X'$ be a morphism of varieties over F. There is a unique homomorphism $f_* : C_r(X) \to C_r(X')$ characterized by the following property: for any $x \in X_{(0)}$, the diagram*

$$
\begin{array}{ccc}
K^M_r F(x) & \to & C_r(X) \\[1em]
\downarrow{\scriptstyle N_{F(x)/F(f(x))}} & & \downarrow{\scriptstyle f_*} \\[1em]
K^M_r F\big(f(x)\big) & \to & C_r(X')
\end{array}
$$

commutes. (Here the upper and lower horizontal maps are the natural map to the x-component and $f(x)$-component, respectively.) This makes C_r a covariant functor on the category of varieties over F.

Proof. Uniqueness is clear by the definition of $C_r(X)$. Functoriality follows from that of the norm map of Milnor K-groups. We prove well-definedness. Take $y \in X_{(1)}$, and put $y' = f(y)$. We need to show that the image of $UK_{r+1}^M F(y)$ in $C_r(X')$ is trivial. We consider two cases separately. First, we assume that $y' \in X'_{(0)}$. With the notation of Section 1.2, we regard $\bar{C}(y)$ as a curve over $F(y')$. Weil reciprocity (see [15, Proposition 7.4.4]) shows that the composition map

$$K_{r+1}^M F(y) \overset{\oplus \partial_x}{\to} \bigoplus_{x \in \bar{C}(y)_{(0)}} K_r^M F(x) \overset{\oplus N_{F(x)/F(y')}}{\to} K_r^M F(y')$$

is the zero map. In view of the definition of $UK_{r+1}^M F(y)$, this proves our assertion in this case. Next, we assume that $y' \in X'_{(1)}$, so that f induces a morphism $f_y : \bar{C}(y) \to \bar{C}(y')$ such that $f_y^{-1}(C_\infty(y')) \subset C_\infty(y)$. It suffices to show that the image of $UK_{r+1}^M F(y)$ by the norm map $K_{r+1}^M F(y) \to K_{r+1}^M F(y')$ is contained in $UK_{r+1}^M F(y')$. This follows from the following basic fact on the tame symbol. (We omit its proof.) □

LEMMA 2.5. *Let L/K be a finite extension of fields. Let v be a discrete valuation on K, and let $\{v_1, \ldots, v_f\}$ be the set of all extensions of v to L. We write C and D_i ($1 \le i \le f$) for the residue field of v and v_i, respectively. Let $r \in \mathbb{Z}_{>0}$. For any $\lambda \in K_r^M L$, we have*

$$\partial_v N_{L/K} \lambda = \sum_{i=1}^{f} N_{D_i/C} \partial_{v_i} \lambda.$$

Proof of Theorem 1.3. The case $r = 0$ is a conjunction of the comparison theorem of motivic homology with Suslin's homology (see [42, Corollary 3.2.7], [26, Proposition 14.18]) and the comparison theorem of Suslin's homology and $C_0(V)$ (see [34, Theorem 5.1]). We proceed by induction on r.

We construct a commutative diagram

$$\bigoplus_{y\in(V\times\mathbb{G}_m)_{(1)}} UK_{r+1}^M F(y) \quad \overset{\partial}{\to} \quad \bigoplus_{x\in(V\times\mathbb{G}_m)_{(0)}} K_r^M F(x)$$

(2.3.1)
$$\downarrow f \qquad\qquad\qquad\qquad \downarrow g$$

$$\bigoplus_{w\in V_{(1)}} UK_{r+2}^M F(w) \quad \overset{\partial}{\to} \quad \bigoplus_{z\in V_{(0)}} K_{r+1}^M F(z)$$

The two horizontal maps ∂ are given by the tame symbol. For $x \in (V \times \mathbb{G}_m)_{(0)}$ and $z \in V_{(0)}$, the (x,z)-component of g is given as follows. It is the zero map if $z \neq p_0(x)$, where $p_0 : V \times \mathbb{G}_m \to V$ is the projection. Suppose that $z = p_0(x)$, so that $F(x)$ is a finite extension of $F(z)$. The composition $x \to V \times \mathbb{G}_m \to \mathbb{G}_m$ defines an element $\xi(x) \in \mathbb{G}_m(F(x)) = F(x)^*$. Now the map in question is given by

$$K_r^M F(x) \overset{\text{mult. by } \xi(x)}{\to} K_{r+1}^M F(x) \overset{N_{F(x)/F(z)}}{\to} K_{r+1}^M F(z).$$

Next, let $y \in (V \times \mathbb{G}_m)_{(1)}$, and let $w \in V_{(1)}$. If $w \neq p_0(y)$, then the (y,w)-component of f is the zero map. Suppose that $w = p_0(y)$, so that $F(y)$ is a finite extension of $F(w)$. The composition $y \to V \times \mathbb{G}_m \to \mathbb{G}_m$ defines an element $\xi(y) \in \mathbb{G}_m(F(y)) = F(y)^*$. Now the (y,w)-component of f is given by

$$UK_{r+1}^M F(y) \overset{\text{mult. by } \xi(y)}{\to} UK_{r+2}^M F(y) \overset{N_{F(y)/F(w)}}{\to} UK_{r+2}^M F(w).$$

We check the commutativity. Let $y \in (V \times \mathbb{G}_m)_{(1)}$, and let $\lambda \in UK_{r+1}^M F(y)$. Put $w = p_0(y)$. When $w \in V_{(0)}$, by definition we have $\partial(f(\lambda)) = 0$, and setting $z = w$, we have

$$g\big(\partial(\lambda)\big) = g\Big(\sum_{x\in C(y)} \partial_x \lambda\Big) = \sum_{x\in\tilde{C}(y)} N_{F(x)/F(z)}\big\{\partial_x\lambda, \xi(x)\big\}$$

$$\overset{(1)}{=} \sum_{x\in\tilde{C}(y)} N_{F(x)/F(z)}\partial_x\big\{\lambda, \xi(y)\big\}$$

$$\overset{(2)}{=} \Big[\sum_{x\in\tilde{C}(y)} + \sum_{x\in C_\infty(y)} \Big]\big(N_{F(x)/F(z)}\partial_x\big\{\lambda, \xi(y)\big\}\big)$$

$$\overset{(3)}{=} 0.$$

Here, at (1) we used the fact that for all $x \in \tilde{C}(y)$ we have $\text{ord}_x \xi(y) = 0$ and $\partial_x\{\xi(y), \pi_x\} = \xi(x)$, where $\pi_x \in k(y)^*$ is a uniformizer at x. At (2), we

used the definition of $UK_{r+1}^M F(y)$ (i.e., $\partial_x(\lambda) = 0$ and $\partial_x\{\lambda, \pi_x\} = 0$ for all $x \in C_\infty(y)$). The equality (3) is Weil reciprocity (see [15, Proposition 7.4.4]). Now we suppose that $w \in V_{(1)}$. We take $z \in V_{(0)}$, and we write $\{x_1, \ldots, x_n\}$ for the set of all points on $\tilde{C}(y)$ above z. We have

$$z\text{-component of } \partial\big(f(\lambda)\big) = \partial_z N_{F(y)/F(w)}\{\lambda, \xi(y)\}$$

$$\overset{(1)}{=} \sum_{i=1}^n N_{F(x_i)/F(z)} \partial_{x_i}\{\lambda, \xi(y)\}$$

$$\overset{(2)}{=} \sum_{i=1}^n N_{F(x_i)/F(z)}\{\partial_{x_i}(\lambda), \xi(x_i)\}$$

$$= z\text{-component of } g\big(\partial(\lambda)\big).$$

At (1), we used Lemma 2.5. The equality (2) follows from the fact that $\mathrm{ord}_x \xi(y) = 0$ and $\partial_x\{\xi(y), \pi_x\} = \xi(x)$ for all $x \in \tilde{C}(y)$. This proves the commutativity.

It can be seen without difficulty that both f and g are surjective. As a consequence, we get a surjection

$$\alpha : C_r(V \times \mathbb{G}_m) \to C_{r+1}(V).$$

It follows from the definition that the composition $C_r(V) = C_r(V \times \{1\}) \to C_r(V \times \mathbb{G}_m) \overset{\alpha}{\to} C_{r+1}(V)$ is the zero map.

Next, we consider a diagram (see [20, (4.3)])

$$(2.3.2) \qquad \begin{array}{ccc} \bigoplus_{x \in (V \times \mathbb{G}_m)_{(0)}} K_r^M F(x) & \to & H_{-r}^M(V \times \mathbb{G}_m, \mathbb{Z}(-r)) \\ \downarrow^g & & \downarrow \\ \bigoplus_{z \in V_{(0)}} K_{r+1}^M F(z) & \to & H_{-r-1}^M(V, \mathbb{Z}(-r-1)) \end{array}$$

Here the horizontal maps are induced by the functoriality of motivic homology and (2.1.6). The right vertical map is the projection with respect to the decomposition (2.1.3). The left vertical map g was defined above.

We denote by $P_r(V)$ the assertion that the diagram (2.3.2) is commutative. We also denote by $Q_r(V)$ the assertion that $\bigoplus_{x \in V_{(0)}} K_r^M F(x) \to H_{-r}^M(V, \mathbb{Z}(-r))$ induces an isomorphism $C_r(V) \cong H_{-r}^M(V, \mathbb{Z}(-r))$ (which is what the theorem claims). We claim the following.

(1) We have $Q_0(V)$ for all V.

(2) Fix $r \in \mathbb{Z}_{\geq 0}$ and a finite extension F'/F. If $Q_r(\mathbb{G}_m \times \operatorname{Spec} F')$ holds, then $P_r(\operatorname{Spec} F')$ holds.

(3) Fix $r \in \mathbb{Z}_{\geq 0}$. If $P_r(\operatorname{Spec} F')$ holds for any finite extension F'/F, then $P_r(V)$ holds for all V.

(4) Fix $r \in \mathbb{Z}_{\geq 0}$ and V. If $P_r(V), Q_r(V)$, and $Q_r(V \times \mathbb{G}_m)$ hold, then $Q_{r+1}(V)$ holds.

Indeed, claim (1) was already remarked at the beginning of the proof. Claim (2) holds because we have $Q_{r+1}(\operatorname{Spec} F')$ by (2.1.6). (By the definition of $C_r(V)$, if one has $Q_r(V \times \mathbb{G}_m)$ and $Q_{r+1}(V)$, then $P_r(V)$ becomes trivial.) Claim (3) is clear from the definition of $P_r(V)$. We prove claim (4). By $P_r(V), Q_r(V \times \mathbb{G}_m)$, and the surjectivity of f, we get a well-defined map (!) in the commutative diagram

$$
\begin{array}{ccc}
C_r(V \times \mathbb{G}_m)/C_r(V) & \xrightarrow{\ \alpha\ } & C_{r+1}(V) \\[2mm]
\downarrow & & \downarrow^{(!)} \\[2mm]
H^M_{-r}\big(V \times \mathbb{G}_m, \mathbb{Z}(-r)\big)/H^M_{-r}\big(V, \mathbb{Z}(-r)\big) & \cong & H^M_{-r-1}\big(V, \mathbb{Z}(-r-1)\big)
\end{array}
$$

By $Q_r(V)$ and $Q_r(V \times \mathbb{G}_m)$, the left vertical map is an isomorphism. It follows that the right vertical map is also an isomorphism, which shows claim (4). Now the theorem follows by induction. $\quad\square$

§3. Étale cohomology with compact support

In this section, F is a field of characteristic zero.

3.1. The map $c^{i,j}_{X,n}$

Let V be a smooth irreducible variety over F of dimension d. Schmidt and Spieß [35, Proposition 3.1 and Remarks, p. 26] constructed a canonical homomorphism

$$(3.1.1) \qquad c^{i,j}_{V,n} : H^M_{2d-i}\big(V, \mathbb{Z}/n(d-j)\big) \to H^i_{\text{ét},c}\big(V, \mathbb{Z}/n(j)\big)$$

for any $i, j \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$, which is functorial in V. If V is projective, $c^{i,j}_{V,n}$ coincides with the map (2.2.2) under the identification (2.1.5) and $H^i_{\text{ét},c}(V, \mathbb{Z}/n(j)) = H^i_{\text{ét}}(V, \mathbb{Z}/n(j))$. The maps $c^{i,j}_{X,n}$ are functorial with respect to the sequences (2.1.7) and the Gysin sequence in étale cohomology. By taking the inductive limit, we also have

$$c^{i,j}_{V,\infty} : H^M_{2d-i}\big(V, \mathbb{Q}/\mathbb{Z}(d-j)\big) \to H^i_{\text{ét},c}\big(V, \mathbb{Q}/\mathbb{Z}(j)\big).$$

The source of the map $c_{V,n}^{2d+r,d+r}$ can be identified with $C_r(V)/n$ by Theorem 1.3.

REMARK 3.1. In [35, Proposition 3.1 and Remarks (b), p. 26], this homomorphism is constructed when k is a finite field. In [35, Remarks (a), p. 26], it is pointed out that the same construction works over any perfect field, by using relative Poincaré duality instead of its absolute version. When k is a p-adic field, one can also use the absolute version (based on $H_{\mathrm{Gal}}^2(k, \mathbb{Z}/n\mathbb{Z}(1)) \cong \mathbb{Z}/n$).

PROPOSITION 3.2. *Let V be a smooth variety over F of dimension d, let $i, j \in \mathbb{Z}_{\geq 0}$, and let $n \in \mathbb{Z}_{>0}$. If $i \leq j$ or $j \geq d + \operatorname{cd} F$, then $c_{V,n}^{i,j}$ is an isomorphism. If $i = j + 1$, then $c_{V,n}^{i,j}$ is an injection.*

Proof. By induction on $\dim V$ and (2.1.7), this can be reduced to the case where V is projective, which is proved in Theorem 2.3. $\square$

3.2. Curves

Let X be a smooth projective irreducible curve over F, and let U be an open dense subscheme of X. Put $Z = X \setminus U$.

LEMMA 3.3. *Let $n \in \mathbb{Z}_{>0} \cup \{\infty\}$.*
(1) *We have $\ker(c_{U,n}^{2,1}) = 0$. The map $\operatorname{coker}(c_{U,n}^{2,1}) \to \operatorname{coker}(c_{X,n}^{2,1})$ is injective.*
(2) *We have $\ker(c_{U,n}^{3,2}) = 0$. If $\operatorname{cd} F \leq 2$, then we have $\operatorname{coker}(c_{U,n}^{3,2}) \cong \operatorname{coker}(c_{X,n}^{3,2})$.*

Proof. We consider a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
H_1^M(X, \mathbb{Z}/n(0)) & \to & C_1(Z)/n & \to & C_0(U)/n & \to & C_0(X)/n & \to & 0 \\
\downarrow c_{X,n}^{1,1} & & \downarrow c_{Z,n}^{1,1} & & \downarrow c_{U,n}^{2,1} & & \downarrow c_{X,n}^{2,1} & & \\
H_{\text{ét}}^1(X, \mathbb{Z}/n(1)) & \to & H_{\text{ét}}^1(Z, \mathbb{Z}/n(1)) & \overset{(*)}{\to} & H_{\text{ét},c}^2(U, \mathbb{Z}/n(1)) & \to & H_{\text{ét}}^2(X, \mathbb{Z}/n(1)). &
\end{array}
$$

The two left vertical maps are bijective and the two right vertical maps injective by Proposition 3.2, and (1) follows. The proof of (2) is similar. $\square$

3.3. Surfaces

Let X be a smooth projective irreducible surface over F. Let $V \subset X$ be an open subvariety such that $\dim(X \setminus V) = 0$. Let $U \subset V$ be an open subvariety such that $V \setminus U$ is a (not necessary connected) smooth curve. Given a smooth surface U, one can always find such V and X. Let C be the smooth compactification of $V \setminus U$ (i.e., the normalization of $X \setminus U$).

PROPOSITION 3.4. *Let* $n \in \mathbb{Z}_{>0} \cup \{\infty\}$.

(1) *There is a canonical homomorphism*

$$\eta_n : \operatorname{coker}(c_{V,n}^{3,2}) \to \operatorname{coker}(c_{V \setminus U,n}^{3,2})$$

for which there are exact sequences

$$\ker(\eta_n) \stackrel{(*)}{\to} \ker(c_{U,n}^{4,2}) \to \ker(c_{X,n}^{4,2}),$$

$$\operatorname{coker}(\eta_n) \stackrel{(**)}{\to} \operatorname{coker}(c_{U,n}^{4,2}) \to \operatorname{coker}(c_{V,n}^{4,2}).$$

If $H_{\text{ét},c}^3(V, \mathbb{Z}/n(2)) \to H_{\text{ét},c}^3(V \setminus U, \mathbb{Z}/n(2))$ *is injective, then* $(*)$ *is injective, too. If* $c_{X,n}^{4,2}$ *is injective, then* $(**)$ *is injective, too. If* $\operatorname{cd} F \leq 2$, *then we have* $\operatorname{coker}(c_{V \setminus U,n}^{3,2}) \cong \operatorname{coker}(c_{C,n}^{3,2})$, $\operatorname{coker}(c_{V,n}^{3,2}) \cong \operatorname{coker}(c_{X,n}^{3,2})$, *and* $\operatorname{coker}(c_{V,n}^{4,2}) \cong \operatorname{coker}(c_{X,n}^{4,2})$. *(That is, one can replace* $V \setminus U$ *and* V *by* C *and* X *in the above sequences.)*

(2) *Suppose that* F *is a finite extension of* $\mathbb{Q}_p$. *Then there is an exact sequence*

$$\operatorname{coker}(c_{X,n}^{4,3}) \to \ker(c_{U,n}^{5,3}) \to \ker(c_{X,n}^{5,3}),$$

and we have $\operatorname{coker}(c_{U,n}^{5,3}) \cong \operatorname{coker}(c_{X,n}^{5,3})$.

Before we give a proof of this proposition, we record a well-known lemma which describes the kernel and cokernel of $c_{X,n}^{i,j}$. We set $\mathcal{H}^i(\mathbb{Z}/n(j)) := R^i \pi_* \mu_n^{\otimes j}$ for $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ and $i, j \in \mathbb{Z}$. (See Theorem 2.3 for the definition of π.)

LEMMA 3.5.

(1) *Let* $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. *There is an exact sequence*

$$0 \to H_1^M(X, \mathbb{Z}/n(0)) \stackrel{c_{X,n}^{3,2}}{\to} H_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2))$$

$$\to H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Z}/n(2))) \to C_0(X)/n \stackrel{c_{X,n}^{4,2}}{\to} H_{\text{ét}}^4(X, \mathbb{Z}/n(2)).$$

(2) *Let* $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. *There is an exact sequence*

$$0 \to H_0^M(X, \mathbb{Z}/n(-1)) \stackrel{c_{X,n}^{4,3}}{\to} H_{\text{ét},c}^4(X, \mathbb{Z}/n(3)) \to H_{\text{Zar}}^0(X, \mathcal{H}^4(\mathbb{Z}/n(3)))$$

$$\to C_1(X)/n \stackrel{c_{X,n}^{5,3}}{\to} H_{\text{ét},c}^5(X, \mathbb{Z}/n(3)) \to H_{\text{Zar}}^1(X, \mathcal{H}^4(\mathbb{Z}/n(3))) \to 0.$$

(3) Let $n \in \mathbb{Z}_{>0}$. The canonical map $H^0_{\mathrm{Zar}}(X, \mathcal{H}^i(\mathbb{Z}/n(i-1))) \to H^0_{\mathrm{Zar}}(X, \mathcal{H}^i(\mathbb{Q}/\mathbb{Z}(i-1)))[n]$ is bijective for $i = 3, 4$.

Proof. Since X is projective, (1) and (2) follow from (2.1.5) and the Bloch–Ogus theory; (3) is a consequence of the Bloch–Kato conjecture. $\square$

Proof of Proposition 3.4(1). We have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
H^M_1(V, \mathbb{Z}/n\mathbb{Z}(0)) & \to & C_1(V \setminus U)/n & \to & C_0(U)/n & \to & C_0(V)/n \to 0 \\
\downarrow c^{3,2}_{V,n} & & \downarrow c^{3,2}_{V \setminus U,n} & & \downarrow c^{4,2}_{U,n} & & \downarrow c^{4,2}_{V,n} \\
H^3_{\mathrm{\acute{e}t},c}(V, \mathbb{Z}/n\mathbb{Z}(2)) & \to & H^3_{\mathrm{\acute{e}t},c}(V \setminus U, \mathbb{Z}/n\mathbb{Z}(2)) & \to & H^4_{\mathrm{\acute{e}t},c}(U, \mathbb{Z}/n\mathbb{Z}(2)) & \to & H^4_{\mathrm{\acute{e}t},c}(V, \mathbb{Z}/n\mathbb{Z}(2))
\end{array}
$$

Proposition 3.2 shows the injectivity of $c^{3,2}_{V,n}$ and $c^{3,2}_{V \setminus U,n}$. If $\mathrm{cd}\, F \leq 2$, then we have $\mathrm{coker}(c^{3,2}_{V \setminus U,n}) \cong \mathrm{coker}(c^{3,2}_{C,n})$ by Lemma 3.3. Using the following lemma, a diagram chase completes the proof. $\square$

LEMMA 3.6. *The map* $\ker(c^{4,2}_{V,n}) \to \ker(c^{4,2}_{X,n})$ *is injective. If* $\mathrm{cd}\, F \leq 2$, *then* $\mathrm{coker}(c^{i,2}_{V,n}) \to \mathrm{coker}(c^{i,2}_{X,n})$ *is bijective for* $i = 3, 4$.

Proof. This follows from Lemma 2.1, Proposition 3.2, and (2.1.7). $\square$

Proof of Proposition 3.4(2). We have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
H^M_0(V, \mathbb{Z}/n\mathbb{Z}(1)) & \to & C_2(V \setminus U)/n & \to & C_1(U)/n & \to & C_1(V)/n & \to 0 \\
\downarrow c^{4,3}_{V,n} & & \downarrow c^{4,3}_{V \setminus U,n} & & \downarrow c^{5,3}_{U,n} & & \downarrow c^{5,3}_{V,n} \\
H^4_{\mathrm{\acute{e}t},c}(V, \mathbb{Z}/n\mathbb{Z}(3)) & \to & H^4_{\mathrm{\acute{e}t},c}(V \setminus U, \mathbb{Z}/n\mathbb{Z}(3)) & \to & H^5_{\mathrm{\acute{e}t},c}(U, \mathbb{Z}/n\mathbb{Z}(3)) & \to & H^5_{\mathrm{\acute{e}t},c}(V, \mathbb{Z}/n\mathbb{Z}(3)) & \to 0
\end{array}
$$

Proposition 3.2 shows the injectivity of the two left vertical maps. In Proposition 4.3 below, we will show that $c^{4,3}_{V \setminus U,n}$ is bijective. Using the following lemma, a diagram chase completes the proof. $\square$

LEMMA 3.7. *The map* $\ker(c^{5,3}_{V,n}) \to \ker(c^{5,3}_{X,n})$ *is injective. If* $\mathrm{cd}\, F \leq 2$, *then* $\mathrm{coker}(c^{i,3}_{V,n}) \to \mathrm{coker}(c^{i,3}_{X,n})$ *is bijective for* $i = 3, 4$.

Proof. This follows from Lemma 2.1, Proposition 3.2, and (2.1.7). $\square$

§4. Varieties over a p-adic field

In this section, k is a finite extension of $\mathbb{Q}_p$. Let X be a smooth geometrically irreducible variety over k of dimension d.

4.1. The Brauer–Manin pairing

By Poincaré duality, we have a canonical isomorphism

$$H_{\text{ét},c}^{2d}\big(X, \hat{\mathbb{Z}}(d)\big) \cong H_{\text{ét}}^2\big(X, \mathbb{Q}/\mathbb{Z}(1)\big)^*.$$

Since $\text{Br}(X)$ is a torsion group, the Kummer sequence implies an exact sequence

$$0 \to \text{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \to H_{\text{ét}}^2\big(X, \mathbb{Q}/\mathbb{Z}(1)\big) \to \text{Br}(X) \to 0;$$

hence, we have an injective homomorphism $\text{Br}(X)^* \to H_{\text{ét},c}^{2d}(X, \hat{\mathbb{Z}}(d))$. We consider a diagram

$$
\begin{array}{ccc}
Z_0(X) & \overset{\tilde{\psi}_X}{\to} & \text{Br}(X)^* \\
{\scriptstyle\text{surj.}}\downarrow & & \downarrow{\scriptstyle\text{inj.}} \\
C_0(X) & \to & H_{\text{ét},c}^{2d}\big(X, \hat{\mathbb{Z}}(d)\big)
\end{array}
$$

where the lower horizontal map is the composition of

$$C_0(X) \to \varprojlim_n C_0(X)/n \overset{(c_{X,n}^{2d,d})_n}{\to} H_{\text{ét},c}^{2d}\big(X, \hat{\mathbb{Z}}(d)\big).$$

This diagram is seen to be commutative by reducing to the case $\dim X = 0$. This shows the first part of Proposition 1.5 (the well-definedness of ψ_X). The same argument shows the following simple lemma.

LEMMA 4.1. *For any $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, we have an isomorphism $\ker(\psi_X/n) \cong \ker(c_{X,n}^{2d,d})$ and an exact sequence*

$$0 \to \text{coker}(\psi_X/n) \to \text{coker}(c_{X,n}^{2d,d}) \to \big(\text{Pic}(X)/n\big)^* \to 0.$$

4.2. Class field theory

By Poincaré duality, we have a canonical isomorphism

$$H_{\text{ét},c}^{2d+1}\big(X, \hat{\mathbb{Z}}(d+1)\big) \cong H_{\text{ét}}^1(X, \mathbb{Q}/\mathbb{Z})^* \cong \pi_1^{\text{ab}}(X).$$

We consider a diagram

$$
\begin{array}{ccc}
Z_0^1(X) & \overset{\tilde{\rho}_X}{\to} & \pi_1^{\text{ab}}(X) \\
{\scriptstyle\text{surj.}}\downarrow & & \downarrow{\scriptstyle\cong} \\
C_1(X) & \to & H_{\text{ét},c}^{2d+1}\big(X, \hat{\mathbb{Z}}(d+1)\big)
\end{array}
$$

where the lower horizontal map is the composition of

$$C_1(X) \to \varprojlim_n C_1(X)/n \xrightarrow{(c_{X,n}^{2d+1,d+1})_n} H_{\text{ét},c}^{2d+1}\big(X, \hat{\mathbb{Z}}(d+1)\big).$$

This diagram is seen to be commutative by reducing to the case $\dim X = 0$. This shows the second part of Proposition 1.5 (the well-definedness of ρ_X). The same argument shows the following simple lemma.

LEMMA 4.2. *For any $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, we have isomorphisms* $\ker(\rho_X/n) \cong \ker(c_{X,n}^{2d+1,d+1})$ *and* $\operatorname{coker}(\rho_X/n) \cong \operatorname{coker}(c_{X,n}^{2d+1,d+1})$.

4.3. Higher degree

We consider the groups $C_r(X)$ when $r \geq 2$. Proposition 4.3 below shows that $C_r(X)$ is uniquely divisible if $r \geq 3$ and that $C_2(X) \to C_2(\operatorname{Spec} k) = K_2(k)$ is a surjective map with uniquely divisible kernel. Note that $K_2(k)$ is the direct sum of a uniquely divisible group and a finite group isomorphic to $\mu(k)$ (see [27]).

PROPOSITION 4.3. *Let X be a smooth geometrically irreducible variety over k. Suppose that there exists a smooth projective variety Y that contains X as an open dense subvariety. Let $i, j \in \mathbb{Z}$, and suppose that $j \leq -2$.*

(1) *If $i < -2$, then $H_i^M(X, \mathbb{Z}(j))$ is uniquely divisible.*

(2) *The map $H_{-2}^M(X, \mathbb{Z}(j)) \to H_{-2}^M(\operatorname{Spec} k, \mathbb{Z}(j))$ induced by the structure morphism is surjective with uniquely divisible kernel. Consequently,*

$$c_{X,n}^{2d+2,d+2} : C_2(X)/n \to H_{\text{ét},c}^{2d+2}\big(X, \mathbb{Z}/n(d+2)\big)$$

is bijective for any $n \in \mathbb{Z}_{>0}$.

Proof. Since $j \leq -2$, Proposition 3.2 shows that

$$(4.3.1) \qquad c_{X,n}^{2d-i,d-j} : H_i^M\big(X, \mathbb{Z}/n(j)\big) \to H_{\text{ét},c}^{2d-i}\big(X, \mathbb{Z}/n(d-j)\big)$$

is an isomorphism for any $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. If $i < -2$, then we have $H_{\text{ét},c}^{2d-i}(X, \mathbb{Z}/n\mathbb{Z}(d-j)) = 0$ since $\operatorname{cd} k = 2$. We also have $H_{\text{ét},c}^{2d+2}(X, \mathbb{Q}/\mathbb{Z}(d-j)) = 0$, because this group is dual to $H_{\text{Gal}}^0(k, \hat{\mathbb{Z}}(j+1))^* = 0$ under Poincaré duality. Now (1) follows from (2.1.2).

Proposition 4.3(2) follows from the four claims (a)–(d) below. In what follows, we write $H_*^M(k, -)$ for $H_*^M(\operatorname{Spec} k, -)$. Put $H_{-2}^M(X, \mathbb{Z}(j))_0 := \ker[H_{-2}^M(X, \mathbb{Z}(j)) \to H_{-2}^M(k, \mathbb{Z}(j))]$. Then we have the following:

(a) $H_{-2}^M(X,\mathbb{Z}(j)) \to H_{-2}^M(k,\mathbb{Z}(j))$ is surjective;

(b) $H_{-2}^M(X,\mathbb{Z}(j))/n \to H_{-2}^M(k,\mathbb{Z}(j))/n$ is bijective for any $n \in \mathbb{Z}_{>0}$;

(c) $H_{-2}^M(X,\mathbb{Z}(j))[n] \to H_{-2}^M(k,\mathbb{Z}(j))[n]$ is surjective for any $n \in \mathbb{Z}_{>0}$;

(d) $H_{-2}^M(X,\mathbb{Z}(j))_0$ is torsion-free.

Indeed, by (a) we have an exact sequence for any $n \in \mathbb{Z}_{>0}$,

$$0 \to H_{-2}^M\big(X,\mathbb{Z}(j)\big)_0[n] \to H_{-2}^M\big(X,\mathbb{Z}(j)\big)[n] \to H_{-2}^M\big(k,\mathbb{Z}(j)\big)[n]$$

$$\to H_{-2}^M\big(X,\mathbb{Z}(j)\big)_0/n \to H_{-2}^M\big(X,\mathbb{Z}(j)\big)/n \to H_{-2}^M\big(k,\mathbb{Z}(j)\big)/n \to 0,$$

which shows (2) in view of (b)–(d).

We prove claims (a)–(d). Since we have shown that $H_{-3}^M(X,\mathbb{Z}(j))_{\mathrm{Tor}} = 0$, (2.1.2) and (4.3.1) yield a commutative diagram

$$
\begin{array}{ccc}
H_{-2}^M\big(X,\mathbb{Z}(j)\big)/n & \cong & H_{\mathrm{\acute{e}t},c}^{2d+2}\big(X,\mathbb{Z}/n\mathbb{Z}(d-j)\big) \\
\downarrow & & \downarrow{\cong} \\
H_{-2}^M\big(k,\mathbb{Z}(j)\big)/n & \cong & H_{\mathrm{Gal}}^{2}\big(k,\mathbb{Z}/n\mathbb{Z}(-j)\big)
\end{array}
$$

which shows (b). Taking a closed point $x \in X$, the cokernel of $H_{-2}^M(X, \mathbb{Z}(j)) \to H_{-2}^M(k,\mathbb{Z}(j))$ is seen to be annihilated by $[k(x):k]$. Using (b) with $n = [k(x):k]$, we get (a). By (2.1.2) we have a commutative diagram with exact rows

$$
\begin{array}{ccc}
H_{\mathrm{\acute{e}t},c}^{2d+1}\big(X,\mathbb{Z}/n\mathbb{Z}(d-j)\big) & \to & H_{-2}^M\big(X,\mathbb{Z}(j)\big)[n] \to 0 \\
{\scriptstyle(*)}\downarrow & & \downarrow \\
H_{\mathrm{Gal}}^{1}\big(k,\mathbb{Z}/n\mathbb{Z}(-j)\big) & \to & H_{-2}^M\big(k,\mathbb{Z}(j)\big)[n] \to 0
\end{array}
$$

The map $(*)$ is induced by the structure morphism $\alpha : X \to \operatorname{Spec} k$. Thus, it is dual to $\alpha^* : H_{\mathrm{Gal}}^1(k,\mathbb{Z}/n\mathbb{Z}(1+j)) \to H_{\mathrm{\acute{e}t}}^1(X,\mathbb{Z}/n\mathbb{Z}(1+j))$, which is injective by Leray's spectral sequence. This shows that $(*)$ is surjective, and (c) follows. We consider a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
& & & & & & & 0 & \\
& & & & & & & \downarrow & \\
& & & & H_{\mathrm{Gal}}^2\big(k,H_{\mathrm{\acute{e}t},c}^{2d-1}(\bar{X},\mathbb{Q}/\mathbb{Z}(d-j))\big) & & H_{-2}^M(X,\mathbb{Z}(j))_{0,\mathrm{Tor}} & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \to & H_{-1}^M(X,\mathbb{Z}(j))\otimes\mathbb{Q}/\mathbb{Z} & \to & H_{\mathrm{\acute{e}t},c}^{2d+1}(X,\mathbb{Q}/\mathbb{Z}(d-j)) & \to & H_{-2}^M(X,\mathbb{Z}(j))_{\mathrm{Tor}} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_{-1}^M(k,\mathbb{Z}(j))\otimes\mathbb{Q}/\mathbb{Z} & \to & H_{\mathrm{Gal}}^{1}(k,\mathbb{Q}/\mathbb{Z}(d-j)) & \to & H_{-2}^M(k,\mathbb{Z}(j))_{\mathrm{Tor}} & \to & 0
\end{array}
$$

Note that the lower left vertical map is surjective since the cokernel of $H_{-1}^M(X, \mathbb{Z}(j)) \to H_{-1}^M(k, \mathbb{Z}(j))$ is torsion (annihilated by $[k(x) : k]$ for any closed point $x \in X$). Now the following lemma completes the proof of (d). □

LEMMA 4.4. *Let X be a variety over k satisfying the same assumption as in Proposition 4.3. If $j \leq -2$, then $H_{\mathrm{Gal}}^2(k, H_{\mathrm{ét},c}^{2d-1}(\bar{X}, \mathbb{Q}/\mathbb{Z}(d-j))) = 0$.*

Proof. By Poincaré duality, it suffices to show that $H_{\mathrm{Gal}}^0(k, H_{\mathrm{ét}}^1(\bar{X}, \mathbb{Z}_l(j+1))) = 0$ for all prime numbers l. There is an exact sequence

$$0 \to H_{\mathrm{ét}}^1(\bar{Y}, \mathbb{Z}_l(j+1)) \to H_{\mathrm{ét}}^1(\bar{X}, \mathbb{Z}_l(j+1)) \to H_{\mathrm{ét}, \bar{Y} \setminus X}^2(\bar{Y}, \mathbb{Z}_l(j+1)).$$

By excision and purity, we have $H_{\mathrm{Gal}}^0(k, H_{\mathrm{ét}, \bar{Y} \setminus X}^2(\bar{Y}, \mathbb{Z}_l(j+1))) = 0$. Thus, we are reduced to showing that $H_{\mathrm{Gal}}^0(k, H_{\mathrm{ét}}^1(\bar{Y}, \mathbb{Z}_l(j+1))) = 0$, but this is a result of Jannsen [17, Theorems 4.2 and 5.3], for the cases $l \neq p$ and $l = p$, respectively. □

4.4. Auxiliary lemmas

For future use, we record a few simple lemmas. The following lemma is often used concurrently with the fact that the groups $\pi_1^{\mathrm{ab}}(X)$ and $\mathrm{Br}(X)^*$ have no nontrivial divisible elements for any smooth variety X (see Section 1.5).

LEMMA 4.5. *Let $f : A \to B$ be a homomorphism of abelian groups.*
(1) *Suppose that $B_{\mathrm{div}} = 0$. Then there is an exact sequence*

$$0 \to A_{\mathrm{div}} \to \ker(f) \to \varprojlim \ker(f/n).$$

(2) *Suppose that there is an $N \in \mathbb{Z}_{>0}$ such that the canonical map $\ker(f/nN) \to \ker(f \otimes \mathbb{Q}/\mathbb{Z})$ is bijective for all $n \in \mathbb{Z}_{>0}$. Then we have $\varprojlim \ker(f/n) = 0$. (Hence, $\ker(f) = A_{\mathrm{div}}$ if $B_{\mathrm{div}} = 0$.)*

Proof. Claim (1) is a direct consequence of the definition. We prove (2). By assumption, the canonical map $\ker(f/mN) \to \ker(f/mnN)$ is bijective for all $n, m \in \mathbb{Z}_{>0}$, which implies that the canonical map $\ker(f/mnN) \to \ker(f/nN)$ is the zero map. Thus, we have $\varprojlim \ker(f/n) = 0$. □

LEMMA 4.6. *Let X be a smooth irreducible variety over k of dimension d, and let $i, j \in \mathbb{Z}$. If $i + 2 \geq j + d$ or $j \geq d + 2$, then we have $H_i^M(X, \mathbb{Z}(j))_{\mathrm{div}} = H_i^M(X, \mathbb{Z}(j))_{\mathrm{Div}}$. In particular, we have $C_r(X)_{\mathrm{div}} = C_r(X)_{\mathrm{Div}}$ for any $r \in \mathbb{Z}_{\geq 0}$ when $d \leq 2$. If $d = 1$, then $C_0(X)_{\mathrm{Div}} = 0$.*

Proof. It suffices to show that $H_i^M(X, \mathbb{Z}(j))[n]$ is finite for any $n \in \mathbb{Z}_{>0}$. Proposition 3.2 and (2.1.2) reduce this to the finiteness of $H_{\text{ét},c}^{2d-i-1}(X, \mathbb{Z}/n(d-j))$, which is well known. For the last statement, we recall that $C_0(X)$ is isomorphic to the relative Picard group (see Remark 1.2), which has no nontrivial divisible subgroup by a theorem of Mattuck [25]. □

§5. Curves over a local field

In this section, k is a finite extension of $\mathbb{Q}_p$. Let X be a smooth projective irreducible curve over k, and let U be an open dense subscheme of X. Put $Z = X \setminus U$.

5.1. The Brauer–Manin pairing

The following theorem was proved by Scheiderer and van Hamel. When $U = X$, this theorem is due to Lichtenbaum [22].

THEOREM 5.1 ([33, (3.5)]). *The homomorphism* $\psi_U : C_0(U) \to \text{Br}(U)^*$ *is an injection with dense image. The induced map* $\psi_U/n : C_0(U)/n \to \text{Br}(U)^*/n$ *is an isomorphism for all* $n \in \mathbb{Z}_{>0}$.

Proof. We deduce this theorem assuming Lichtenbaum's result. We may assume that X is geometrically irreducible over k. Let $n \in \mathbb{Z}_{>0}$. Lemmas 3.3 and 4.1 show the injectivity of ψ_U/n. By Lemma 4.1, we have a commutative diagram

$$
\begin{array}{ccccccc}
0 & \to & \text{coker}(\psi_U/n) \to & \text{coker}(c_{U,n}^{2,1}) & \to & \big(\text{Pic}(U)/n\big)^* & \to & 0 \\
 & & & \downarrow^{(*)} & & \downarrow & & \\
 & & & \text{coker}(c_{X,n}^{2,1}) & \cong & \big(\text{Pic}(X)/n\big)^* & &
\end{array}
$$

The bottom horizontal map is bijective because $\text{coker}(\psi_X/n) = 0$ by Lichtenbaum's result. By Lemma 3.3, $(*)$ is injective; thus, $\text{coker}(\psi_U/n) = 0$. The rest of the assertion follows from Lemmas 4.5 and 4.6. □

5.2. Class field theory

We recall a result for projective curves due to Bloch and Saito.

THEOREM 5.2 ([1], [29]).

(1) *The kernel of the homomorphism* $\rho_X : C_1(X)(= SK_1(X)) \to \pi_1^{\text{ab}}(X)$ *coincides with* $C_1(X)_{\text{Div}}$. *Moreover,* ρ_X/n *is injective for all* $n \in \mathbb{Z}_{>0}$.

(2) *The group $D(X) := \pi_1^{\mathrm{ab}}(X)/\overline{\mathrm{Im}(\rho_X)}$ is isomorphic to $\hat{\mathbb{Z}}^{\oplus r(X)}$ for some $r = r(X) \in \mathbb{Z}_{\geq 0}$. We have $\mathrm{coker}(\rho_X/n) \cong D(X)/n$ for any $n \in \mathbb{Z}_{>0}$.*

REMARK 5.3. If X has potentially good reduction, then $r(X) = 0$. In general, we have an inequality $r(X) \leq g(X)$, where $g(X)$ is the genus of X (see [29, (6.2)]). In particular, we have $D(X) = 0$ when $g(X) = 0$.

The following generalization follows from Lemmas 3.3, 4.2, 4.5, and 4.6.

THEOREM 5.4.

(1) *The kernel of the homomorphism $\rho_U : C_1(U) \to \pi_1^{\mathrm{ab}}(U)$ coincides with $C_1(U)_{\mathrm{Div}}$. Moreover, ρ_U/n is injective for all $n \in \mathbb{Z}_{>0}$.*
(2) *Set $D(U) := \pi_1^{\mathrm{ab}}(U)/\overline{\mathrm{Im}(\rho_U)}$. The canonical maps $D(U) \to D(X)$ and $\mathrm{coker}(\rho_U/n) \to \mathrm{coker}(\rho_X/n)(\cong D(X)/n)$ are bijective for any $n \in \mathbb{Z}_{>0}$.*

REMARK 5.5. Following Hiranouchi [16], we define

$$C_1^w(U) := \mathrm{coker}\Big[K_2\big(k(X)\big) \to \Big(\bigoplus_{x \in U_{(0)}} k(x)^* \Big) \oplus \Big(\bigoplus_{z \in Z} K_2\big(k(X)_z\big) \Big)\Big],$$

where $k(X)_z$ is the completion of $k(X)$ at z. Hiranouchi constructed the reciprocity map $\rho_U' : C_1^w(U) \to \pi_1^{\mathrm{ab}}(U)$ and showed that the kernel of ρ_U' coincides with $C_1^w(U)_{\mathrm{Div}}$ and that $\pi_1^{\mathrm{ab}}(U)/\overline{\mathrm{Im}(\rho_U')} \cong D(X)$. It is easy to see the following.

• The natural projection $\pi : C_1^w(U) \to C_1(U)$ fits into an exact sequence

$$\bigoplus_{z \in Z} UK_2 k(X)_z \to C_1^w(U) \xrightarrow{\pi} C_1(U) \to 0.$$

Here we put $UK_2 k(X)_z = \ker[K_2 k(X)_z \to k(z)^* \oplus K_2 k(z)]$, where the map is defined by $a \mapsto (\partial_z(a), \partial_z(\{\pi_z\} \cup a)$ for some uniformizer $\pi_z \in k(X)_z$ at z (and ∂_z is the tame symbol). Note that $UK_2 k(X)_z$ is uniquely divisible.
• We have $\rho_U' = \rho_U \circ \pi$.

Hence, Hiranouchi's result (see [16]) can be recovered by Theorem 5.4.

§6. Surfaces over a local field

We keep assuming that k is a finite extension of $\mathbb{Q}_p$. Let X be a smooth projective geometrically connected surface over k. Let $U \subset V \subset X$ be open dense subsets such that $\dim(X \setminus V) = 0$ and such that $V \setminus U$ is a smooth curve. Let C be the normalization of $X \setminus U$, and set $D(C) = \bigoplus_i D(C_i)$, where C_i are the irreducible components of C, and $D(C_i)$ is the group defined in Theorem 5.2.

6.1. $\mathbb{P}^2$ minus curves

As the first example, we show the following.

PROPOSITION 6.1. *Suppose that $X \cong \mathbb{P}^2$. Let d be the greatest common divisor of the degrees (as a subvariety of $\mathbb{P}^2$) of the irreducible components of $X \setminus U$. Then, there is an exact sequence*

$$0 \to C_0(U)/nd \overset{\psi_U/nd}{\to} \mathrm{Br}(U)^*/nd \to D(C)/nd \to 0$$

for any $n \in \mathbb{Z}_{>0}$. We have $\ker(\psi_U) = C_0(U)_{\mathrm{Div}}$ and $\mathrm{Br}(U)^*/\overline{\mathrm{Im}(\psi_U)} \cong D(C)$.

For the map $\rho_U : C_1(U) \to \pi_1^{\mathrm{ab}}(U)$, see Theorem 6.5 below.

Proof. Let $n \in \mathbb{Z}_{>0}$. Note that $\psi_X/n : C_0(X)/n \to \mathrm{Br}(X)^*/n$ is bijective and that $H^0_{\mathrm{Zar}}(X, \mathcal{H}^i(\mathbb{Q}/\mathbb{Z}(i-1))) = 0$ for $i = 3, 4$. By Lemmas 3.5 and 4.1 and Proposition 3.4, we have $\ker(\psi_U/n) = 0$. By Lemmas 4.5 and 4.6, we get $\ker(\psi_U) = C_0(U)_{\mathrm{Div}}$.

Note that $\mathbb{Z} = \mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is a surjection with kernel $d\mathbb{Z}$. By Lemma 4.1, we get a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \to & \mathrm{coker}(\psi_U/nd) & \to & \mathrm{coker}(c_{U,nd}^{4,2}) & \to & \mathbb{Z}/d\mathbb{Z} & \to & 0 \\
& & & & \downarrow & & \downarrow^{\mathrm{inj.}} & & \\
0 & = & \mathrm{coker}(\psi_X/nd) & \to & \mathrm{coker}(c_{X,nd}^{4,2}) & \cong & \mathbb{Z}/nd & &
\end{array}
$$

By Proposition 3.4 and Lemma 3.5, we get an exact sequence

$$0 \to \mathrm{coker}(c_{C,nd}^{3,2}) \to \mathrm{coker}(c_{U,nd}^{4,2}) \to \mathrm{coker}(c_{X,nd}^{4,2}).$$

It follows that $\mathrm{coker}(\psi_U/nd) \cong \mathrm{coker}(c_{C,nd}^{3,2})$, which is isomorphic to $D(C)/nd$ by Lemma 4.2 and Theorem 5.2. This completes the proof. $\square$

6.2. Rational surface

Suppose that $\bar{X} := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$ is a rational surface over $\bar{k}$ (i.e., $\bar{X}$ is birational to $\mathbb{P}^2_{\bar{k}}$) and that $X(k) \neq \phi$. In this situation, the Chow group $C_0(X) = \mathrm{CH}_0(X)$ and related maps such as ψ_X and $c_{X,n}^{4,2,}$ have been studied by several authors (see, e.g., [2]–[5], [8], [24]). We will briefly recall a few results among them.

We set $A_0(X) := \ker[C_0(X) = \mathrm{CH}_0(X) \overset{\mathrm{deg}}{\to} \mathbb{Z}]$. Let $S = \mathrm{Hom}(NS(\bar{X}), \bar{k}^*)$ be the Néron–Severi torus of X. Recall that we have the Bloch map (see [2], [8])

$$\phi_X : A_0(X) \to H^1_{\mathrm{Gal}}(k, S).$$

It is known by [3] that ϕ_X is injective. Note that $H^1_{\mathrm{Gal}}(k, S)$ is finite (hence, so is $A_0(X)$). We will also need the following result.

THEOREM 6.2 (Saito [30]; see also [4]). *There is an $N = N_X \in \mathbb{Z}_{>0}$ such that $c^{4,2}_{X,nN}$ is injective for all $n \in \mathbb{Z}_{>0}$. Moreover, we have* $\ker(\psi_X) = 0$.

Now we state our main result.

THEOREM 6.3. *We assume the following condition:*

> (∗) *the irreducible components of $\bar{X} \setminus \bar{U}$ generate $NS(\bar{X})$.*

Then we have the following:
(1) $\ker(\psi_U) = C_0(U)_{\mathrm{Div}}$;
(2) $F_0(U) := \ker(\psi_U \otimes \mathbb{Q}/\mathbb{Z})$ *is finite, and there is an $N \in \mathbb{Z}_{>0}$ such that for all $n \in \mathbb{Z}_{>0}$ the canonical map $\ker(\psi_U/nN) \to F_0(U)$ is bijective;*
(3) *there is an injection $F_0(U) \hookrightarrow \mathrm{coker}(\phi_X)$, which is bijective if $D(C) = 0$.*

In Examples 6.8 and 6.10 below, we will provide concrete examples for which $F_0(U) \neq 0$. Theorem 1.6(1) is a consequence of this theorem and the examples. Note that Parimala and Suresh [28] have constructed smooth projective surfaces X and X' such that $\ker(\psi_X) \neq C_0(X)_{\mathrm{Div}}$ and $\ker(\psi_{X'}/2) \neq 0$. It seems difficult to control the cokernel of ψ_U, because, as Proposition 6.1 shows, it can easily become bigger and bigger. If we fix X and vary U, the above theorem immediately implies the following.

PROPOSITION 6.4. *There exists an open dense subvariety $U_0 \subset X$ such that, for any open dense subvariety $U \subset U_0$, the natural map $F_0(U) \to F_0(U_0)$ is an isomorphism.*

Proof. We take $U_* \subset X$ that satisfies (∗). By Theorem 6.3(3), the map $F_0(U) \to F_0(U_*)$ is injective for any $U \subset U_*$. Since $F_0(U_*)$ is finite, there exists $U_0 \subset U_*$ such that

$$F_0(U_0) = \bigcap_{U \subset U_*} F_0(U).$$

The proposition holds with this U_0. □

As for the map ρ_U, we prove the following result. Theorem 1.6(2) follows from this.

THEOREM 6.5. *The map $\rho_U/n : C_1(U)/n \to \pi_1^{\mathrm{ab}}(U)/n$ is an isomorphism for any $n \in \mathbb{Z}_{>0}$. We have $\ker(\rho_U) = C_1(U)_{\mathrm{Div}}$ and $\pi_1^{\mathrm{ab}}(U) = \overline{\mathrm{Im}(\rho_U)}$.*

Note that Sato [32] has constructed smooth projective $K3$ surfaces X, X' such that $\ker(\rho_X) \neq C_1(X)_{\mathrm{Div}}$ and $\ker(\rho_{X'}/2n) \neq 0$ for all $n \in \mathbb{Z}_{>0}$.

Before we start the proof of Theorems 6.3 and 6.5, we introduce a few lemmas.

LEMMA 6.6. *We have a canonical isomorphism*

$$H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)\big) \cong \mathrm{coker}\big[\phi_X : A_0(X) \to H^1_{\mathrm{Gal}}(k, S)\big].$$

In particular, $H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$ is finite.

Proof. By the Kummer sequence, we get an exact sequence for any $n \in \mathbb{Z}_{>0}$

$$0 \to S(k)/n \to H^1_{\mathrm{Gal}}\big(k, S[n]\big) \to H^1_{\mathrm{Gal}}(k, S)[n] \to 0.$$

Using the facts that $H^1_{\mathrm{Gal}}(k, S)$ is a torsion group and that the Hochschild–Serre spectral sequence induces an isomorphism $H^3_{\mathrm{ét}}(X, \mathbb{Q}/\mathbb{Z}(2)) \cong H^1_{\mathrm{Gal}}(k, S_{\mathrm{Tor}})$, we get an exact sequence which fits into the lower row in the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & H^3_M\big(X, \mathbb{Z}(2)\big) \otimes \mathbb{Q}/\mathbb{Z} & \to & H^3_M\big(X, \mathbb{Q}/\mathbb{Z}(2)\big) & \to & \mathrm{CH}_0(X)_{\mathrm{Tor}} & \to & 0 \\
& & \downarrow^{\cong} & & \downarrow^{c^{3,2}_{X,\infty}} & & \downarrow^{\phi_X} & & \\
0 & \to & S(k) \otimes \mathbb{Q}/\mathbb{Z} & \to & H^3_{\mathrm{ét}}\big(X, \mathbb{Q}/\mathbb{Z}(2)\big) & \to & H^1_{\mathrm{Gal}}(k, S) & \to & 0
\end{array}
$$

(Note that $\mathrm{CH}_0(X)_{\mathrm{Tor}} = A_0(X)$.) Here the upper row is exact by (2.1.2). The left vertical isomorphism is given by [3, Theorem C]. (Here we used the assumption that $X(k) \neq \phi$.) Now the assertion follows from Theorem 6.2 and Lemma 3.5. $\qquad \Box$

LEMMA 6.7. *If $(\ast)$ is satisfied, then $H^3_{\mathrm{ét},c}(V, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{\mathrm{ét},c}(V \setminus U, \mathbb{Q}/\mathbb{Z}(2))$ is injective.*

Proof. It suffices to show that $H^3_{\mathrm{ét},c}(U, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{\mathrm{ét},c}(V, \mathbb{Q}/\mathbb{Z}(2))$ is the zero map. By Poincaré duality, this amounts to showing that $H^3_{\mathrm{ét}}(V, \hat{\mathbb{Z}}(1)) \to H^3_{\mathrm{ét}}(U, \hat{\mathbb{Z}}(1))$ is the zero map. By the Gysin sequence $H^3_{\mathrm{ét}}(X, \hat{\mathbb{Z}}(1)) \to H^3_{\mathrm{ét}}(V, \hat{\mathbb{Z}}(1)) \to H^0_{\mathrm{ét}}(X \setminus V, \hat{\mathbb{Z}}(-1)) = 0$, it suffices to show that $H^3_{\mathrm{ét}}(X, \hat{\mathbb{Z}}(1)) \to H^3_{\mathrm{ét}}(U, \hat{\mathbb{Z}}(1))$ is the zero map. By the Hochschild–Serre spectral sequence, we have $H^3_{\mathrm{ét}}(X, \hat{\mathbb{Z}}(1)) = H^1_{\mathrm{Gal}}(k, H^2_{\mathrm{ét}}(\bar{X}, \hat{\mathbb{Z}}(1)))$, and we are reduced to showing that $NS(\bar{X}) \otimes \hat{\mathbb{Z}} = H^2_{\mathrm{ét}}(\bar{X}, \hat{\mathbb{Z}}(1)) \to H^2_{\mathrm{ét}}(\bar{U}, \hat{\mathbb{Z}}(1))$ is the zero map, but this follows from the assumption $(\ast)$. $\qquad \Box$

Proof of Theorem 6.3. Let $N_1 = N_X \in \mathbb{Z}_{>0}$ be the natural number appearing in Theorem 6.2, and let N_2 be the order of $H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$. Put $N = N_1 N_2$, and take any $n \in \mathbb{Z}_{>0}$. By Theorem 6.2 and Lemma 3.5, we have $\mathrm{coker}(c^{3,2}_{X,nN}) \cong H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Z}/nN(2)))$. Hence, we have a commutative diagram which contains the map η_n appearing in Proposition 3.4:

$$
\begin{array}{ccccc}
H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^3\big(\mathbb{Z}/nN(2)\big)\big) & \overset{\eta_{nN}}{\to} & \mathrm{coker}(c^{3,2}_{C,nN}) & \cong & (\mathbb{Z}/nN)^{\oplus r(C)} \\
\| & & \downarrow & & \cap \\
H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^3\big(\mathbb{Q}/\mathbb{Z}(2)\big)\big) & \overset{\eta_\infty}{\to} & \mathrm{coker}(c^{3,2}_{C,\infty}) & \cong & (\mathbb{Q}/\mathbb{Z})^{\oplus r(C)}
\end{array}
$$

where the two right horizontal isomorphisms are given by Theorem 5.2. We get $\ker(\eta_{nN}) \cong \ker(\eta_\infty)$. Then by Theorem 6.2 and Proposition 3.4, we have a commutative diagram,

$$
\begin{array}{ccc}
\ker(\eta_{nN}) & \twoheadrightarrow & \ker(\psi_U/nN) \\
\downarrow^{\cong} & & \downarrow \\
\ker(\eta_\infty) & \cong & \ker(\psi_U \otimes \mathbb{Q}/\mathbb{Z}) = F_0(U)
\end{array}
$$

in which the top horizontal map is surjective. Hence, all the maps in this diagram are bijective. This proves (2). If $D(C) = 0$, then we have $\ker(\eta_\infty) = H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$, and Lemma 6.6 proves (3). Claim (1) follows from Lemma 4.5. $\qquad\square$

Proof of Theorem 6.5. By Proposition 3.4 and Lemma 4.1, it suffices to show the bijectivity of $c^{i,3}_{X,n}$ for $i = 4, 5$ and $n \in \mathbb{Z}_{>0}$. Probably this is well known to the specialists, but the author could not find a suitable reference. For the sake of completeness, we include a proof here.

Let $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. The injectivity of $c^{4,3}_{X,n}$ follows from Proposition 3.2. We consider the commutative diagram with exact rows

$$
(6.2.1) \quad
\begin{array}{ccccccccc}
0 & \to & H^M_0\big(X, \mathbb{Z}(-1)\big)/n & \to & H^M_0\big(X, \mathbb{Z}/n(-1)\big) & \to & C_1(X)[n] & \to & 0 \\
& & \downarrow & & \downarrow^{c^{4,3}_{X,n}} & & \downarrow & & \\
0 & \to & H^2_{\mathrm{Gal}}\big(k, NS(\bar{X}) \otimes \mathbb{Z}/n(2)\big) & \to & H^4_{\text{ét}}\big(X, \mathbb{Z}/n(3)\big) & \to & H^0_{\mathrm{Gal}}\big(k, \mathbb{Z}/n(1)\big) & \to & 0
\end{array}
$$

Here the upper and lower rows are given by (2.1.2) and by the Hochschild–Serre spectral sequence, respectively. The right vertical map is given by

$$
C_1(X)[n] \to C_1(\mathrm{Spec}\, k)[n] = k^*[n] = H^0_{\mathrm{Gal}}\big(k, \mathbb{Z}/n(1)\big).
$$

The left vertical map is defined by the commutativity of this diagram. The existence of a k-rational point of X shows that both rows are split exact. It also shows the surjectivity of the right vertical map. We show the surjectivity of the left vertical map. Since the corestriction map

$$H^2_{\mathrm{Gal}}\big(k', NS(\bar{X}) \otimes \mathbb{Z}/n(2)\big) \to H^2_{\mathrm{Gal}}\big(k, NS(\bar{X}) \otimes \mathbb{Z}/n(2)\big)$$

is surjective for any finite extension k'/k, it suffices to show this surjectivity when X is a *split* rational surface (i.e., $k(X)$ is purely transcendental over k). This follows from Lemma 3.5 because $H^0_{\mathrm{Zar}}(X, \mathcal{H}^4(\mathbb{Z}/n(3))) = 0$ when X is a split rational surface. (More explicitly, one can directly show the surjectivity of the composite map

$$\mathrm{Pic}(X) \otimes K_2 k \overset{\mathrm{prod.}}{\to} H^4_M\big(X, \mathbb{Z}(3)\big)$$
$$\cong\ H^M_0\big(X, \mathbb{Z}(-1)\big) \to H^2_{\mathrm{Gal}}\big(k, NS(\bar{X}) \otimes \mathbb{Z}/n(2)\big),$$

where the first map is given by the product structure.)

Next, we consider $c^{5,3}_{X,n}$. By the Hochschild–Serre spectral sequence, we have an isomorphism

$$H^5_{\text{ét}}\big(X, \mathbb{Z}/n(3)\big) \cong H^1_{\mathrm{Gal}}\big(k, \mathbb{Z}/n(1)\big) \cong k^*/n.$$

It suffices to show that the structure morphism induces an isomorphism $C_1(X) \to C_1(\mathrm{Spec}\, k) = k^*$. By the existence of a k-rational point of X, one knows that this map is split surjective. We set $V(X) := \ker[C_1(X) \to C_1(\mathrm{Spec}\, k) = k^*]$. Since $V(X') = 0$ if X' is a split rational surface, the norm argument shows that $V(X)$ is a torsion group. It suffices to show that $V(X)_{\mathrm{Tor}} = 0$. This follows from a commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \to & V(X)_{\mathrm{Tor}} & \to & C_1(X)_{\mathrm{Tor}} & \overset{(!)}{\to} & \mu(k) \to 0 \\
 & & & & \uparrow{\scriptstyle\mathrm{surj.}} & & \| \\
 & & & & H^M_0\big(X, \mathbb{Q}/\mathbb{Z}(-1)\big) & \overset{(!)}{\to} & H^M_0\big(\mathrm{Spec}\, k, \mathbb{Q}/\mathbb{Z}(1)\big) \\
 & & & & \downarrow{\scriptstyle c^{4,3}_{X,\infty}} & & \| \\
 & & & & H^4_{\text{ét}}\big(X, \mathbb{Q}/\mathbb{Z}(3)\big) & \cong & H^0_{\mathrm{Gal}}\big(k, \mathbb{Q}/\mathbb{Z}(1)\big)
\end{array}
$$

Here the maps (!) are split surjective, and $c^{4,3}_{X,\infty}$ is injective by Proposition 3.2. The lower horizontal map is bijective because the first term in the

bottom row of (6.2.1) vanishes when $n = \infty$. The middle upper vertical map is given by (2.1.2). □

EXAMPLE 6.8. Let $a, b \in k^*$. Suppose that $p \neq 2$, $a \neq b$, and $\mathrm{ord}_k(a) = \mathrm{ord}_k(b) = r$. We also take $d \in k^*$ such that $k(\sqrt{d})/k$ is a nontrivial unramified extension. Let X be a smooth projective surface with function field $K := k(x, y)[z]/(x^2 - dy^2 - z(z - a)(z - b))$ (called a *Châtelet surface*). Let C be a divisor on X, and put $U = X \setminus C$. We assume that the irreducible components of $\bar{C}$ generate $NS(\bar{X})$ and that $D(C) = 0$. (When X is minimal with respect to the fibration corresponding to the field extension $K/k(z)$, we can take C to be the union of the four singular fibers and the "∞-section," so that each component has genus zero (see Remark 5.3; see [7] for details).) We claim the following.

(1) If r is even and $\mathrm{ord}_k(a - b) = r$, then $F_0(U) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
(2) If r is even and $\mathrm{ord}_k(a - b) > r$, then $F_0(U) \cong \mathbb{Z}/2$.
(3) If r is odd, then $F_0(U) = 0$.

It is shown in [7] that $H^1_{\mathrm{Gal}}(k, S) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Hence, the claim follows from Theorem 6.3 and the following theorem.

THEOREM 6.9 ([9, (4.7)]).

(1) *If r is even and $\mathrm{ord}_k(a - b) = r$, then $A_0(X) = 0$.*
(2) *If r is even and $\mathrm{ord}_k(a - b) > r$, then $A_0(X) \cong \mathbb{Z}/2$.*
(3) *If r is odd, then $A_0(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.*

One can extend this example to the case where $k(\sqrt{d})/k$ is a ramified extension, by using a result of Dalawat [10].

EXAMPLE 6.10. Let $a \in k^*$, and let X be a cubic surface defined by $T_0^3 + T_1^3 + T_2^3 + aT_3^3 = 0$ in $\mathbb{P}_k^3$. Let $\zeta \in \bar{k}$ be a primitive cubic root of unity. Set $r = \mathrm{ord}_k(a)$. Let C be a divisor on X, and put $U = X \setminus C$. We assume that the irreducible components of $\bar{C}$ generate $NS(\bar{X})$ and that $D(C) = 0$. (For example, we can take C to be the union of the 27 lines on X; see Remark 5.3.) We claim the following.

(1) If $p \neq 3$ and $r \equiv 0 \mod 3$, then $F_0(U) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$.
(2) If $p \neq 3, r \not\equiv 0 \mod 3$, and $\zeta \notin k$, then $F_0(U) \cong \mathbb{Z}/3$.
(3) Suppose that $\zeta \in k$. If $p \neq 3$, assume that $r \not\equiv 0 \mod 3$. If $p = 3$, assume that $r \equiv 1 \mod 3$. Then $F_0(X) = 0$.

Manin [24] has observed that $H^1_{\mathrm{Gal}}(k, S) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$. Hence, the claim follows from Theorem 6.3 and the following theorem.

THEOREM 6.11 ([31, (5.1.1)]).

(1) *If $p \neq 3$ and $r \equiv 0 \mod 3$, then $A_0(X) = 0$.*
(2) *If $p \neq 3, r \not\equiv 0 \mod 3$, and $\zeta \notin k$, then $A_0(X) \cong \mathbb{Z}/3$.*
(3) *Suppose that $\zeta \in k$. If $p \neq 3$, assume that $r \not\equiv 0 \mod 3$. If $p = 3$, assume that $r \equiv 1 \mod 3$. Then $A_0(X) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$.*

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