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ROUQUIER BLOCKS OF THE CYCLOTOMIC HECKE ALGEBRAS OF $G(de, e, r)$

MARIA CHLOUVERAKI

Abstract. The Rouquier blocks of the cyclotomic Hecke algebras, introduced by Rouquier, are a substitute for the families of characters defined by Lusztig for Weyl groups, which can be applied to all complex reflection groups. In this article, we determine them for the cyclotomic Hecke algebras of the groups of the infinite series $G(de, e, r)$, thus completing their calculation for all complex reflection groups.

Introduction

Until recently, the lack of Kazhdan-Lusztig bases for the non-Coxeter complex reflection groups did not allow the generalization of the notion of families of characters from Weyl groups to all complex reflection groups. However, thanks to the results of Gyoja [12] and Rouquier [21], we have obtained a substitute for the families of characters that can be applied to all complex reflection groups. In particular, Rouquier has proved that the families of characters of a Weyl group W coincide with the Rouquier blocks of the Iwahori-Hecke algebra of W, that is, its blocks over a suitable coefficient ring. This definition generalizes to all complex reflection groups, and we are grateful for this for the following reasons.

On the one hand, since the families of characters of a Weyl group play an essential role in the definition of the families of unipotent characters of the corresponding finite reductive group (see [14]), the families of characters of the cyclotomic Hecke algebras could play a key role in the organization of families of unipotent characters in general. On the other hand, for some (non-Coxeter) complex reflection groups W, we have data that seem to indicate that behind the group W, there exists another mysterious object—the *Spets* (see [3], [18])—that could play the role of the "series of finite reductive groups of Weyl group W."

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In [2], Broué and Kim presented an algorithm for the determination of the Rouquier blocks of the cyclotomic Hecke algebras of the groups $G(d,1,r)$. Using the generalization of some classic results, known as *Clifford theory*, they were able to obtain the Rouquier blocks for $G(d,d,r)$ from those of $G(d,1,r)$. Later, Kim [13] generalized the methods used in [2] in order to obtain the Rouquier blocks of the cyclotomic Hecke algebras of $G(de,e,r)$ from those of $G(de,1,r)$.

As far as the exceptional complex reflection groups are concerned, some special cases were treated by Malle and Rouquier in [19]. Finally, in [5], the author gives the complete classification of the Rouquier blocks of the cyclotomic Hecke algebras for all exceptional complex reflection groups.

However, recently it was realized that the algorithm of [2] for $G(d,1,r)$ does not work, unless d is a power of a prime number. In [7], the author gives the correct algorithm, which is more complicated than the one in [2]. Now, it remains to recalculate the Rouquier blocks of the cyclotomic Hecke algebras of $G(de,e,r)$ in order to complete the determination of the Rouquier blocks for all complex reflection groups.

Using the same idea as in [13], we apply Clifford theory in order to obtain the Rouquier blocks for $G(de,e,r)$ from those of $G(de,1,r)$. However, there is one case where this is not possible, that is, when $r=2$ and e is even. In that case, we apply the same methods as in [5] in order to determine the Rouquier blocks of the cyclotomic Hecke algebras of $G(de,2,2)$, and then we apply Clifford theory in order to obtain the Rouquier blocks for $G(de,e,2)$.

Finally, to every irreducible character of a cyclotomic Hecke algebra of a complex reflection group we can attach integers a and A, as Lusztig has done for Weyl groups. In [15], Lusztig shows that these integers are constant on families. Here, we complete the proof that a and A are constant on the Rouquier blocks of the cyclotomic Hecke algebras of all irreducible complex reflection groups, having already shown it for the exceptional ones (cf. [6]) and $G(d,1,r)$ (cf. [7]).

§1. Blocks of symmetric algebras

All the results of this section are presented here for the convenience of the reader. Their proofs can be found in [5, Chapter 2].

1.1. Generalities on blocks

Let us assume that $\mathcal{O}$ is a commutative integral domain with field of fractions F and that A is an $\mathcal{O}$-algebra, free and finitely generated as an $\mathcal{O}$-module.

DEFINITION 1.1. The *block-idempotents (blocks)* of A are the primitive idempotents of ZA.

Let K be a field extension of F. Suppose that the K-algebra $KA := K \otimes_{\mathcal{O}} A$ is semisimple. Then there exists a bijection between the set $\mathrm{Irr}(KA)$ of irreducible characters of KA and the set $\mathrm{Bl}(KA)$ of blocks of KA:

$$\mathrm{Irr}(KA) \leftrightarrow \mathrm{Bl}(KA),$$

$$\chi \mapsto e_\chi.$$

The following theorem establishes a relation between the blocks of the algebra A and the blocks of KA.

THEOREM 1.2. *There exists a unique partition* $\mathrm{Bl}(A)$ *of* $\mathrm{Irr}(KA)$ *such that*

(1) *for all* $B \in \mathrm{Bl}(A)$, *the idempotent* $e_B := \sum_{\chi \in B} e_\chi$ *is a block of* A; *and*

(2) *for every central idempotent* e *of* A, *there exists a subset* $\mathrm{Bl}(A, e)$ *of* $\mathrm{Bl}(A)$ *such that*

$$e = \sum_{B \in \mathrm{Bl}(A,e)} e_B.$$

In particular, the set $\{e_B\}_{B \in \mathrm{Bl}(A)}$ *is the set of all the blocks of* A.

If $\chi \in B$ for some $B \in \mathrm{Bl}(A)$, then we say that χ belongs to the block e_B.

1.2. Symmetric algebras

From now on, we make the following assumptions.

ASSUMPTIONS 1.3.

(int) *The ring* $\mathcal{O}$ *is a Noetherian and integrally closed domain with field of fractions* F, *and* A *is an* $\mathcal{O}$*-algebra that is free and finitely generated as an* $\mathcal{O}$*-module.*

(spl) *The field* K *is a finite Galois extension of* F, *and the algebra* KA *is split (i.e., for every simple* KA*-module* V, $\mathrm{End}_{KA}(V) \simeq K$*) semisimple.*

DEFINITION 1.4. We say that a linear map $t : A \to \mathcal{O}$ is a *symmetrizing form* on A or that A is a *symmetric algebra* if

(a) t is a trace function, that is, $t(ab) = t(ba)$ for all $a, b \in A$; and

(b) the morphism

$$\hat{t} : A \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}), a \mapsto \big(x \mapsto \hat{t}(a)(x) := t(ax)\big)$$

is an isomorphism of A-modules-A.

EXAMPLE 1.5. In the case where $\mathcal{O} = \mathbb{Z}$ and $A = \mathbb{Z}[G]$ (G a finite group), we can define the following symmetrizing form (*canonical*) on A:

$$t : \mathbb{Z}[G] \to \mathbb{Z}, \ \sum_{g \in G} a_g g \mapsto a_1,$$

where $a_g \in \mathbb{Z}$ for all $g \in G$.

From now on, let us suppose that A is a symmetric algebra with symmetrizing form t. By [9], we have the following results.

THEOREM 1.6. (1) *We have*

$$t = \sum_{\chi \in \mathrm{Irr}(KA)} \frac{1}{s_\chi} \chi,$$

where s_χ is the Schur element of χ with respect to t.

(2) *For all $\chi \in \mathrm{Irr}(KA)$, the central primitive idempotent associated to χ is*

$$e_\chi = \frac{1}{s_\chi} \sum_{i \in I} \chi(e_i') e_i,$$

where $(e_i)_{i \in I}$ is a basis of A over $\mathcal{O}$ and $(e_i')_{i \in I}$ is the dual basis with respect to t (i.e., $t(e_i e_j') = \delta_{ij}$).

COROLLARY 1.7. *The blocks of A are the nonempty subsets B of $\mathrm{Irr}(KA)$ minimal with respect to the property*

$$\sum_{\chi \in B} \frac{1}{s_\chi} \chi(a) \in \mathcal{O}, \quad \textit{for all } a \in A.$$

Let us suppose now that $\mathcal{O}$ is a discrete valuation ring with unique prime ideal $\mathfrak{p}$ and that K is the field of fractions of $\mathcal{O}$. Then the following result gives a criterion for a block to be a singleton.

PROPOSITION 1.8. *Let $\chi \in \mathrm{Irr}(KA)$. The character χ is a block of A by itself if and only if $s_\chi \notin \mathfrak{p}$.*

Proof. If $s_\chi \notin \mathfrak{p}$, then $1/s_\chi \in \mathcal{O}$ and Corollary 1.7 implies that the character χ is a block of A by itself. The inverse is a consequence of a theorem by Geck and Rouquier (see [10, Proposition 4.4]). ∎

1.3. Twisted symmetric algebras of finite groups

Let A be an $\mathcal{O}$-algebra such that Assumptions 1.3 are satisfied with a symmetrizing form t. Let $\bar{A}$ be a subalgebra of A free and of finite rank as an $\mathcal{O}$-module.

DEFINITION 1.9. We say that $\bar{A}$ is a *symmetric subalgebra* of A if it satisfies the following two conditions:

(1) $\bar{A}$ is free (of finite rank) as an $\mathcal{O}$-module and the restriction $\mathrm{Res}^A_{\bar{A}}(t)$ of the form t to $\bar{A}$ is a symmetrizing form on $\bar{A}$; and
(2) A is free (of finite rank) as an $\bar{A}$-module for the action of left multiplication by the elements of $\bar{A}$.

We denote by

$$\mathrm{Ind}^A_{\bar{A}} : {}_{\bar{A}}\mathbf{mod} \to {}_A\mathbf{mod} \qquad \text{and} \qquad \mathrm{Res}^A_{\bar{A}} : {}_A\mathbf{mod} \to {}_{\bar{A}}\mathbf{mod}$$

the functors defined as usual by

$$\mathrm{Ind}^A_{\bar{A}} := A \otimes_{\bar{A}}, \quad \text{where } A \text{ is viewed as an } A\text{-module-}\bar{A}$$

and

$$\mathrm{Res}^A_{\bar{A}} := A \otimes_A, \quad \text{where } A \text{ is viewed as an } \bar{A}\text{-module-}A.$$

In the next sections, we will work on the Hecke algebras of complex reflection groups, which are symmetric. Sometimes the Hecke algebra of a group W appears as a symmetric subalgebra of the Hecke algebra of another group W', which contains W. Since we are mostly interested in the determination of the blocks of these algebras, it would be helpful to obtain the blocks of the former from the blocks of the latter. This is possible with the use of a generalization of some classic results, already introduced above as Clifford theory (see, e.g., [8]), to the twisted symmetric algebras of finite groups and, more precisely, of finite cyclic groups.

DEFINITION 1.10. We say that a symmetric $\mathcal{O}$-algebra (A,t) is the *twisted symmetric algebra of a finite group G over the subalgebra $\bar{A}$* if the following conditions are satisfied.

- $\bar{A}$ is a symmetric subalgebra of A.
- There exists a family $\{A_g | g \in G\}$ of $\mathcal{O}$-submodules of A such that
  (a) $A = \bigoplus_{g\in G} A_g$;
  (b) $A_g A_h = A_{gh}$ for all $g, h \in G$;
  (c) $A_1 = \bar{A}$;
  (d) $t(A_g) = 0$ for all $g \in G, g \neq 1$;
  (e) $A_g \cap A^\times \neq \emptyset$ for all $g \in G$ (where $A^\times$ is the set of units of A).

  In particular, if $a_g \in A_g \cap A^\times$, then we have $A_g = a_g\bar{A} = \bar{A}a_g$.

Action of G on $Z\bar{A}$

From now on, we assume that (A,t) is the twisted symmetric algebra of a finite group G over $\bar{A}$ and that K is an extension of F such that the algebras KA, $K\bar{A}$, and KG are split semisimple.

THEOREM–DEFINITION 1.11. *Let $\bar{a} \in Z\bar{A}$, and let $g \in G$. There exists a unique element $g(\bar{a})$ of $\bar{A}$ satisfying*

$$g(\bar{a})\mathfrak{g} = \mathfrak{g}\bar{a} \quad \textit{for all } \mathfrak{g} \in A_g.$$

If $a_g \in A^\times$ such that $A_g = a_g\bar{A}$, then

$$g(\bar{a}) = a_g \bar{a} a_g^{-1}.$$

The map $\bar{a} \mapsto g(\bar{a})$ defines an action of G as ring automorphism of $Z\bar{A}$.

Induction and restriction of KA-modules and $K\bar{A}$-modules

For all $\bar{\chi} \in \mathrm{Irr}(K\bar{A})$, we denote by $\bar{e}(\bar{\chi})$ the block-idempotent of $K\bar{A}$ associated to $\bar{\chi}$. If $g \in G$, then $g(\bar{e}(\bar{\chi}))$ is also a block of $K\bar{A}$. Since $K\bar{A}$ is split semisimple, it must be associated to an irreducible character $g(\bar{\chi})$ of $K\bar{A}$. Thus, we can define an action of G on $\mathrm{Irr}(K\bar{A})$ such that for all $g \in G$, $\bar{e}(g(\bar{\chi})) = g(\bar{e}(\bar{\chi}))$. We denote by $G_{\bar{\chi}}$ the stabilizer of the character $\bar{\chi}$ in G, and we denote by $\bar{\Omega}$ the orbit of $\bar{\chi}$ under the action of G. We have $|\bar{\Omega}| = |G|/|G_{\bar{\chi}}|$. We define

$$\bar{e}(\bar{\Omega}) := \sum_{g\in G/G_{\bar{\chi}}} \bar{e}\big(g(\bar{\chi})\big) = \sum_{g\in G/G_{\bar{\chi}}} g\big(\bar{e}(\bar{\chi})\big).$$

Case where G is cyclic. Since the group G is abelian, the set $\mathrm{Irr}(KG)$ forms a group denoted by $G^\vee$. The application $\psi \mapsto \psi \cdot \xi$, where $\psi \in \mathrm{Irr}(KA)$ and $\xi \in G^\vee$, defines an action of $G^\vee$ on $\mathrm{Irr}(KA)$. Then we have the following result.

PROPOSITION 1.12. *If the group G is cyclic, there exists a bijection*

$$\mathrm{Irr}(K\bar{A})/G \xleftrightarrow{\sim} \mathrm{Irr}(KA)/G^\vee,$$
$$\bar{\Omega} \leftrightarrow \Omega$$

such that

$$\bar{e}(\bar{\Omega}) = e(\Omega), \qquad |\bar{\Omega}||\Omega| = |G|,$$

and

$$\begin{cases} \forall \chi \in \Omega, & \mathrm{Res}^{KA}_{K\bar{A}}(\chi) = \sum_{\bar{\chi} \in \bar{\Omega}} \bar{\chi}, \\ \forall \bar{\chi} \in \bar{\Omega}, & \mathrm{Ind}^{KA}_{K\bar{A}}(\bar{\chi}) = \sum_{\chi \in \Omega} \chi. \end{cases}$$

Moreover, for all $\chi \in \Omega$ and $\bar{\chi} \in \bar{\Omega}$, we have

$$s_\chi = |\Omega| s_{\bar{\chi}}.$$

Blocks of A and blocks of $\bar{A}$

Denote by $\mathrm{Bl}(A)$ the set of blocks of A, and denote by $\mathrm{Bl}(\bar{A})$ the set of blocks of $\bar{A}$. For $\bar{b} \in \mathrm{Bl}(\bar{A})$, we set

$$\mathrm{Tr}(G, \bar{b}) := \sum_{g \in G/G_{\bar{b}}} g(\bar{b}).$$

The algebra $(Z\bar{A})^G$ is contained in both $Z\bar{A}$ and ZA, and the set of its blocks is

$$\mathrm{Bl}\big((Z\bar{A})^G\big) = \big\{ \mathrm{Tr}(G, \bar{b}) \mid \bar{b} \in \mathrm{Bl}(\bar{A})/G \big\}.$$

Moreover, $\mathrm{Tr}(G, \bar{b})$ is a sum of blocks of A, and we define the subset $\mathrm{Bl}(A, \bar{b})$ of $\mathrm{Bl}(A)$ as follows:

$$\mathrm{Tr}(G, \bar{b}) := \sum_{b \in \mathrm{Bl}(A, \bar{b})} b.$$

LEMMA 1.13. *Let $\bar{b}$ be a block of $\bar{A}$ and $\bar{B} := \mathrm{Irr}(K\bar{A}\bar{b})$. Then*

(1) *for all* $\bar{\chi} \in \bar{B}$, *we have* $G_{\bar{\chi}} \subseteq G_{\bar{b}}$;
(2) *we have*

$$\mathrm{Tr}(G, \bar{b}) = \sum_{\bar{\chi} \in \bar{B}/G} \mathrm{Tr}\big(G, \bar{e}(\bar{\chi})\big) = \sum_{\{\bar{\Omega} | \bar{\Omega} \cap \bar{B} \neq \emptyset\}} \bar{e}(\bar{\Omega}).$$

Now let $G^\vee := \mathrm{Hom}(G, K^\times)$. We suppose that $K = F$. The multiplication of the characters of KA by the characters of KG defines an action of the group $G^\vee$ on $\mathrm{Irr}(KA)$. This action is induced by the operation of $G^\vee$ on the algebra A, which is defined in the following way:

$$\xi \cdot (\bar{a} a_g) := \xi(g) \bar{a} a_g \quad \text{for all } \xi \in G^\vee, \bar{a} \in \bar{A}, g \in G.$$

In particular, $G^\vee$ acts on the set of blocks of A. Let b be a block of A. Denote by $\xi \cdot b$ the product of ξ and b, and denote by $(G^\vee)_b$ the stabilizer of b in $G^\vee$. We set

$$\mathrm{Tr}(G^\vee, b) := \sum_{\xi \in G^\vee/(G^\vee)_b} \xi \cdot b.$$

The set of blocks of the algebra $(ZA)^{G^\vee}$ is given by

$$\mathrm{Bl}\big((ZA)^{G^\vee}\big) = \big\{\mathrm{Tr}(G^\vee, b) \mid b \in \mathrm{Bl}(A)/G^\vee\big\}.$$

The following lemma is the analogue of Lemma 1.13.

LEMMA 1.14. *Let* b *be a block of* A *and* $B := \mathrm{Irr}(KAb)$. *Then,*
(1) *for all* $\chi \in B$, *we have* $(G^\vee)_\chi \subseteq (G^\vee)_b$;
(2) *we have*

$$\mathrm{Tr}(G^\vee, b) = \sum_{\chi \in B/G^\vee} \mathrm{Tr}\big(G^\vee, e(\chi)\big) = \sum_{\{\Omega | \Omega \cap B \neq \emptyset\}} e(\Omega).$$

Case where G *is cyclic.* For every orbit $\mathcal{Y}$ of $G^\vee$ on $\mathrm{Bl}(A)$, we denote by $b(\mathcal{Y})$ the block of $(ZA)^{G^\vee}$ defined by

$$b(\mathcal{Y}) := \sum_{b \in \mathcal{Y}} b.$$

For every orbit $\bar{\mathcal{Y}}$ of G on $\mathrm{Bl}(\bar{A})$, we denote by $\bar{b}(\bar{\mathcal{Y}})$ the block of $(Z\bar{A})^G$ defined by

$$\bar{b}(\bar{\mathcal{Y}}) := \sum_{\bar{b} \in \bar{\mathcal{Y}}} \bar{b}.$$

The following proposition results from Proposition 1.12 and Lemmas 1.13 and 1.14.

PROPOSITION 1.15. *If the group G is cyclic, there exists a bijection*

$$\mathrm{Bl}(\bar{A})/G \xleftrightarrow{\sim} \mathrm{Bl}(A)/G^\vee,$$
$$\bar{\mathcal{Y}} \leftrightarrow \mathcal{Y}$$

such that

$$\bar{b}(\bar{\mathcal{Y}}) = b(\mathcal{Y});$$

that is,

$$\mathrm{Tr}(G,\bar{b}) = \mathrm{Tr}(G^\vee, b) \quad \textit{for all } \bar{b} \in \bar{\mathcal{Y}} \textit{ and } b \in \mathcal{Y}.$$

In particular, the algebras $(Z\bar{A})^G$ and $(ZA)^{G^\vee}$ have the same blocks.

COROLLARY 1.16. *If the blocks of A are stable by the action of $G^\vee$, then the blocks of A coincide with the blocks of $(Z\bar{A})^G$.*

§2. Hecke algebras of complex reflection groups

2.1. Generic Hecke algebras

Let μ_∞ be the group of all the roots of unity in $\mathbb{C}$, and let K be a number field contained in $\mathbb{Q}(\mu_\infty)$. We denote by $\mu(K)$ the group of all the roots of unity of K. For every integer $d>1$, we set $\zeta_d := \exp(2\pi i/d)$ and we denote by μ_d the group of all the dth roots of unity.

Let V be a K-vector space of finite dimension r. Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo)reflections acting irreducibly on V. Let us denote by $\mathcal{A}$ the set of the reflecting hyperplanes of W. We set $\mathcal{M} := \mathbb{C}\otimes V - \bigcup_{H\in\mathcal{A}} \mathbb{C}\otimes H$. For $x_0 \in \mathcal{M}$, let $P := \Pi_1(\mathcal{M}, x_0)$ and let $B := \Pi_1(\mathcal{M}/W, x_0)$. Then there exists a short exact sequence (see [4]):

$$\{1\} \to P \to B \to W \to \{1\}.$$

We denote by τ the central element of P defined by the loop

$$[0,1] \to \mathcal{M}, \qquad t \mapsto \exp(2\pi i t) x_0.$$

For every orbit $\mathcal{C}$ of W on $\mathcal{A}$, we denote by $e_\mathcal{C}$ the common order of the subgroups W_H, where H is any element of $\mathcal{C}$ and W_H is the subgroup formed by id_V and all the reflections fixing the hyperplane H.

We choose a set of indeterminates $\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\le j\le e_\mathcal{C}-1)}$, and we denote by $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ the Laurent polynomial ring in all the indeterminates

$\mathbf{u}$. We define the *generic Hecke algebra* $\mathcal{H}$ of W to be the quotient of the group algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ by the ideal generated by the elements of the form

$$(\mathbf{s} - u_{\mathcal{C},0})(\mathbf{s} - u_{\mathcal{C},1}) \cdots (\mathbf{s} - u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

where $\mathcal{C}$ runs over the set $\mathcal{A}/W$ and $\mathbf{s}$ runs over the set of monodromy generators around the images in $\mathcal{M}/W$ of the elements of the hyperplane orbit $\mathcal{C}$.

We make some assumptions for the algebra $\mathcal{H}$. Note that they have been verified for all but a finite number of irreducible complex reflection groups [3, remarks before (1.17), §2]; [11].

ASSUMPTIONS 2.1. *The algebra $\mathcal{H}$ is a free $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-module of rank $|W|$. Moreover, there exists a linear form $t : \mathcal{H} \to \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ with the following properties.*

(1) *The map t is a symmetrizing form on $\mathcal{H}$.*
(2) *Via the specialization $u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^{j}$, the form t becomes the canonical symmetrizing form on the group algebra $\mathbb{Z}_K W$.*
(3) *If we denote by $\alpha \mapsto \alpha^*$ the automorphism of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ consisting of the simultaneous inversion of the indeterminates, then for all $b \in B$, we have*

$$t(b^{-1})^* = \frac{t(b\tau)}{t(\tau)}.$$

We know that the form t is unique [3, (2.1)]. From now on, let us suppose that the Assumptions 2.1 are satisfied. Then we have the following result by Malle [17, (5.2)].

THEOREM 2.2. *Let $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}/W)(0 \le j \le e_{\mathcal{C}}-1)}$ be a set of $\sum_{\mathcal{C} \in \mathcal{A}/W} e_{\mathcal{C}}$ indeterminates such that, for every $\mathcal{C}, j$, we have $v_{\mathcal{C},j}^{|\mu(K)|} = \zeta_{e_{\mathcal{C}}}^{-j} u_{\mathcal{C},j}$. Then the $K(\mathbf{v})$-algebra $K(\mathbf{v})\mathcal{H}$ is split semisimple.*

By Tits's deformation theorem (cf., e.g., [3, (7.2)]), it follows that the specialization $v_{\mathcal{C},j} \mapsto 1$ induces a bijection $\chi \mapsto \chi_{\mathbf{v}}$ from the set $\mathrm{Irr}(K(\mathbf{v})\mathcal{H})$ of absolutely irreducible characters of $K(\mathbf{v})\mathcal{H}$ to the set $\mathrm{Irr}(W)$ of absolutely irreducible characters of W.

The following result concerning the form of the Schur elements associated with the irreducible characters of $K(\mathbf{v})\mathcal{H}$ is proved in [5, Theorem 4.2.5], using case-by-case analysis.

THEOREM 2.3. *The Schur element $s_\chi(\mathbf{v})$ associated with the character $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}$ is an element of $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ of the form*

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i\in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}},$$

where

- *ξ_χ is an element of $\mathbb{Z}_K$;*
- *$N_\chi = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{b_{\mathcal{C},j}}$ is a monomial in $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ such that $\sum_{j=0}^{e_\mathcal{C}-1} b_{\mathcal{C},j} = 0$ for all $\mathcal{C}\in\mathcal{A}/W$;*
- *I_χ is an index set;*
- *$(\Psi_{\chi,i})_{i\in I_\chi}$ is a family of K-cyclotomic polynomials in one variable (i.e., minimal polynomials of the roots of unity over K);*
- *$(M_{\chi,i})_{i\in I_\chi}$ is a family of monomials in $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$, and if $M_{\chi,i} = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$, then $\gcd(a_{\mathcal{C},j}) = 1$ and $\sum_{j=0}^{e_\mathcal{C}-1} a_{\mathcal{C},j} = 0$ for all $\mathcal{C}\in\mathcal{A}/W$; and*
- *$(n_{\chi,i})_{i\in I_\chi}$ is a family of positive integers.*

This factorization is unique in $K[\mathbf{v},\mathbf{v}^{-1}]$. Moreover, the monomials $(M_{\chi,i})_{i\in I_\chi}$ are unique up to inversion, whereas the coefficient ξ_χ is unique up to multiplication by a root of unity.

Let $A := \mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$, and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$.

DEFINITION 2.4. Let $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a monomial in A such that $\gcd(a_{\mathcal{C},j}) = 1$. We say that M is *$\mathfrak{p}$-essential for a character* $\chi\in\mathrm{Irr}(W)$ if there exists a K-cyclotomic polynomial Ψ such that

- $\Psi(M)$ divides $s_\chi(\mathbf{v})$, and
- $\Psi(1)\in\mathfrak{p}$.

We say that M is *$\mathfrak{p}$-essential* for W if there exists a character $\chi\in\mathrm{Irr}(W)$ such that M is $\mathfrak{p}$-essential for χ.

The following proposition (see [5, Proposition 3.1.3]) gives a characterization of $\mathfrak{p}$-essential monomials, which plays an essential role in the proof of Theorem 2.11.

PROPOSITION 2.5. *Let $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a monomial in A such that $\gcd(a_{\mathcal{C},j}) = 1$. We set $\mathfrak{q}_M := (M-1)A + \mathfrak{p}A$. Then*

(1) *the ideal $\mathfrak{q}_M$ is a prime ideal of A,*
(2) *M is $\mathfrak{p}$-essential for $\chi\in\mathrm{Irr}(W)$ if and only if $s_\chi(\mathbf{v})/\xi_\chi\in\mathfrak{q}_M$.*

If M is a $\mathfrak{p}$-essential monomial for W, then Theorem 2.11 establishes a relation between the blocks of the algebra $A_{\mathfrak{q}_M}\mathcal{H}$ and the Rouquier blocks. The following results concerning the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ are proven in [5, Propositions 3.2.3 and 3.2.5].

PROPOSITION 2.6. *Let $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a monomial in A such that $\gcd(a_{\mathcal{C},j}) = 1$ and $\mathfrak{q}_M := (M-1)A + \mathfrak{p}A$. Then*

(1) *if two irreducible characters are in the same block of $A_{\mathfrak{p}A}\mathcal{H}$, then they are in the same block of $A_{\mathfrak{q}_M}\mathcal{H}$;*
(2) *if C is a block of $A_{\mathfrak{p}A}\mathcal{H}$ and M is not $\mathfrak{p}$-essential for any irreducible character in C, then C is a block of $A_{\mathfrak{q}_M}\mathcal{H}$.*

2.2. Cyclotomic Hecke algebras

Let y be an indeterminate. We set $q := y^{|\mu(K)|}$.

DEFINITION 2.7. A *cyclotomic specialization* of $\mathcal{H}$ is a $\mathbb{Z}_K$-algebra morphism $\phi : \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ with the following properties:

- $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$, where $n_{\mathcal{C},j} \in \mathbb{Z}$ for all $\mathcal{C}$ and j;
- for all $\mathcal{C} \in \mathcal{A}/W$, and assuming that z is another indeterminate, the element of $\mathbb{Z}_K[y, y^{-1}, z]$ defined by

$$\Gamma_{\mathcal{C}}(y, z) := \prod_{j=0}^{e_{\mathcal{C}}-1} (z - \zeta_{e_{\mathcal{C}}}^{j} y^{n_{\mathcal{C},j}})$$

is invariant by the action of $\mathrm{Gal}(K(y)/K(q))$.

If ϕ is a cyclotomic specialization of $\mathcal{H}$, the corresponding *cyclotomic Hecke algebra* is the $\mathbb{Z}_K[y, y^{-1}]$-algebra, denoted by $\mathcal{H}_\phi$, which is obtained as the specialization of the $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$-algebra $\mathcal{H}$ via the morphism ϕ. It also has a symmetrizing form t_ϕ defined as the specialization of the canonical form t.

REMARK. Sometimes we describe the morphism ϕ by the formula

$$u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^{j} q^{n_{\mathcal{C},j}}.$$

The following result is proved in [5, Proposition 4.3.4].

PROPOSITION 2.8. *The algebra $K(y)\mathcal{H}_\phi$ is split semisimple.*

For $y=1$ this algebra specializes to the group algebra KW (the form t_ϕ becoming the canonical form on the group algebra). Thus, by Tits's deformation theorem, the specialization $v_{\mathcal{C},j} \mapsto 1$ induces the following bijections:

$$\begin{array}{ccccc} \mathrm{Irr}\big(K(\mathbf{v})\mathcal{H}\big) & \leftrightarrow & \mathrm{Irr}\big(K(y)\mathcal{H}_\phi\big) & \leftrightarrow & \mathrm{Irr}(W), \\ \chi_{\mathbf{v}} & \mapsto & \chi_\phi & \mapsto & \chi. \end{array}$$

2.3. Rouquier blocks of the cyclotomic Hecke algebras

DEFINITION 2.9. We call the Rouquier ring of K, and we denote by $\mathcal{R}_K(y)$, the $\mathbb{Z}_K$-subalgebra of $K(y)$

$$\mathcal{R}_K(y) := \mathbb{Z}_K[y, y^{-1}, (y^n-1)_{n\geq 1}^{-1}].$$

Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization, and let $\mathcal{H}_\phi$ be the corresponding cyclotomic Hecke algebra. The *Rouquier blocks* of $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi$.

REMARK. If we set $q := y^{|\mu(K)|}$, then the corresponding cyclotomic Hecke algebra $\mathcal{H}_\phi$ can be considered either over the ring $\mathbb{Z}_K[y, y^{-1}]$ or over the ring $\mathbb{Z}_K[q, q^{-1}]$. We define the *Rouquier blocks* of $\mathcal{H}_\phi$ to be the blocks of $\mathcal{H}_\phi$ defined over the Rouquier ring $\mathcal{R}_K(y)$ in $K(y)$. However, in other texts (e.g., [2]), the Rouquier blocks are determined over the Rouquier ring $\mathcal{R}_K(q)$ in $K(q)$. Since $\mathcal{R}_K(y)$ is the integral closure of $\mathcal{R}_K(q)$ in $K(y)$, [2, Proposition 1.12] establishes a relation between the blocks of $\mathcal{R}_K(y)\mathcal{H}_\phi$ and the blocks of $\mathcal{R}_K(q)\mathcal{H}_\phi$. Moreover, in the case where $\mathcal{H}$ is an Ariki-Koike algebra (see Section 3.2), they coincide (see [7, Proposition 3.6]).

Set $\mathcal{O} := \mathcal{R}_K(y)$, and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. The ring $\mathcal{O}$ is a Dedekind ring (see, e.g., [5, Proposition 4.4.2]), and hence its localization $\mathcal{O}_{\mathfrak{p}\mathcal{O}}$ at the prime ideal generated by $\mathfrak{p}$ is a discrete valuation ring. Following [7, Proposition 2.14], we have the following.

PROPOSITION 2.10. *Two characters $\chi, \psi \in \mathrm{Irr}(W)$ are in the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of prime ideals of $\mathbb{Z}_K$ such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$;*
- *for all j $(1 \leq j \leq n)$, the characters χ_{j-1} and χ_j belong to the same block of $\mathcal{O}_{\mathfrak{p}_j\mathcal{O}}\mathcal{H}_\phi$.*

The above proposition implies that if we know the blocks of the algebra $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ for every prime ideal of $\mathbb{Z}_K$, then we know the Rouquier blocks of

$\mathcal{H}_\phi$. In order to determine the former, we can use the following theorem [5, Theorem 3.3.2].

THEOREM 2.11. *Let $A := \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$, and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. Let $M_1, \dots, M_k$ be all the $\mathfrak{p}$-essential monomials for W such that $\phi(M_j) = 1$ for all $j = 1, \dots, k$. Set $\mathfrak{q}_0 := \mathfrak{p}A$, $\mathfrak{q}_j := \mathfrak{p}A + (M_j - 1)A$ for $j = 1, \dots, k$, and set $\mathcal{Q} := \{\mathfrak{q}_0, \mathfrak{q}_1, \dots, \mathfrak{q}_k\}$. Two irreducible characters $\chi, \psi \in \mathrm{Irr}(W)$ are in the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\varphi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \dots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{q}_{j_1}, \dots, \mathfrak{q}_{j_n} \in \mathcal{Q}$ such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$;*
- *for all i $(1 \le i \le n)$, the characters χ_{i-1} and χ_i are in the same block of $A_{\mathfrak{q}_{j_i}}\mathcal{H}$.*

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$, and let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization. If $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ is a $\mathfrak{p}$-essential monomial for W, then

$$\phi(M) = 1 \Leftrightarrow \sum_{\mathcal{C},j} a_{\mathcal{C},j} n_{\mathcal{C},j} = 0.$$

Set $m := \sum_{\mathcal{C} \in \mathcal{A}/W} e_{\mathcal{C}}$. The hyperplane defined in $\mathbb{C}^m$ by the relation

$$\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0,$$

where $(t_{\mathcal{C},j})_{\mathcal{C},j}$ is a set of m indeterminates, is called a $\mathfrak{p}$-*essential hyperplane* for W. A hyperplane in $\mathbb{C}^m$ is called *essential* for W if it is $\mathfrak{p}$-essential for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$. Respectively, a monomial is called *essential* for W if it is $\mathfrak{p}$-essential for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$.

DEFINITION 2.12. Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization such that the integers $n_{\mathcal{C},j}$ belong to only one essential hyperplane H (resp., to no essential hyperplane). We say that ϕ is a *cyclotomic specialization associated with the essential hyperplane H* (resp., *with no essential hyperplane*). We call *Rouquier blocks associated with the hyperplane H* (resp., *with no essential hyperplane*), and denote by $\mathcal{B}^H$ (resp., $\mathcal{B}^\emptyset$), the partition of $\mathrm{Irr}(W)$ into Rouquier blocks of $\mathcal{H}_\phi$.

With the help of the above definition and thanks to Proposition 2.10 and Theorem 2.11, we obtain the following characterization for the Rouquier blocks of a cyclotomic Hecke algebra.

PROPOSITION 2.13. *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization. If the integers $n_{\mathcal{C},j}$ belong to no essential hyperplane, then the Rouquier blocks of the cyclotomic Hecke algebra $\mathcal{H}_\phi$ coincide with the partition $\mathcal{B}^\emptyset$. Otherwise, two irreducible characters $\chi, \psi \in \mathrm{Irr}(W)$ belong to the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $H_1, \ldots, H_n$ of essential hyperplanes that the $n_{\mathcal{C},j}$ belong to such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$;*
- *for all i $(1 \le i \le n)$, the characters χ_{i-1} and χ_i belong to $\mathcal{B}^{H_i}$.*

2.4. Functions a and A

Following the notation in [3, (6B)], for every element $P(y) \in \mathbb{C}(y)$, we call

- *valuation of $P(y)$ at y*, and denote by $\mathrm{val}_y(P)$, the order of $P(y)$ at 0 (we have $\mathrm{val}_y(P) < 0$ if 0 is a pole of $P(y)$ and $\mathrm{val}_y(P) > 0$ if 0 is a zero of $P(y)$), and
- *degree of $P(y)$ at y*, and denote by $\deg_y(P)$, the opposite of the valuation of $P(1/y)$.

Moreover, if $q := y^{|\mu(K)|}$, then

$$\mathrm{val}_q(P) := \frac{\mathrm{val}_y(P)}{|\mu(K)|} \qquad \text{and} \qquad \deg_q(P) := \frac{\deg_y(P)}{|\mu(K)|}.$$

For $\chi \in \mathrm{Irr}(W)$, we define

$$a_{\chi_\phi} := \mathrm{val}_q\big(s_{\chi_\phi}(y)\big) \qquad \text{and} \qquad A_{\chi_\phi} := \deg_q\big(s_{\chi_\phi}(y)\big).$$

The following result is proved in [2, Proposition 2.9].

PROPOSITION 2.14. *Let $\chi, \psi \in \mathrm{Irr}(W)$. If χ_ϕ and ψ_ϕ belong to the same Rouquier block, then*

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

The values of the functions a and A can be calculated from the generic Schur elements. In order to explain how, we need to introduce the following symbols.

DEFINITION 2.15. Let $n \in \mathbb{Z}$. We set

- $n^+ := \begin{cases} n, & \text{if } n > 0, \\ 0, & \text{if } n \le 0, \end{cases}$ and $(y^n)^+ := n^+$;

- $n^- = \begin{cases} n, & \text{if } n < 0, \\ 0, & \text{if } n \geq 0, \end{cases}$ and $(y^n)^- := n^-$.

Now let us fix $\chi \in \mathrm{Irr}(W)$. Following the notations of Theorem 2.3, the generic Schur element $s_\chi(\mathbf{v})$ associated to χ is an element of $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ of the form

$$(\dagger) \qquad s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}.$$

We fix the factorization (†) for $s_\chi(\mathbf{v})$. The following result is used in [6] in order to obtain that the functions a and A are constant on the Rouquier blocks of the cyclotomic Hecke algebras of the exceptional complex reflection groups.

PROPOSITION 2.16. *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization. Then*

- $\mathrm{val}_y(s_{\chi_\phi}(y)) = \phi(N_\chi)^+ + \phi(N_\chi)^- + \sum_{i \in I_\chi} n_{\chi,i} \deg(\Psi_{\chi,i})(\phi(M_{\chi,i}))^-$;
- $\deg_y(s_{\chi_\phi}(y)) = \phi(N_\chi)^+ + \phi(N_\chi)^- + \sum_{i \in I_\chi} n_{\chi,i} \deg(\Psi_{\chi,i})(\phi(M_{\chi,i}))^+$.

§3. Rouquier blocks of the cyclotomic Hecke algebras of $G(de, e, r)$, $r > 2$

In [13], Kim determined the Rouquier blocks for the cyclotomic Hecke algebras of $G(de, e, r)$ following the method used in [2] for $G(e, e, r)$. More specifically, she applied Clifford theory to obtain the blocks of $G(de, e, r)$ from the blocks of $G(de, 1, r)$. However, due to the incorrect determination of the Rouquier blocks for $G(de, 1, r)$ in [2] and further small mistakes in [13], we will proceed here to some modifications to the results and their proofs. Moreover, in the next section, the author explains why we have to distinguish the case where $r = 2$ (more precisely, where $r = 2$ and e is even).

3.1. Combinatorics

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a partition, that is, a finite decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h \geq 1$. The integer $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_h$ is called *the size of* λ. We also say that λ *is a partition of* $|\lambda|$. The integer h is called *the height of* λ, and we set $h_\lambda := h$. To each partition λ we associate its *β-number*, $\beta_\lambda = (\beta_1, \beta_2, \ldots, \beta_h)$, defined as follows:

$$\beta_1 := h + \lambda_1 - 1, \beta_2 := h + \lambda_2 - 2, \ldots, \beta_h := h + \lambda_h - h.$$

Multipartitions

From now on, let d be a positive integer. Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ be a d-partition (i.e., a family of d partitions indexed by the set $\{0, 1, \dots, d-1\}$). We set

$$h^{(a)} := h_{\lambda^{(a)}}, \qquad \beta^{(a)} := \beta_{\lambda^{(a)}},$$

and we have

$$\lambda^{(a)} = (\lambda_1^{(a)}, \lambda_2^{(a)}, \dots, \lambda_{h^{(a)}}^{(a)}).$$

The integer

$$|\lambda| := \sum_{a=0}^{d-1} |\lambda^{(a)}|$$

is called *the size of* λ. We also say that λ *is a* d*-partition of* $|\lambda|$.

Ordinary symbols

If $\beta = (\beta_1, \beta_2, \dots, \beta_h)$ is a sequence of positive integers such that $\beta_1 > \beta_2 > \cdots > \beta_h$ and if m is a positive integer, then the m-"shifted" of β is the sequence of numbers defined by

$$\beta[m] = (\beta_1 + m, \beta_2 + m, \dots, \beta_h + m, m-1, m-2, \dots, 1, 0).$$

Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ be a d-partition. We call d*-height of* λ the family $(h^{(0)}, h^{(1)}, \dots, h^{(d-1)})$, and we define the *height of* λ to be the integer

$$h_\lambda := \max\{h^{(a)} \mid (0 \le a \le d-1)\}.$$

DEFINITION 3.1. The *ordinary standard symbol* of λ is the family of numbers defined by $B_\lambda = (B^{(0)}, B^{(1)}, \dots, B^{(d-1)})$, where we have, for all cases of a $(0 \le a \le d-1)$,

$$B^{(a)} := \beta^{(a)}[h_\lambda - h^{(a)}].$$

The *ordinary content* of a d-partition of ordinary standard symbol B_λ is the multiset

$$\mathrm{Cont}_\lambda = B^{(0)} \cup B^{(1)} \cup \cdots \cup B^{(d-1)}.$$

Charged symbols

Assume that we have a given weight system, that is, a family of integers

$$m := (m^{(0)}, m^{(1)}, \dots, m^{(d-1)}).$$

Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ be a d-partition. We call (d, m)*-charged height of* λ the family $(hc^{(0)}, hc^{(1)}, \dots, hc^{(d-1)})$, where

$$hc^{(0)} := h^{(0)} - m^{(0)}, hc^{(1)} := h^{(1)} - m^{(1)}, \dots, hc^{(d-1)} := h^{(d-1)} - m^{(d-1)}.$$

We define the *m-charged height of* λ to be the integer

$$hc_\lambda := \max\{hc^{(a)} \mid (0 \leq a \leq d-1)\}.$$

DEFINITION 3.2. The *m-charged standard symbol* of λ is the family of numbers defined by $Bc_\lambda = (Bc^{(0)}, Bc^{(1)}, \dots, Bc^{(d-1)})$, where we have, for all cases of a $(0 \leq a \leq d-1)$,

$$Bc^{(a)} := \beta^{(a)}[hc_\lambda - hc^{(a)}].$$

REMARK. The ordinary standard symbol corresponds to the weight system

$$m^{(0)} = m^{(1)} = \cdots = m^{(d-1)} = 0.$$

The *m-charged content* of a d-partition of m-charged standard symbol Bc_λ is the multiset

$$\mathrm{Contc}_\lambda = Bc^{(0)} \cup Bc^{(1)} \cup \cdots \cup Bc^{(d-1)}.$$

3.2. Ariki-Koike algebras

The group $G(d,1,r)$ is the group of all monomial $r \times r$ matrices with entries in μ_d. It is isomorphic to the wreath product $\mu_d \wr \mathfrak{S}_r$, and its field of definition is $K := \mathbb{Q}(\zeta_d)$. Its irreducible characters are indexed by the d-partitions of r. If λ is a d-partition of r, then we denote by χ_λ the corresponding irreducible character of $G(d,1,r)$.

The *generic Ariki-Koike algebra* is the algebra $\mathcal{H}_{d,r}$ generated over the Laurent polynomial ring in $d+1$ indeterminates

$$\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \dots, u_{d-1}, u_{d-1}^{-1}, x, x^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{r-1}$ satisfying the relations

- $\mathbf{s}\mathbf{t}_1\mathbf{s}\mathbf{t}_1 = \mathbf{t}_1\mathbf{s}\mathbf{t}_1\mathbf{s}$, $\mathbf{s}\mathbf{t}_j = \mathbf{t}_j\mathbf{s}$ for $j \neq 1$;
- $\mathbf{t}_j\mathbf{t}_{j+1}\mathbf{t}_j = \mathbf{t}_{j+1}\mathbf{t}_j\mathbf{t}_{j+1}$, $\mathbf{t}_i\mathbf{t}_j = \mathbf{t}_j\mathbf{t}_i$ for $|i-j| > 1$; and
- $(\mathbf{s}-u_0)(\mathbf{s}-u_1)\cdots(\mathbf{s}-u_{d-1}) = (\mathbf{t}_j - x)(\mathbf{t}_j + 1) = 0$.

Let

$$\phi : \begin{cases} u_j \mapsto \zeta_d^j q^{m_j} & (0 \leq j < d), \\ x \mapsto q^n \end{cases}$$

be a cyclotomic specialization for $\mathcal{H}_{d,r}$. Thanks to Proposition 2.13, in order to determine the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$ for any ϕ, it suffices to determine the Rouquier blocks associated with no essential hyperplane and those

associated with each essential hyperplane for $G(d,1,r)$. Following [7], the essential hyperplanes for $G(d,1,r)$ are

- $kN + M_s - M_t = 0$, where $-r < k < r$ and $0 \le s < t < d$ such that $\zeta_d^s - \zeta_d^t$ belongs to a prime ideal of $\mathbb{Z}[\zeta_d]$, and
- $N = 0$.

We have proved the following (see [7, Propositions 3.12, 3.15, 3.17]).

THEOREM 3.3.

(1) *The Rouquier blocks associated with no essential hyperplane are trivial.*
(2) *Two irreducible characters χ_λ and χ_μ belong to the same Rouquier block associated with the essential hyperplane $kN + M_s - M_t = 0$ if and only if the following two conditions are satisfied:*
   - *we have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s,t\}$;*
   - *if $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system $(0,k)$.*
(3) *Two irreducible characters χ_λ and χ_μ belong to the same Rouquier block associated with the essential hyperplane $N = 0$ if and only if $|\lambda^{(a)}| = |\mu^{(a)}|$ for all $a = 0,1,\dots,d-1$.*

Following Proposition 2.13, the above theorem gives us an algorithm for the determination of the Rouquier blocks of any cyclotomic Ariki-Koike algebra (see [7, Theorem 3.18]).

3.3. Rouquier blocks for $G(de,e,r)$, $r > 2$

The group $G(de,e,r)$ is the group of all $r \times r$ monomial matrices with entries in μ_{de} such that the product of all nonzero entries lies in μ_d.

Following Ariki [1], we define the Hecke algebra of $G(de,e,r)$, $r > 2$, to be the algebra $\mathcal{H}_{de,e,r}$ generated over the Laurent polynomial ring in $d+1$ indeterminates

$$\mathbb{Z}[x_0, x_0^{-1}, x_1, x_1^{-1}, \dots, x_{d-1}, x_{d-1}^{-1}, z, z^{-1}]$$

by the elements $a_0, a_1, \dots, a_r$ satisfying the relations

- $(a_0 - x_0)(a_0 - x_1)\cdots(a_0 - x_{d-1}) = (a_j - z)(a_j + 1) = 0$ for $j = 1,\dots,r$;
- $a_1a_3a_1 = a_3a_1a_3$, $a_ja_{j+1}a_j = a_{j+1}a_ja_{j+1}$ for $j = 2,\dots,r-1$;
- $a_1a_2a_3a_1a_2a_3 = a_3a_1a_2a_3a_1a_2$;
- $a_1a_j = a_ja_1$ for $j = 4,\dots,r$;
- $a_ia_j = a_ja_i$ for $2 \le i < j \le r$ with $j - i > 1$;

- $a_0a_1a_2 = (z^{-1}a_1a_2)^{2-e}a_2a_0a_1 + (z-1)\sum_{k=1}^{e-2}(z^{-1}a_1a_2)^{1-k}a_0a_1 = a_1a_2a_0$; and
- $a_0a_j = a_ja_0$ for $j = 3, \dots, r$.

Let

$$\vartheta : \begin{cases} x_j \mapsto \zeta_d^j q^{m_j} & (0 \leq j < d), \\ y \mapsto q^n \end{cases}$$

be a cyclotomic specialization for $\mathcal{H}_{de,e,r}$. In order to determine the Rouquier blocks of $(\mathcal{H}_{de,e,r})_\vartheta$, we might as well consider the cyclotomic specialization

$$\phi : \begin{cases} x_j \mapsto \zeta_d^j q^{em_j} & (0 \leq j < d), \\ y \mapsto q^{en}. \end{cases}$$

Since the integers $\{(m_j)_{0\leq j<d}, n\}$ and $\{(em_j)_{0\leq j<d}, en\}$ belong to the same essential hyperplanes for $G(de,e,r)$, Proposition 2.13 implies that the Rouquier blocks of $(\mathcal{H}_{de,e,r})_\vartheta$ coincide with the Rouquier blocks of $(\mathcal{H}_{de,e,r})_\phi$.

We now consider the generic Ariki-Koike algebra $\mathcal{H}_{de,r}$ generated over the ring

$$\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \dots, u_{de-1}, u_{de-1}^{-1}, x, x^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{r-1}$ satisfying the relations described in Section 3.2. Let

$$\phi' : \begin{cases} u_j \mapsto \zeta_{de}^j q^{n_j} & (0 \leq j < de, n_j := m_{j \bmod d}), \\ x \mapsto q^{en} \end{cases}$$

be the "corresponding" cyclotomic specialization for $\mathcal{H}_{de,r}$, that is, the specialization with respect to the weight system

$$(m_0, m_1, \dots, m_{d-1}, m_0, m_1, \dots, m_{d-1}, \dots, m_0, m_1, \dots, m_{d-1}).$$

Set $\mathcal{H} := (\mathcal{H}_{de,r})_{\phi'}$, and let $\bar{\mathcal{H}}$ be the subalgebra of $\mathcal{H}$ generated by

$$\mathbf{s}^e, \tilde{\mathbf{t}}_1 := \mathbf{s}^{-1}\mathbf{t}_1\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{r-1}.$$

We have

$$\prod_{j=0}^{d-1}(\mathbf{s}^e - \zeta_d^j q^{em_j}) = (\tilde{\mathbf{t}}_1 - q^{en})(\tilde{\mathbf{t}}_1 + 1) = (\mathbf{t}_i - q^{en})(\mathbf{t}_i + 1) = 0$$

for $i=1,\dots,r-1$. Then, by [1, Proposition 1.16], we know that the algebra $(\mathcal{H}_{de,e,r})_\phi$ is isomorphic to the algebra $\bar{\mathcal{H}}$ via the morphism

$$a_0 \mapsto \mathbf{s}^e, \qquad a_1 \mapsto \tilde{\mathbf{t}}_1, a_j \mapsto \mathbf{t}_{j-1} \quad (2 \le j \le r).$$

We have the following result (see [13, Proposition 3.1]).

PROPOSITION 3.4. *The algebra $\mathcal{H}$ is a free $\bar{\mathcal{H}}$-module of rank e with basis $\{1, \mathbf{s}, \dots, \mathbf{s}^{e-1}\}$; that is,*

$$\mathcal{H} = \bar{\mathcal{H}} \oplus \mathbf{s}\bar{\mathcal{H}} \oplus \cdots \oplus \mathbf{s}^{e-1}\bar{\mathcal{H}}.$$

By [3, Proposition 1.18], the algebra $\mathcal{H}$ is symmetric and $\bar{\mathcal{H}}$ is a symmetric subalgebra of $\mathcal{H}$. In particular, following Definition 1.10, $\mathcal{H}$ is the twisted symmetric algebra of the cyclic group of order e over $\bar{\mathcal{H}}$ (since $\mathbf{s}$ is a unit in $\mathcal{H}$). Therefore, we can apply Proposition 1.15 and obtain the following (using the notations of Section 1.3).

PROPOSITION 3.5. *If G is the cyclic group of order e and $K := \mathbb{Q}(\zeta_{de})$, then the block-idempotents of $(Z\mathcal{R}_K(q)\bar{\mathcal{H}})^G$ coincide with the block-idempotents of $(Z\mathcal{R}_K(q)\mathcal{H})^{G^\vee}$, where $\mathcal{R}_K(q)$ is the Rouquier ring of K.*

The action of the cyclic group $G^\vee$ of order e on $\mathrm{Irr}(K(q)\mathcal{H})$ corresponds to the action generated by the cyclic permutation by d-packages on the de-partitions (see, e.g., [17, Section 4.A]):

$$\begin{aligned}\tau_d : (\lambda^{(0)}, \dots, \lambda^{(d-1)}, \lambda^{(d)}, \dots, \lambda^{(2d-1)}, \dots, \lambda^{(ed-d)}, \dots, \lambda^{(ed-1)}) \\ \mapsto (\lambda^{(ed-d)}, \dots, \lambda^{(ed-1)}, \lambda^{(0)}, \dots, \lambda^{(d-1)}, \dots, \lambda^{(ed-2d)}, \dots, \lambda^{(ed-d-1)}).\end{aligned}$$

More generally, the symmetric group $\mathfrak{S}_{de}$ acts naturally on the set of de-partitions of r: if $\tau \in \mathfrak{S}_{de}$ and $\nu = (\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(de-1)})$ is a de-partition of r, then $\tau(\nu) := (\nu^{(\tau(0))}, \nu^{(\tau(1))}, \dots, \nu^{(\tau(de-1))})$. The group $G^\vee$ is the cyclic subgroup of $\mathfrak{S}_{de}$ generated by the element

$$\tau_d = \prod_{j=0}^{d-1} \prod_{k=1}^{e-1} (j, j+kd).$$

Recall that $\mathcal{H}$ is the cyclotomic Ariki-Koike algebra of $G(de,1,r)$ corresponding to the weight system

$$(m_0, m_1, \dots, m_{d-1}, m_0, m_1, \dots, m_{d-1}, \dots, m_0, m_1, \dots, m_{d-1}).$$

Following Proposition 2.13, the Rouquier blocks of $\mathcal{H}$ are unions of the Rouquier blocks associated with the essential hyperplanes of the form

$$M_{j+kd} = M_{j+ld} \quad (0 \leq j < d)\ (0 \leq k < l < e).$$

In order to show that the Rouquier blocks of $\mathcal{H}$ are stable under the action of $G^\vee$, it suffices to prove the following lemma.

LEMMA 3.6. *Let λ be a de-partition of r, let $j \in \{0, \dots, d-1\}$, and let $k \in \{1, \dots, e-1\}$. If $\mu = (j, j+kd)\lambda$, then χ_λ and χ_μ belong to the same Rouquier block of $\mathcal{H}$.*

Proof. Suppose that $e = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where p_i are prime numbers such that $p_s \neq p_t$ for $s \neq t$. For $s \in \{1, 2, \dots, m\}$, we set $c_s := e/p_s^{a_s}$. Then $\gcd(c_s) = 1$ and, by Bezout's theorem, there exist integers $(b_s)_{1 \leq s \leq m}$ such that $\sum_{s=1}^m b_s c_s = 1$. Consequently, $k = \sum_{s=1}^m k b_s c_s$. We set $k_s := k b_s c_s$.

For all $s \in \{1, 2, \dots, m\}$, the element $1 - \zeta_e^{c_s}$ belongs to the prime ideal of $\mathbb{Z}[\zeta_{de}]$ lying over the prime number p_s. So does $1 - \zeta_e^{k_s}$. Now set

$$l_0 := 0 \qquad \text{and} \qquad l_s := \sum_{t=1}^{s} k_t \pmod e.$$

We have that the element $\zeta_{de}^{j+l_{s-1}d} - \zeta_{de}^{j+l_s d} = \zeta_{de}^{j+l_{s-1}d}(1 - \zeta_e^{k_s})$ belongs to the prime ideal of $\mathbb{Z}[\zeta_{de}]$ lying over the prime number p_s. Therefore, the hyperplane $M_{j+l_{s-1}d} = M_{j+l_s d}$ is essential for $G(de, 1, r)$. Following the characterization of the Rouquier blocks associated with that hyperplane by Theorem 3.3 and the fact that the ordinary content is stable under the action of a transposition, we obtain that the Rouquier blocks of $\mathcal{H}$ are stabilized by the action of $\sigma_s := (j + l_{s-1}d, j + l_s d)$. Set

$$\sigma := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{m-1} \circ \sigma_m \circ \sigma_{m-1} \circ \cdots \circ \sigma_2 \circ \sigma_1.$$

Then the characters χ_λ and $\chi_{\sigma(\lambda)}$ belong to the same Rouquier block of $\mathcal{H}$. It is easy to check that $\sigma(\lambda) = \mu$. ∎

Now the following result is immediate.

PROPOSITION 3.7. *If λ is a de-partition of r, then the characters χ_λ and $\chi_{\tau_d(\lambda)}$ belong to the same Rouquier block of $\mathcal{H}$. Therefore, the blocks of $\mathcal{R}_K(q)\mathcal{H}$ are stable under the action of $G^\vee$.*

Thanks to the above result, Proposition 3.5 now reads as follows.

COROLLARY 3.8. *The block-idempotents of* $(Z\mathcal{R}_K(q)\bar{\mathcal{H}})^G$ *coincide with the block-idempotents of* $\mathcal{R}_K(q)\mathcal{H}$.

Before we state our main result on the determination of the Rouquier blocks of $\mathcal{H}$, we will introduce the notion of "d-stuttering de-partition," following [13].

DEFINITION 3.9. Let λ be a de-partition of r. We say that λ is *d-stuttering* if it is fixed by the action of $G^\vee$, that is, if it is of the form

$$\lambda = (\lambda^{(0)}, \dots, \lambda^{(d-1)}, \lambda^{(0)}, \dots, \lambda^{(d-1)}, \dots, \lambda^{(0)}, \dots, \lambda^{(d-1)}),$$

where the first d partitions are repeated e times.

We are now ready to prove the main result.

THEOREM 3.10. *Let* λ *be a* de*-partition of* r*, and let* χ_λ *be the corresponding irreducible character of* $G(de,1,r)$. *We define* $\mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda$ *to be the subset of* $\mathrm{Irr}(K(q)\bar{\mathcal{H}})$ *with the property*

$$\mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}\chi_\lambda = \sum_{\bar{\chi}\in\mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda} \bar{\chi}.$$

Then

(1) *if* λ *is* d*-stuttering and* χ_λ *is a block of* $\mathcal{R}_K(q)\mathcal{H}$ *by itself, then there are* e *irreducible characters* $(\bar{\chi})_{\bar{\chi}\in\mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda}$, *each of which is a block of* $\mathcal{R}_K(q)\bar{\mathcal{H}}$ *by itself;*

(2) *the other blocks of* $\mathcal{R}_K(q)\mathcal{H}$ *are in bijection with the blocks of* $\mathcal{R}_K(q)\bar{\mathcal{H}}$ *via the map of Proposition 1.15; that is, the corresponding block-idempotents of* $\mathcal{R}_K(q)\mathcal{H}$ *coincide with the remaining block-idempotents of* $\mathcal{R}_K(q)\bar{\mathcal{H}}$.

Proof. We will use here the notations of Propositions 1.12 and 1.15.

If λ is a d-stuttering partition, then it is the only element in its orbit Ω under the action of $G^\vee$. We have that $|\Omega||\bar{\Omega}| = |G| = e$, whence there exist e elements in $\bar{\Omega} = \mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda$. If $\bar{\chi}\in\bar{\Omega}$, then its Schur element $s_{\bar{\chi}}$ is equal to the Schur element s_λ of χ_λ. If χ_λ is a block of $\mathcal{R}_K(q)\mathcal{H}$ by itself, then, by Propositions 2.10 and 1.8, s_λ is invertible in $\mathcal{R}_K(q)$ and so is $s_{\bar{\chi}}$. Thus, $\bar{\chi}$ is a block of $\mathcal{R}_K(q)\bar{\mathcal{H}}$ by itself.

If λ is not a d-stuttering partition and if b is the block containing χ_λ, then, in order to establish the desired bijection, we have to show that the

block $\bar{b}$ of $\mathcal{R}_K(q)\bar{\mathcal{H}}$ that contains a character in $\mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda$ is fixed by the action of G, that is, that $\bar{b} = \mathrm{Tr}(G, \bar{b})$. Thanks to Lemma 3.11, for all prime divisors p of e, there exists a de-partition $\lambda(p)$ of r such that $\chi_{\lambda(p)}$ belongs to b and the order of $G^\vee_{\chi_{\lambda(p)}}$ is not divisible by p. By Proposition 1.12, we know that for each $\bar{\chi} \in \mathrm{Irr}(K(q)\bar{\mathcal{H}})_{\lambda(p)}$, we have $|G^\vee_{\chi_{\lambda(p)}}||G_{\bar{\chi}}| = e$. Thus, $|G_{\bar{\chi}}|$ is divisible by the largest power of p dividing e. Since $b = \mathrm{Tr}(G, \bar{b})$, the elements of $\mathrm{Irr}(K(q)\bar{\mathcal{H}})_{\lambda(p)}$ belong to blocks of $\mathcal{R}_K(q)\bar{\mathcal{H}}$ conjugate of $\bar{b}$ by G, whose stabilizer is $G_{\bar{b}}$. By Lemma 1.13(1), we obtain that, for every prime number p, $|G_{\bar{b}}|$ is divisible by the largest power of p dividing e. Thus, $G_{\bar{b}} = G$ and $\mathrm{Tr}(G, \bar{b}) = \bar{b}$.

It remains to show that if λ is a d-stuttering partition and χ_λ is not a block of $\mathcal{R}_K(q)\mathcal{H}$ by itself, then there exists a partition μ such that χ_λ and χ_μ belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$ and μ is not d-stuttering. Then the second case described above covers our needs.

If λ is a d-stuttering partition, then the description of the Schur elements for $\mathcal{H}$ (see, e.g., [20, Corollary 6.5]) implies that the essential hyperplanes of the form

$$M_{j+kd} = M_{j+ld} \quad (0 \leq j < d)\ (0 \leq k < l < e)$$

are not essential for χ_λ. If now χ_λ is not a block of $\mathcal{R}_K(q)\mathcal{H}$ by itself, then, by Proposition 2.13, there exists a de-partition $\mu \neq \lambda$ such that χ_λ and χ_μ belong to the same Rouquier block associated with another essential hyperplane H for $G(de, 1, r)$ such that the integers $\{(n_j)_{0 \leq j < de}, en\}$ belong to H.

If H is $N = 0$, then, by Theorem 3.3, we have $|\lambda^{(a)}| = |\mu^{(a)}|$ for all $a = 0, 1, \ldots, de-1$. Since $\lambda \neq \mu$, there exists $s \in \{0, 1, \ldots, de-1\}$ such that $\lambda^{(s)} \neq \mu^{(s)}$. If ν is the partition obtained from λ by exchanging $\lambda^{(s)}$ and $\mu^{(s)}$, then χ_λ and χ_ν belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$ and ν is not d-stuttering.

If H is of the form $kN + M_s - M_t = 0$, where $-r < k < r$ and $0 \leq s < t < de$, then $\lambda^{(a)} = \mu^{(a)}$ for all $a \neq s, t$. If $s \not\equiv t \bmod d$ or $e > 2$, then μ cannot be d-stuttering. Suppose now that $s \equiv t \bmod d$ and $e = 2$. As mentioned above, the hyperplane $M_s = M_t$ is not essential for χ_λ, whence $k \neq 0$. Since the integers $\{(n_j)_{0 \leq j < de}, en\}$ belong to H and $n_s = n_t$, we must have $n = 0$. If μ is d-stuttering, then $\mu^{(s)} = \mu^{(t)}$, whence we deduce that $|\mu^{(s)}| = |\mu^{(t)}| = |\lambda^{(t)}| = |\lambda^{(s)}|$. Let ν be the de-partition obtained from λ by replacing $\lambda^{(t)}$ with $\mu^{(t)}$. Then ν is not d-stuttering and the characters χ_λ and χ_ν belong to the same Rouquier block associated with the essential hyperplane $N = 0$.

Since $n=0$, Proposition 2.13 implies that χ_λ and χ_ν belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$. □

LEMMA 3.11. *If λ is not a d-stuttering partition of r and p is a prime divisor of e, then there exists a de-partition $\lambda(p)$ of r such that χ_λ and $\chi_{\lambda(p)}$ belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$ and the order of $G^\vee_{\chi_{\lambda(p)}}$ is not divisible by p.*

Proof. If $\lambda = (\lambda^{(0)},\dots,\lambda^{(d-1)},\lambda^{(d)},\dots,\lambda^{(2d-1)},\dots,\lambda^{(ed-d)},\dots,\lambda^{(ed-1)})$, then, for $i=0,1,\dots,e-1$, we define the d-partition λ_i as follows:

$$\lambda_i := (\lambda^{(id)},\lambda^{(id+1)},\dots,\lambda^{(id+d-1)}).$$

Then $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_{e-1})$. Since λ is not d-stuttering, there exists $m\in\{0,1,\dots,e-1\}$ such that $\lambda_0\neq\lambda_m$. We denote by $\lambda(p)$ the partition obtained from λ by exchanging λ_m and $\lambda_{e/p}$. Due to Lemma 3.6, the characters χ_λ and $\chi_{\lambda(p)}$ belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$. Moreover, by construction, the de-partition $\lambda(p)$ is not fixed by the generator of the unique subgroup of order p of $G^\vee$, which proves that the order of its stabilizer is prime to p. □

Functions a and A

- The description of the Rouquier blocks of $\bar{\mathcal{H}}$ by Theorem 3.10,
- the relation between the Schur elements of $\bar{\mathcal{H}}$ and the Schur elements of $\mathcal{H}$ given by Proposition 1.12, and
- the invariance of the integers a_χ and A_χ on the Rouquier blocks of $\mathcal{H}$, resulting from [2, Proposition 3.18] and [7, Proposition 3.21], imply the following.

PROPOSITION 3.12. *The valuations $a_{\bar\chi}$ and the degrees $A_{\bar\chi}$ of the Schur elements are constant on the Rouquier blocks of $\bar{\mathcal{H}}$.*

§4. Rouquier blocks of the cyclotomic Hecke algebras of $G(de,e,2)$

If the integer e is odd, then the Hecke algebra of the group $G(de,e,2)$ can be viewed as a symmetric subalgebra of a Hecke algebra of the group $G(de,1,2)$, and all the results of the previous section hold.

If e is even, this cannot be done because there exist three orbits of reflecting hyperplanes under the action of the group. Following [1, Proposition 1.16], Malle shows [16, Proposition 3.9] that the Hecke algebra of the

group $G(de,e,2)$ can be viewed as a symmetric subalgebra of a Hecke algebra of the group $G(de,2,2)$, and thus we can apply Clifford theory in order to obtain the blocks of the former from the blocks of the latter.

4.1. Rouquier blocks for $G(2d,2,2)$

Let $d \geq 1$. The group $G(2d,2,2)$ has $4d$ irreducible characters of degree 1,

$$\chi_{ijk} \quad (0 \leq i,j \leq 1)\ (0 \leq k < d),$$

and d^2-d irreducible characters of degree 2,

$$\chi^1_{kl}, \chi^2_{kl} \quad (0 \leq k \neq l < d),$$

with $\chi^{1,2}_{kl} = \chi^{1,2}_{lk}$.

The generic Hecke algebra of the group $G(2d,2,2)$ is the algebra $\mathcal{H}_d$ generated over the Laurent polynomial ring in $d+4$ indeterminates

$$\mathbb{Z}[x_0, x_0^{-1}, x_1, x_1^{-1}, y_0, y_0^{-1}, y_1, y_1^{-1}, z_0, z_0^{-1}, z_1, z_1^{-1}, \dots, z_{d-1}, z_{d-1}^{-1}]$$

by the elements s,t,u satisfying the relations

- $stu = tus = ust$,
- $(s-x_0)(s-x_1) = (t-y_0)(t-y_1) = (u-z_0)(u-z_1)\cdots(u-z_{d-1}) = 0$.

The following theorem (see [16, Theorem 3.11]) gives a description of the generic Schur elements for $G(2d,2,2)$.

THEOREM 4.1. *Let us denote by Φ_1 the first $\mathbb{Q}$-cyclotomic polynomial (i.e., $\Phi_1(q) = q-1$). The generic Schur elements for $\mathcal{H}_d$ are given by*

$$\Phi_1(x_i x_{1-i}^{-1}) \cdot \Phi_1(y_j y_{1-j}^{-1}) \cdot \prod_{l=0,\, l\neq k}^{d-1} \big(\Phi_1(z_k z_l^{-1}) \cdot \Phi_1(x_i x_{1-i}^{-1} y_j y_{1-j}^{-1} z_k z_l^{-1})\big)$$

for the linear characters χ_{ijk}, and

$$-2 \cdot \prod_{m=0,\, m\neq k,l}^{d-1} \big(\Phi_1(z_k z_m^{-1}) \cdot \Phi_1(z_l z_m^{-1})\big) \times \prod_{i=0}^{1} \big(\Phi_1(X_i X_{1-i}^{-1} Y_i Y_{1-i}^{-1} Z_k Z_l^{-1}) \cdot \Phi_1(X_i X_{1-i}^{-1} Y_{1-i} Y_i^{-1} Z_l Z_k^{-1})\big),$$

with $X_i^2 := x_i$, $Y_j^2 := y_j$, $Z_k^2 := z_k$ for the characters $\chi^{1,2}_{kl}$ of degree 2.

The field of definition of $G(2d,2,2)$ is $K := \mathbb{Q}(\zeta_{2d})$. Following Theorem 2.2, if we set

$$\mathcal{X}_i^{|\mu(K)|} := (-1)^{-i}x_i \quad \text{for } i=0,1, \qquad \mathcal{Y}_j^{|\mu(K)|} := (-1)^{-j}y_j \quad \text{for } j=0,1,$$

and

$$\mathcal{Z}_k^{|\mu(K)|} := \zeta_d^{-k}z_j \quad \text{for } k=0,1,\dots,d-1,$$

then the algebra $K(\mathcal{X}_0,\mathcal{X}_1,\mathcal{Y}_0,\mathcal{Y}_1,\mathcal{Z}_0,\mathcal{Z}_1,\dots,\mathcal{Z}_{d-1})\mathcal{H}_d$ is split semisimple.

Let $\mathfrak{I}$ be the prime ideal of $\mathbb{Z}[\zeta_{2d}]$ lying over 2. The description of the generic Schur elements by Theorem 4.1 implies that the essential monomials for $G(2d,2,2)$ are

- $\mathcal{X}_0\mathcal{X}_1^{-1}$ ($\mathfrak{I}$-essential);
- $\mathcal{Y}_0\mathcal{Y}_1^{-1}$ ($\mathfrak{I}$-essential);
- $\mathcal{Z}_k\mathcal{Z}_l^{-1}$, where $0\le k<l<d$ are such that $\zeta_d^k-\zeta_d^l$ belongs to a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_{2d}]$ ($\mathfrak{p}$-essential);
- $\mathcal{X}_i\mathcal{X}_{1-i}^{-1}\mathcal{Y}_j\mathcal{Y}_{1-j}^{-1}\mathcal{Z}_k\mathcal{Z}_l^{-1}$, where $0\le i,j\le 1$ and $0\le k<l<d$ are such that $\zeta_d^k-\zeta_d^l$ belongs to a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_{2d}]$ ($\mathfrak{p}$-essential).

Let ϕ be a cyclotomic specialization for $\mathcal{H}_d$, that is, a $\mathbb{Z}_K$-algebra morphism of the form

$$\phi : \mathcal{X}_i \mapsto y^{a_i}, \qquad \mathcal{Y}_j \mapsto y^{b_j}, \qquad \mathcal{Z}_k \mapsto y^{c_k}.$$

Set $q := y^{|\mu(K)|}$. Then ϕ can be described as follows:

$$\phi : x_i \mapsto (-1)^i q^{a_i}, \qquad y_j \mapsto (-1)^j q^{b_j}, \qquad z_k \mapsto \zeta_d^k q^{c_k}.$$

Due to Proposition 2.8, Tits's deformation theorem implies that the specialization $y\mapsto 1$ induces a bijection

$$\operatorname{Irr}\big(K(y)(\mathcal{H}_d)_\phi\big) \leftrightarrow \operatorname{Irr}\big(G(2d,2,2)\big),$$
$$\chi_\phi \mapsto \chi.$$

For $\chi\in\operatorname{Irr}(G(2d,2,2))$, let s_{χ_ϕ} be the corresponding cyclotomic Schur element. As in Section 2.4, we set

$$a_{\chi_\phi} := \operatorname{val}_q(s_{\chi_\phi}(y)) = \frac{\operatorname{val}_y(s_{\chi_\phi}(y))}{|\mu(K)|}$$

and

$$A_{\chi_\phi} := \deg_q\big(s_{\chi_\phi}(y)\big) = \frac{\deg_y(s_{\chi_\phi}(y))}{|\mu(K)|}.$$

Then, by Proposition 2.14, we have that if two irreducible characters χ_ϕ and ψ_ϕ belong to the same Rouquier block of $(\mathcal{H}_d)_\phi$, then

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

Thanks to the formulas of Proposition 2.16, the following result derives immediately from the description of the generic Schur elements by Theorem 4.1.

PROPOSITION 4.2. *Let $\chi \in \mathrm{Irr}(G(2d,2,2))$. If χ is a linear character χ_{ijk}, then*

$$a_{\chi_\phi} + A_{\chi_\phi} = d(a_i - a_{1-i} + b_j - b_{1-j} + 2c_k) - 2\sum_{l=0}^{d-1} c_l.$$

If χ is a character $\chi_{kl}^{1,2}$ of degree 2, then

$$a_{\chi_\phi} + A_{\chi_\phi} = d(c_k + c_l) - 2\sum_{m=0}^{d-1} c_m.$$

Following Proposition 2.13, in order to determine the Rouquier blocks of the cyclotomic Hecke algebras of $G(2d,2,2)$, it suffices to determine the Rouquier blocks associated with its essential hyperplanes.

THEOREM 4.3. *For the group $G(2d,2,2)$, we have the following.*

(1) *The nontrivial Rouquier blocks associated with no essential hyperplane are*

$$\{\chi_{kl}^1, \chi_{kl}^2\} \quad \text{for all } 0 \le k < l < d.$$

(2) *The nontrivial Rouquier blocks associated with the $\mathfrak{I}$-essential hyperplane $A_0 = A_1$ are*

$$\{\chi_{0jk}, \chi_{1jk}\} \quad \text{for all } 0 \le j \le 1 \text{ and } 0 \le k < d,$$
$$\{\chi_{kl}^1, \chi_{kl}^2\} \quad \text{for all } 0 \le k < l < d.$$

(3) *The nontrivial Rouquier blocks associated with the $\mathfrak{I}$-essential hyperplane $B_0 = B_1$ are*

$$\{\chi_{i0k}, \chi_{i1k}\} \quad \text{for all } 0 \le i \le 1 \text{ and } 0 \le k < d,$$
$$\{\chi_{kl}^1, \chi_{kl}^2\} \quad \text{for all } 0 \le k < l < d.$$

(4) *The nontrivial Rouquier blocks associated with the* $\mathfrak{p}$*-essential hyperplane* $C_k = C_l$ $(0 \le k < l < d)$ *are*

$$\begin{aligned} &\{\chi_{ijk}, \chi_{ijl}\} && \text{for all } 0 \le i,j \le 1,\\ &\{\chi^1_{km}, \chi^2_{km}, \chi^1_{lm}, \chi^2_{lm}\} && \text{for all } 0 \le m < d \text{ with } m \notin \{k,l\},\\ &\{\chi^1_{kl}, \chi^2_{kl}\}, &&\\ &\{\chi^1_{rs}, \chi^2_{rs}\} && \text{for all } 0 \le r < s < d \text{ with } r,s \notin \{k,l\}. \end{aligned}$$

(5) *The nontrivial Rouquier blocks associated with the* $\mathfrak{p}$*-essential hyperplane* $A_i - A_{1-i} + B_j - B_{1-j} + C_k - C_l = 0$ $(0 \le i,j \le 1)$ $(0 \le k < l < d)$ *are*

$$\begin{aligned} &\{\chi_{ijk}, \chi_{1-i,1-j,l}, \chi^1_{kl}, \chi^2_{kl}\},\\ &\{\chi^1_{rs}, \chi^2_{rs}\} \quad \text{for all } 0 \le r < s < d \text{ with } (r,s) \ne (k,l). \end{aligned}$$

Proof. Following Definition 2.12, in each case we need to determine the Rouquier blocks of a cyclotomic Hecke algebra obtained via a specialization associated with the corresponding essential hyperplane. We recall that, due to Proposition 1.8, if a hyperplane is essential for an irreducible character χ, then χ is not a Rouquier block by itself. Moreover, Proposition 2.6(1) implies that the Rouquier blocks associated with an essential hyperplane are unions of the Rouquier blocks associated with no essential hyperplane.

(1) Let ϕ be any cyclotomic specialization associated with no essential hyperplane. Due to Proposition 1.8, each linear character is a Rouquier block by itself, whereas any character of degree 2 is not. Now, by Proposition 2.14, we have that if two irreducible characters χ_ϕ and ψ_ϕ belong to the same Rouquier block of $(\mathcal{H}_d)_\phi$, then $a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}$. The formulas of Proposition 4.2 imply that the character χ^1_{kl} $(0 \le k < l < d)$ can be in the same block only with the character χ^2_{kl}.

(2) Let ϕ be any cyclotomic specialization associated with the $\mathfrak{I}$-essential hyperplane $A_0 = A_1$. Since this is not an essential hyperplane for the characters of degree 2, Proposition 2.6 implies that $\{\chi^1_{kl}, \chi^2_{kl}\}$ is a Rouquier block of $(\mathcal{H}_d)_\phi$ for all $0 \le k < l < d$. Now, the hyperplane $A_0 = A_1$ is $\mathfrak{I}$-essential for all characters of degree 1, and thus, by Proposition 1.8, the linear characters do not form blocks by themselves. Due to Proposition 2.14, the formulas of Proposition 4.2 imply that the character χ_{0jk} $(0 \le j \le 1, 0 \le k < d)$ can be in the same block only with the character χ_{1jk}.

(3) For the $\mathfrak{I}$-essential hyperplane $B_0 = B_1$, we use the same method as in the previous case.

(4) Let ϕ be a cyclotomic specialization associated with the $\mathfrak{p}$-essential hyperplane $C_k = C_l$, where $0 \leq k < l < d$. Since the Rouquier blocks associated with an essential hyperplane are unions of the Rouquier blocks associated with no essential hyperplane, the characters χ^1_{rs} and χ^2_{rs} are in the same Rouquier block of $(\mathcal{H}_d)_\phi$ for all $0 \leq r < s < d$.

The hyperplane $C_k = C_l$ is $\mathfrak{p}$-essential for the linear characters

$$\chi_{ijk}, \chi_{ijl} \quad \text{for all } 0 \leq i, j \leq 1,$$

and the characters of degree 2

$$\chi^1_{km}, \chi^2_{km}, \chi^1_{lm}, \chi^2_{lm} \quad \text{for all } 0 \leq m < d \text{ with } m \notin \{k, l\}.$$

Due to Proposition 2.14, the formulas of Proposition 4.2 imply that

- the character χ_{ijk} $(0 \leq i, j \leq 1)$ can be in the same block only with the character χ_{ijl}, and
- the character χ^1_{km} $(0 \leq m < d$ and $m \notin \{k, l\})$ can be in the same block only with the characters $\chi^2_{km}, \chi^1_{lm}, \chi^2_{lm}$.

Let $m \in \{0, 1, \ldots, d-1\} \setminus \{k, l\}$. We have that the characters χ^1_{km} and χ^2_{km} are in the same Rouquier block of $(\mathcal{H}_d)_\phi$. The same holds for the characters χ^1_{lm} and χ^2_{lm}. Therefore, in order to obtain the desired result, it is enough to show that $\{\chi^1_{km}, \chi^2_{km}\}$ is not a Rouquier block of $(\mathcal{H}_d)_\phi$. Following [16, Table 3.10], there exists an element T_1 of $\mathcal{H}_d$ such that

$$\chi^1_{km}(T_1) = \chi^2_{km}(T_1) = x_0 + x_1.$$

Let $\mathcal{O}$ be the Rouquier ring of K. Suppose that $\{\chi^1_{km}, \chi^2_{km}\}$ is a block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_d)_\phi$. Then, by Corollary 1.7, we must have

$$\frac{\phi(\chi^1_{km}(T_1))}{\phi(s_{\chi^1_{km}})} + \frac{\phi(\chi^2_{km}(T_1))}{\phi(s_{\chi^2_{km}})} = \phi(x_0 + x_1) \cdot \Big(\frac{1}{\phi(s_{\chi^1_{km}})} + \frac{1}{\phi(s_{\chi^2_{km}})} \Big) \in \mathcal{O}_{\mathfrak{p}\mathcal{O}}.$$

Since ϕ is associated with the hyperplane $C_k = C_l$, we have that

$$\phi(x_0 + x_1) \notin \mathfrak{p}\mathcal{O},$$

and thus we obtain that

$$\frac{1}{\phi(s_{\chi^1_{km}})} + \frac{1}{\phi(s_{\chi^2_{km}})} \in \mathcal{O}_{\mathfrak{p}\mathcal{O}}.$$

Using the formulas of Theorem 4.1, we can easily calculate that the above element does not belong to $\mathcal{O}_{\mathfrak{p}\mathcal{O}}$.

(5) Let ϕ be a cyclotomic specialization associated with the $\mathfrak{p}$-essential hyperplane $A_i - A_{1-i} + B_j - B_{1-j} + C_k - C_l = 0$, where $0 \leq i, j \leq 1$ and $0 \leq k < l < d$. This hyperplane is $\mathfrak{p}$-essential for the following characters:

$$\chi_{ijk}, \chi_{1-i,1-j,l} \qquad \text{and} \qquad \chi^1_{kl} \text{ or } \chi^2_{kl}.$$

Let $\mathcal{O}$ be the Rouquier ring of K. If the hyperplane is essential for only three characters, then, due to Proposition 1.8, these three characters are in the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_d)_\phi$. Otherwise, using the same argument as in the previous case, we can prove that all four characters are in the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_d)_\phi$. Now, by Proposition 2.10, the Rouquier blocks of $(\mathcal{H}_d)_\phi$ are unions of the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_d)_\phi$ and $\mathcal{O}_{\mathfrak{J}\mathcal{O}}(\mathcal{H}_d)_\phi$. Therefore, the nontrivial Rouquier blocks of $(\mathcal{H}_d)_\phi$ are

$$\{\chi_{ijk}, \chi_{1-i,1-j,l}, \chi^1_{kl}, \chi^2_{kl}\},$$
$$\{\chi^1_{rs}, \chi^2_{rs}\} \quad \text{for all } 0 \leq r < s < d \text{ with } (r,s) \neq (k,l).$$

□

We are now going to prove the following desired result about the functions a and A.

PROPOSITION 4.4. *Let $\phi : x_i \mapsto (-1)^i q^{a_i}, y_j \mapsto (-1)^j q^{b_j}, z_k \mapsto \zeta_d^k q^{c_k}$ be a cyclotomic specialization for $\mathcal{H}_d$. If the irreducible characters χ_ϕ and ψ_ϕ belong to the same Rouquier block of $(\mathcal{H}_d)_\phi$, then*

$$a_{\chi_\phi} = a_{\psi_\phi} \qquad \text{and} \qquad A_{\chi_\phi} = A_{\psi_\phi}.$$

Proof. Thanks to Proposition 2.13, it suffices to show that the valuations a_{χ_ϕ} and the degrees A_{χ_ϕ} of the Schur elements are constant on the Rouquier blocks associated with an essential hyperplane H (resp., no essential hyperplane), when the integers a_i, b_j, c_k belong to the hyperplane H (resp., no essential hyperplane).

First, due to the description of the Schur elements by Theorem 4.1 and the formulas of Proposition 2.16, we can deduce that the Schur elements of the characters χ^1_{kl} and χ^2_{kl} $(0 \leq k < l < d)$ have the same valuation and the same degree for any cyclotomic specialization ϕ.

For the same reasons, we have that

- if $a_0 = a_1$, then

$$a_{\chi_{0jk}} = a_{\chi_{1jk}} \qquad \text{and} \qquad A_{\chi_{0jk}} = A_{\chi_{1jk}} \quad \text{for all } 0 \leq j \leq 1, 0 \leq k < d,$$

- if $b_0 = b_1$, then

$$a_{\chi_{i0k}} = a_{\chi_{i1k}} \qquad \text{and} \qquad A_{\chi_{i0k}} = A_{\chi_{i1k}} \quad \text{for all } 0 \leq i \leq 1, 0 \leq k < d,$$

- if $c_k = c_l$ $(0 \leq k < l < d)$, then

$$a_{\chi_{ijk}} = a_{\chi_{ijl}} \qquad \text{and} \qquad A_{\chi_{ijk}} = A_{\chi_{ijl}} \quad \text{for all } 0 \leq i, j \leq 1,$$
$$a_{\chi^{1,2}_{km}} = a_{\chi^{1,2}_{lm}} \qquad \text{and} \qquad A_{\chi^{1,2}_{km}} = A_{\chi^{1,2}_{lm}} \quad \text{for all } m \in \{0, 1, \ldots, d-1\} \setminus \{k, l\}.$$

Now let us suppose that $a_i - a_{1-i} + b_j - b_{1-j} + c_k - c_l = 0$, with $i, j \in \{0, 1\}$, $k, l \in \{0, 1, \ldots, d-1\}$, and $k < l$. We have to show that

$$a_{\chi_{ijk}} = a_{\chi_{1-i,1-j,l}} = a_{\chi^{1,2}_{kl}} \qquad \text{and} \qquad A_{\chi_{ijk}} = A_{\chi_{1-i,1-j,l}} = A_{\chi^{1,2}_{kl}}.$$

Due to Proposition 2.14, it suffices to show that

$$a_{\chi_{ijk}} = a_{\chi_{1-i,1-j,l}} = a_{\chi^{1,2}_{kl}}.$$

Using the notations of Proposition 2.16, Theorem 4.1 implies that

$$\begin{aligned}
a_{\chi_{ijk}} &= (a_i - a_{1-i})^- + (b_j - b_{1-j})^- \\
&\quad + \sum_{m=0,\, m\neq k}^{d-1} [(c_k - c_m)^- + (a_i - a_{1-i} + b_j - b_{1-j} + c_k - c_m)^-], \\
a_{\chi_{1-i,1-j,l}} &= (a_{1-i} - a_i)^- + (b_{1-j} - b_j)^- \\
&\quad + \sum_{m=0,\, m\neq l}^{d-1} [(c_l - c_m)^- + (a_{1-i} - a_i + b_{1-j} - b_j + c_l - c_m)^-], \\
a_{\chi^{1,2}_{kl}} &= \sum_{m=0,\, m\neq k,l}^{d-1} [(c_k - c_m)^- + (c_l - c_m)^-] \\
&\quad + (1/2) \cdot \sum_{h=0}^{1} [(a_h - a_{1-h} + b_h - b_{1-h} + c_k - c_l)^- \\
&\qquad + (a_h - a_{1-h} + b_{1-h} - b_h + c_l - c_k)^-].
\end{aligned}$$

Since $a_i - a_{1-i} + b_j - b_{1-j} + c_k - c_l = 0$, the above relations give

$$a_{\chi_{ijk}} = (a_i - a_{1-i})^- + (b_j - b_{1-j})^- + \sum_{m=0,\, m\neq k}^{d-1} [(c_k - c_m)^- + (c_l - c_m)^-],$$

$$a_{\chi_{1-i,1-j,l}} = (a_{1-i} - a_i)^- + (b_{1-j} - b_j)^- + \sum_{m=0,\, m\neq l}^{d-1} [(c_l - c_m)^- + (c_k - c_m)^-],$$

$$a_{\chi^{1,2}_{kl}} = \sum_{m=0,\, m\neq k,l}^{d-1} [(c_k - c_m)^- + (c_l - c_m)^-] + D,$$

where

$$D := \begin{cases} (a_i - a_{1-i})^- + (b_j - b_{1-j})^- + (c_k - c_l)^-, & \text{if } i = j, \\ (a_{1-i} - a_i)^- + (b_{1-j} - b_j)^- + (c_l - c_k)^-, & \text{if } i \neq j. \end{cases}$$

Obviously, if $i = j$, then $a_{\chi^{1,2}_{kl}} = a_{\chi_{ijk}}$, and if $i \neq j$, then $a_{\chi^{1,2}_{kl}} = a_{\chi_{1-i,1-j,l}}$. Therefore, it is enough to show that $a_{\chi_{ijk}} = a_{\chi_{1-i,1-j,l}}$, that is, that

$$\begin{aligned} &(a_i - a_{1-i})^- + (b_j - b_{1-j})^- + (c_k - c_l)^- \\ &\qquad = (a_{1-i} - a_i)^- + (b_{1-j} - b_j)^- + (c_l - c_k)^-. \end{aligned}$$

Since $n^- - (-n)^- = n$, for all $n \in \mathbb{Z}$ and $a_i - a_{1-i} + b_j - b_{1-j} + c_k - c_l = 0$, the above equality holds. ∎

4.2. Rouquier blocks for $G(2pd, 2p, 2)$

Let $p, d \geq 1$. We denote by $\mathcal{H}_{2pd,2p,2}$ the generic Hecke algebra of $G(2pd, 2p, 2)$ generated over the Laurent polynomial ring in $d+4$ indeterminates

$$\mathbb{Z}[X_0, X_0^{-1}, X_1, X_1^{-1}, Y_0, Y_0^{-1}, Y_1, Y_1^{-1}, Z_0, Z_0^{-1}, Z_1, Z_1^{-1}, \ldots, Z_{d-1}, Z_{d-1}^{-1}],$$

by the elements S, T, U satisfying the relations

- $(S - X_0)(S - X_1) = (T - Y_0)(T - Y_1) = (U - Z_0)(U - Z_1)\cdots(U - Z_{d-1}) = 0$,
- $STU = UST$, $TUS(TS)^{p-1} = U(ST)^p$.

Let

$$\vartheta : \begin{cases} X_i \mapsto (-1)^i q^{a_i} & (0 \leq i \leq 1), \\ Y_j \mapsto (-1)^j q^{b_j} & (0 \leq j \leq 1), \\ Z_k \mapsto \zeta_d^k q^{c_k} & (0 \leq k < d) \end{cases}$$

be a cyclotomic specialization for $\mathcal{H}_{2pd,2p,2}$. In order to determine the Rouquier blocks of $(\mathcal{H}_{2pd,2p,2})_\vartheta$, we might as well consider the cyclotomic specialization

$$\phi : \begin{cases} X_i \mapsto (-1)^i q^{pa_i} & (0 \leq i \leq 1), \\ Y_j \mapsto (-1)^j q^{pb_j} & (0 \leq j \leq 1), \\ Z_k \mapsto \zeta_d^k q^{pc_k} & (0 \leq k < d). \end{cases}$$

Since the integers $\{a_i, b_j, c_k\}$ and $\{pa_i, pb_j, pc_k\}$ belong to the same essential hyperplanes for $G(2pd, 2p, 2)$, Proposition 2.13 implies that the Rouquier blocks of $(\mathcal{H}_{2pd,2p,2})_\vartheta$ coincide with the Rouquier blocks of $(\mathcal{H}_{2pd,2p,2})_\phi$.

We now consider the generic Hecke algebra $\mathcal{H}_{pd}$ of $G(2pd, 2, 2)$ generated over the ring

$$\mathbb{Z}[x_0, x_0^{-1}, x_1, x_1^{-1}, y_0, y_0^{-1}, y_1, y_1^{-1}, z_0, z_0^{-1}, z_1, z_1^{-1}, \ldots, z_{pd-1}, z_{pd-1}^{-1}]$$

by the elements s, t, u satisfying the relations described in the beginning of Section 4.1. Let

$$\phi' : \begin{cases} x_i \mapsto (-1)^i q^{pa_i} & (0 \leq i \leq 1), \\ y_j \mapsto (-1)^j q^{pb_j} & (0 \leq j \leq 1), \\ z_k \mapsto \zeta_{pd}^k q^{e_k} & (0 \leq k < pd, e_k := c_{k \bmod d}) \end{cases}$$

be the "corresponding" cyclotomic specialization for $\mathcal{H}_{pd}$. Set $\mathcal{H} := (\mathcal{H}_{pd})_{\phi'}$, and let $\bar{\mathcal{H}}$ be the subalgebra of $\mathcal{H}$ generated by s, t, and u^p. We have

$$(s - q^{pa_0})(s + q^{pa_1}) = (t - q^{pb_0})(t + q^{pb_1}) = \prod_{k=0}^{d-1} (u^p - \zeta_d^k q^{pc_k}) = 0.$$

Then (as stated in [16, Proposition 3.9]) [1, Proposition 1.16] implies that the algebra $(\mathcal{H}_{2pd,2p,2})_\phi$ is isomorphic to the algebra $\bar{\mathcal{H}}$ via the morphism

$$S \mapsto s, T \mapsto t, U \mapsto u^p.$$

Under Assumptions 2.1, the algebra $\mathcal{H}$ is of rank $(2pd)^2$, whereas the algebra $\bar{\mathcal{H}}$ is of rank $(2pd)^2/p$. The following is immediate.

PROPOSITION 4.5. *The algebra $\mathcal{H}$ is a free $\bar{\mathcal{H}}$-module with basis $\{1, u, \ldots, u^{p-1}\}$; that is,*

$$\mathcal{H} = \bar{\mathcal{H}} \oplus u\bar{\mathcal{H}} \oplus \cdots \oplus u^{p-1}\bar{\mathcal{H}}.$$

Again under Assumptions 2.1, the algebra $\mathcal{H}$ is symmetric and $\bar{\mathcal{H}}$ is a symmetric subalgebra of $\mathcal{H}$. In particular, following Definition 1.10, $\mathcal{H}$ is the twisted symmetric algebra of the cyclic group of order p over $\bar{\mathcal{H}}$ (since u is a unit in $\mathcal{H}$). Therefore, we can apply Proposition 1.15 and obtain (using the notation of Section 1.3) the following.

PROPOSITION 4.6. *If G is the cyclic group of order p and $K := \mathbb{Q}(\zeta_{2pd})$, then the block-idempotents of $(Z\mathcal{R}_K(q)\bar{\mathcal{H}})^G$ coincide with the block-idempotents of $(Z\mathcal{R}_K(q)\mathcal{H})^{G^\vee}$, where $\mathcal{R}_K(q)$ is the Rouquier ring of K.*

The action of the cyclic group $G^\vee$ of order p on $\mathrm{Irr}(K(q)\mathcal{H})$ corresponds to the action

$$\chi_{i,j,k} \mapsto \chi_{i,j,k+d} \quad (0 \le i,j \le 1)\ (0 \le k < pd),$$
$$\chi^{1,2}_{k,l} \mapsto \chi^{1,2}_{k+d,l+d} \quad (0 \le k < l < pd),$$

where all the indexes are considered mod pd. With the help of the following lemma, we will show that the Rouquier blocks of $\mathcal{H}$ are stable under the action of $G^\vee$. Here the results of Theorem 4.3 will be used as definitions.

LEMMA 4.7. *Let k_1, k_2, k_3 be three distinct elements of $\{0,1,\dots,pd-1\}$. If the blocks of $\mathcal{R}_K(q)\mathcal{H}$ are unions of the Rouquier blocks associated with the (not necessarily essential) hyperplanes $C_{k_1} = C_{k_2}$ and $C_{k_2} = C_{k_3}$, then they are also unions of the Rouquier blocks associated with the (not necessarily essential) hyperplane $C_{k_1} = C_{k_3}$.*

Proof. We only need to show that

(a) the characters χ_{i,j,k_1} and χ_{i,j,k_3} are in the same block of $\mathcal{R}_K(q)\mathcal{H}$ for all $0 \le i,j \le 1$, and
(b) the characters $\chi^{1,2}_{k_1,m}$ and $\chi^{1,2}_{k_3,m}$ are in the same block of $\mathcal{R}_K(q)\mathcal{H}$ for all $0 \le m < pd$ with $m \notin \{k_1,k_3\}$.

Since the blocks of $\mathcal{R}_K(q)\mathcal{H}$ are unions of the Rouquier blocks associated with the hyperplanes $C_{k_1} = C_{k_2}$ and $C_{k_2} = C_{k_3}$, Theorem 4.3 yields that

(1) the characters χ_{i,j,k_1} and χ_{i,j,k_2} are in the same block of $\mathcal{R}_K(q)\mathcal{H}$ for all $0 \le i,j \le 1$;
(2) the characters χ_{i,j,k_2} and χ_{i,j,k_3} are in the same block of $\mathcal{R}_K(q)\mathcal{H}$ for all $0 \le i,j \le 1$;
(3) the characters $\chi^{1,2}_{k_1,m}$ and $\chi^{1,2}_{k_2,m}$ are in the same block of $\mathcal{R}_K(q)\mathcal{H}$ for all $0 \le m < pd$ with $m \notin \{k_1,k_2\}$; and

(4) the characters $\chi^{1,2}_{k_2,m}$ and $\chi^{1,2}_{k_3,m}$ are in the same block of $\mathcal{R}_K(q)\mathcal{H}$ for all $0 \leq m < pd$ with $m \notin \{k_2, k_3\}$.

We immediately deduce (a) for all $0 \leq i, j \leq 1$, and (b) for all $0 \leq m < pd$ with $m \notin \{k_1, k_2, k_3\}$. Finally, (3) implies that the characters $\chi^{1,2}_{k_1,k_3}$ and $\chi^{1,2}_{k_2,k_3}$ are in the same block of $\mathcal{R}_K(q)\mathcal{H}$, whereas by (4), $\chi^{1,2}_{k_1,k_2}$ and $\chi^{1,2}_{k_1,k_3}$ are also in the same block of $\mathcal{R}_K(q)\mathcal{H}$. Thus, the characters $\chi^{1,2}_{k_1,k_2}$ and $\chi^{1,2}_{k_2,k_3}$ belong to the same Rouquier block of $\mathcal{H}$. ∎

THEOREM 4.8. *The blocks of $\mathcal{R}_K(q)\mathcal{H}$ are stable under the action of $G^\vee$.*

Proof. Following Proposition 2.13, the Rouquier blocks of $\mathcal{H}$ are unions of the Rouquier blocks associated with all the essential hyperplanes of the form

$$C_{h+md} = C_{h+nd} \quad (0 \leq h < d, 0 \leq m < n < p).$$

Recall that the hyperplane $C_{h+md} = C_{h+nd}$ is actually essential for $G(2pd, 2, 2)$ if and only if the element $\zeta_{pd}^{h+md} - \zeta_{pd}^{h+nd}$ belongs to a prime ideal of $\mathbb{Z}[\zeta_{2pd}]$, that is, if and only if the element $\zeta_p^m - \zeta_p^n$ belongs to a prime ideal of $\mathbb{Z}[\zeta_{2pd}]$.

Suppose that $p = p_1^{t_1} p_2^{t_2} \cdots p_r^{t_r}$, where the p_i are distinct prime numbers. For $s \in \{1, 2, \ldots, r\}$, we set $h_s := p/p_s^{t_s}$. Then $\gcd(h_s) = 1$ and, by Bezout's theorem, there exist integers $(g_s)_{1 \leq s \leq r}$ such that $\sum_{s=1}^r g_s h_s = 1$. The element $1 - \zeta_p^{g_s h_s}$ belongs to all the prime ideals of $\mathbb{Z}[\zeta_{2pd}]$ lying over the prime number p_s. Let $h \in \{0, 1, \ldots, d-1\}$, and let $m \in \{0, 1, \ldots, p-2\}$. Set

$$l_0 := m \qquad \text{and} \qquad l_s := (l_{s-1} + g_s h_s) \bmod p, \quad \text{for all } s \ (1 \leq s \leq r).$$

We have that the element $\zeta_p^{l_{s-1}} - \zeta_p^{l_s} = \zeta_p^{l_{s-1}}(1 - \zeta_p^{g_s h_s})$ belongs to all the prime ideals of $\mathbb{Z}[\zeta_{2pd}]$ lying over the prime number p_s. Therefore, the hyperplane $C_{h+l_{s-1}d} = C_{h+l_s d}$ is essential for $G(2pd, 2, 2)$ for all s $(1 \leq s \leq r)$. Since $l_0 = m$ and $l_r = m+1$, Lemma 4.7 implies that the Rouquier blocks of $\mathcal{H}$ are unions of the Rouquier blocks associated with the (not necessarily essential) hyperplane

$$C_{h+md} = C_{h+(m+1)d},$$

following their description by Theorem 4.3. Since this holds for all m such that $0 \leq m \leq p-2$, Lemma 4.7 again implies that the Rouquier blocks of $\mathcal{H}$ are unions of the Rouquier blocks associated with all the hyperplanes of the form

$$C_{h+md} = C_{h+nd} \quad (0 \leq m < n < p),$$

for all $h (0 \leq h < d)$. We deduce that

(1) the characters $(\chi_{i,j,h+md})_{0\leq m<p}$ belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$, for all $0 \leq i,j \leq 1$ and $0 \leq h < d$;
(2) the characters $(\chi^{1,2}_{h+md,h+nd})_{0\leq m<n<p}$ belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$, for all $0 \leq h < d$; and
(3) the characters $(\chi^{1,2}_{h+md,h'+nd})_{0\leq m,n<p}$ belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$, for all $0 \leq h < h' < d$.

Hence, the blocks of $\mathcal{R}_K(q)\mathcal{H}$ are stable under the action of $G^\vee$. ∎

Following Theorem 4.8, Proposition 4.6 now gives the following.

COROLLARY 4.9. *If G is the cyclic group of order p and $K := \mathbb{Q}(\zeta_{2pd})$, then the block-idempotents of $(Z\mathcal{R}_K(q)\bar{\mathcal{H}})^G$ coincide with the block-idempotents of $\mathcal{R}_K(q)\mathcal{H}$.*

Now, let $\bar{\chi} \in \mathrm{Irr}(K(q)\bar{\mathcal{H}})$. Using the notation of Proposition 1.12, we have that $|\Omega||\bar{\Omega}| = p$. Since $|\Omega| = p$, we obtain that $|\bar{\Omega}| = 1$, and thus $e(\bar{\chi})$ is fixed by the action of G. Therefore, the block-idempotents of $\mathcal{R}_K(q)\bar{\mathcal{H}}$ are also fixed by the action of G. Consequently, we obtain the following.

PROPOSITION 4.10. *The block-idempotents of $\mathcal{R}_K(q)\bar{\mathcal{H}}$ coincide with the block-idempotents of $\mathcal{R}_K(q)\mathcal{H}$.*

Thanks to the above result, in order to determine the Rouquier blocks of $\bar{\mathcal{H}}$, it suffices to calculate the Rouquier blocks of $\mathcal{H}$ and restrict all the characters to $\bar{\mathcal{H}}$. The Rouquier blocks of $\mathcal{H}$ can be obtained with the use of Theorem 4.3.

Now,

- the description of the Rouquier blocks of $\bar{\mathcal{H}}$ by Proposition 4.9,
- the relation between the Schur elements of $\bar{\mathcal{H}}$ and the Schur elements of $\mathcal{H}$ given by Proposition 1.12, and
- the invariance of the integers a_χ and A_χ on the Rouquier blocks of $\mathcal{H}$, resulting from Proposition 4.4, imply the following.

PROPOSITION 4.11. *The valuations $a_{\bar{\chi}}$ and the degrees $A_{\bar{\chi}}$ of the Schur elements are constant on the Rouquier blocks of $\bar{\mathcal{H}}$.*

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University of Edinburgh
School of Mathematics
JCMB, King's Buildings
Edinburgh
EH9 3JZ, UK
maria.chlouveraki@ed.ac.uk