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ALTERNATIVE POLARIZATIONS OF BOREL FIXED IDEALS

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Abstract. For a monomial ideal I of a polynomial ring S, a *polarization* of I is a square-free monomial ideal J of a larger polynomial ring $\widetilde{S}$ such that S/I is a quotient of $\widetilde{S}/J$ by a (linear) regular sequence. We show that a Borel fixed ideal admits a nonstandard polarization. For example, while the usual polarization sends $xy^2 \in S$ to $x_1y_1y_2 \in \widetilde{S}$, ours sends it to $x_1y_2y_3$. Using this idea, we recover/refine the results on *square-free operation* in the shifting theory of simplicial complexes. The present paper generalizes a result of Nagel and Reiner, although our approach is very different.

§1. Introduction

Let both $S := \Bbbk[x_1, \dots, x_n]$ and $\widetilde{S} := \Bbbk[x_{i,j} \mid 1 \le i \le n, 1 \le j \le d]$ be polynomial rings over a field $\Bbbk$. Any monomial $\mathsf{m} \in S$ has a unique expression

$$\mathsf{m} = \prod_{i=1}^{e} x_{\alpha_i} \quad \text{with } 1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_e \le n. \tag{1.1}$$

If $\deg(\mathsf{m})(= e) \le d$, we set

$$\mathsf{b\text{-}pol}(\mathsf{m}) = \prod_{i=1}^{e} x_{\alpha_i, i} \in \widetilde{S}. \tag{1.2}$$

Note that $\mathsf{b\text{-}pol}(\mathsf{m})$ is a square-free monomial. For a monomial ideal $I \subset S$, $G(I)$ denotes the set of minimal (monomial) generators of I. If $\deg(\mathsf{m}) \le d$ for all $\mathsf{m} \in G(I)$, we set

$$\mathsf{b\text{-}pol}(I) := \big(\mathsf{b\text{-}pol}(\mathsf{m}) \bigm| \mathsf{m} \in G(I)\big) \subset \widetilde{S}.$$

In Theorem 3.4, we will show that if I is *Borel fixed* (i.e., $\mathsf{m} \in I$, $x_i | \mathsf{m}$, and $j < i$ imply that $(x_j/x_i) \cdot \mathsf{m} \in I$), then $J := \mathsf{b\text{-}pol}(I)$ is a polarization

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of I; that is, $\Theta := \{x_{i,1} - x_{i,j} \mid 1 \le i \le n, 2 \le j \le d\} \subset \widetilde{S}$ forms an $\widetilde{S}/J$-regular sequence with the canonical isomorphism $\widetilde{S}/(J + (\Theta)) \cong S/I$. For general monomial ideals, the corresponding statement is not true. Even for a Borel fixed ideal, $\mathsf{b\text{-}pol}$ is essentially different from the standard polarization (see Example 2.3). Recall that Borel fixed ideals play an important role in Gröbner basis theory and many related areas, since they appear as the *generic initial ideals* of homogeneous ideals (see [5, Section 15.9]).

The idea of $\mathsf{b\text{-}pol}(I)$ first appeared in Nagel and Reiner [10], although they did not give a specific name to this construction. Among other things, under the additional assumption that all elements of $G(I)$ have the same degree, they have shown the above result (it is not directly stated there, but follows from [10, Theorem 3.13]). Inspired by this, Lohne [8] undertakes a study of all possible polarizations of certain monomial ideals. He calls $\mathsf{b\text{-}pol}(I)$ the *box polarization*, since combinatorial objects called "boxes" are used in [10]. While the name "box" is no longer natural in our case, we use the symbol $\mathsf{b\text{-}pol}$.

To prove Theorem 3.4, we show that $\widetilde{S}/J$ has a *pretty clean filtration* introduced by Herzog and Popescu [6] and is sequentially Cohen-Macaulay. Moreover, since J is square-free, the simplicial complex associated with $\widetilde{S}/J$ is *nonpure shellable* in the sense of Björner and Wachs [3].

Inspired by Kalai's theory on the *algebraic shifting* of simplicial complexes (see [7]), Aramova, Herzog, and Hibi [2] introduced the operation sending a monomial $\mathsf{m} \in S$ of (1.1) to the square-free monomial

$$\mathsf{m}^\sigma := \prod_{i=1}^{e} x_{\alpha_i + i - 1}$$

in a polynomial ring $T := \Bbbk[x_1, \dots, x_N]$ with $N \gg 0$. If $I \subset S$ is a Borel fixed monomial ideal, we can define the square-free monomial ideal $I^\sigma \subset T$ in the natural way. (This construction works for general monomial ideals, but is important for Borel fixed ideals.) This operation has the remarkable property that $\beta^S_{i,j}(I) = \beta^T_{i,j}(I^\sigma)$ for all i, j, as shown in [2]. Here $\beta_{i,j}(-)$ denotes the graded Betti number, as usual.

In Section 4, we will study I^σ through our polarization $J := \mathsf{b\text{-}pol}(I)$. In fact, $\Theta_1 := \{x_{i,j} - x_{i+1,j-1} \mid 1 \le i < n, 1 < j \le d\}$ also forms an $\widetilde{S}/J$-regular sequence, and we have $\widetilde{S}/(J + (\Theta_1)) \cong T/I^\sigma$ (if we set the number N of the variables of T to be $n + d - 1$). Hence, we get a new proof of the equation $\beta^S_{i,j}(I) = \beta^T_{i,j}(I^\sigma)$. Moreover, we have $\beta^T_{i,j}(\operatorname{Ext}^k_T(T/I^\sigma, T)) =$

$\beta^S_{i,j}(\mathrm{Ext}^k_S(S/I,S))$ for all i,j,k. Murai ([9]) has generalized the operation $(-)^\sigma$ so that the equations on the Betti numbers remain true. We can also understand his operation using $\mathsf{b\text{-}pol}$. In fact, it is enough to change an $\widetilde{S}/J$-regular sequence $\Theta' \subset \widetilde{S}$.

§2. Preparation

We introduce the conventions and notation used throughout the paper. Let $S = \Bbbk[x_1,\dots,x_n]$ be a polynomial ring over a field $\Bbbk$. The ith coordinate of $\mathbf{a} \in \mathbb{N}^n$ is denoted by a_i (i.e., we change the font). For $\mathbf{a} \in \mathbb{N}^n$, $x^{\mathbf{a}}$ denotes the monomial $\prod_{i=1}^n x_i^{a_i} \in S$. For a monomial $\mathsf{m} := x^{\mathbf{a}}$, set $\deg(\mathsf{m}) := \sum_{i=1}^n a_i$, and set $\deg_i(\mathsf{m}) := a_i$. We define the order $\succeq$ on $\mathbb{N}^n$ so that $\mathbf{a} \succeq \mathbf{b}$ if $a_i \geq b_i$ for all i. We refer to [4] and [5] for unexplained terminology.

Take $\mathbf{d} \in \mathbb{N}^n$ with $d_i \geq 1$ for all i, and set

$$\widetilde{S} := \Bbbk[x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d_i].$$

Note that

$$\Theta := \{x_{i,1} - x_{i,j} \mid 1 \leq i \leq n, 2 \leq j \leq d_i\} \subset \widetilde{S}$$

forms a regular sequence with $\widetilde{S}/(\Theta) \cong S$. Here the isomorphism is induced by the ring homomorphism $\phi : \widetilde{S} \to S$ with $\phi(x_{i,j}) = x_i$. Throughout this paper, $\widetilde{S}$ and Θ are used in this meaning, while the choice of $\mathbf{d} \in \mathbb{N}^n$ depends on the context.

Definition 2.1. For a monomial ideal $I \subset S$, a *polarization* of I is a square-free monomial ideal $J \subset \widetilde{S}$ satisfying the following conditions.
(i) Through the isomorphism $S \to \widetilde{S}/(\Theta)$, we have $\widetilde{S}/(\Theta) \otimes_{\widetilde{S}} \widetilde{S}/J \cong S/I$.
(ii) Θ forms an $\widetilde{S}/J$-regular sequence.

Clearly, condition (i) holds if and only if $\phi(J) = I$. The following is a well-known fact, and a proof is found in [10, Lemma 6.9].

Lemma 2.2 (see [10, Lemma 6.9]). *Let I and J be monomial ideals of S and $\widetilde{S}$, respectively. Assume that Definition 2.1(i) is satisfied. Then condition (ii) is equivalent to the following:*
(ii′)

$$\beta^{\widetilde{S}}_{i,j}(J) = \beta^S_{i,j}(I) \quad \textit{for all } i,j.$$

While the proof in [10] concerns only the case $\#\Theta = 1$, it works in the general case. If Θ does not form an $\widetilde{S}/J$-regular sequence, the relation between

$\beta_{i,j}^{\widetilde{S}}(J)$ and $\beta_{i,j}^{S}(I)$ is not simple. So it is better to compare the Hilbert series of $\widetilde{S}/J$ with that of S/I (recall that the Hilbert series is determined by the Betti numbers).

For a monomial $x^{\mathbf{a}}$ with $\mathbf{a} \preceq \mathbf{d}$, set

$$\mathsf{pol}(x^{\mathbf{a}}) := \prod_{1 \le i \le n} x_{i,1}x_{i,2}\cdots x_{i,a_i} \in \widetilde{S}.$$

Let $I \subset S$ be a monomial ideal with $\mathbf{a} \preceq \mathbf{d}$ for all $x^{\mathbf{a}} \in G(I)$. Here $G(I)$ denotes the set of minimal (monomial) generators of I. Then it is well known that

$$\mathsf{pol}(I) = \big(\mathsf{pol}(x^{\mathbf{a}}) \bigm| x^{\mathbf{a}} \in G(I)\big)$$

gives a polarization of I, which is called the *standard polarization*. (If the reader is nervous about the choice of $\mathbf{d} \in \mathbb{N}^n$, take it so that $x^{\mathbf{d}}$ is the least common multiple of the minimal generators of I. For the properties considered in this paper, the choice of $\mathbf{d}$ is not essential.) While all monomial ideals have the standard polarizations, some have alternative ones.

Let d be a positive integer, and set

$$\widetilde{S} := \Bbbk[x_{i,j} \mid 1 \le i \le n, 1 \le j \le d]. \tag{2.1}$$

For a monomial $x^{\mathbf{a}} \in S$ with $e := \deg(x^{\mathbf{a}}) \le d$, set $b_i := \sum_{j=1}^{i} a_j$ for each $i \ge 0$ (here $b_0 = 0$), and set

$$\mathsf{b\text{-}pol}(x^{\mathbf{a}}) := \prod_{\substack{1 \le i \le n \\ b_{i-1}+1 \le j \le b_i}} x_{i,j} \in \widetilde{S}.$$

If $a_i = 0$, then $b_{i-1} = b_i$ and $x_{i,j}$ does not divide $\mathsf{b\text{-}pol}(x^{\mathbf{a}})$ for all j. If $\mathsf{m} = x^{\mathbf{a}} \in S$ is the monomial of (1.1), then we have $b_i = \max\{j \mid \alpha_j \le i\}$, and the above definition of $\mathsf{b\text{-}pol}(x^{\mathbf{a}})$ coincides with the one given in (1.2).

Let $I \subset S$ be a monomial ideal with $\deg(x^{\mathbf{a}}) \le d$ for all $x^{\mathbf{a}} \in G(I)$. Set

$$\mathsf{b\text{-}pol}(I) := \big(\mathsf{b\text{-}pol}(x^{\mathbf{a}}) \bigm| x^{\mathbf{a}} \in G(I)\big) \subset \widetilde{S}.$$

Occasionally, this ideal gives a polarization of I. Note that Definition 2.1(i) is always satisfied, and the problem is condition (ii).

In the remainder of this article, when we treat $\mathsf{b\text{-}pol}(I)$, we assume that $\widetilde{S}$ is the one in (2.1) and that $\deg(\mathsf{m}) \le d$ for all $\mathsf{m} \in G(I)$.

EXAMPLE 2.3. (1) For $I = (x^2, xy, xz, y^2, yz) \subset \Bbbk[x, y, z]$, we have

$$\mathsf{b\text{-}pol}(I) = (x_1x_2, x_1y_2, x_1z_2, y_1y_2, y_1z_2),$$

and it gives a polarization. In fact, since I is Borel fixed, we can use Theorem 3.4 below. It is essentially different from the standard polarization

$$\mathsf{pol}(I) = (x_1x_2, x_1y_1, x_1z_1, y_1y_2, y_1z_1).$$

More precisely, $\mathsf{b\text{-}pol}(I)$ and $\mathsf{pol}(I)$ are different even after permutation of variables.

(2) In general, $\mathsf{b\text{-}pol}(I)$ does not give a polarization. For example, if $I = (xyz, x^2y, xy^2, x^3)$, then $\mathsf{b\text{-}pol}(I) = (x_1y_2z_3, x_1x_2y_3, x_1y_2y_3, x_1x_2x_3)$, and it is not a polarization. To see this, use Lemma 2.2. Note that I is a *stable* monomial ideal, and Borel fixed ideals are nothing other than *strongly stable* monomial ideals (see [1] for the definitions).

DEFINITION 2.4. We say that a polarization J of I is *faithful* if Θ forms an $\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})$-regular sequence for all i.

If a polarization J of I is faithful, then we have

$$\widetilde{S}/(\Theta) \otimes_{\widetilde{S}} \operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S}) \cong \operatorname{Ext}^i_S(S/I, S).$$

In fact, the long exact sequences of $\operatorname{Ext}^\bullet_{\widetilde{S}}(-, \widetilde{S})$ yield

$$\widetilde{S}/(\Theta) \otimes_{\widetilde{S}} \operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S}) \cong \operatorname{Ext}^{i+(\#\Theta)}_{\widetilde{S}}\big(\widetilde{S}/(J + (\Theta)), \widetilde{S}\big).$$

Since $\Theta \subset \widetilde{S}$ forms an $\widetilde{S}$-regular sequence with $\widetilde{S}/(J + (\Theta)) \cong S/I$, we have

$$\operatorname{Ext}^{i+(\#\Theta)}_{\widetilde{S}}\big(\widetilde{S}/(J + (\Theta)), \widetilde{S}\big) \cong \operatorname{Ext}^i_S(S/I, S).$$

Hence, if J is faithful, $\operatorname{Ext}^i_S(S/I, S)$ and $\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})$ have the same degree and Betti numbers. So S/I and $\widetilde{S}/J$ have the same arithmetic degree in this case.

REMARK 2.5. For any I, the standard polarization is always faithful by [11, Corollary 4.10] (see also [13, Theorem 4.4]). It is an easy exercise to show that if S/I is Cohen-Macaulay, then any polarization of I is faithful. In Lemma 2.8 below, we will generalize this fact.

EXAMPLE 2.6. For the ideal $I := (x^2y, x^2z, xyz, xz^2, y^3, y^2z, yz^2)$ of $S := \Bbbk[x,y,z]$, $J := \textsf{b-pol}(I) \subset \widetilde{S}$ gives a polarization. (To see this, compute the Betti numbers.) However, $\deg \operatorname{Ext}^3_S(S/I, S) = 6$ and $\deg \operatorname{Ext}^3_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S}) = 5$. Hence, J is not faithful.

Let M be a finitely generated S-module. We say that M is *sequentially Cohen-Macaulay* if $\operatorname{Ext}^{n-i}_S(M, S)$ is either a Cohen-Macaulay module of dimension i or the 0-module for all i. The original definition is given by the existence of a certain filtration (see [12, III, Definition 2.9]); however, it is equivalent to the above by [12, III, Theorem 2.11].

LEMMA 2.7. *Let M be a sequentially Cohen-Macaulay S-module, and let $y \in S$ be a nonzero divisor of M. Then y is a nonzero divisor of $\operatorname{Ext}^i_S(M, S)$ for all i, and M/yM is a sequentially Cohen-Macaulay module with*

$$\operatorname{Ext}^{i+1}_S(M/yM, S) \cong \operatorname{Ext}^i_S(M, S)/y \cdot \operatorname{Ext}^i_S(M, S).$$

Moreover, we have

$$\begin{aligned}&\operatorname{Ass}(M/yM)\\ &\quad = \bigl\{\mathfrak{p} \bigm| \mathfrak{p} \text{ is a minimal prime of } \mathfrak{p}' + (y) \text{ for some } \mathfrak{p}' \in \operatorname{Ass}(M)\bigr\}.\end{aligned}$$

If $y \in S_1$, and all associated primes of M are generated by elements in S_1, then

$$\operatorname{Ass}(M/yM) = \bigl\{\mathfrak{p}' + (y) \bigm| \mathfrak{p}' \in \operatorname{Ass}(M)\bigr\}.$$

To prove this lemma, recall the following basic properties of a finitely generated module N over S (see [4, Theorem 8.1.1]):

(1) $\dim_S(\operatorname{Ext}^i_S(N, S)) \le n - i$ for all i;

(2) for a prime ideal $\mathfrak{p} \subset S$ of codimension c, $\mathfrak{p} \in \operatorname{Ass}(N)$ if and only if $\mathfrak{p}$ is an associated (equivalently, minimal) prime of $\operatorname{Ext}^c_S(N, S)$.

Proof. By the above remark, we have $\operatorname{Ass}(M) = \bigcup_i \operatorname{Ass}(\operatorname{Ext}^i_S(M, S))$. Hence, the first half of the lemma is easy. To see the next assertion, let $\mathfrak{p} \subset S$ be a prime ideal of codimension c. Then we have

$$\begin{aligned}\mathfrak{p} \in \operatorname{Ass}(M/yM) &\Longleftrightarrow \mathfrak{p}S_{\mathfrak{p}} \in \operatorname{Ass}_{S_{\mathfrak{p}}}\bigl(\operatorname{Ext}^c_S(M/yM, S) \otimes_S S_{\mathfrak{p}}\bigr)\\ &\Longleftrightarrow \dim_{S_{\mathfrak{p}}}\bigl(\operatorname{Ext}^{c-1}_S(M, S) \otimes_S S_{\mathfrak{p}}\bigr) = n - c + 1 \text{ and } y \in \mathfrak{p}\\ &\Longleftrightarrow \exists \mathfrak{p}' \in \operatorname{Ass}\bigl(\operatorname{Ext}^{c-1}_S(M, S)\bigr) \text{ with } \operatorname{codim} \mathfrak{p}' = c - 1,\end{aligned}$$

$$\begin{aligned} &\mathfrak{p}' \subset \mathfrak{p}, \text{ and } y \in \mathfrak{p} \\ \iff &\exists \mathfrak{p}' \in \operatorname{Ass}(M) \text{ with } \operatorname{codim} \mathfrak{p}' = c-1, \mathfrak{p}' \subset \mathfrak{p}, \text{ and } y \in \mathfrak{p} \\ \iff &\exists \mathfrak{p}' \in \operatorname{Ass}(M) \text{ such that } \mathfrak{p} \text{ is a minimal prime of} \\ &\mathfrak{p}' + (y). \end{aligned}$$

The last assertion of the lemma is clear now, since $\mathfrak{p}' + (y)$ is a prime ideal for all $\mathfrak{p}' \in \operatorname{Ass}(M)$ in this case. ∎

LEMMA 2.8. *Let J be a polarization of I. If $\widetilde{S}/J$ is sequentially Cohen-Macaulay, then so is S/I, and J is faithful.*

Proof. This follows from the first assertion of Lemma 2.7. ∎

REMARK 2.9. Even if S/I is sequentially Cohen-Macaulay, a polarization J is not necessarily faithful. In fact, S/I of Example 2.6 is sequentially Cohen-Macaulay.

DEFINITION 2.10. Let M be an S-module, and let

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_t = M$$

be a prime filtration; that is, there is a prime ideal $\mathfrak{p}_i$ such that $M_i/M_{i-1} \cong S/\mathfrak{p}_i$ for each $1 \le i \le t$. Herzog and Popescu [6] call the filtration $\mathcal{F}$ *pretty clean* if $i < j$ and $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ imply that $\mathfrak{p}_i = \mathfrak{p}_j$.

For example, if $\operatorname{codim} \mathfrak{p}_i \ge \operatorname{codim} \mathfrak{p}_j$ for all i, j with $i < j$, then $\mathcal{F}$ is pretty clean. By [6, Theorem 4.1, Corollary 3.4], if M admits a pretty clean filtration $\mathcal{F}$, then M is sequentially Cohen-Macaulay and $\operatorname{Ass} M = \{\mathfrak{p}_i \mid 1 \le i \le t\}$.

§3. Main results

We say that a monomial ideal I is *Borel fixed* if $\mathsf{m} \in I$, $x_i | \mathsf{m}$, and $j < i$ imply that $(x_j/x_i) \cdot \mathsf{m} \in I$. If $\operatorname{char}(\Bbbk) > 0$, this terminology is unnatural (see [5, Section 15.9.2] for details), and the terms *0-Borel fixed ideals* or *strongly stable monomial ideals* are also used in the literature. However, we will call it a *Borel fixed ideal* for simplicity.

For a monomial $\mathsf{m} \in S$, set

$$\nu(\mathsf{m}) := \max\{i \mid x_i \text{ divides } \mathsf{m}\}.$$

Similarly, for a monomial ideal $I \subset S$, set $\nu(I) := \max\{\nu(\mathsf{m}) \mid \mathsf{m} \in G(I)\}$. If I is Borel fixed, it is well known that $\nu(I) = \operatorname{proj.dim}_S(S/I)$ (see [5, Corollary 15.25]), although we do not use this fact.

LEMMA 3.1. *If I is a Borel fixed ideal (with* $\deg(\mathsf{m}) \le d$ *for all* $\mathsf{m} \in G(I)$*), then*

$$\text{b-pol}(I) = \big(\text{b-pol}(\mathsf{m}) \bigm| \mathsf{m} \in I \ \textit{with} \ \deg(\mathsf{m}) \le d\big).$$

Proof. Since the inclusion "$\subseteq$" is clear, it suffices to show the converse. For the contrary, assume that there is some $\mathsf{m} \in I$ with $\deg(\mathsf{m}) \le d$ and $\text{b-pol}(\mathsf{m}) \notin \text{b-pol}(I)$. Take m so that it has the smallest degree among these monomials. It is clear that $\mathsf{m} \notin G(I)$. Hence, there is some i with $x_i|\mathsf{m}$ and $\mathsf{m}' := \mathsf{m}/x_i \in I$. Set $l := \nu(\mathsf{m})$. Since I is Borel fixed, we have $\mathsf{m}'' := \mathsf{m}/x_l = (x_i/x_l)\cdot \mathsf{m}' \in I$. Since $\deg(\mathsf{m}'') < \deg(\mathsf{m}) =: e$, we have $\text{b-pol}(\mathsf{m}'') \in \text{b-pol}(I)$. Hence, $\text{b-pol}(\mathsf{m}) = x_{l,e} \cdot \text{b-pol}(\mathsf{m}'') \in \text{b-pol}(I)$. This is a contradiction. □

As shown in [6, Proposition 5.2], the quotient S/I of a Borel fixed ideal I has a pretty clean filtration. The next result states that the same is true for $J := \text{b-pol}(I)$. Moreover, since J is a radical ideal, $\widetilde{S}/J$ actually admits a clean filtration by [6, Corollary 3.5]. Hence, the simplicial complex associated with J is nonpure shellable.

THEOREM 3.2. *Let I be a Borel fixed ideal, and set $J := \text{b-pol}(I)$. Then $\widetilde{S}/J$ has a pretty clean filtration; in particular, $\widetilde{S}/J$ is sequentially Cohen-Macaulay.*

Proof. Set $l := \nu(I)$. Then $\{\mathsf{m} \in G(I) \mid \nu(\mathsf{m}) = l\}$ is nonempty. Let m be the maximum element of this set with respect to the lexicographic order. If $\mathsf{m} = x_l$, then I (resp., J) is a prime ideal $(x_1, \dots, x_l)$ (resp., $(x_{1,1}, x_{2,1}, \dots, x_{l,1})$) and there is nothing to prove. So we may assume that $\mathsf{m} \ne x_l$, and we set $\mathsf{m}_1 := \mathsf{m}/x_l$. Since $\mathsf{m} \in G(I)$, we have $\mathsf{m}_1 \notin I$.

CLAIM 1. *The ideal $I_1 := I + (\mathsf{m}_1)$ is Borel fixed.*

Proof of Claim 1. It suffices to show that $x_i|\mathsf{m}_1$ and $j < i$ imply that $(x_j/x_i)\cdot \mathsf{m}_1 \in I$. Note that $\mathsf{m}' := x_l \cdot (x_j/x_i)\cdot \mathsf{m}_1 = (x_j/x_i)\cdot \mathsf{m} \in I$ and $\mathsf{m}' > \mathsf{m}$ with respect to the lexicographic order. From our choice of m, we have $\mathsf{m}' \notin G(I)$. Hence, there is some k such that $x_k|\mathsf{m}'$ and $\mathsf{m}'/x_k \in I$. If $k = l$, then we have $(x_j/x_i)\cdot \mathsf{m}_1 = \mathsf{m}'/x_k \in I$. So we may assume that $k \ne l$ and $\nu(\mathsf{m}'/x_k) = l$. Since I is Borel fixed, we have $(x_j/x_i)\cdot \mathsf{m}_1 = \mathsf{m}'/x_l = (x_k/x_l)\cdot (\mathsf{m}'/x_k) \in I$. □

If $\mathsf{m}_1 = \prod_{i=1}^{l} x_i^{a_i}$, then

$$\mathsf{n} := \text{b-pol}(\mathsf{m}_1) = \prod_{\substack{1 \le i \le l \\ b_{i-1}+1 \le j \le b_i}} x_{i,j},$$

where $b_i := \sum_{j=1}^{i} a_j$ for each $i \geq 0$ (here $b_0 = 0$). Note that $b_l = \deg(\mathsf{m}_1) = \deg(\mathsf{n})$.

Claim 2. *With the above notation, we have* $J : \mathsf{n} = (x_{i,b_i+1} \mid 1 \leq i \leq l)$.

Proof of Claim 2. First we prove that $x_{i,b_i+1} \cdot \mathsf{n} \in J$ for $1 \leq i \leq l$. Note that $x_i \cdot \mathsf{m}_1 = (x_i/x_l) \cdot \mathsf{m} \in I$. Since $\deg(x_i \cdot \mathsf{m}_1) = \deg(\mathsf{m}) \leq d$, we have $\mathsf{b\text{-}pol}(x_i \cdot \mathsf{m}_1) \in J$ by Lemma 3.1. If $\nu(\mathsf{m}_1) \leq i$, then we have $b_i = \deg(\mathsf{n})$ and $x_{i,b_i+1} \cdot \mathsf{n} = \mathsf{b\text{-}pol}(x_i \cdot \mathsf{m}_1) \in J$. Hence, we may assume that $\nu(\mathsf{m}_1) > i$, and we can take $k := \min\{j \mid a_j > 0, j > i\}$. Since $\mathsf{m}' := (x_i/x_k) \cdot \mathsf{m}_1$ is in I by Claim 1, we have $\mathsf{b\text{-}pol}(\mathsf{m}') \in J$ by Lemma 3.1. Hence, $x_{i,b_i+1} \cdot \mathsf{n} = x_{k,b_i+1} \cdot \mathsf{b\text{-}pol}(\mathsf{m}') \in J$.

Next we prove that $J : \mathsf{n} \subseteq (x_{i,b_i+1} \mid 1 \leq i \leq l)$. For the contrary, assume that there is a monomial $\mathsf{n}' \in \widetilde{S} \setminus (x_{i,b_i+1} \mid 1 \leq i \leq l)$ satisfying $\mathsf{n}' \cdot \mathsf{n} \in J$. Then there is a monomial $\mathsf{m}'' = \prod x_i^{c_i} \in G(I)$ such that $\mathsf{b\text{-}pol}(\mathsf{m}'')$ divides $\mathsf{n}' \cdot \mathsf{n}$. By the present assumption, we have that $\mathsf{b\text{-}pol}(\mathsf{m}'') \notin (x_{i,b_i+1} \mid 1 \leq i \leq l)$. Under this assumption, we have the following.

Claim 2.1. *Set* $d_i := \sum_{j=1}^{i} c_j$. *Then* $b_i \geq d_i$ *for all* i.

The above fact completes the proof of Claim 2. To see this, take the expression $\mathsf{m}_1 := \prod_{i=1}^{e} x_{\alpha_i}$ as in (1.1), where $e = \deg(\mathsf{m}_1)$. We have $e = b_l \geq d_l = \deg(\mathsf{m}'') =: f$. Moreover, since I is Borel fixed and $b_i \geq d_i$ for all i, $\mathsf{m}'' \in I$ implies that $\prod_{i=1}^{f} x_{\alpha_i} \in I$. It follows that $\mathsf{m}_1 \in I$, which is a contradiction.

Proof of Claim 2.1. Clearly, $b_0 = d_0 = 0$. Hence, if the claim does not hold, there is some $i \geq 1$ such that $(b_i \geq) b_{i-1} \geq d_{i-1}$ and $b_i < d_i$. Note that $x_{i,j}$ divides $\mathsf{b\text{-}pol}(\mathsf{m}'')$ if and only if $d_{i-1} + 1 \leq j \leq d_i$. Hence, under the present assumption, x_{i,b_i+1} divides $\mathsf{b\text{-}pol}(\mathsf{m}'')$. This is a contradiction. $\square$

Continuation of the proof of Theorem 3.2. Set $J_1 := J + (\mathsf{n})$, and set $\mathfrak{p} := (x_{i,b_i+1} \mid 1 \leq i \leq l)$. Then $J_1/J \cong (\widetilde{S}/\mathfrak{p})$ up to degree shift, and $\mathsf{b\text{-}pol}(I_1) = J_1$. If I_1 is not a prime ideal, applying the above argument to I_1, we get a Borel fixed ideal $I_2 (\supset I_1)$ such that $\mathsf{b\text{-}pol}(I_2)/J_1$ satisfies the similar property to J_1/J. Repeating this procedure, we have a sequence of Borel fixed ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_t$$

of S such that $J_i := \mathsf{b\text{-}pol}(I_i)$ satisfies $J_i/J_{i-1} \cong \widetilde{S}/\mathfrak{p}_i$ up to degree shift for all $i \geq 1$. Here $\mathfrak{p}_i \subset \widetilde{S}$ is a prime ideal of the form $(x_{j,c_{i,j}} \mid 1 \leq j \leq l_i)$ for some $l_i, c_{i,j} \in \mathbb{N}$. By the Noetherian property of S, the procedure eventually terminates; that is, I_t will become a prime ideal. In this case, $J_t = \mathsf{b\text{-}pol}(I_t)$ is also a prime ideal, and we have a prime filtration

$$0 \subset J_1/J \subset J_2/J \subset \cdots \subset J_t/J \subset \widetilde{S}/J.$$

This is a pretty clean filtration. In fact, $\nu(I_1) \leq \nu(I)$ by the construction. Similarly, $\nu(I_j) \leq \nu(I_i)$ holds for all i, j with $j \geq i$. On the other hand, we have $\operatorname{codim} \mathfrak{p}_i = l_i = \nu(I_i)$. Hence, $\operatorname{codim} \mathfrak{p}_j \leq \operatorname{codim} \mathfrak{p}_i$ for all $j \geq i$. Now recall the remark after Definition 2.10. ∎

REMARK 3.3. By the above proof, we see that any associated prime of J is of the form $(x_{i,c_i} \mid 1 \leq i \leq m)$ for some $m, c_i \in \mathbb{N}$ with $c_1 \leq c_2 \leq \cdots \leq c_m$.

THEOREM 3.4. *If $I \subset S$ is a Borel fixed ideal, then $J := \mathsf{b\text{-}pol}(I)$ gives a polarization of I, which is faithful.*

Proof. To see that J is a polarization, it suffices to show that Θ forms an $\widetilde{S}/J$-regular sequence. So, assuming that a subset Θ' of Θ forms an $\widetilde{S}/J$-regular sequence, we show that $\Theta' \cup \{x_{i,1} - x_{i,j}\}$ is also an $\widetilde{S}/J$-regular sequence for $x_{i,1} - x_{i,j} \in \Theta \setminus \Theta'$. Since $\widetilde{S}/J$ is sequentially Cohen-Macaulay and Θ' is assumed to be a regular sequence, $\widetilde{S}/(J + (\Theta'))$ is also sequentially Cohen-Macaulay and

$$\operatorname{Ass}_S\big(\widetilde{S}/(J + (\Theta'))\big) = \big\{\mathfrak{p} + (\Theta') \bigm| \mathfrak{p} \in \operatorname{Ass}(\widetilde{S}/J)\big\}$$

by the repeated use of Lemma 2.7. Since all $\mathfrak{p} \in \operatorname{Ass}(\widetilde{S}/J)$ are of the form $(x_{k,c_k} \mid 1 \leq k \leq m)$, $x_{i,1} - x_{i,j}$ is $\widetilde{S}/(J + (\Theta'))$-regular.

The faithfulness follows from Lemma 2.8. ∎

Murai told us that Theorem 3.4 can be shown by using his [9, Proposition 1.9]. We will explain this idea in Remark 4.3 below, since it requires *(generalized) square-free operations* introduced in the next section.

However, this second proof does not give a pretty clean filtration of $\widetilde{S}/\mathsf{b\text{-}pol}(I)$ (equivalently, the nonpure shellability of the associated simplicial complex) and the following generalization of Theorem 3.4. Moreover, in the next section, we will show a new proof of [9, Proposition 1.9] using $\mathsf{b\text{-}pol}(I)$ and give a new perspective to the square-free operations.

THEOREM 3.5. *Let A be a subset of $\{1, 2, \ldots, n\}$. For a monomial $\mathsf{m} = x^{\mathbf{a}} \in S$, set $\mathsf{m}_A := \prod_{i \in A} x_i^{a_i}$, $\mathsf{m}_{-A} := \prod_{i \notin A} x_i^{a_i}$, and*

$$\mathsf{b\text{-}pol}_A(\mathsf{m}) := \mathsf{b\text{-}pol}(\mathsf{m}_A) \cdot \mathsf{pol}(\mathsf{m}_{-A}) \in \widetilde{S}.$$

(We set $\widetilde{S} := \Bbbk[x_{i,j} \mid 1 \le i \le n, 1 \le j \le d]$, where $d := \max\{\deg(\mathsf{m}) \mid \mathsf{m} \in G(I)\}$.) If I is Borel fixed, then $\widetilde{S}/\,\mathsf{b\text{-}pol}_A(I)$ has a pretty clean filtration, where

$$\mathsf{b\text{-}pol}_A(I) := \big(\mathsf{b\text{-}pol}_A(\mathsf{m}) \,\big|\, \mathsf{m} \in G(I)\big).$$

Moreover, $\mathsf{b\text{-}pol}_A(I)$ gives a faithful polarization of I.

By the above theorem, we see that Borel fixed ideals have many alternative polarizations.

LEMMA 3.6. *In the situation of Theorem 3.5, we have*

$$\mathsf{b\text{-}pol}_A(I) = \big(\mathsf{b\text{-}pol}_A(\mathsf{m}) \,\big|\, \mathsf{m} \in I \textit{ with } \deg(\mathsf{m}) \le d\big).$$

Clearly, this is a generalization of Lemma 3.1.

Proof. It suffices to prove "$\supseteq$". Set $J := \mathsf{b\text{-}pol}_A(I)$. For the contrary, assume that there is some $\mathsf{m} = x^{\mathbf{a}} \in I$ with $\deg(\mathsf{m}) \le d$ and $\mathsf{b\text{-}pol}_A(\mathsf{m}) \notin J$. Since $\mathsf{m} \notin G(I)$, there is some i with $x_i | \mathsf{m}$ and $\mathsf{m}' := \mathsf{m}/x_i \in I$. If $i \notin A$, then it is easy to see that $\mathsf{b\text{-}pol}_A(\mathsf{m}) = x_{i,a_i} \cdot \mathsf{b\text{-}pol}_A(\mathsf{m}') \in J$. Hence, we have $i \in A$. If we replace $\nu(\mathsf{m})$ by $\nu_A(\mathsf{m}) := \max\{i \in A \mid a_i > 0\}$, the last part of the proof of Lemma 3.1 works verbatim, except that $\mathsf{b\text{-}pol}_A(\mathsf{m}) = x_{\nu_A(\mathsf{m}),f} \cdot \mathsf{b\text{-}pol}_A(\mathsf{m}'')$ with $f := \sum_{i \in A} a_i$. ∎

Proof of Theorem 3.5. For the former assertion, we imitate the proof of Theorem 3.2. First, take the same $\mathsf{m} \in \widetilde{S}$ as in the proof of Theorem 3.2. (Here $\nu(\mathsf{m}) = \nu(I) =: l$, and $\nu_A(I)$ is *not* used.) As shown in Claim 1 of the original proof, $I + (\mathsf{m}_1)$ is Borel fixed.

For the statement corresponding to Claim 2, we need modification. If $\mathsf{m} \ne x_l$, set $\mathsf{m}_1 := \mathsf{m}/x_l = \prod_{i=1}^n x^{a_i}$, and set $\mathsf{n} = \mathsf{b\text{-}pol}_A(\mathsf{m}_1)$. For each $i \in A$, set

$$b_i := \sum_{j \in A, j \le i} a_j.$$

Next, we will show that $J : \mathsf{n} = \mathfrak{p}$, where

$$\mathfrak{p} := (x_{i,b_i+1} \mid i \in A, i \le l) + (x_{i,a_i+1} \mid i \notin A, i \le l).$$

Note that $x_i \cdot \mathsf{m}_1 = (x_i/x_l) \cdot \mathsf{m} \in I$ for $i \le l$. If $i \notin A$, then we have $x_{i,a_i+1} \cdot \mathsf{n} = \mathsf{b\text{-}pol}_A(x_i \cdot \mathsf{m}_1) \in J$. If $i \in A$, then we can show that $x_{i,b_i+1} \cdot \mathsf{n} \in J$ by a similar argument to the proof of Claim 2, while we have to replace $\min\{j \mid a_j > 0, j > i\}$ by $\min\{j \in A \mid a_j > 0, j > i\}$. Hence, we have $J : \mathsf{n} \supset \mathfrak{p}$.

To prove the converse, assume that a monomial $\mathsf{n}' \in \widetilde{S}$ satisfies $\mathsf{n}' \cdot \mathsf{n} \in J$. Then there is a monomial $\mathsf{m}'' \in G(I)$ such that $\mathsf{b\text{-}pol}_A(\mathsf{m}'')$ divides $\mathsf{n}' \cdot \mathsf{n}$. If $\mathsf{n}' \notin (x_{i,a_i+1} \mid i \notin A, i \leq l)$, then $\mathsf{b\text{-}pol}_A(\mathsf{m}'') \notin (x_{i,a_i+1} \mid i \notin A, i \leq l)$ also. It means that $\deg_i(\mathsf{m}'') \leq a_i = \deg_i(\mathsf{m}_1)$ for all $i \notin A$. Now, concentrating our attention to the variables x_i with $i \in A$ and $i \leq l$, we can use the proof of Claim 2 (almost) verbatim, and we see that the assumption $\mathsf{n}' \notin \mathfrak{p}$ implies that $\mathsf{m}_1 \in I$. This is a contradiction.

Hence, we have $J : \mathsf{n} = \mathfrak{p}$, and a pretty clean filtration can be constructed as in (the final step of) the proof of Theorem 3.2.

The above argument shows that any associated prime of $\widetilde{S}/\mathsf{b\text{-}pol}(I)$ is of the form $(x_{i,c_i} \mid 1 \leq i \leq m)$ (but we lose the relation $c_1 \leq c_2 \leq \cdots \leq c_m$ here). Hence, by an argument similar to the proof of Theorem 3.4, we can show that J is a faithful polarization. ∎

§4. Application to square-free operation

Throughout this section, let $\{a_i\}_{i \in \mathbb{N}}$ be a nondecreasing sequence of non-negative integers. We also assume that $a_0 = 0$ for convenience.

Let $T = \Bbbk[x_1, \ldots, x_N]$ be a polynomial ring with $N \gg 0$. For a monomial $\mathsf{m} \in S = \Bbbk[x_1, \ldots, x_n]$, take the expression $\mathsf{m} = \prod_{i=1}^{e} x_{\alpha_i}$ as (1.1). Murai [9] defined the operation $(-)^{\sigma(a)}$ which sends m to

$$\mathsf{m}^{\sigma(a)} := \prod_{i=1}^{e} x_{\alpha_i + a_{i-1}} \in T.$$

For a monomial ideal $I \subset S$, he also set

$$I^{\sigma(a)} := \big(\mathsf{m}^{\sigma(a)} \bigm| \mathsf{m} \in G(I)\big) \subset T.$$

(In [9], the symbol "α^a" is used for this operation. However, we change the notation, since the letter α has been used already.)

If $a_{i+1} > a_i$ for all i, then $\mathsf{m}^{\sigma(a)}$ is a square-free monomial. In particular, if $a_i = i$ for all i, then $(-)^{\sigma(a)}$ coincides with the *square-free operation* $(-)^{\sigma}$, which plays an important role in the construction of the *symmetric shifting* of a simplicial complex (see [2]; see also [7] for the original form of the shifting theory).

Let L_a be the linear subspace of S_1 spanned by

$$X_a := \{x_{i,j} - x_{i',j'} \mid i + a_{j-1} = i' + a_{j'-1}\},$$

and take a subset $\Theta_a \subset X_a$ so that it forms a basis of L_a. For example, if $a_i = i$ for all i, then we can take

$$\{x_{i,j} - x_{i+1,j-1} \mid 1 \le i < n, 1 < j \le d\}$$

as Θ_a. Clearly, Θ_a is an $\widetilde{S}$-regular sequence, and the ring homomorphism $\psi : \widetilde{S} \to T(= \Bbbk[x_1, \dots, x_N])$ defined by $\widetilde{S} \ni x_{i,j} \mapsto x_{i+a_{j-1}} \in T$ induces the isomorphism $\widetilde{S}/(\Theta_a) \cong T$ (if we adjust the number N).

Proposition 4.1. *Let $I \subset S$ be a Borel fixed ideal, and set $J := \mathsf{b\text{-}pol}(I)$. Then Θ_a forms an $\widetilde{S}/J$-regular sequence, and we have $\widetilde{S}/(\Theta_a) \otimes_{\widetilde{S}} \widetilde{S}/J \cong T/I^{\sigma(a)}$ through the isomorphism $S/(\Theta_a) \to T$ (i.e., we have $\psi(J) = I^{\sigma(a)}$).*

Proof. The latter assertion is clear by expression (1.2), and it suffices to prove the former. Recall that $\widetilde{S}/J$ is sequentially Cohen-Macaulay, and any associated prime of $\widetilde{S}/J$ is of the form $(x_{i,c_i} \mid 1 \le i \le m)$ with $c_1 \le c_2 \le \cdots \le c_m$. If $x_{i,j} - x_{i',j'} \in \Theta_a$ and $i < i'$, then $a_{j-1} - a_{j'-1} = i' - i > 0$. Since $\{a_k\}_{k \in \mathbb{N}}$ is a nondecreasing sequence, we have $j > j'$. Hence, Θ_a forms an $\widetilde{S}/J$-regular sequence by the same argument as in the proof of Theorem 3.4. ∎

Corollary 4.2 ([9, Proposition 1.9]). *Let I be a Borel fixed ideal. Then,*

$$\beta^S_{i,j}(I) = \beta^T_{i,j}(I^{\sigma(a)})$$

for all i, j.

Proof. The left (resp., right) side of the equation equals $\beta^{\widetilde{S}}_{i,j}(J)$ by Theorem 3.4 (resp., Proposition 4.1). ∎

The original proof in [9] uses a formula given in [1], and is very different from ours.

Remark 4.3. Murai told us that Corollary 4.2 (i.e., his [9, Proposition 1.9]) can be used to prove Theorem 3.4. In fact, if $a_i = i \cdot n$ for each i, $(-)^{\sigma(a)}$ corresponds to our $\mathsf{b\text{-}pol}$. To see this, assign our variable $x_{i,j}$ to his $x_{(j-1)\cdot n+i}$. Since $(-)^{\sigma(a)}$ preserves the Betti numbers of a Borel fixed ideal, it gives a polarization by Lemma 2.2. However, our proof has advantages, as mentioned before in Theorem 3.5, and we can refine Corollary 4.2 as follows. That is, Theorem 3.4 (the polarization $\mathsf{b\text{-}pol}(I)$) and Corollary 4.2 (generalized square-free operation) imply each other, but our analysis of $\mathsf{b\text{-}pol}(I)$ contains more precise information.

Corollary 4.4. *With the situation of Proposition 4.1,* Θ_a *forms an* $\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J,\widetilde{S})$*-regular sequence for all* i*, and*

$$\widetilde{S}/(\Theta_a)\otimes_{\widetilde{S}}\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J,\widetilde{S})\cong\operatorname{Ext}^i_T(T/I^{\sigma(a)},T).$$

Hence, we have

$$\beta^T_{i,j}\bigl(\operatorname{Ext}^k_T(T/I^{\sigma(a)},T)\bigr)=\beta^S_{i,j}\bigl(\operatorname{Ext}^k_S(S/I,S)\bigr)$$

for all i,j,k*. Similarly,* $\deg(\operatorname{Ext}^i_T(T/I^{\sigma(a)},T))=\deg(\operatorname{Ext}^i_S(S/I,S))$ *for all* i*, and hence* S/I *and* $T/I^{\sigma(a)}$ *have the same arithmetic degree.*

Proof. Since Θ_a is an $\widetilde{S}/J$-regular sequence and $\widetilde{S}/J$ is sequentially Cohen-Macaulay, the former assertion follows from iterated use of Lemma 2.7 (see also the argument after Definition 2.4). The equation on the Betti numbers holds, since both sides equal $\beta^{\widetilde{S}}_{i,j}(\operatorname{Ext}^k_{\widetilde{S}}(\widetilde{S}/J,\widetilde{S}))$. The equations on the degrees can be proved in a similar way. ∎

Proposition 4.5. *If* $I\subset S$ *is a Borel fixed ideal, then* $T/I^{\sigma(a)}$ *has a pretty clean filtration. In particular, if* $I^{\sigma(a)}$ *is square-free (e.g., if* $a_{i+1}>a_i$ *for all* i*), then the corresponding simplicial complex of* $T/I^{\sigma(a)}$ *is nonpure shellable.*

Proof. Take the pretty clean filtration $0\subset J_1/J\subset J_2/J\subset\cdots\subset J_t/J\subset\widetilde{S}/J$ ($J_0=J$) constructed in the proof of Theorem 3.2. Recall that $J_i/J_{i-1}\cong\widetilde{S}/\mathfrak{p}_i$ up to degree shift for each $i\geq 1$. Since $\mathfrak{p}_i\in\operatorname{Ass}(\widetilde{S}/J)$, Θ_a forms an $\widetilde{S}/\mathfrak{p}_i$-regular sequence by the same argument as in the proof of Proposition 4.1. Moreover, $\widetilde{S}/(\Theta_a)\otimes_{\widetilde{S}}\widetilde{S}/\mathfrak{p}_i\cong T/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i\subset T$ with $\operatorname{codim}\mathfrak{p}_i=\operatorname{codim}\mathfrak{q}_i$. From the exact sequence $0\to J_{i-1}/J\to J_i/J\to\widetilde{S}/\mathfrak{p}_i\to 0$, we have the exact sequence

$$0\to\widetilde{S}/(\Theta_a)\otimes_{\widetilde{S}}J_{i-1}/J\to\widetilde{S}/(\Theta_a)\otimes_{\widetilde{S}}J_i/J\to\widetilde{S}/(\Theta_a)\otimes_{\widetilde{S}}\widetilde{S}/\mathfrak{p}_i\to 0$$

by [4, Proposition 1.1.4]. Set $M_i:=\widetilde{S}/(\Theta_a)\otimes_{\widetilde{S}}J_i/J_{i-1}$. Then $0\subset M_1\subset\cdots\subset M_t\subset T/I^{\sigma(a)}$ is a pretty clean filtration. ∎

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