Nagoya Math. J. **207** (2012), 1–45
DOI 10.1215/00277630-1630014

SYMPLECTIC FILLINGS OF LINKS OF QUOTIENT SURFACE SINGULARITIES

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Abstract. We study symplectic deformation types of minimal symplectic fillings of links of quotient surface singularities. In particular, there are only finitely many symplectic deformation types for each quotient surface singularity.

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§1. Introduction

In recent years, the geometry of contact structures on 3-manifolds has been a subject of intensive studies. In particular, tight contact structures have been the focus of interest. For instance, tight contact structures on lens spaces have now been classified by Giroux [3] and Honda [7]. The link L of an isolated surface singularity (V,O) provides examples of tight contact in 3-manifolds. Namely, the complex tangency to the link gives a codimension 1 distribution $\xi = TL \cap J(TL)$ which is completely nonintegrable, hence a contact structure. Here J is the complex structure on $V \setminus O$. Let $\pi\colon \widetilde{V} \to V$ be a resolution of the singularity, and let U be a neighborhood of O in V such that $\partial U = L$. Then $\pi^{-1}(U)$ is a so-called *symplectic filling* of (L,ξ) and ξ is

Received November 7, 2007. Revised March 28, 2011. Accepted March 30, 2011.
2010 Mathematics Subject Classification. Primary 53D35; Secondary 57R17, 32S25.
Bhupal's work partially supported by a Japan Society for the Promotion of Science Postdoctoral Fellowship for Foreign Researchers.
Ono's work partially supported by the Japan Society for the Promotion of Science grants (14340019) and (21244002).

a symplectically fillable contact structure, which implies that ξ is tight by a theorem of Eliashberg [4, Corollary 4.5] and Gromov [6, Section 2.4.D_2'(b)].

It is also interesting to classify symplectic fillings of the links of certain classes of isolated surface singularities. For the case of cyclic quotient singularities of $A_{n,1}$-type, McDuff [15] classified symplectic deformation classes of minimal symplectic fillings. H. Ohta and the second named author investigated the cases of simple singularities [20] and simple elliptic singularities [19]. Meanwhile, Lisca [11] presents a classification for the case of cyclic quotient singularities. In this paper, we study the case of quotient surface singularities $\mathbb{C}/\Gamma$, where Γ is a finite subgroup of $\mathrm{GL}(2;\mathbb{C})$. Note that this class contains all simple singularities, which is the case where $\Gamma \subset \mathrm{SL}(2;\mathbb{C})$. Simple singularities are characterized as isolated surface singularities which are described by both quotient singularities and hypersurface singularities. Thus, we can use both aspects in the argument. Namely, since they are quotient singularities, the link is a spherical space form. In particular, they carry a metric of positive scalar curvature. This fact is one of main ingredients in [18]. They are also hypersurface singularities with explicit defining equations. This enables us to describe the compactifications of their Milnor fibers in appropriate weighted projective spaces (see [23]). The results in [20] are some of the main ingredients in this paper. Since the situation here is more complicated than in the case of simple singularities, we have to study rational curves with negative self-intersection numbers carefully. The main theorem is the following.

THEOREM 1.1. *A symplectic filling of the link of a quotient surface singularity is symplectic deformation equivalent to the complement of a certain divisor in an iterated blowup of* $\mathbb{C}P^2$ *or* $\mathbb{C}P^1 \times \mathbb{C}P^1$.

A detailed description of the symplectic fillings is given later. In particular, we get finiteness of symplectic deformation types of minimal symplectic fillings for each quotient surface singularity.

§2. Preliminaries

Although a complete understanding of symplectic 4-manifolds is far from being realized, we have a good understanding of closed symplectic 4-manifolds containing a pseudoholomorphic embedded sphere of nonnegative self-intersection number thanks to works by Gromov [6] and McDuff [15] (see Theorem 2.1 below). These are the so-called *rational* or *ruled* symplectic

4-manifolds, and they are symplectomorphic to blowups of rational or ruled surfaces, in the sense of complex surface theory, equipped with some Kähler form (see [9]). Note that blowing up and down can be performed in the symplectic category. The existence of such pseudoholomorphic spheres is a highly nontrivial issue. Shortly after the discovery of Seiberg-Witten monopole invariants, Taubes [24] established a striking theory relating Seiberg-Witten invariants and Gromov invariants, which count pseudoholomorphic curves in a certain way. In particular, the Seiberg-Witten invariant is nonzero for a spin^c structure if and only if there exists a pseudoholomorphic curve in the corresponding homology class as long as $b_2^+ > 1$. His argument can be applied in the case that $b_2^+ = 1$ under suitable conditions as in Theorem 2.2 below.

We adopt the argument in [20] and [19] to determine the symplectic fillings of quotient surface singularities. We explain our strategy in the following.

Let (L, ξ) be a closed contact manifold. A compact symplectic manifold $X = (X, \omega)$ with boundary is called a *strong symplectic filling* (resp., *strong concave filling*) of (L, ξ) if $\partial X = L$ and if there exists a Liouville vector field v, that is, $\mathcal{L}_v\omega = \omega$, on a collar neighborhood of ∂X such that v points outward (resp., inward) along ∂X and satisfies $\xi = \ker(\iota(v)\omega|_L)$. There is also a notion of a weak symplectic filling. When L is a rational homology sphere, we can modify a weak symplectic filling to a strong symplectic filling (see, e.g., [18]). In the case of the link of a quotient surface singularity, the link is a rational homology 3-sphere; hence, we simply call it a *symplectic filling.*

If X^- and X^+ are a strong symplectic filling and a strong concave filling, respectively, after possibly rescaling the symplectic structure either on X^- or on X^+ and inserting a suitable subset of the symplectization, we can glue them to obtain a closed symplectic manifold.

Now let (L, ξ) be the contact link of a quotient surface singularity (V, O). First, we realize the singularity (V, O) in a successive blowup of the complex projective plane. Then the complement of a small neighborhood of the singularity gives a strong concave filling. We will find a suitable realization, hence a suitable concave filling Y. In fact, Y is a regular neighborhood of the *compactifying divisor* (see Section 3). For a given symplectic filling X, we glue X and Y to obtain a closed symplectic 4 manifold Z.

Next we apply the criterion in Theorem 2.2 and an elementary topological consideration to conclude that the Z is a successive blowup of the complex projective plane. The original X is the complement of Y in Z, since the embedding of Y in Z can be seen from the embedding of the compactifying divisor in Z. The compactifying divisor is a treelike configuration of holomorphic spheres. To detect such holomorphic spheres, we use some results on holomorphic spheres in rational symplectic 4-manifolds, which are explained in the remainder of this section. Note that the holomorphic spheres appearing in the compactifying divisor Z have negative self-intersection numbers, hence, they are not realized by J-holomorphic spheres for a generic choice of J. We are thus forced to use a nongeneric J to find such a configuration of holomorphic spheres.

We now present several facts which are necessary to carry out these steps. First of all, we recall some basic results for rational and ruled symplectic 4-manifolds.

THEOREM 2.1 (see McDuff [15]). *Let (M,ω) be a closed symplectic 4-manifold. If M contains a symplectically embedded 2-sphere C of nonnegative self-intersection number k, then M is either a rational symplectic 4-manifold or a blowup of a ruled symplectic 4-manifold. In particular, if $k=0$ (resp., 1), M becomes a ruled symplectic 4-manifold (resp., the complex projective plane) after blowing down symplectic (-1)-curves away from C.*

Here a rational symplectic 4-manifold means a symplectic blowup of the complex projective plane at some points, and a ruled symplectic 4-manifold means a 2-sphere bundles over an oriented surface with a symplectic structure which is nondegenerate on each fiber. Combining Theorem 2.1 and Taubes's theorem "SW = Gr," we get the following.

THEOREM 2.2 (see [17], [13]). *Let (M,ω) be a closed symplectic 4-manifold such that $\int_M c_1(M)\wedge\omega > 0$. Then (M,ω) is either a rational symplectic 4-manifold or a (blowup of a) ruled symplectic 4-manifold.*

If the pseudoholomorphic curve C is singular, we have the following result as a byproduct of uniqueness of minimal symplectic fillings of the link of a simple singularity (see [20]).

THEOREM 2.3 (see [21]). *Let M be a closed symplectic 4-manifold containing a pseudoholomorphic rational curve C with a $(2,3)$-cusp point. Suppose that C is nonsingular away from the $(2,3)$-cusp point. If the self-intersection number C^2 of C is positive, then M must be a rational symplectic 4-manifold and C^2 is at most 9. Moreover, if $M\setminus C$ does not contain any symplectic (-1)-curves, then C represents the Poincaré dual to $c_1(M)$,* *that is, an anticanonical divisor. When $C^2 = 9$, $M = \mathbb{C}P^2$ and C is a pseudoholomorphic cuspidal cubic curve.*

For the statement that C represents an anticanonical class if $M \setminus C$ does not contain any symplectic (-1)-curves, we use the following fact: since the condition that a divisor D is an anticanonical divisor is preserved under the blowup at a regular point of D and the blowdown of a (-1)-curve E such that $E \cdot D = 1$, we find that C is an anticanonical divisor.

We call a homology class $e \in H_2(M;\mathbb{Z})$ a *symplectic* (-1)-*class* if e is represented by a symplectically embedded 2-sphere of self-intersection number -1. A symplectic (-1)-curve class is represented by a J-holomorphic sphere for a generic tame almost complex structure J. However, if we restrict the class of tame almost complex structure, this may not be the case. Here we have the following (essentially [21, Proposition 4.1]).

PROPOSITION 2.4. *Let M be a symplectic 4-manifold, and let $C_1, \dots, C_k$ be irreducible J_0-holomorphic curves in M with respect to a tame almost complex structure J_0. Suppose that each of $C_1, \dots, C_k$ is either nondegenerate, singular, or of higher genus. Then, for a generic J among tame almost complex structures for which $C_1, \dots, C_k$ are pseudoholomorphic, any symplectic (-1)-curve has a unique J-holomorphic representative.*

Here we call C_i *nondegenerate* if the linearized operator of the pseudoholomorphic curve equation is surjective at C_i. Now we collect a series of observations.

If a pseudoholomorphic curve C intersects a (-1)-curve transversally, [20, Lemma 4.1] ensures that the image under the blowing-down map is also pseudoholomorphic with respect to a suitable almost complex structure. Transversality of intersections can be achieved by a small perturbation of the almost complex structure.

LEMMA 2.5. *Let M be a closed symplectic 4-manifold, and let L be a symplectically embedded 2-sphere of self-intersection number 1. Then any irreducible singular or higher genus pseudoholomorphic curve C in M satisfies $C \cdot L \geq 3$. In particular, neither an irreducible singular nor a higher genus pseudoholomorphic curve is contained in $M \setminus L$.*

Proof. If necessary, we perturb the almost complex structure slightly in such a way that the (-1)-curves do not pass through the singular points of C. We then blow down a maximal disjoint family of pseudoholomorphic

(-1)-curves away from L. Here we can assume that L and C are also pseudoholomorphic with respect to the same almost complex structure (Proposition 2.4). Then M becomes the complex projective plane and L becomes a line. Since C is singular and irreducible or of higher genus, the image $\overline{C}$ has degree at least 3. Thus we have $C \cdot L = \overline{C} \cdot L \geq 3$. ∎

LEMMA 2.6. *Let M be a closed symplectic 4-manifold, and let C be a pseudoholomorphic rational curve with a $(2,3)$-cusp point as a unique singularity. Suppose that the self-intersection number of C is positive. Then neither an irreducible singular nor a higher genus pseudoholomorphic curve is contained in $M \setminus C$.*

Proof. Suppose that D is such a singular or a higher genus pseudoholomorphic curve. Let J be a tame almost complex structure on M with respect to which C and D are pseudoholomorphic. By Proposition 2.4, we may assume that all symplectic (-1)-classes are represented by J-holomorphic (-1)-curves. We blow down a maximal family of J-holomorphic (-1)-curves in $M \setminus C$. Thus, we may assume that $M \setminus C$ does not contain any symplectic (-1)-curves. By Theorem 2.3, C is an anticanonical divisor, and we have $c_1(M)[D] = 0$. If D is singular or of higher genus, the adjunction formula tells us that $D \cdot D \geq 0$. On the other hand, the intersection form on $M \setminus C$ is negative definite. Hence, $[D]$ is homologous to zero, which is absurd. ∎

LEMMA 2.7. *Let M and C be as in Lemma 2.6. Suppose that the self-intersection number of C is at least 2. Then there does not exist a pseudoholomorphic curve A such that A is either singular and irreducible or of higher genus and such that $A \cdot C = 1$.*

Proof. If such a curve A exists, we blow up M at the intersection point of A and C. Then the proper transform of A violates the conclusion of Lemma 2.6. ∎

If the self-intersection number of C is 1, then there exist singular or genus 1 pseudoholomorphic curves A such that $A \cdot C = 1$. In addition, if $M \setminus C$ is minimal, it turns out that A is homologous to C in M.

LEMMA 2.8. *Let M be a closed symplectic 4 manifold, and let L be a symplectically embedded sphere of self-intersection number 1. Then no symplectically embedded sphere of nonnegative self-intersection number is contained in $M \setminus L$. Pseudoholomorphic (-1)-curves in $M \setminus L$ are mutually disjoint.*

LEMMA 2.9. *Let M be a closed symplectic 4-manifold, and let C be an irreducible singular or higher genus pseudoholomorphic curve. Then no symplectically embedded sphere of nonnegative self-intersection number is contained in $M \setminus C$.*

Proof. Let A be a symplectically embedded sphere in $M \setminus C$. Set $k = A \cdot A$. If $k = 1$, the result follows from Lemma 2.5. If $k > 1$, blow up M at $k-1$ points on A. The proper transform of A has self-intersection number 1. So this case is reduced to the case where $k = 1$. If $k = 0$, we blow down a maximal disjoint family of pseudoholomorphic (-1)-curves away from A. Note that Proposition 2.4 guarantees that these (-1)-curves and C are pseudoholomorphic with respect to the same almost complex structure. The blown-down manifold is a ruled symplectic 4-manifold, and A is a fiber. Since C is singular and irreducible or of higher genus, its image $\overline{C}$ under the blowing-down map is also singular and irreducible or of higher genus. Thus, it is not a fiber of the ruling. Since fibers sweep out the whole space, $\overline{C}$ should intersect a fiber. This contradicts the fact that $\overline{C} \cdot A = C \cdot A = 0$. ∎

Similarly, we get the following.

LEMMA 2.10. *Let M be a closed symplectic 4-manifold, and let C be a singular pseudoholomorphic curve. Then there is no symplectically embedded sphere A of nonnegative self-intersection number such that $A \cdot C = 1$.*

Proof. If $k = A \cdot A$ is positive, we blow up M at k points on $A \setminus C$. So we may assume that $k = 0$. We blow down a maximal disjoint family of pseudoholomorphic (-1)-curves away from A to get a ruled symplectic 4-manifold. Then the image $\overline{C}$ of C under the blowing-down map satisfies $\overline{C} \cdot A = 1$. However, there exists another fiber A' passing through a singular point of C, for which we have $\overline{C} \cdot A' \geq 2$. This is a contradiction. ∎

The following lemma is a consequence of [18, Theorem 1].

LEMMA 2.11. *Let X be a symplectic filling of the link of a quotient surface singularity. Then pseudoholomorphic (-1)-curves in X are mutually disjoint.*

Proof. Suppose that there are two pseudoholomorphic (-1)-curves E and E' which intersect each other. Contract E, and denote the blowing-down map by $\pi\colon X \to X'$. Then the homology class $\pi_*[E']$ has nonnegative self-intersection number. Note that X' is also a symplectic filling of ∂X. Also note that the link of a quotient surface singularity is a spherical space form;

hence, it carries a metric of positive scalar curvature. This contradicts the fact that any symplectic filling of a contact manifold with a metric of positive scalar curvature has negative definite intersection form ([18, Theorem 1], [10, Theorem 1.4]). ∎

REMARK 2.12. In Lemmas 2.5–2.7, we showed the nonexistence of higher genus pseudoholomorphic curves in the complement of a certain divisor D. These arguments also imply the nonexistence of a cycle of pseudoholomorphic spheres in the complement of D. Indeed, a cycle of pseudoholomorphic spheres is a stable map of genus 1. By gluing the adjacent components around the nodes, we get an irreducible symplectically embedded surface of genus 1 which is pseudoholomorphic with respect to a tame almost complex structure which coincides with the original almost complex structure outside of a neighborhood of the nodes. But Lemmas 2.5–2.7 prohibit the existence of such a pseudoholomorphic curve of genus 1.

We now state some further lemmas that will be useful when we discuss symplectic fillings of links of quotient surface singularities.

LEMMA 2.13. *Let L, $C_1,\dots,C_k$ be a collection of symplectically embedded 2-spheres in a closed symplectic 4-manifold M with $L\cdot L = 1$, $C_i\cdot C_i \leq 0$ $(i = 1,\dots,k)$. Suppose that J is a tame almost complex structure for which L, $C_1,\dots,C_k$ are pseudoholomorphic. Then there exists at least one J-holomorphic (-1)-curve in $M\setminus L$.*

Proof. If one of the C_i which are disjoint from L is a symplectic (-1)-curve, there is nothing to prove. Suppose that none of the C_i which are disjoint from L are symplectic (-1)-curves. By Theorem 2.1, M is a rational symplectic 4-manifold. Since $C_1\cdot C_1 \leq 0$, M is not minimal. So, by Theorem 2.1, there are symplectic (-1)-curves in $M\setminus L$. After blowing them down, we get the complex projective plane. Denote by E a symplectic (-1)-curve in $M\setminus L$. If the homology class $[E]$ is represented by a J-holomorphic sphere, we are done. Suppose that $[E]$ is not represented by a J-holomorphic sphere. Pick a sequence of tame almost complex structures J_n such that L is J_n-holomorphic, $[E]$ is represented by a J_n-holomorphic sphere E_n, and $\{J_n\}$ converges to J. Then E_n converges to the image of a J-stable map. Let $A_1,\dots,A_l$ be its irreducible components. Since $c_1(M)[E] = 1$, there is a component A_j with $c_1(M)[A_j] > 0$. Note that A_j is disjoint from L. (Otherwise, $E\cdot L$ must be positive.) If A_j is a multiply covered component, take the underlying reduced curve E'. Then Lemma 2.5 implies that A_j or E' is an embedded pseudoholomorphic sphere. Also, Lemma 2.8 implies that A_j or E' has self-intersection less than 0. Since $c_1(M)[A_j] > 0$, by the adjunction formula, A_j or E' is a J-holomorphic (-1)-curve. ∎

LEMMA 2.14. *Let M be a closed symplectic 4-manifold with a tame almost complex structure J_0 containing a pseudoholomorphic rational curve D with a $(2,3)$-cusp point and an embedded pseudoholomorphic 2-sphere A with self-intersection number 0 intersecting D at the cusp point. Suppose that $A \cdot D = 2$. If $M \setminus (A \cup D)$ is minimal, then D represents the anticanonical class of M.*

Proof. By Proposition 2.4, we may find a tame almost complex structure J with respect to which A and D are pseudoholomorphic and any symplectic (-1)-curve has a unique J-holomorphic representative. If we blow down all (-1)-curves away from A, we get either $\mathbb{C}P^1 \times \mathbb{C}P^1$ or the 1-point blowup of $\mathbb{C}P^2$ (these are $\mathbb{C}P^1$-bundles over $\mathbb{C}P^1$). (Theorem 2.1 implies that M blows down to a ruled symplectic 4-manifold; the existence of a cuspidal pseudoholomorphic rational curve implies that it is not irrationally ruled.) We show that J-holomorphic representatives of such (-1)-curves intersect D once transversally. Suppose that E is a (-1)-curve in the complement of A which intersects D with multiplicity > 1. After perturbing the almost complex structure, we can make the intersection of E and D transversal. So after blowing down all (-1)-curves, including E, away from A, we obtain a rational ruled surface with A being a ruling fiber. Pick a ruling fiber F through the image p of E under the blowing-down map, through which $\overline{D}$, the image D, passes. We can assume that, locally around p, $\overline{D}$ consists of a bunch of pseudoholomorphic curves intersecting at a point in such a way that any two branches intersect transversally. (We may call such a point a *simple multiple point.*) Since A and $\overline{D}$ intersect with multiplicity 2, F and $\overline{D}$ intersect only at p with multiplicity 2. Before blowing down E, F must be a (-1)-curve which is disjoint from $A \cup D$. If we assume that $M \setminus (A \cup D)$ is minimal, then such a situation does not arise. So D becomes a bidegree $(2,2)$-cuspidal curve in $\mathbb{C}P^1 \times \mathbb{C}P^1$ or the proper transform of the cuspidal cubic curve in $\mathbb{C}P^2$ blown up at a regular point of the cuspidal curve. Then the original D is a proper transform of such a standard object (which is an anticanonical divisor in these surfaces) under blowing up at regular points of the cuspidal curve. Hence, D represents the anticanonical class. ∎

LEMMA 2.15. *Let M be a closed symplectic 4-manifold with a tame almost complex structure J_0 containing a pseudoholomorphic rational curve D with* *a* $(2,3)$*-cusp point and embedded pseudoholomorphic* 2*-spheres* A *and* B *both with self-intersection number* 0 *intersecting as depicted in Figure 7. If* $M\setminus(A\cup B\cup D)$ *is minimal, then* D *represents the anticanonical class of* M*.*

Proof. Suppose that $M\setminus(A\cup B\cup D)$ is minimal. If $M\setminus(A\cup D)$ is also minimal, then the conclusion follows from Lemma 2.14. If $M\setminus(A\cup D)$ is not minimal, then there must be a (-1)-curve E in the complement of $A\cup D$ such that $E\cdot B\geq 1$. By Proposition 2.4, we may find a tame almost complex structure J with respect to which A, B, and D are pseudoholomorphic and E has a unique J-holomorphic representative. By perturbing the almost complex structure J, if necessary, we may assume that E and B intersect transversally. If $E\cdot B>1$, then blowing down E contradicts Lemma 2.9. If $E\cdot B=1$, then blowing down E contradicts Lemma 2.5. ∎

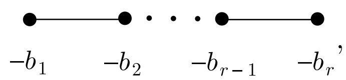
Remark 2.16. In [20], we used the fact that the canonical bundles of the minimal symplectic fillings of the links of simple singularities are trivial. In the cases of types E_6, E_7, and E_8, we used K3 surfaces to find an appropriate compactification. This argument is also applied to the cases of types A_n and D_n with n small such that the corresponding configuration of (-2)-curves is realized in some K3 surface. For general A_n and D_n, this fact was shown by Kanda in [8]. Here we note that in the cases of type D_n, the compactification contains a pseudoholomorphic rational curve A of self-intersection number 0 and a pseudoholomorphic rational curve D with a $(2,3)$-cusp intersecting as described in Lemma 2.14. It follows from Lemma 2.14 that the canonical bundle of the complement of the compactifying divisor is trivial. For the cases of type A_n, the compactification contains a pair of pseudoholomorphic rational curves A and B both having self-intersection number 0 and a pseudoholomorphic rational curve D with a $(2,3)$-cusp intersecting as depicted in Figure 7. In this case, Lemma 2.15 shows that the canonical bundle of the complement of the compactifying divisor is trivial.

§3. Quotient singularities

We consider germs of quotient singularities $(\mathbb{C}^2/G,0)$, where G is a finite subgroup of $\mathrm{GL}(2,\mathbb{C})$. It is known that every such quotient singularity is isomorphic to the quotient of $\mathbb{C}^2$ by a small group $G<\mathrm{GL}(2,\mathbb{C})$, where *small* means that G does not contain any reflections. Also, it is known that, for small groups G_1, G_2, the singularity $(\mathbb{C}^2/G_1,0)$ is analytically isomorphic to $(\mathbb{C}^2/G_2,0)$ if and only if G_1 is conjugate to G_2. Hence, the problem of classifying quotient singularities $(\mathbb{C}^2/G, 0)$ is reduced to the problem of classifying small subgroups of $\mathrm{GL}(2,\mathbb{C})$ up to conjugation. Since G is finite, we may assume that $G \subset \mathrm{U}(2)$. The action of G on $\mathbb{C}^2$ lifts to an action on the blowup of $\mathbb{C}^2$ at the origin: $\pi\colon \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$. The exceptional divisor $E = \pi^{-1}(0)$ is stable under the G action which is induced by $G \subset \mathrm{U}(2) \to \mathrm{PU}(2) \cong \mathrm{SO}(3)$. The image of G in $\mathrm{SO}(3)$ is (conjugate to) either a cyclic subgroup, a dihedral subgroup, the tetrahedral subgroup, the octahedral subgroup, or the icosahedral subgroup. If G is the cyclic group $C_{n,1}$ whose generator $\varphi_{n,1}$ acts on $\mathbb{C}^2$ via $(u,v) \mapsto (\zeta_n u, \zeta_n v)$, where $\zeta_n = e^{2\pi i/n}$, then the quotient space $\widetilde{\mathbb{C}}^2/G$ is smooth with the image of E a $(-n)$-curve. Otherwise, let F be the set of points of E which have nontrivial stabilizer subgroups. Then the quotient space $\widetilde{\mathbb{C}}^2/G$ has isolated singularities at F/G, each of which is a cyclic quotient singularity. The end result of this is given in [2]. Briefly, quotient singularities can be divided into five families: cyclic quotient singularities, dihedral singularities, tetrahedral singularities, octahedral singularities, and icosahedral singularities. Presently, we describe the possible minimal resolutions that occur for quotient singularities together with compactifying divisors which are convenient from our point of view. The latter can be obtained by the method of McCarthy and Wolfson [14]; for more information on quotient singularities, see [22].

3.1. Cyclic quotient singularities, $A_{n,q}$, where $0 < q < n$ and $(n,q) = 1$

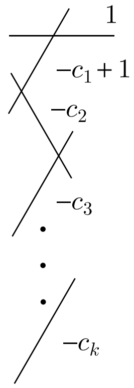
It is well known that the minimal resolution of $A_{n,q}$ is given by



where the b_i are defined by the Hirzebruch-Jung continued fraction:

$$\frac{n}{q} = [b_1, b_2, \ldots, b_r] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}, \quad b_i \geq 2 \text{ for all } i.$$

It is not difficult to check that the following configuration of curves gives a compactifying divisor for $A_{n,q}$:

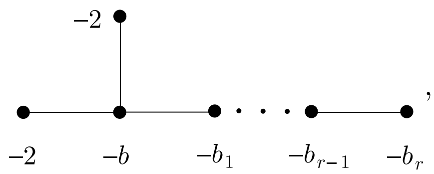


where the c_i are given by

$$\frac{n}{n-q} = [c_1, c_2, \ldots, c_k], \quad c_i \geq 2 \text{ for all } i.$$

3.2. Dihedral singularities, $D_{n,q}$, where $1 < q < n$ and $(n,q) = 1$

The minimal resolution is given by



where $b, b_i, i = 1, \ldots, r$ are defined by

$$\frac{n}{q} = [b, b_1, \ldots, b_r], \quad b \geq 2, b_i \geq 2 \text{ for all } i.$$

In this case, one can check that a compactifying divisor is given by

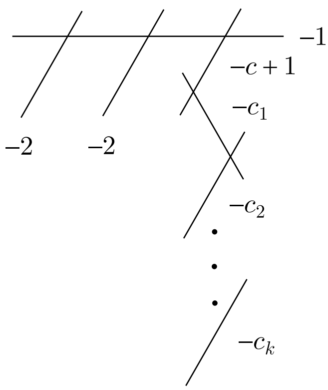
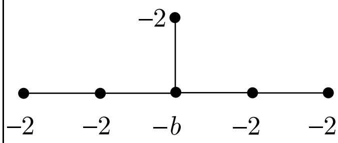
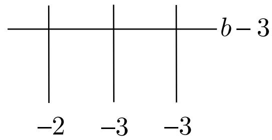
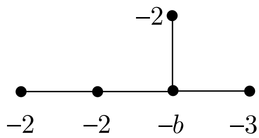
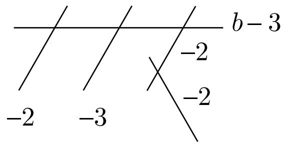
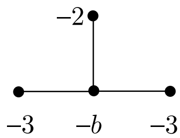
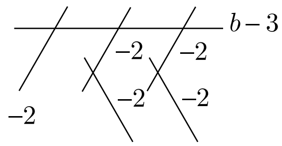


Table 1: Dual resolution graphs and compactifying divisors for tetrahedral singularities

Tetrahedral singularity	Dual resolution graph	Compactifying divisor
$m=6(b-2)+1$		
$m=6(b-2)+3$		
$m=6(b-2)+5$		

where $c, c_i, i=1,\dots,k$ are given by

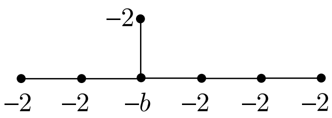
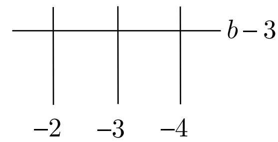
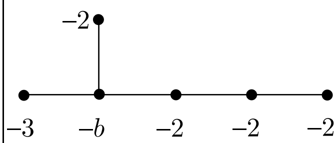
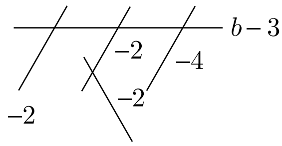
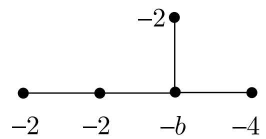
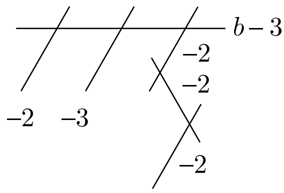
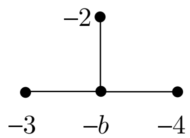
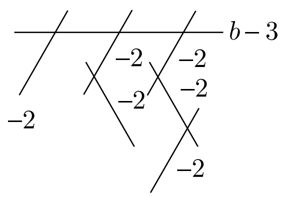
$$\frac{n}{n-q}=[c,c_1,\dots,c_k], \quad c\geq 2, c_i\geq 2 \text{ for all } i.$$

3.3. Tetrahedral singularities, T_m, where $m=1,3,5 \mod 6$
The dual resolution graphs and compactifying divisors are given in Table 1.

3.4. Octahedral singularities, O_m, where $(m,6)=1$
The dual resolution graphs and compactifying divisors are given in Table 2.

3.5. Icosahedral singularities, I_m, where $(m,30)=1$
The dual resolution graphs and compactifying divisors are given in Table 3.

Table 2: Dual resolution graphs and compactifying divisors for octahedral singularities

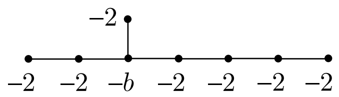
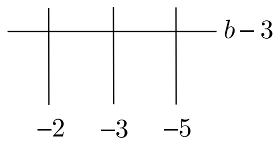
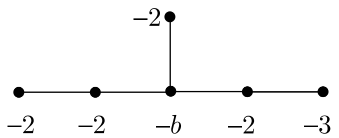
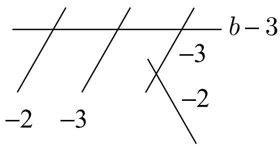
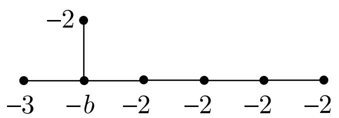
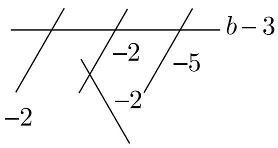
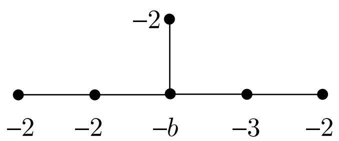
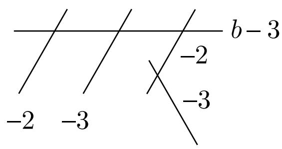
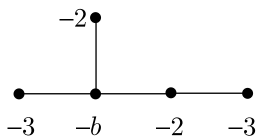
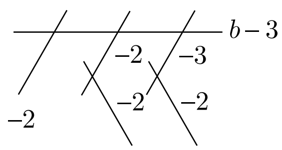
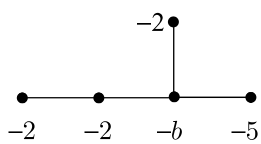
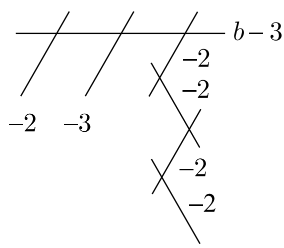
Octahedral singularity	Dual resolution graph	Compactifying divisor
$m = 12(b-2)+1$		
$m = 12(b-2)+5$		
$m = 12(b-2)+7$		
$m = 12(b-2)+11$		

§4. Compactification of symplectic fillings

Let X be a symplectic filling of the link of a quotient surface singularity. Without loss of generality, we may assume that X is minimal, that is, that X does not contain any symplectically embedded spheres of self-intersection number -1. Denote by Y a regular neighborhood of the compactifying divisor K presented in Section 3. We may take Y so that it is a strong concave filling of $\partial Y \cong \partial X$ (see [1], [21]). Gluing X and Y, we get a closed symplectic manifold Z. The classification problem of symplectic fillings reduces to the symplectic deformation classification of the pair (Z, K).

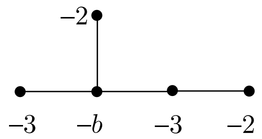
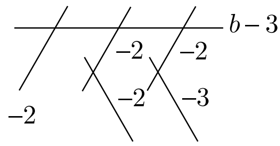
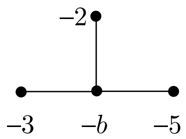
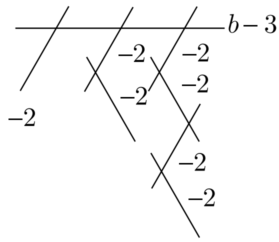
For a configuration $\mathcal{C}$ of finitely many pseudoholomorphic curves, we denote by $\mathcal{J}_{\mathcal{C}}$ the space of tamed almost complex structures J, for which all

Table 3: Dual resolution graphs and compactifying divisors for icosahedral singularities

Icosahedral singularity	Dual resolution graph	Compactifying divisor
$m = 30(b-2)+1$		
$m = 30(b-2)+7$		
$m = 30(b-2)+11$		
$m = 30(b-2)+13$		
$m = 30(b-2)+17$		
$m = 30(b-2)+19$		

(continued)

Table 3: Continued

<table>
<tr><th>Icosahedral singularity</th><th>Dual resolution graph</th><th>Compactifying divisor</th></tr>
<tr><td>$m = 30(b-2)+23$</td><td>
</td><td>
</td></tr>
<tr><td>$m = 30(b-2)+29$</td><td>
</td><td>
</td></tr>
</table>

the curves in $\mathcal{C}$ are J-holomorphic. For a generic choice of $J \in \mathcal{J}_{\mathcal{C}}$, the maximal number of mutually disjoint J-holomorphic (-1)-curves away from $\mathcal{C}$ is equal to the maximal number of mutually disjoint symplectic (-1)-spheres away from $\mathcal{C}$.

4.1. Cyclic quotient singularities

Let $\mathcal{D} = L \cup C_1 \cup \cdots \cup C_k$ be a string of symplectically embedded 2-spheres in a closed symplectic 4-manifold M with $L \cdot L = 1$. Here by a *string* of spheres we mean a configuration of spheres whose dual graph is an unbranched tree. We assume that the vertices corresponding to L and C_k are the leaves. The main examples are the compactifying divisors for cyclic quotient singularities given in Section 3. Note that $C_1 \cdot C_1 \leq 0$ and that $C_i \cdot C_i < 0$ $(i = 2, \ldots, k)$. Note also that M is a blowup of $\mathbb{C}P^2$ by Theorem 2.1.

Let J be a tame almost complex structure for which L, $C_1, \ldots, C_k$ are pseudoholomorphic. By Lemma 2.13, there exists at least one J-holomorphic (-1)-curve in $M \setminus L$. In fact, by successively blowing down, the compactifying divisor is reduced to two complex projective lines in the complex projective plane as Lisca [11] claimed.* Here we give a proof based on Lemma 2.13 for the sake of completeness.

*After we finished this manuscript, Lisca's paper [12] appeared.

We call a configuration $\mathcal{D} = L \cup C_1 \cup \cdots \cup C_k$ of rational curves *admissible* (for symplectic fillings of links of cyclic quotient singularities) if it is the total transform of two distinct lines in $\mathbb{C}P^2$ under some iterated blowup.

Suppose that $M \setminus (L \cup C_1 \cup \cdots \cup C_k)$ is minimal. Denote by $\mathcal{J}_\mathcal{D}$ the set of tame almost complex structures with respect to which $\mathcal{D} = L \cup C_1 \cup \cdots \cup C_k$ is pseudoholomorphic. We will blow down a maximal family $\{E_i\}$ of pseudoholomorphic (-1)-curves in $M \setminus L$ to reduce the configuration of rational curves to an admissible configuration. Note that these (-1)-curves are mutually disjoint (Lemma 2.8).

Proposition 4.1. *Let J be a tame almost complex structure, which is generic in $\mathcal{J}_\mathcal{D}$. Denote by M' the symplectic 4-manifold obtained by blowing down all J-holomorphic (-1)-curves $\{E_j\}$ away from L, and denote by C_i' the image of C_i. Then $\{L, C_i'\}$ is an admissible configuration for a symplectic filling of a link of cyclic quotient singularity.*

Proof. First, we note that any pseudoholomorphic (-1)-curve in $M' \setminus L$ is one of the C_i' $(i \geq 2)$. Indeed, assume to the contrary that there is a pseudoholomorphic (-1)-curve E in $M' \setminus L$ which is not one of the C_i'. Perturbing the almost complex structure slightly, we may assume that E does not pass through the images of the blown-down (-1)-curves E_j. Hence, we may assume that E is actually a pseudoholomorphic (-1)-curve already in $M \setminus L$, which contradicts the maximality of $\{E_j\}$.

Next we note that each C_i' is embedded. It is enough to show that each E_j intersects exactly one of the C_i with $E_j \cdot C_i = 1$ for that i. Perturbing the almost complex structure away from $\mathcal{D}$, we may assume that E_j intersects each C_i transversally. Suppose that E_j intersects C_i with $E_j \cdot C_i \geq 2$. After contracting E_j, C_i becomes a nodal curve, which is singular. The existence of such a curve is prohibited by Lemma 2.5. Similarly, Lemma 2.5 and Remark 2.12 exclude the possibility that E_j intersects at least two of the C_i.

Thus, Lemma 2.13 implies that one of the C_i' is a symplectic (-1)-curve. After blowing it down, we get a new configuration of rational curves $L \cup C_1'' \cup \cdots \cup C_{k-1}''$. Let M'' denote the resulting ambient symplectic 4-manifold.

We claim that there are no symplectic (-1)-curves in $M'' \setminus L$ except for some of the C_i''. Suppose that E is such a pseudoholomorphic (-1)-curve. A similar argument as above shows that E can be lifted to a pseudoholomorphic (-1)-curve in $M \setminus L$. This contradicts the maximality of $\{E_j\}$.

Continuing this process, we can successively blow down (-1)-curves until we obtain the complex projective plane and C_1 has been transformed into a complex projective line L' transversal to L. The other C_i are contracted to a point on L' in the process. ∎

4.2. Dihedral singularities

Let $\widetilde{M}$ be a closed symplectic 4-manifold containing a configuration of symplectically embedded 2-spheres intersecting in the manner shown in the first picture in Figure 1. Here $c, c_i \geq 2, i = 1, \ldots, k$. By a sequence of blow-downs and a blowup, as in [20] in the simple dihedral case, we can transform this configuration into a configuration containing a cusp curve (see Figure 1). Let M denote the resulting ambient manifold.

To obtain a classification of fillings of links of dihedral singularities, we proceed via a process of blowing down symplectic (-1)-curves in M. From the proof of Lemma 2.14, it follows that M, and hence $\widetilde{M}$, is rational.

Consider now, generally, a closed symplectic 4-manifold Z containing a configuration of rational curves $\mathcal{D} = A \cup D \cup C_1 \cup \cdots \cup C_k$ as depicted in Figure 2. Here D is a singular curve with a $(2,3)$-cusp, A is an embedded 0-curve intersecting D at the cusp point, and $C_1, \ldots, C_k$ are embedded curves. By [21], $D \cdot D \leq 8$. Also, by Lemmas 2.9 and 2.10, $C_i \cdot C_i \leq -1$ for all i.

Lemma 4.2. *Assume that the string $C_1, \ldots, C_k$ is nonempty. Let J be a tame almost complex structure for which $A, D, C_1, \ldots, C_k$ are pseudoholomorphic. Then C_1 is a (-1)-curve or there exists a J-holomorphic (-1)-curve in $Z \setminus (A \cup D)$.*

Proof. First suppose that the complement of $A \cup D$ is minimal. Then, by Lemma 2.14, we know that the anticanonical class of Z is represented by D. Since $C_1 \cdot D = 1$, it follows that C_1 is a (-1)-curve.

Suppose now that $Z \setminus (A \cup D)$ is not minimal. Then there exists a symplectic (-1)-curve E in $Z \setminus (A \cup D)$. If $[E]$ is represented by a J-holomorphic curve, we are done. If $[E]$ cannot be represented by a J-holomorphic curve, then pick a sequence of tame almost complex structures J_n converging to J such that $[E]$ is represented by a J_n-holomorphic curve E_n for each n. By the compactness theorem, after taking a subsequence, E_n converges to the image of a stable map. Let $A_1, \ldots, A_l$ denote the irreducible components of this stable map. Arguing as in the proof of [21, Proposition 4.1], it can be shown that none of the A_i coincide with (or multiply covers) D. Hence, it also follows that none of the A_i intersect D. Note also that A_j is disjoint from A. Since $c_1(Z)[E] = 1$, it follows that $c_1(Z)[A_j] > 0$ for some j. By replacing A_j by the underlying simple curve, if A_j is multiply covered, assume that A_j is a simple curve. By Lemma 2.6, A_j is an embedded

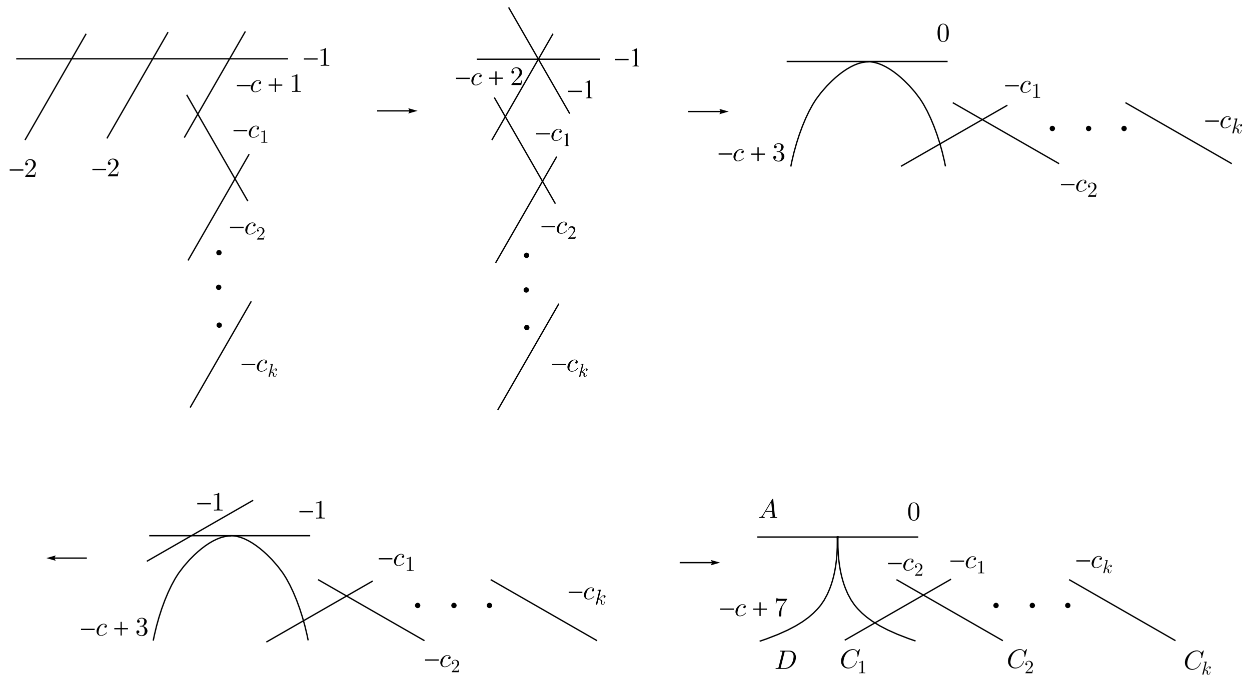


Figure 1: Sequence of blowdowns and a blowup transforming the compactifying divisor of a dihedral singularity into a configuration containing a cusp curve

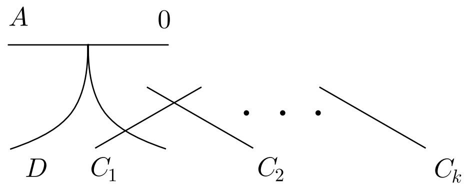


Figure 2: General configuration of rational curves considered in case of symplectic fillings of links of dihedral singularities

J-holomorphic sphere. Also, by Lemma 2.9, $A_j \cdot A_j < 0$. By the adjunction formula, it now follows that A_j is a J-holomorphic (-1)-curve. ∎

In general, we call a configuration of rational curves $\mathcal{D} = A \cup D \cup C_1 \cup \cdots \cup C_k$ as in Figure 2, in a closed symplectic 4-manifold Z, *admissible* (for symplectic fillings of links of dihedral singularities) if it can be obtained as the total transform of an iterated blowup of either

(a) a union of a cuspidal rational curve of bidegree $(2,2)$ and a 0-curve intersecting only at the cusp point in the ruled surface $\mathbb{C}P^1 \times \mathbb{C}P^1$, or
(b) a union of a singular rational curve representing the class $3\mathbb{C}P^1 - L$ and a 0-curve intersecting only at the cusp point in the 1-point blowup of $\mathbb{C}P^2$, where L represents the exceptional curve of the blowup.

We call $A \cup D \cup C_1 \cup \cdots \cup C_k$ a *preadmissible* configuration if it becomes admissible after possibly blowing down some (-1)-curves intersecting only D.

Assume now that $M \setminus (A \cup D \cup C_1 \cup \cdots \cup C_k)$ is minimal. The following proposition shows that after blowing down a maximal family of pseudoholomorphic (-1)-curves in $M \setminus (A \cup D)$, the configuration $A \cup D \cup C_1 \cup \cdots \cup C_k$ is reduced to a preadmissible configuration. Note that by Lemma 2.11 these (-1)-curves are necessarily disjoint. Note also that if these (-1)-curves are not contained in the string $C_1, \ldots, C_k$, then they can intersect it at most once; that is, for any such (-1)-curve E, $\sum E \cdot C_i \leq 1$. Indeed, suppose that there is a (-1)-curve E such that $\sum E \cdot C_i > 1$; then, after contracting E, we get either a singular pseudoholomorphic curve or a cycle of pseudoholomorphic spheres whose intersection number with A is 0. This contradicts Lemma 2.9 and Remark 2.12. Now let $\mathcal{J}_{\mathcal{C}}$ denote the set of tame almost complex structures with respect to which $\mathcal{C} = A \cup D \cup C_1 \cup \cdots \cup C_k$ is pseudoholomorphic.

PROPOSITION 4.3. *Let J be a tame almost complex structure which is generic in $\mathcal{J}_{\mathcal{C}}$. Denote by M' the symplectic 4-manifold obtained by blowing down all J-holomorphic (-1)-curves in $M \setminus (A \cup D)$, and denote by C_i' the image of C_i. Then $\{A, D, C_i'\}$ is a preadmissible configuration for a symplectic fillings of a link of a dihedral singularity.*

Proof. We first show that one of the C_i' is a (-1)-curve. If C_1' is a (-1)-curve, we are done. If C_1' is not a (-1)-curve, then, by Lemma 4.2, there exists a (-1)-curve in $M' \setminus (A \cup D)$. We claim that this (-1)-curve must be one of the C_i' $(i > 1)$. Suppose that it is not. Then it can intersect only one of the C_i' at exactly one point transversally. Hence, it must already have been in $M \setminus (A \cup D)$, contradicting the fact that we blew down all such (-1)-curves. We now blow down this curve to obtain a new configuration $\{A, D', C_i''\}$. If the string $\{C_i''\}$ is not empty, we can argue in a similar way to show that it must also contain a (-1)-curve. Blowing down this (-1)-curve also and continuing in this way, we can show that the whole string $\{C_i'\}$ can eventually be blown down. Thus, we are left with A and the image $\overline{D}$ of D such that the complement of $A \cup \overline{D}$ is minimal. Now, as in the proof of Lemma 2.14, we can see that any pseudoholomorphic (-1)-curve in the complement of A intersects $\overline{D}$ once transversally. After contracting any such (-1)-curves, we get a configuration of type (a) or (b). As we have noticed before, such (-1)-curves, if they exist, must exist in the original manifold M. Changing the order of the blowing-down processes, we find that $\{A, D, C_i'\}$ is a preadmissible configuration. ∎

4.3. Tetrahedral, octahedral, and icosahedral singularities

For the purposes of classification of symplectic fillings of tetrahedral, octahedral, and icosahedral singularities, it is convenient to divide these singularities into two sets: those with a branch consisting of two (-2)-curves intersecting the central curve in the minimal resolution, and those with a branch consisting of a single (-3)-curve intersecting the central curve. We designate these singularities as type $(3, 2)$ singularities and type $(3, 1)$ singularities, respectively. There is exactly one class of quotient singularities which lies in both sets, namely, the tetrahedral singularities $T_{6(b-2)+3}$.

We begin by discussing the classification of fillings of type $(3, 2)$ singularities. Note that, given a singularity Γ of type $(3, 2)$, we can always choose a compactifying divisor such that its central curve has self-intersection number -1. Namely, if $b = 2$, the central curve given in Section 2 is a (-1)-curve. If $b \geq 3$, we blow up the compactifying divisor given earlier at the transversal intersection of the central curve and the third branch—that is, the branch that does not consist of a single (-2)- or (-3)-curve (except in case $\Gamma = T_{6(b-2)+1}$, in which case we blow up at the intersection of the central curve and one of the (-3)-curves)—repeatedly until the self-intersection number of the central curve has dropped to -1. Now let $\widetilde{M}$ be a closed symplectic 4-manifold containing a configuration of symplectically embedded 2-spheres intersecting in the manner shown in the first picture in Figure 3. Here $a \geq 1$ and $c_i \geq 2$ for $i = 1, \ldots, k$. Note that $a = 1$, when $b \geq 3$. When $b = 2$, $2 \leq a \leq 5$ (see Tables 1–3). It will be convenient to transform this configuration of symplectically embedded 2-spheres into one containing a cusp curve. We achieve this by a sequence of blowdowns, as in [20, cases E_6, E_7, E_8] (see Figure 3). Let M denote the resulting closed symplectic 4-manifold, let D denote the cusp curve, and let $C_1, \ldots, C_k$ denote the string of curves attached to D. As the self-intersection number of the cusp curve D is always positive, we can immediately conclude, by Theorem 2.3, that M, and hence $\widetilde{M}$, is a rational symplectic manifold.

Consider now, generally, a closed symplectic 4-manifold Z containing a configuration of rational curves $\mathcal{D} = D \cup C_1 \cup \cdots \cup C_k$ as depicted in Figure 4 with $D \cdot D > 0$. Here, D is a singular curve with a $(2,3)$-cusp, and $C_1, \ldots, C_k$ are embedded curves. By Theorem 2.3, $D \cdot D \leq 9$. Also, by Lemmas 2.9 and 2.10, $C_i \cdot C_i \leq -1$ for all i.

Lemma 4.4. *Assume that the string $C_1, \ldots, C_k$ is nonempty. Let J be a tame almost complex structure for which $D, C_1, \ldots, C_k$ are pseudoholomorphic. Then C_1 is a (-1)-curve or there exists a J-holomorphic (-1)-curve in $Z \setminus D$.*

Proof. Suppose that the complement of D is minimal. Since $D \cdot D > 0$, Theorem 2.3 implies that an anticanonical divisor is given by D. It follows that C_1 is a (-1)-curve. The remainder of the proof proceeds in a similar way to the proof of Lemma 4.2. ∎

Denote by $D_0 \subset \mathbb{C}P^2$ a cuspidal cubic curve. Following the process of blowup and blowdown from (M, D) to $(\mathbb{C}P^2, D_0)$, we find that D is an anticanonical divisor when $M \setminus D$ does not contain symplectic (-1)-curves.

In general, we call a configuration of rational curves $\mathcal{D} = D \cup C_1 \cup \cdots \cup C_k$ as in Figure 4, in a closed symplectic 4-manifold Z, *admissible* (for symplectic fillings of links of tetrahedral, octahedral, and icosahedral singularities of type $(3,2)$) if it can be obtained as the total transform of an iterated blowup

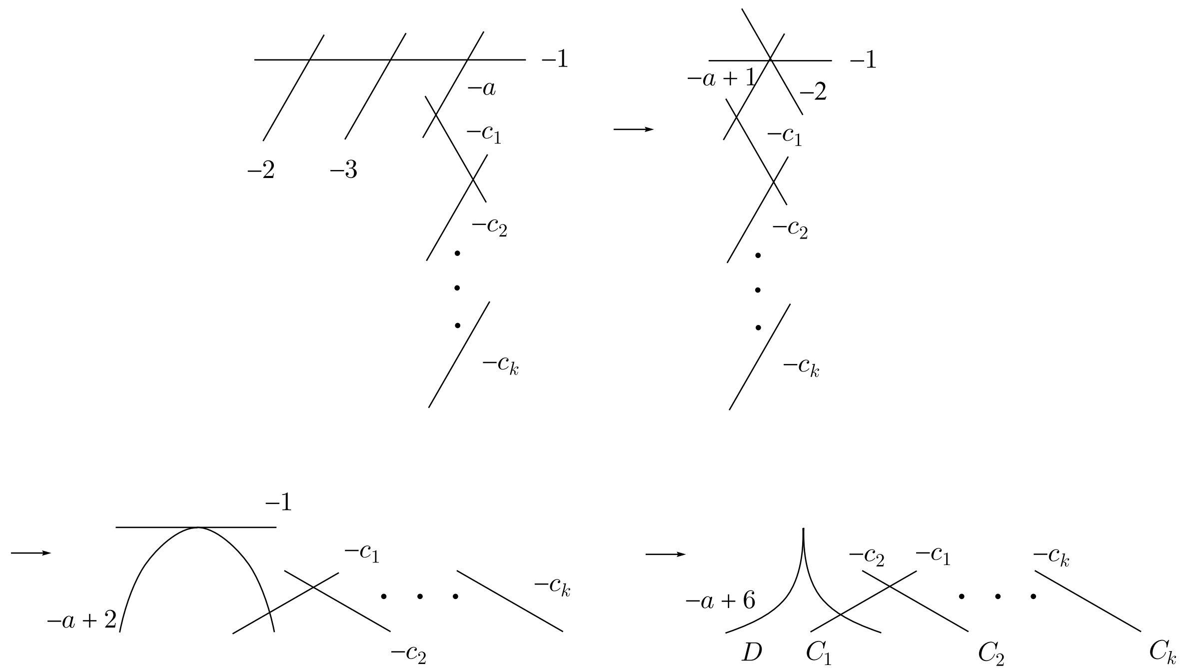


Figure 3: Sequence of blowdowns transforming the compactifying divisor of a tetrahedral, octahedral, or icosahedral singularity of type $(3, 2)$ into a configuration containing a cusp curve

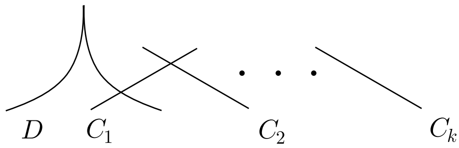


Figure 4: General configuration of rational curves considered in case of symplectic fillings of links of tetrahedral, octahedral, and icosahedral singularities of type $(3, 2)$

of the cuspidal cubic curve in $\mathbb{C}P^2$, or of the cuspidal curve of bidegree $(2, 2)$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$. In [20], we can always blow down a maximal family of (-1)-curves to get a cuspidal cubic curve in $\mathbb{C}P^2$. In fact, if $b_2(M) \geq 3$, one can also contract another maximal family of (-1)-curves to get $\mathbb{C}P^1 \times \mathbb{C}P^1$. In our present situation, we do not blow down (-1)-curves intersecting both D and some C_i. Hence, we may arrive not at $\mathbb{C}P^2$ but at $\mathbb{C}P^1 \times \mathbb{C}P^1$. Again, we call such a configuration *preadmissible* if it becomes admissible after possibly blowing down some (-1)-curves intersecting only D.

Now assume that $M \setminus (D \cup C_1 \cup \cdots \cup C_k)$ is minimal. The following proposition shows that, after blowing down a maximal family of pseudoholomorphic (-1)-curves in $M \setminus D$, the configuration $D \cup C_1 \cup \cdots \cup C_k$ is reduced to a preadmissible configuration. By Lemma 2.11, these (-1)-curves are necessarily disjoint. Also, if these (-1)-curves are not contained in the string $C_1, \ldots, C_k$, then they can intersect it at most once. Indeed, suppose that there is such a (-1)-curve E such that $\sum E \cdot C_i \geq 2$; then, after contracting E, the image of all the curves C_i contains a singular pseudoholomorphic curve or a cycle of pseudoholomorphic spheres. This contradicts Lemma 2.6, Lemma 2.7, or Remark 2.12, thus proving the assertion. (Note that for a type $(3, 2)$ singularity Γ for which the string $C_1, \ldots, C_k$ is nonempty, the self-intersection number $D \cdot D \geq 3$.) In summary, after contracting any pseudoholomorphic (-1)-curve in $M \setminus D$, we again get a configuration consisting of D and a string of embedded spheres.

Let $\mathcal{J}_{\mathcal{C}}$ denote the set of tame almost complex structures with respect to which $\mathcal{C} = D \cup C_1 \cup \cdots \cup C_k$ is pseudoholomorphic. We can prove the following proposition in a similar way to the proof of Proposition 4.3.

Proposition 4.5. *Let J be a tame almost complex structure which is generic in $\mathcal{J}_{\mathcal{C}}$. Denote by M' the symplectic 4-manifold obtained by blowing down all J-holomorphic (-1)-curves in $M \setminus D$, and denote by C_i' the image* *of C_i. Then $\{D, C_i'\}$ is a preadmissible configuration for a symplectic filling of a link of a tetrahedral, octahedral, or icosahedral singularity of type* $(3,2)$.

We now turn to the classification of fillings of type $(3,1)$ singularities. Again, given a singularity Γ of type $(3,1)$, as in the case of type $(3,2)$ singularities, we can always choose a compactifying divisor whose central curve has self-intersection number -1. Namely, as before, when $b \geq 3$, we blow up the compactifying divisor given earlier at the transversal intersection of the central curve and the third branch, that is, the branch that does not consist of one or two (-2)-curves (except in case $\Gamma = T_{6(b-2)+5}$, in which case we blow up at the intersection of the central curve and one of the branches consisting of two (-2)-curves) repeatedly until the self-intersection number of the central curve has dropped to -1. Now let $\widetilde{M}$ be a closed symplectic 4-manifold containing a configuration of symplectically embedded 2-spheres intersecting in the manner shown in the first picture in Figure 5. Here $a \geq 1$ and $c_i \geq 2$ for $i = 1, \ldots, k$. Again, it will be convenient to transform this configuration of symplectically embedded 2-spheres into one containing a cusp curve. We achieve this by a sequence of blowdowns and a blowup (see Figure 5). Let M denote the resulting closed symplectic 4-manifold, D the cuspidal curve, A the 0-curve, B the (-1)-curve, and $C_1, \ldots, C_k$ the string of curves intersecting D. From the proof of Lemma 2.14, it follows that M, and hence $\widetilde{M}$, is rational. Let $\mathcal{J}_{\mathcal{C}}$ denote the set of tame almost complex structures with respect to which $\mathcal{C} = A \cup B \cup D \cup C_1 \cup \cdots \cup C_k$ is pseudoholomorphic.

LEMMA 4.6. *Let J be generic in $\mathcal{J}_{\mathcal{C}}$. Then there exists a J-holomorphic (-1)-curve E in $M \setminus (A \cup D)$ such that $B \cdot E = 1$. Moreover, such a curve E is unique.*

Proof. Let $\mathcal{J}_{\mathcal{D}}$ denote the set of tame almost complex structures with respect to which $\mathcal{D} = A \cup B \cup D$ is pseudoholomorphic. By Proposition 2.4 applied to the curves $A \cup B \cup D$, for a generic almost complex structure $J' \in \mathcal{J}_{\mathcal{D}}$, any symplectic (-1)-curve in M has a unique J'-holomorphic representative. In particular, for any maximal disjoint family $E_1, \ldots, E_N$ of symplectic (-1)-curves in the complement of $A \cup D$, we have a unique family of J'-holomorphic representatives $E_1', \ldots, E_N'$. It follows, by the results of [20], that an anticanonical divisor of M is given by $D - \sum E_i'$. Now since $c_1(M)[B] = 1$ and $B \cdot D = 2$, it follows that there is exactly one E_i' such that $B \cdot E_i' = 1$ and $B \cdot E_j' = 0$ if $j \neq i$. Assume, without loss of generality, that

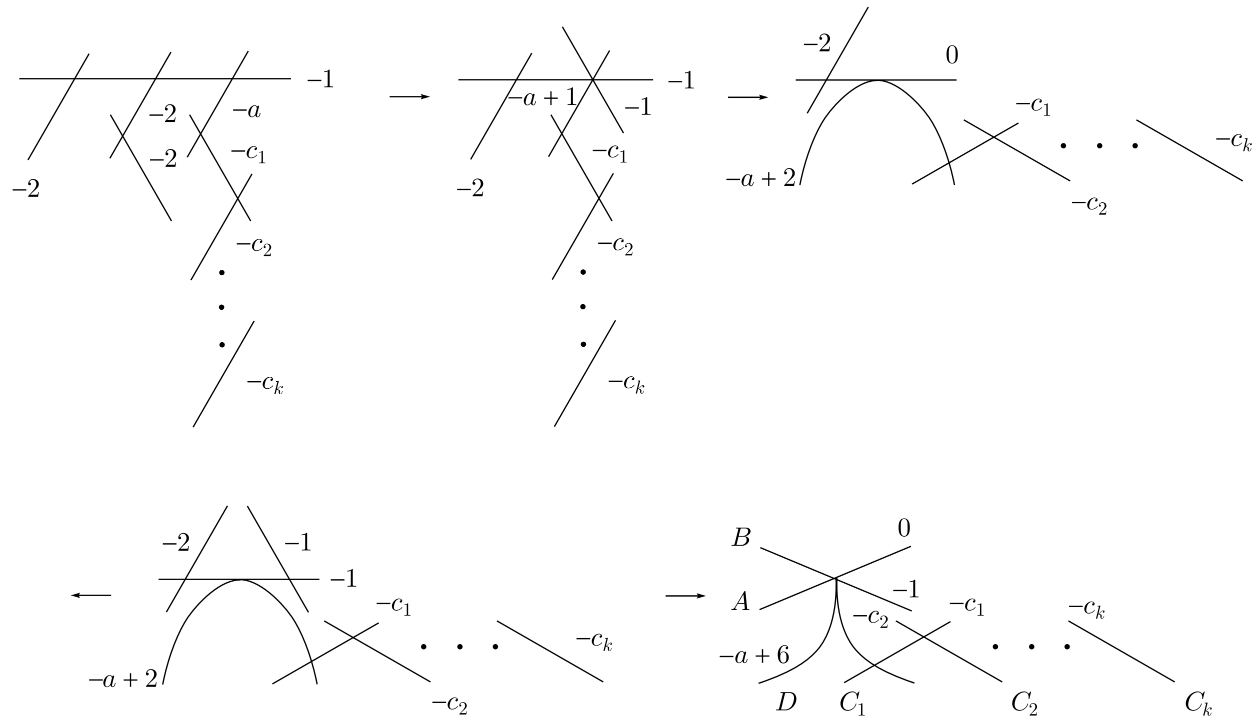


Figure 5: Sequence of blowdowns and a blowup transforming the compactifying divisor of a tetrahedral, octahedral, or icosahedral singularity of type $(3, 1)$ into a configuration containing a cusp curve

$B \cdot E_1' = 1$. Now consider a sequence of generic almost complex structures $J_n \in \mathcal{J}_{\mathcal{D}}$ converging to J such that $[E_1']$ is represented by a J_n-holomorphic curve B_n for each n. By the compactness theorem, B_n converges to the image of a stable map. Let $A_1, \dots, A_l$ denote the irreducible components of this stable map. By the proof of [21, Proposition 4.1], no component of this stable map coincides with (or is a multiple cover of) D. Hence, in particular, no component can intersect D. Since $[B] \cdot [E_1'] = 1$, there is a component A_i such that $A_i \cdot B = 1$. It follows that A_i is a simple curve. Since A_i is disjoint from D, by Lemma 2.9, A_i is a rational curve of negative self-intersection. Now the fact that J is generic away from the configuration $\mathcal{C}$ allows us to conclude that A_i is in fact a (-1)-curve. (The virtual dimension of the moduli space of singular pseudoholomorphic curves of negative self-intersection number is negative.) Taking $E = A_i$ gives the required J-holomorphic (-1)-curve.

If E and E' are pseudoholomorphic (-1)-curves such that $E \cdot B = E' \cdot B = 1$, then, for a generic $J' \in \mathcal{J}_{\mathcal{D}}$, both $[E]$ and $[E']$ are represented by J'-holomorphic (-1)-curves. By Lemma 2.11, they are mutually disjoint, and we may assume that E and E' are contained in $\{E_1', \dots, E_N'\}$. However, as we saw, there is exactly one E_i' such that $B \cdot E_i' = 1$. Hence, we find uniqueness; that is, $E = E'$. ∎

Note that E can intersect at most one of the C_i (see Remark 2.12). In a similar way, $E \cdot C_i = 1$ if E intersects C_i. There are now two cases to consider: the case where E is disjoint from the string $C_1, \dots, C_k$, and the case where E intersects precisely one member of the string $C_1, \dots, C_k$.

Case I: $E \cdot C_i = 0$ for all i. In this case, blow down the (-1)-curve E and denote the image of B under the blowing-down map by B'. Then B' is a 0-curve, and the resulting configuration $\mathcal{C}' = A \cup B' \cup D \cup C_1 \cup \cdots \cup C_k$ is as in Figure 6. Let M' denote the resulting symplectic 4-manifold. We will show that, after blowing down (-1)-curves in $M' \setminus (A \cup B' \cup D)$, the string $C_1, \dots, C_k$ is transformed into one which can be sequentially blown down.

Consider, generally, a closed symplectic 4-manifold Z containing a configuration of rational curves $\mathcal{D} = A \cup B \cup D \cup C_1 \cup \cdots \cup C_k$ as depicted in Figure 7. Here, D is a singular curve with a $(2,3)$-cusp, A and B are embedded 0-curves intersecting transversely at the cusp point of D, and $C_1, \dots, C_k$ are embedded curves. By [21], $D \cdot D \leq 8$. Also, by Lemmas 2.9 and 2.10, $C_i \cdot C_i \leq -1$ for all i.

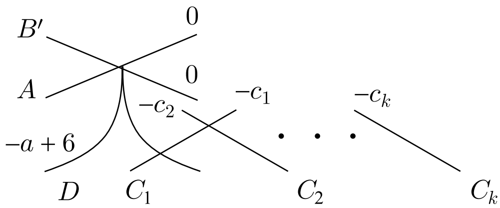


Figure 6: Image of compactifying divisor after blowing down the (-1)-curve E in Case I

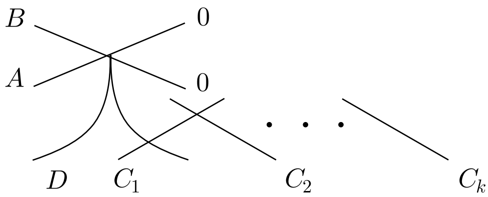


Figure 7: General configuration of rational curves considered in Case I symplectic fillings of links of tetrahedral, octahedral, and icosahedral singularities of type $(3,1)$

LEMMA 4.7. *Assume that the string $C_1, \ldots, C_k$ is nonempty. Let J be a tame almost complex structure for which $A, B, D, C_1, \ldots, C_k$ are pseudoholomorphic. Then C_1 is a (-1)-curve or there exists a J-holomorphic (-1)-curve in $Z \setminus (A \cup B \cup D)$.*

Proof. Suppose that the complement of $A \cup B \cup D$ is minimal. Then, by Lemma 2.15, we know that an anticanonical divisor is given by D. It follows that C_1 is a (-1)-curve. The remainder of the proof proceeds in a similar way to the proof of Lemma 4.2. ∎

In general, we call a configuration of rational curves $\mathcal{D} = A \cup B \cup D \cup C_1 \cup \cdots \cup C_k$ as in Figure 7, in a closed symplectic 4-manifold Z, *admissible* (for case I symplectic fillings of links of tetrahedral, octahedral, and icosahedral singularities of type $(3,1)$) if it can be obtained as the total transform of an iterated blowup of a union of a cuspidal rational curve of bidegree $(2,2)$ and two 0-curves intersecting transversely at its cusp point in $\mathbb{C}P^1 \times \mathbb{C}P^1$. Again, we call such a configuration *preadmissible* if it becomes admissible after possibly blowing down some (-1)-curves intersecting D only.

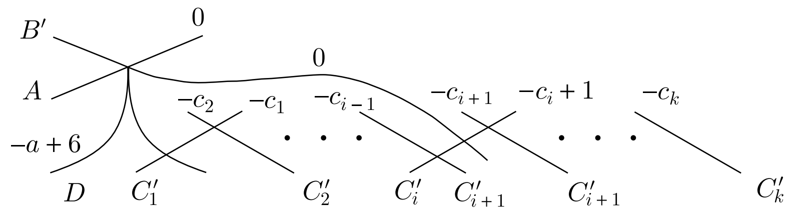


Figure 8: Image of compactifying divisor after blowing down the (-1)-curve E in Case II

Assume now that $M' \setminus (A \cup B' \cup D \cup C_1 \cup \cdots \cup C_k)$ is minimal. The following proposition shows that, after blowing down a maximal family of (-1)-curves in $M' \setminus (A \cup B' \cup D)$, the configuration $\mathcal{C}' = A \cup B' \cup D \cup C_1 \cup \cdots \cup C_k$ is reduced to a preadmissible configuration. Note that, by construction, $C_2, \ldots, C_k$ are not (-1)-curves and that C_1 intersects D. By Lemma 2.9, these (-1)-curves are necessarily disjoint. Again, by Lemma 2.6, Lemma 2.7, and Remark 2.12, any such (-1)-curve, if it is not contained in the string $C_1, \ldots, C_k$, can intersect it at most once. Let $\mathcal{J}_{\mathcal{C}'}$ denote the set of tame almost complex structures with respect to which all the irreducible components of the configuration $\mathcal{C}'$ are pseudoholomorphic.

Proposition 4.8. *Let J be a tame almost complex structure which is generic in $\mathcal{J}_{\mathcal{C}'}$. Denote by M'' the symplectic 4-manifold obtained by blowing down all J-holomorphic (-1)-curves in $M' \setminus (A \cup B' \cup D)$, and denote by C_i' the image of C_i. Then $\{A, B', D, C_i'\}$ is a preadmissible configuration for a Case I symplectic filling of a link of a tetrahedral, octahedral, or icosahedral singularity of type $(3,1)$.*

Case II: $E \cdot C_i = 1$ for some i. Again, begin by blowing down E. Denote the resulting symplectic 4-manifold by M', the image of B by B', and the image of C_j by C_j' for $j = 1, \ldots, k$. Then B' is a 0-curve, $C_i' \cdot C_i' = -c_i + 1$, and the resulting configuration $\mathcal{C}' = A \cup B' \cup D \cup C_1' \cup \cdots \cup C_k'$ is as in Figure 8. As in other cases, we blow down some pseudoholomorphic (-1)-curves and reduce the compactifying divisor to a standard form. For each pseudoholomorphic (-1)-curve F in the complement of $A \cup B'$, F intersects at most one of the C_j'. Moreover, their intersection number is 1. After contracting such (-1)-curves, $C_1', \ldots, C_k'$ remain symplectically embedded spheres, whose dual graph is a string. The image C_j'' of C_j' $(j \neq i)$ is contained in the complement of $A \cup B'$ and hence has negative self-intersection number. Note

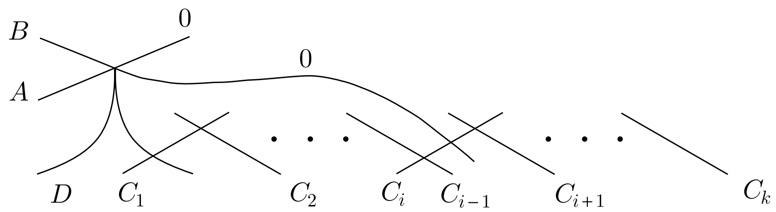


Figure 9: General configuration of rational curves considered in Case II symplectic fillings of links of tetrahedral, octahedral, and icosahedral singularities of type $(3,1)$

that F may also intersect D. In such a case, we have $F\cdot D=1$. If $F\cdot D>1$, after contracting F, the image of D contains at least two singular points. However, the intersection with A (resp., B') remains the same. Thus, after contracting other (-1)-curves in the complement of $A\cup B'$, the image of D should represent a homology class of bidegree $(2,2)$. This is absurd.

Consider now, generally, a closed symplectic 4-manifold Z containing a configuration of rational curves $\mathcal{D}=A\cup B\cup D\cup C_1\cup\cdots\cup C_k$ intersecting each other as in Figure 9, but with the possibility that there might be more intersections between the string $C_1,\ldots,C_k$ and the singular curve D than indicated in the figure. Here D is a singular curve with a $(2,3)$-cusp, A and B are embedded 0-curves intersecting transversely at the cusp point of D, and $C_1,\ldots,C_k$ are embedded curves. By [21], $D\cdot D\leq 8$. Also, by Lemmas 2.9 and 2.10, $C_j\cdot C_j\leq -1$ for all j.

Lemma 4.9. *Let J be a tame almost complex structure for which the irreducible components of the configuration $\mathcal{D}$ are pseudoholomorphic. Then there exists a J-holomorphic (-1)-curve in $Z\setminus(A\cup B)$, unless $k=1$, $C_1\cdot C_1=0$ and $D\cdot D=8$.*

Proof. By [20], we know that after collapsing a maximal family of (-1)-curves in the complement of $A\cup B$, the manifold Z is reduced to $\mathbb{C}P^1\times\mathbb{C}P^1$ with the image D being a pseudoholomorphic cuspidal rational curve of bidegree $(2,2)$. If $k>1$, there are at least two C_j, one of which is contained in the complement of $A\cup B$. Hence, its self-intersection number is negative. Suppose that C_1 is the only member of the string and that $C_1\cdot C_1\neq 0$. Since C_1 does not intersect A, the self-intersection number of C_1 must be negative. Otherwise, the self-intersection number of D is less than 8. In each

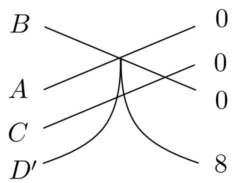


Figure 10: Arrangement of curves in $\mathbb{C}P^1 \times \mathbb{C}P^1$ giving rise to admissible configurations for Case II symplectic fillings of links of tetrahedral, octahedral or icosahedral singularities of type $(3,1)$

case, we find that the complement of $A \cup B$ is not minimal. The proof now proceeds in a similar way to the proof of Lemma 4.2. ∎

In general, we say that a configuration of rational curves $\mathcal{D} = A \cup B \cup D \cup C_1 \cup \cdots \cup C_k$ in a closed symplectic 4-manifold Z, where $A \cup B \cup D$ are as in Figure 9 and $C_1, \ldots, C_k$ is a string of embedded curves, is *admissible* (for Case II symplectic fillings of links of tetrahedral, octahedral, or icosahedral singularities of type $(3,1)$) if it can be obtained as a total transform, under an iterated blowup, of a configuration $\mathcal{D}' = A \cup B \cup C \cup D'$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$, intersecting as depicted in Figure 10. Here D' is a cuspidal rational curve of bidegree $(2,2)$ and A, B, C are ruling fibers. (Note that D necessarily intersects the string $C_1 \cup \cdots \cup C_k$ twice in an admissible configuration.)

Assume now that the complement of the configuration $\mathcal{C}' = A \cup B' \cup D \cup C_1' \cup \cdots \cup C_k'$ in M' is minimal, and let $\mathcal{J}_{\mathcal{C}'}$ denote the set of tame almost complex structures with respect to which $\mathcal{C}'$ is pseudoholomorphic.

Proposition 4.10. *Let J be a tame almost complex structure which is generic in $\mathcal{J}_{\mathcal{C}'}$. Denote by M'' the symplectic 4-manifold obtained by blowing down all J-holomorphic (-1)-curves in $M' \setminus (A \cup B')$, denote by D' the image of D, and denote by C_l'' the image of C_l'. Then $\{A, B', D', \{C_l''\}\}$ is an admissible configuration for a Case II symplectic filling of a link of a tetrahedral, octahedral, or icosahedral singularity of type $(3,1)$.*

Proof.

Claim 1. *There is a J-holomorphic (-1)-curve F in $M' \setminus (A \cup B')$ such that $F \cdot D = 1$, $F \cdot C_j' = 1$ for some j, and $F \cdot C_l' = 0$, $l \neq j$.*

Proof of Claim 1. Suppose that there is no such curve F. Note that any J-holomorphic (-1)-curve in the complement of $A \cup B'$ cannot intersect

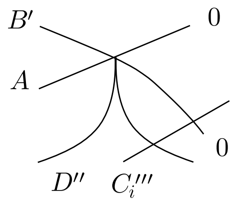


Figure 11: See proof of Claim 1

the string $C_1' \cup \cdots \cup C_k'$ with intersection number greater than 1 for the same reason as mentioned in the dihedral singularities case. Thus, any J-holomorphic (-1)-curve in $M' \setminus (A \cup B')$ is disjoint either from D or from $C_1' \cup \cdots \cup C_k'$. After blowing down all such (-1)-curves, denote the image of D by D' and denote the image of C_l' by C_l''. We now appeal to Lemma 4.9 to find further (-1)-curves away from $A \cup B'$ in the resulting symplectic 4-manifold M''. Arguing as in other cases, any such (-1)-curve must be one of the C_l'', $l \neq i$. After iteratively blowing down all such (-1)-curves, we arrive at the situation depicted in Figure 11. But this situation cannot be minimal since, if it were, C_i''' would have to be homologous to A and hence would have to intersect D'' twice. Hence, arguing as in the proof of Lemma 4.9, there must still be a (-1)-curve in the resulting symplectic 4-manifold M''' away from $A \cup B'$. But this (-1)-curve must have already existed in $M' \setminus (A \cup B')$, which is a contradiction since we are assuming that we blew down all such (-1)-curves. ∎

Claim 2. *Let F be as in Claim 1. Then $j \geq i$.*

Proof of Claim 2. Suppose that $j < i$; then, after blowing down all (-1)-curves in $M' \setminus (A \cup B')$, let M'' denote the resulting symplectic 4-manifold, and denote by D' the image of D and by C_l'' the image of C_l'. Arguing as in the proof of Claim 1, we can now iteratively blow down all the curves C_l'', $l \neq i$. Let M''' denote the resulting symplectic 4-manifold, and denote by D'' the image of D'. Since the image of the string $C_1', \ldots, C_{i-1}'$ in M'' intersects D' at least twice, D'' will have a singular point away from the cusp point. Now, after contracting a maximal family of pseudoholomorphic (-1)-curves in $M''' \setminus (A \cup B')$, we obtain a $(2,2)$-curve in $\mathbb{C}P^1 \times \mathbb{C}P^1$ with at least two singular points, which is impossible. ∎

Claim 3. *There is at most one such curve F as in Claim 1.*

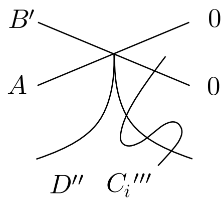


Figure 12: See proof of Claim 3

Proof of Claim 3. Suppose there is more than one such curve, and denote these curves $F_1, \ldots, F_s$. Assume that $F_l \cdot C'_{j(l)} = 1$ for $l = 1, \ldots, s$. Then, by Claim 2, $j(l) \geq i$ for all l. First, suppose that $\sharp\{l \mid j(l) > i\} \geq 2$. Then after contracting all (-1)-curves in $M' \setminus (A \cup B')$, the image of the string $C'_{i+1}, \ldots, C'_k$ in the resulting symplectic 4-manifold M'' intersects the image of D at least twice. The proof is now as in the proof of Claim 2. Now suppose that $\sharp\{l \mid j(l) > i\} \leq 1$. Then, after contracting all (-1)-curves in $M' \setminus (A \cup B')$ and then iteratively contracting the images of the curves C'_l, $l \neq i$, denote by M''' the resulting symplectic 4-manifold, by D'' the image of D, and by C'''_i the image of C'_i (see Figure 12 for the case $s = 2$). Note that this situation does not occur in $\mathbb{C}P^1 \times \mathbb{C}P^1$ since C'''_i is a smoothly embedded rational curve which is disjoint from A and hence must be homologous to A but $C'''_i \cdot D'' \geq 3$. Since blowing down (-1)-curves away from $A \cup B'$ can only increase the intersection number $C'''_i \cdot D''$, it follows that M''' can also not be a blowup of $\mathbb{C}P^1 \times \mathbb{C}P^1$, which is absurd. ∎

We prove the proposition. After contracting all (-1)-curves in $M' \setminus (A \cup B')$ it follows from Claim 3 that the image of the string $C'_1, \ldots, C'_k$ intersects the image of D exactly twice. Namely, C''_1 and C''_j for some $j \geq i$ intersect D'. One can now, using Lemma 4.9, iteratively blow down the curves C''_l, $l \neq i$ to obtain the configuration given in Figure 10. This shows that the configuration $\{A, B', D', \{C''_l\}\}$ is an admissible configuration. ∎

§5. Conclusion

In Section 4, we reduced the compactification to a standard configuration of rational curves in either $\mathbb{C}P^2$ or $\mathbb{C}P^1 \times \mathbb{C}P^1$ as follows.

(1) Cyclic quotient singularities: two distinct lines in $\mathbb{C}P^2$.

(2) Dihedral singularities: a union of a 0-curve and a cuspidal curve of bidegree $(2,2)$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$ which intersect at the cusp point, or the proper transform of a union of a line and a cuspidal cubic curve in $\mathbb{C}P^2$ which meet at the cusp point under the blowup of their transversal intersection point (that is, another intersection point of them).
(3) Tetrahedral, octahedral, and icosahedral singularities of type $(3,2)$: a cuspidal curve of degree 3 (resp., of bidegree $(2,2)$) in $\mathbb{C}P^2$ (resp., $\mathbb{C}P^1 \times \mathbb{C}P^1$).
(4) Tetrahedral, octahedral, and icosahedral singularities of type $(3,1)$. Two kinds of configurations appear:
   (i) a union of $\mathbb{C}P^1 \times \{\mathrm{pt}\}$, $\{\mathrm{pt}\} \times \mathbb{C}P^1$ and a cuspidal curve of bidegree $(2,2)$, which meet at the cusp point;
   (ii) a union of $\mathbb{C}P^1 \times \{\mathrm{pt}\}$, $\{\mathrm{pt}\} \times \mathbb{C}P^1$, a cuspidal curve of bidegree $(2,2)$, which meet at the cusp point, and another rational curve homologous to $\mathbb{C}P^1 \times \{\mathrm{pt}\}$.

To recover the symplectic filling X, we first sequentially blow up the manifold at points on the total transform of the divisor in the above list and blow down, if necessary, to get a closed symplectic 4-manifold Z containing the compactifying divisor K. Then we get X as the complement of a regular neighborhood of K in Z. To see that X is indeed a symplectic filling of the relevant quotient singularity link, we argue as follows.

First, observe that K contains a "central divisor" (often of self-intersection number $b-3$, but in the cyclic case it has self-intersection number 1, and in the dihedral case -1). Collapse, repeatedly, all (-1)-curves away from the central divisor in the cyclic quotient and dihedral case. Then the compactifying divisor will consist of a central divisor and one or three strings of rational curves of negative self-intersection, which are exceptional divisors of some cyclic quotient singularities. Blow down these strings to obtain orbifold singularities. Here we take the weighted blowdown in the orbifold category (according to Godinho [5]) to get symplectic cyclic orbifold singularities. Then the central divisor becomes an orbifold curve of positive self-intersection number (which may be a rational number). So the tubular neighborhood is a disk bundle in an orbifold complex line bundle with positive Chern number. Consequently, it has concave boundary. To prove the latter, one can proceed as follows. Since we have the notion of the differential forms and de Rham theory in the orbifold setting, the theory of connections and curvature for orbibundles is developed in the same way as for bundles on manifolds. Hence, McDuff's [16] argument can be applied to show that the boundary of a (sufficiently small) disk bundle in the orbifold complex line bundle with positive Chern number is concave.

For classification up to symplectic deformation equivalence, we need uniqueness of symplectic deformation types of the standard configurations, which we can prove as in [20].

It is not difficult to see that for cyclic quotient singularities and dihedral singularities, we can find links with arbitrarily many nondiffeomorphic symplectic fillings. (For cyclic quotient singularities, this fact was noted by Lisca [11].) However, for tetrahedral, octahedral, and icosahedral singularities, the number of symplectic fillings for each class of singularities in Tables 1–3 is bounded above by a number independent of b. We give a list of all symplectic fillings in these cases. To aid this, for case (4ii) above, we note the following constraints:

(a) if $i > 1$, then $c_i \neq 2$,
(b) if $j < k$, then $c_j \neq 2$,
(c) $b \leq \max\{5, c_{b-2}\}$,

where i is as in Figure 8 and j is as in the proof of Proposition 4.10. In particular, since $c_{b-2} \leq 6$ for quotient singularities, there are only a finite number of symplectic filling which fall into case (4ii) above.

The list we give below is the list of compactifications Z and compactifying divisors K of minimal symplectic fillings. There may be symplectically deformation equivalent fillings in the list. To get a list of minimal symplectic fillings, we should describe the contactomorphisms up to contact isotopies. We leave this as a topic for future research.

Symplectic fillings of links of tetrahedral, octahedral, and icosahedral singularities of type $(3, 2)$

We use the notation $(m; D \cdot D, -c_1, \ldots, -c_k; a_1 \times i_1, \ldots, a_l \times i_l)$ to denote the symplectic filling of the link of T_m, O_m, or I_m given as the complement of a regular neighborhood of the compactifying divisor $K = D \cup C_1 \cup \cdots \cup C_k$ given in Figure 4. Here $-c_1, \ldots, -c_k$ denote the self-intersections of the curves $C_1, \ldots, C_k$, and $a_j \times i_j$ denotes the existence of a_j distinct (-1)-curves intersecting C_{i_j} in Z; if there are no such (-1)-curves, then we put $\emptyset$. We abbreviate $1 \times i_j = i_j$. In each case we indicate whether the pair (Z, K) is given by blowing up ($\mathbb{C}P^2$, cuspidal cubic curve of degree 3) or ($\mathbb{C}P^1 \times \mathbb{C}P^1$, cuspidal curve of bidegree $(2, 2)$). Note that when we blow down the compactification Z of a symplectic filling of the link of a singularity of type $(3, 2)$, we can always guarantee that we end up with $\mathbb{C}P^2$ unless the image of D under the blowing-down map has self-intersection number 8. In that case we may also end up with $\mathbb{C}P^1 \times \mathbb{C}P^1$. There are 16 cases where this occurs.

Tetrahedral, T_m

1. E_6
2. $(6(3-2)+1; 5, -4; 3\times 1), \mathbb{C}P^2$
3. $(6(4-2)+1; 5, -2, -4; 3\times 2), \mathbb{C}P^2$
4. $(6(4-2)+1; 5, -2, -4; 1, 2\times 2), \mathbb{C}P^2$
5. $(6(5-2)+1; 5, -2, -2, -4; 3\times 3), \mathbb{C}P^2$
6. $(6(5-2)+1; 5, -2, -2, -4; 1, 2\times 3), \mathbb{C}P^2$
7. $(6(5-2)+1; 5, -2, -2, -4; 1, 2\times 3), \mathbb{C}P^1\times\mathbb{C}P^1$
8. $(6(5-2)+1; 5, -2, -2, -4; 2, 3), \mathbb{C}P^2$
9. $(6(6-2)+1; 5, -2, -2, -2, -4; 3\times 4), \mathbb{C}P^2$
10. $(6(6-2)+1; 5, -2, -2, -2, -4; 1, 2\times 4), \mathbb{C}P^2$
11. $(6(6-2)+1; 5, -2, -2, -2, -4; 3), \mathbb{C}P^2$
12. $(6(b-2)+1, b\geq 7; 5, \underbrace{-2, \dots, -2}_{b-3}, -4; 3\times k), k=b-2, \mathbb{C}P^2$
13. $(6(2-2)+3; 4, -2; 1), \mathbb{C}P^2$
14. $(6(3-2)+3; 5, -3, -2; 1, 2), \mathbb{C}P^2$
15. $(6(3-2)+3; 5, -3, -2; 2\times 1), \mathbb{C}P^2$
16. $(6(4-2)+3; 5, -2, -3, -2; 2, 3), \mathbb{C}P^2$
17. $(6(4-2)+3; 5, -2, -3, -2; 1, 3), \mathbb{C}P^2$
18. $(6(4-2)+3; 5, -2, -3, -2; 1, 2), \mathbb{C}P^2$
19. $(6(4-2)+3; 5, -2, -3, -2; 1, 2), \mathbb{C}P^1\times\mathbb{C}P^1$
20. $(6(5-2)+3; 5, -2, -2, -3, -2; 3, 4), \mathbb{C}P^2$
21. $(6(5-2)+3; 5, -2, -2, -3, -2; 1, 4), \mathbb{C}P^2$
22. $(6(5-2)+3; 5, -2, -2, -3, -2; 1, 4), \mathbb{C}P^1\times\mathbb{C}P^1$
23. $(6(5-2)+3; 5, -2, -2, -3, -2; 1, 3), \mathbb{C}P^2$
24. $(6(5-2)+3; 5, -2, -2, -3, -2; 2), \mathbb{C}P^2$
25. $(6(5-2)+3; 5, -2, -2, -3, -2; 2), \mathbb{C}P^1\times\mathbb{C}P^1$
26. $(6(6-2)+3; 5, -2, -2, -2, -3, -2; 4, 5), \mathbb{C}P^2$
27. $(6(6-2)+3; 5, -2, -2, -2, -3, -2; 1, 5), \mathbb{C}P^2$
28. $(6(b-2)+3, b\geq 7; 5, \underbrace{-2, \dots, -2}_{b-3}, -3, -2; k-1, k), k=b-1, \mathbb{C}P^2$

Octahedral, O_m

29. E_7
30. $(12(3-2)+1; 5, -5; 4 \times 1), \mathbb{C}P^2$
31. $(12(4-2)+1; 5, -2, -5; 4 \times 2), \mathbb{C}P^2$
32. $(12(4-2)+1; 5, -2, -5; 1, 3 \times 2), \mathbb{C}P^2$
33. $(12(5-2)+1; 5, -2, -2, -5; 4 \times 3), \mathbb{C}P^2$
34. $(12(5-2)+1; 5, -2, -2, -5; 1, 3 \times 3), \mathbb{C}P^2$
35. $(12(5-2)+1; 5, -2, -2, -5; 1, 3 \times 3), \mathbb{C}P^1 \times \mathbb{C}P^1$
36. $(12(5-2)+1; 5, -2, -2, -5; 2, 2 \times 3), \mathbb{C}P^2$
37. $(12(6-2)+1; 5, -2, -2, -2, -5; 4 \times 4), \mathbb{C}P^2$
38. $(12(6-2)+1; 5, -2, -2, -2, -5; 1, 3 \times 4), \mathbb{C}P^2$
39. $(12(6-2)+1; 5, -2, -2, -2, -5; 3, 4), \mathbb{C}P^2$
40. $(12(7-2)+1; 5, -2, -2, -2, -2, -5; 4 \times 5), \mathbb{C}P^2$
41. $(12(7-2)+1; 5, -2, -2, -2, -2, -5; 4), \mathbb{C}P^2$
42. $(12(b-2)+1, b \geq 8; 5, \underbrace{-2, \ldots, -2}_{b-3}, -5; 4 \times k), k = b-2, \mathbb{C}P^2$
43. $(12(2-2)+7; 4, -2, -2; 2), \mathbb{C}P^2$
44. $(12(2-2)+7; 4, -2, -2; 1), \mathbb{C}P^2$
45. $(12(3-2)+7; 5, -3, -2, -2; 1, 3), \mathbb{C}P^2$
46. $(12(3-2)+7; 5, -3, -2, -2; 2 \times 1), \mathbb{C}P^2$
47. $(12(3-2)+7; 5, -3, -2, -2; 2 \times 1), \mathbb{C}P^1 \times \mathbb{C}P^1$
48. $(12(3-2)+7; 5, -3, -2, -2; 2), \mathbb{C}P^2$
49. $(12(4-2)+7; 5, -2, -3, -2, -2; 2, 4), \mathbb{C}P^2$
50. $(12(4-2)+7; 5, -2, -3, -2, -2; 1, 2), \mathbb{C}P^2$
51. $(12(4-2)+7; 5, -2, -3, -2, -2; 1, 4), \mathbb{C}P^2$
52. $(12(4-2)+7; 5, -2, -3, -2, -2; 3), \mathbb{C}P^2$
53. $(12(5-2)+7; 5, -2, -2, -3, -2, -2; 3, 5), \mathbb{C}P^2$
54. $(12(5-2)+7; 5, -2, -2, -3, -2, -2; 1, 5), \mathbb{C}P^2$
55. $(12(5-2)+7; 5, -2, -2, -3, -2, -2; 1, 5), \mathbb{C}P^1 \times \mathbb{C}P^1$
56. $(12(5-2)+7; 5, -2, -2, -3, -2, -2; 4), \mathbb{C}P^2$
57. $(12(5-2)+7; 5, -2, -2, -3, -2, -2; 2), \mathbb{C}P^2$
58. $(12(6-2)+7; 5, -2, -2, -2, -3, -2, -2; 4, 6), \mathbb{C}P^2$
59. $(12(6-2)+7; 5, -2, -2, -2, -3, -2, -2; 1, 6), \mathbb{C}P^2$
60. $(12(6-2)+7; 5, -2, -2, -2, -3, -2, -2; 5), \mathbb{C}P^2$
61. $(12(b-2)+7, b \geq 7; 5, \underbrace{-2, \ldots, -2}_{b-3}, -3, -2, -2; k-2, k), k = b, \mathbb{C}P^2$

62. $(12(b-2)+7, b \geq 7; 5, \underbrace{-2, \ldots, -2}_{b-3}, -3, -2, -2; k-1), k = b, \mathbb{C}P^2$

Icosahedral, I_m

63. E_8
64. $(30(3-2)+1; 5, -6; 5 \times 1), \mathbb{C}P^2$
65. $(30(4-2)+1; 5, -2, -6; 5 \times 2), \mathbb{C}P^2$
66. $(30(4-2)+1; 5, -2, -6; 1, 4 \times 2), \mathbb{C}P^2$
67. $(30(5-2)+1; 5, -2, -2, -6; 5 \times 3), \mathbb{C}P^2$
68. $(30(5-2)+1; 5, -2, -2, -6; 1, 4 \times 3), \mathbb{C}P^2$
69. $(30(5-2)+1; 5, -2, -2, -6; 1, 4 \times 3), \mathbb{C}P^1 \times \mathbb{C}P^1$
70. $(30(5-2)+1; 5, -2, -2, -6; 2, 3 \times 3), \mathbb{C}P^2$
71. $(30(6-2)+1; 5, -2, -2, -2, -6; 5 \times 4), \mathbb{C}P^2$
72. $(30(6-2)+1; 5, -2, -2, -2, -6; 1, 4 \times 4), \mathbb{C}P^2$
73. $(30(6-2)+1; 5, -2, -2, -2, -6; 3, 2 \times 4), \mathbb{C}P^2$
74. $(30(7-2)+1; 5, -2, -2, -2, -2, -6; 5 \times 5), \mathbb{C}P^2$
75. $(30(7-2)+1; 5, -2, -2, -2, -2, -6; 4, 5), \mathbb{C}P^2$
76. $(30(8-2)+1; 5, -2, -2, -2, -2, -2, -6; 5 \times 6), \mathbb{C}P^2$
77. $(30(8-2)+1; 5, -2, -2, -2, -2, -2, -6; 5), \mathbb{C}P^2$
78. $(30(b-2)+1, b \geq 9; 5, \underbrace{-2, \ldots, -2}_{b-3}, -6; 5 \times k), k = b-2, \mathbb{C}P^2$
79. $(30(2-2)+7; 3, -2; 1), \mathbb{C}P^2$
80. $(30(3-2)+7; 5, -4, -2; 2 \times 1, 2), \mathbb{C}P^2$
81. $(30(3-2)+7; 5, -4, -2; 3 \times 1), \mathbb{C}P^2$
82. $(30(4-2)+7; 5, -2, -4, -2; 2 \times 2, 3), \mathbb{C}P^2$
83. $(30(4-2)+7; 5, -2, -4, -2; 1, 2 \times 2), \mathbb{C}P^2$
84. $(30(4-2)+7; 5, -2, -4, -2; 1, 2 \times 2), \mathbb{C}P^1 \times \mathbb{C}P^1$
85. $(30(4-2)+7; 5, -2, -4, -2; 1, 2, 3), \mathbb{C}P^2$
86. $(30(5-2)+7; 5, -2, -2, -4, -2; 2 \times 3, 4), \mathbb{C}P^2$
87. $(30(5-2)+7; 5, -2, -2, -4, -2; 1, 2 \times 3), \mathbb{C}P^2$
88. $(30(5-2)+7; 5, -2, -2, -4, -2; 2, 3), \mathbb{C}P^2$
89. $(30(5-2)+7; 5, -2, -2, -4, -2; 2, 3), \mathbb{C}P^1 \times \mathbb{C}P^1$
90. $(30(5-2)+7; 5, -2, -2, -4, -2; 2, 4), \mathbb{C}P^2$
91. $(30(5-2)+7; 5, -2, -2, -4, -2; 1, 3, 4), \mathbb{C}P^2$
92. $(30(5-2)+7; 5, -2, -2, -4, -2; 1, 3, 4), \mathbb{C}P^1 \times \mathbb{C}P^1$
93. $(30(6-2)+7; 5, -2, -2, -2, -4, -2; 2 \times 4, 5), \mathbb{C}P^2$
94. $(30(6-2)+7; 5, -2, -2, -2, -4, -2; 1, 4, 5), \mathbb{C}P^2$
95. $(30(6-2)+7; 5, -2, -2, -2, -4, -2; 3), \mathbb{C}P^2$

96. $(30(6-2)+7;5,-2,-2,-2,-4,-2;3),\mathbb{C}P^1\times\mathbb{C}P^1$
97. $(30(b-2)+7,b\geq 7;5,\underbrace{-2,\ldots,-2}_{b-3},-4,-2;2\times(k-1),k),k=b-1,\mathbb{C}P^2$
98. $(30(2-2)+13;4,-3;2\times 1),\mathbb{C}P^2$
99. $(30(3-2)+13;5,-3,-3;1,2\times 2),\mathbb{C}P^2$
100. $(30(3-2)+13;5,-3,-3;2\times 1,2),\mathbb{C}P^2$
101. $(30(4-2)+13;5,-2,-3,-3;2,2\times 3),\mathbb{C}P^2$
102. $(30(4-2)+13;5,-2,-3,-3;1,2,3),\mathbb{C}P^2$
103. $(30(4-2)+13;5,-2,-3,-3;1,2,3),\mathbb{C}P^1\times\mathbb{C}P^1$
104. $(30(4-2)+13;5,-2,-3,-3;1,2\times 3),\mathbb{C}P^2$
105. $(30(4-2)+13;5,-2,-3,-3;2\times 2),\mathbb{C}P^2$
106. $(30(5-2)+13;5,-2,-2,-3,-3;3,2\times 4),\mathbb{C}P^2$
107. $(30(5-2)+13;5,-2,-2,-3,-3;1,2\times 4),\mathbb{C}P^2$
108. $(30(5-2)+13;5,-2,-2,-3,-3;1,2\times 4),\mathbb{C}P^1\times\mathbb{C}P^1$
109. $(30(5-2)+13;5,-2,-2,-3,-3;1,3,4),\mathbb{C}P^2$
110. $(30(5-2)+13;5,-2,-2,-3,-3;2,4),\mathbb{C}P^2$
111. $(30(5-2)+13;5,-2,-2,-3,-3;2,4),\mathbb{C}P^1\times\mathbb{C}P^1$
112. $(30(6-2)+13;5,-2,-2,-2,-3,-3;4,2\times 5),\mathbb{C}P^2$
113. $(30(6-2)+13;5,-2,-2,-2,-3,-3;1,2\times 5),\mathbb{C}P^2$
114. $(30(b-2)+13,b\geq 7;5,\underbrace{-2,\ldots,-2}_{b-3},-3,-3;k-1,2\times k),k=b-1,\mathbb{C}P^2$
115. $(30(2-2)+19;4,-2,-2,-2;3),\mathbb{C}P^2$
116. $(30(2-2)+19;4,-2,-2,-2;1),\mathbb{C}P^2$
117. $(30(3-2)+19;5,-3,-2,-2,-2;1,4),\mathbb{C}P^2$
118. $(30(3-2)+19;5,-3,-2,-2,-2;2\times 1),\mathbb{C}P^2$
119. $(30(4-2)+19;5,-2,-3,-2,-2,-2;2,5),\mathbb{C}P^2$
120. $(30(4-2)+19;5,-2,-3,-2,-2,-2;1,5),\mathbb{C}P^2$
121. $(30(5-2)+19;5,-2,-2,-3,-2,-2,-2;3,6),\mathbb{C}P^2$
122. $(30(5-2)+19;5,-2,-2,-3,-2,-2,-2;1,6),\mathbb{C}P^2$
123. $(30(5-2)+19;5,-2,-2,-3,-2,-2,-2;1,6),\mathbb{C}P^1\times\mathbb{C}P^1$
124. $(30(6-2)+19;5,-2,-2,-2,-3,-2,-2,-2;4,7),\mathbb{C}P^2$
125. $(30(6-2)+19;5,-2,-2,-2,-3,-2,-2,-2;1,7),\mathbb{C}P^2$
126. $(30(b-2)+19,b\geq 7;5,\underbrace{-2,\ldots,-2}_{b-3},-3,-2,-2,-2;k-3,k),k=b+1,\mathbb{C}P^2$

Symplectic fillings of links of tetrahedral, octahedral, and icosahedral singularities of type $(3,1)$

Case I. Here refer we to the final picture in Figure 5. We use the notation $(m; D\cdot D, -c_1, \dots, -c_k; a_1 \times i_1, \dots, a_l \times i_l)$ to denote the Case I symplectic filling of the link of T_m, O_m, or I_m given as the complement of a regular neighborhood of the compactifying divisor $K = A \cup B \cup D \cup C_1 \cup \cdots \cup C_k$ in Z. The notation is as for symplectic fillings of links of singularities of type $(3,2)$.

Tetrahedral, T_m

127. $(6(2-2)+5; 4, -2; 1)$
128. $(6(3-2)+5; 5, -3, -2; 1, 2)$
129. $(6(3-2)+5; 5, -3, -2; 2\times 1)$
130. $(6(4-2)+5; 5, -2, -3, -2; 2, 3)$
131. $(6(4-2)+5; 5, -2, -3, -2; 1, 3)$
132. $(6(4-2)+5; 5, -2, -3, -2; 1, 2)$
133. $(6(5-2)+5; 5, -2, -2, -3, -2; 3, 4)$
134. $(6(5-2)+5; 5, -2, -2, -3, -2; 1, 4)$
135. $(6(5-2)+5; 5, -2, -2, -3, -2; 2)$
136. $(6(b-2)+5, b\geq 6; 5, \underbrace{-2, \dots, -2}_{b-3}, -3, -2; k-1, k), k = b-1$

Octahedral, O_m

137. $(12(2-2)+5; 2; \emptyset)$
138. $(12(3-2)+5; 5, -5; 4\times 1)$
139. $(12(4-2)+5; 5, -2, -5; 4\times 2)$
140. $(12(4-2)+5; 5, -2, -5; 1, 3\times 2)$
141. $(12(5-2)+5; 5, -2, -2, -5; 4\times 3)$
142. $(12(5-2)+5; 5, -2, -2, -5; 1, 3\times 3)$
143. $(12(5-2)+5; 5, -2, -2, -5; 2, 2\times 3)$
144. $(12(6-2)+5; 5, -2, -2, -2, -5; 4\times 4)$
145. $(12(6-2)+5; 5, -2, -2, -2, -5; 3, 4)$
146. $(12(7-2)+5; 5, -2, -2, -2, -2, -5; 4\times 5)$
147. $(12(7-2)+5; 5, -2, -2, -2, -2, -5; 4)$
148. $(12(b-2)+5, b\geq 8; 5, \underbrace{-2, \dots, -2}_{b-3}, -5; 4\times k), k = b-2$
149. $(12(2-2)+11; 4, -2, -2; 2)$
150. $(12(2-2)+11; 4, -2, -2; 1)$
151. $(12(3-2)+11; 5, -3, -2, -2; 1, 3)$

152. $(12(3-2)+11; 5, -3, -2, -2; 2 \times 1)$
153. $(12(3-2)+11; 5, -3, -2, -2; 2)$
154. $(12(4-2)+11; 5, -2, -3, -2, -2; 2, 4)$
155. $(12(4-2)+11; 5, -2, -3, -2, -2; 1, 4)$
156. $(12(4-2)+11; 5, -2, -3, -2, -2; 3)$
157. $(12(5-2)+11; 5, -2, -2, -3, -2, -2; 3, 5)$
158. $(12(5-2)+11; 5, -2, -2, -3, -2, -2; 1, 5)$
159. $(12(5-2)+11; 5, -2, -2, -3, -2, -2; 4)$
160. $(12(b-2)+11, b \geq 6; 5, \underbrace{-2, \ldots, -2}_{b-3}, -3, -2, -2; k-2, k), k = b$
161. $(12(b-2)+11, b \geq 6; 5, \underbrace{-2, \ldots, -2}_{b-3}, -3, -2, -2; k-1), k = b$

Icosahedral, I_m

162. $(30(2-2)+11; 1; \emptyset)$
163. $(30(3-2)+11; 5, -6; 5 \times 1)$
164. $(30(4-2)+11; 5, -2, -6; 5 \times 2)$
165. $(30(4-2)+11; 5, -2, -6; 1, 4 \times 2)$
166. $(30(5-2)+11; 5, -2, -2, -6; 5 \times 3)$
167. $(30(5-2)+11; 5, -2, -2, -6; 1, 4 \times 3)$
168. $(30(5-2)+11; 5, -2, -2, -6; 2, 3 \times 3)$
169. $(30(6-2)+11; 5, -2, -2, -2, -6; 5 \times 4)$
170. $(30(6-2)+11; 5, -2, -2, -2, -6; 3, 2 \times 4)$
171. $(30(7-2)+11; 5, -2, -2, -2, -2, -6; 5 \times 5)$
172. $(30(7-2)+11; 5, -2, -2, -2, -2, -6; 4, 5)$
173. $(30(8-2)+11; 5, -2, -2, -2, -2, -2, -6; 5 \times 6)$
174. $(30(8-2)+11; 5, -2, -2, -2, -2, -2, -6; 5)$
175. $(30(b-2)+11, b \geq 9; 5, \underbrace{-2, \ldots, -2}_{b-3}, -6; 5 \times k), k = b-2$
176. $(30(2-2)+17; 3, -2; 1)$
177. $(30(3-2)+17; 5, -4, -2; 2 \times 1, 2)$
178. $(30(3-2)+17; 5, -4, -2; 3 \times 1)$
179. $(30(4-2)+17; 5, -2, -4, -2; 2 \times 2, 3)$
180. $(30(4-2)+17; 5, -2, -4, -2; 1, 2 \times 2)$
181. $(30(4-2)+17; 5, -2, -4, -2; 1, 2, 3)$
182. $(30(5-2)+17; 5, -2, -2, -4, -2; 2 \times 3, 4)$
183. $(30(5-2)+17; 5, -2, -2, -4, -2; 2, 3)$
184. $(30(5-2)+17; 5, -2, -2, -4, -2; 2, 4)$

185. $(30(5-2)+17; 5, -2, -2, -4, -2; 1, 3, 4)$
186. $(30(6-2)+17; 5, -2, -2, -2, -4, -2; 2 \times 4, 5)$
187. $(30(6-2)+17; 5, -2, -2, -2, -4, -2; 3)$
188. $(30(b-2)+17, b \geq 7; 5, \underbrace{-2, \ldots, -2}_{b-3}, -4, -2; 2 \times (k-1), k), k = b-1$
189. $(30(2-2)+23; 4, -3; 2 \times 1)$
190. $(30(3-2)+23; 5, -3, -3; 1, 2 \times 2)$
191. $(30(3-2)+23; 5, -3, -3; 2 \times 1, 2)$
192. $(30(4-2)+23; 5, -2, -3, -3; 2, 2 \times 3)$
193. $(30(4-2)+23; 5, -2, -3, -3; 1, 2, 3)$
194. $(30(4-2)+23; 5, -2, -3, -3; 1, 2 \times 3)$
195. $(30(4-2)+23; 5, -2, -3, -3; 2 \times 2)$
196. $(30(5-2)+23; 5, -2, -2, -3, -3; 3, 2 \times 4)$
197. $(30(5-2)+23; 5, -2, -2, -3, -3; 1, 2 \times 4)$
198. $(30(5-2)+23; 5, -2, -2, -3, -3; 2, 4)$
199. $(30(b-2)+23, b \geq 6; 5, \underbrace{-2, \ldots, -2}_{b-3}, -3, -3; k-1, 2 \times k), k = b-1$
200. $(30(2-2)+29; 4, -2, -2, -2; 3)$
201. $(30(2-2)+29; 4, -2, -2, -2; 1)$
202. $(30(3-2)+29; 5, -3, -2, -2, -2; 1, 4)$
203. $(30(4-2)+29; 5, -2, -3, -2, -2, -2; 2, 5)$
204. $(30(4-2)+29; 5, -2, -3, -2, -2, -2; 1, 5)$
205. $(30(5-2)+29; 5, -2, -2, -3, -2, -2, -2; 3, 6)$
206. $(30(5-2)+29; 5, -2, -2, -3, -2, -2, -2; 1, 6)$
207. $(30(b-2)+29, b \geq 7; 5, \underbrace{-2, \ldots, -2}_{b-3}, -3, -2, -2, -2; k-3, k), k = b+1$

Case II. Again we refer to the final picture in Figure 5. We use the notation $(m; D \cdot D, -c_1, \ldots, -c_k; i, j; a_1 \times i_1, \ldots, a_l \times i_l)$ to denote the Case II symplectic filling of the link of T_m, O_m, or I_m given as the complement of a regular neighborhood of the compactifying divisor $K = A \cup B \cup D \cup C_1 \cup \cdots \cup C_k$ in Z. Here the numbers i and j denote the existence of (-1)-curves intersecting B and C_i and D and C_j, respectively.

Tetrahedral, T_m

208. $(6(2-2)+5; 4, -2; 1, 1; \emptyset)$
209. $(6(3-2)+5; 5, -3, -2; 1, 1; 2)$
210. $(6(3-2)+5; 5, -3, -2; 1, 2; 1)$
211. $(6(4-2)+5; 5, -2, -3, -2; 1, 2; 3)$

212. $(6(4-2)+5;5,-2,-3,-2;1,3;2)$
213. $(6(4-2)+5;5,-2,-3,-2;2,3;1)$
214. $(6(5-2)+5;5,-2,-2,-3,-2;1,3;4)$

Octahedral, O_m

215. $(12(3-2)+5;5,-5;1,1;3\times 1)$
216. $(12(4-2)+5;5,-2,-5;1,2;3\times 2)$
217. $(12(4-2)+5;5,-2,-5;2,2;1,2\times 2)$
218. $(12(5-2)+5;5,-2,-2,-5;1,3;3\times 3)$
219. $(12(5-2)+5;5,-2,-2,-5;3,3;1,2\times 3)$
220. $(12(5-2)+5;5,-2,-2,-5;3,3;2,3)$
221. $(12(6-2)+5;5,-2,-2,-2,-5;4,4;3)$
222. $(12(2-2)+11;4,-2,-2;1,2)$
223. $(12(3-2)+11;5,-3,-2,-2;1,1;3)$
224. $(12(3-2)+11;5,-3,-2,-2;1,3;1)$
225. $(12(4-2)+11;5,-2,-3,-2,-2;1,2;4)$
226. $(12(5-2)+11;5,-2,-2,-3,-2,-2;1,3;5)$

Icosahedral, I_m

227. $(30(3-2)+11;5,-6;1,1;4\times 1)$
228. $(30(4-2)+11;5,-2,-6;1,2;4\times 2)$
229. $(30(4-2)+11;5,-2,-6;2,2;1,3\times 2)$
230. $(30(5-2)+11;5,-2,-2,-6;1,3;4\times 3)$
231. $(30(5-2)+11;5,-2,-2,-6;3,3;1,3\times 3)$
232. $(30(5-2)+11;5,-2,-2,-6;3,3;2,2\times 3)$
233. $(30(6-2)+11;5,-2,-2,-2,-6;4,4;3,4)$
234. $(30(7-2)+11;5,-2,-2,-2,-2,-6;5,5;4)$
235. $(30(2-2)+17;3,-2;1,1;\emptyset)$
236. $(30(3-2)+17;5,-4,-2;1,1;1,2)$
237. $(30(3-2)+17;5,-4,-2;1,2;2\times 1)$
238. $(30(4-2)+17;5,-2,-4,-2;1,2;2,3)$
239. $(30(4-2)+17;5,-2,-4,-2;1,3;2\times 2)$
240. $(30(4-2)+17;5,-2,-4,-2;2,2;1,3)$
241. $(30(4-2)+17;5,-2,-4,-2;2,3;1,2)$
242. $(30(5-2)+17;5,-2,-2,-4,-2;1,3;3,4)$
243. $(30(5-2)+17;5,-2,-2,-4,-2;3,3;1,4)$
244. $(30(5-2)+17;5,-2,-2,-4,-2;3,4;2)$
245. $(30(2-2)+23;4,-3;1,1;1)$
246. $(30(3-2)+23;5,-3,-3;1,1;2\times 2)$

247. $(30(3-2)+23;5,-3,-3;1,2;1,2)$
248. $(30(3-2)+23;5,-3,-3;2,2;2\times 1)$
249. $(30(4-2)+23;5,-2,-3,-3;1,2;2\times 3)$
250. $(30(4-2)+23;5,-2,-3,-3;1,3;2,3)$
251. $(30(4-2)+23;5,-2,-3,-3;2,3;1,3)$
252. $(30(4-2)+23;5,-2,-3,-3;3,3;1,2)$
253. $(30(5-2)+23;5,-2,-2,-3,-3;1,3;2\times 4)$
254. $(30(2-2)+29;4,-2,-2,-2;1,3)$
255. $(30(3-2)+29;5,-3,-2,-2,-2;1,1;4)$
256. $(30(4-2)+29;5,-2,-3,-2,-2,-2;1,2;5)$
257. $(30(5-2)+29;5,-2,-2,-3,-2,-2,-2;1,3;6)$

Acknowledgment. The main part of this work was carried out during the first author's stay at Hokkaido University, and he thanks its mathematics department for its hospitality.

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