

GEOMETRY OF G_2 ORBITS AND ISOPARAMETRIC HYPERSURFACES

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Abstract. We characterize the adjoint G_2 orbits in the Lie algebra \mathfrak{g} of G_2 as fibered spaces over S^6 with fibers given by the complex Cartan hypersurfaces. This combines the isoparametric hypersurfaces of case $(g, m) = (6, 2)$ with case $(3, 2)$. The fibrations on two singular orbits turn out to be diffeomorphic to the twistor fibrations of S^6 and $G_2/SO(4)$. From the symplectic point of view, we show that there exists a 2-parameter family of Lagrangian submanifolds on every orbit.

§1. Introduction

The exceptional compact Lie group G_2 plays an important role in various fields of geometry. Here we are concerned with the adjoint orbits of G_2 in S^{13} , where G_2 acts on its Lie algebra $\mathfrak{g} \cong \mathbb{R}^{14}$ as an isometry with respect to the bi-invariant metric. They are the unique isoparametric hypersurfaces with six principal curvatures of multiplicity 2 (see [M4]). Those with multiplicity 1 are obtained by the inverse image of the real Cartan hypersurfaces $C_{\mathbb{R}}^3$ in S^4 under the Hopf fibration $\pi : S^7 \rightarrow S^4$ (see [M1]). The purpose of this paper is to characterize the multiplicity 2 case in conjunction with the complex Cartan hypersurfaces $C_{\mathbb{C}}^6$ in S^7 (the dimension of a hypersurface is always given in real). The difference is, however, that there is no fibration between S^{13} and S^7 . On the other hand, since $\pi^{-1}(C_{\mathbb{R}}^3) \cong C_{\mathbb{R}}^3 \times S^3$, by interchanging the fiber and the base manifold we succeed in obtaining the following theorems.

THEOREM 1.1. *Let M be a principal G_2 orbit in S^{13} , and let M_{\pm} be the singular orbits. Then M is diffeomorphic to G_2/T^2 , and M_{\pm} are both diffeomorphic to $\mathbb{Q}^5 = G_2/U(2)$, the complex quadratic. Each orbit has a*

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Kähler structure with respect to the induced metric and, moreover, has a Kähler fibration:

(i) $M \rightarrow S^6$ with the fiber $C_{\mathbb{C}}^6 = SU(3)/T^2$, the complex Cartan hypersurface;

(ii) $M_+ \rightarrow S^6$ with the fiber $\mathbb{C}P^2 = SU(3)/S(U(2) \times U(1))$, a fibration that is diffeomorphic to the twistor fibration on S^6 ;

(iii) $M_- \rightarrow G_2/SO(4)$ with the fiber $\mathbb{C}P^1 = SU(2)/S(U(1) \times U(1))$, which fibration is diffeomorphic to the twistor fibration on the quaternionic Kähler manifold $G_2/SO(4)$.

Hence, M_+ is not congruent to M_- in S^{13} , but the fibrations are converted from one to the other through the fibration on the principal orbits.

THEOREM 1.2. *Let M and M_{\pm} be as in Theorem 1.1. Then at each point of M , there exists a 2-parameter family of Lagrangian submanifolds transferred from an $SO(4)$ suborbit $N^6 \cong C_{\mathbb{R}}^3 \times S^3$, which collapses into $N_{\pm}^5 \cong \mathbb{R}P^2 \times S^3$ on M_{\pm} . These are minimal Lagrangian submanifolds of M_{\pm} and of M_0 , where the latter is the minimal principal orbit.*

Theorem 1.1 is not a formal factorization of a homogeneous space but has a significant application, say, a reduction of analysis on M to that on the factored spaces (see [MO]).

Isoparametric hypersurfaces in a real space form \overline{M} are hypersurfaces with constant principal curvatures. They consist of a 1-parameter family of parallel hypersurfaces which sweeps out \overline{M} with focal submanifold(s) at the end. There are rich examples in $\overline{M} = S^n$, where the number of principal curvatures g takes values in $\{1, 2, 3, 5, 6\}$ (see [Mü]). Typical examples are given by homogeneous hypersurfaces which have been classified as the linear isotropy orbits of rank 2 symmetric spaces (see [HL]). Other than hyperspheres ($g = 1$) and the Clifford hypersurfaces ($g = 2$), those with $g = 3$ were found by Cartan and called the *Cartan hypersurfaces* $C_{\mathbb{F}}$ (see [C]). They are tubes over the standard embedding of $\mathbb{F}P^2$ in S^{3d+1} , where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and \mathbb{H} are Cayley numbers and $d = 1, 2, 4, 8$, respectively. The case $g = 4$ is exceptional, as there *exist* infinitely many nonhomogeneous isoparametric hypersurfaces (see [OT], [FKM]) where the classification problem (see [Y]) still remains open (see [CCJ], [I]).

When $g = 6$, the multiplicity of each principal curvature coincides, which takes values $m = 1, 2$ (see [A]). For $m = 1$, the hypersurfaces are homogeneous and given by the isotropy orbits of $G_2/SO(4)$ (see [DN], [M2]). Homogeneous hypersurfaces M^{12} with $(g, m) = (6, 2)$ are unique; that is,

the G_2 orbits (see [HL]). Dorfmeister and Neher [DN] conjectured that the isoparametric hypersurfaces with $(g, m) = (6, 2)$ are homogeneous (see [M4] for the affirmative answer).

The paper is organized as follows. In Section 2, we review some basic facts of isoparametric hypersurfaces, and in Section 3, we compute basic data of G_2 orbits in terms of the root and root vectors. Finally, we prove our theorem in a refined way in Section 4.

§2. Preliminaries

We refer readers to [Th] for a survey of isoparametric hypersurfaces. Here we review fundamental facts and the notation of [M1] and [M3]. Let M be an isoparametric hypersurface in the unit sphere S^{n+1} . Let ξ be a unit normal vector field. We denote the Riemannian connection on S^{n+1} by $\tilde{\nabla}$, and we denote the induced connection on M by ∇ . Let $\lambda_1 \geq \dots \geq \lambda_n$ be the principal curvatures of M , and let $D_\lambda(p)$ be the curvature distribution of $\lambda \in \{\lambda_1, \dots, \lambda_n\}$ with multiplicity m_λ . Then D_λ is completely integrable, and a leaf L_λ is an m_λ -dimensional sphere of S^{n+1} . Choose a local orthonormal frame e_1, \dots, e_n consisting of unit principal vectors corresponding to $\lambda_1, \dots, \lambda_n$. We express

$$(1) \quad \tilde{\nabla}_{e_\alpha} e_\beta = \Lambda_{\alpha\beta}^\sigma e_\sigma + \lambda_\alpha \delta_{\alpha\beta} \xi, \quad \Lambda_{\alpha\beta}^\gamma = -\Lambda_{\alpha\gamma}^\beta$$

where $1 \leq \alpha, \beta, \sigma \leq n$, using the Einstein convention. From the equation of Codazzi, we obtain for distinct $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$,

$$(2) \quad \Lambda_{\alpha\beta}^\gamma (\lambda_\beta - \lambda_\gamma) = \Lambda_{\gamma\alpha}^\beta (\lambda_\alpha - \lambda_\beta) = \Lambda_{\beta\gamma}^\alpha (\lambda_\gamma - \lambda_\alpha).$$

Because λ_α is constant on M , we can see that

$$(3) \quad \Lambda_{aa}^\gamma = 0 = \Lambda_{ab}^\gamma, \quad \text{if } \lambda_a = \lambda_b \neq \lambda_\gamma \text{ and } a \neq b.$$

Now, consider the case $(g, m) = (6, 2)$. As is well known, we can express

$$(4) \quad \lambda_i = \cot\left(\theta_1 + \frac{(i-1)\pi}{6}\right), \quad 0 < \theta_1 < \frac{\pi}{6}, 1 \leq i \leq 6.$$

Note that if we choose $\theta_1 = \pi/12 = -\theta_6$, we have a minimal case with

$$(5) \quad \lambda_1 = -\lambda_6 = 2 + \sqrt{3}, \quad \lambda_2 = -\lambda_5 = 1, \quad \lambda_3 = -\lambda_4 = 2 - \sqrt{3}.$$

Denote $D_i = D_{\lambda_i}$. We take a local frame field $e_1, e_{\bar{1}}, \dots, e_6, e_{\bar{6}}$, where $e_i, e_{\bar{i}}$ is an orthonormal frame of D_i . For convenience, we put $\lambda_{\bar{i}} = \lambda_i$, and \underline{i} always

stands for i or \bar{i} . Each leaf $L_i = L_i(p)$ of D_i is a 2-sphere, and M has a structure of an iterated S^2 bundle over S^2 . For $a = 6$ or 1 , define the focal map $f_a: M \rightarrow S^{13}$ by

$$f_a(p) = \cos \theta_a p + \sin \theta_a \xi_p,$$

which makes $L_a(p)$ collapse into a point $\bar{p} = f_a(p)$. Then we have

$$(6) \quad df_a(e_j) = \sin \theta_a (\lambda_a - \lambda_j) e_j \quad \text{and} \quad df_a(e_{\bar{j}}) = \sin \theta_a (\lambda_a - \lambda_j) e_{\bar{j}},$$

where the right-hand sides are considered as vectors in $T_{\bar{p}}S^{13}$ by a parallel translation of S^{13} . In the following, we always use such identification. The rank of f_a is constant, and we obtain the focal submanifold M_a of M :

$$M_a = \{ \cos \theta_a p + \sin \theta_a \xi_p \mid p \in M \}.$$

We denote $M_+ = M_6$ and $M_- = M_1$. It follows that $T_{\bar{p}}M_a = \bigoplus_{j \neq a} D_j(q)$ from (6) for any $q \in f_a^{-1}(\bar{p})$. An orthonormal basis of the normal space of M_a at \bar{p} is given by

$$\eta_q = -\sin \theta_a q + \cos \theta_a \xi_q, \quad \zeta_q = e_a(q) \quad \text{and} \quad \bar{\zeta}_q = e_{\bar{a}}(q),$$

for any $q \in L_a(p) = f_a^{-1}(\bar{p})$. By a standard argument, we obtain the following (see [M2], [M4]).

LEMMA 2.1. *When we identify $T_{\bar{p}}M_a$ with $\bigoplus_{j=1}^5 D_{a+j}(p)$, where the indices are modulo 6, the shape operators B_{η_p} , B_{ζ_p} , and $B_{\bar{\zeta}_p}$ at \bar{p} with respect to the basis $e_{a+1}, e_{\bar{a}+1}, \dots, e_{a+5}, e_{\bar{a}+5}$ at p are given, respectively, by the symmetric matrices*

$$B_{\eta_p} = \begin{pmatrix} \sqrt{3}I & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}}I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{3}}I & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3}I \end{pmatrix},$$

$$B_{\zeta_p} = \begin{pmatrix} 0 & B_{a+1a+2} & B_{a+1a+3} & B_{a+1a+4} & B_{a+1a+5} \\ B_{a+2a+1} & 0 & B_{a+2a+3} & B_{a+2a+4} & B_{a+2a+5} \\ B_{a+3a+1} & B_{a+3a+2} & 0 & B_{a+3a+4} & B_{a+3a+5} \\ B_{a+4a+1} & B_{a+4a+2} & B_{a+4a+3} & 0 & B_{a+4a+5} \\ B_{a+5a+1} & B_{a+5a+2} & B_{a+5a+3} & B_{a+5a+4} & 0 \end{pmatrix},$$

$$B_{\zeta_p} = \begin{pmatrix} 0 & \bar{B}_{a+1a+2} & \bar{B}_{a+1a+3} & \bar{B}_{a+1a+4} & \bar{B}_{a+1a+5} \\ \bar{B}_{a+2a+1} & 0 & \bar{B}_{a+2a+3} & \bar{B}_{a+2a+4} & \bar{B}_{a+2a+5} \\ \bar{B}_{a+3a+1} & \bar{B}_{a+3a+2} & 0 & \bar{B}_{a+3a+4} & \bar{B}_{a+3a+5} \\ \bar{B}_{a+4a+1} & \bar{B}_{a+4a+2} & \bar{B}_{a+4a+3} & 0 & \bar{B}_{a+4a+5} \\ \bar{B}_{a+5a+1} & \bar{B}_{a+5a+2} & \bar{B}_{a+5a+3} & \bar{B}_{a+5a+4} & 0 \end{pmatrix},$$

where I (resp., zero) is the 2×2 unit (resp., zero) matrix and

$$(7) \quad \begin{aligned} B_{jk} &= \frac{1}{\sin \theta_a(\lambda_j - \lambda_a)} \begin{pmatrix} \Lambda_{ja}^k & \bar{\Lambda}_{ja}^k \\ \Lambda_{ja}^k & \bar{\Lambda}_{ja}^k \end{pmatrix} = {}^t B_{kj}, \\ \bar{B}_{jk} &= \frac{1}{\sin \theta_a(\lambda_j - \lambda_a)} \begin{pmatrix} \Lambda_{j\bar{a}}^k & \bar{\Lambda}_{j\bar{a}}^k \\ \Lambda_{j\bar{a}}^k & \bar{\Lambda}_{j\bar{a}}^k \end{pmatrix} = {}^t \bar{B}_{kj}. \end{aligned}$$

In particular, we have $B_{\eta_p}(e_j) = \mu_j e_j$, where

$$(8) \quad \mu_j = \frac{1 + \lambda_j \lambda_a}{\lambda_a - \lambda_j} \in \left\{ \pm\sqrt{3}, \pm\frac{1}{\sqrt{3}}, 0 \right\}.$$

Since any unit normal can be expressed as η_q for some $q \in L_6(p)$, all the shape operators have the same eigenvalues with multiplicity 2.

§3. Geometric data of G_2 orbits

In this section, we investigate an adjoint G_2 orbit M in S^{13} , which is the same as an isotropy orbit of the symmetric space $G_2 \times G_2/G_2$. Here, G_2 is the automorphism group of the Cayley numbers \mathcal{C} . Let \mathcal{C} be generated by $\{e_0, e_1, \dots, e_7\}$ satisfying

$$\begin{cases} e_0 = 1, \\ e_i^2 = -1, & 1 \leq i \leq 7, \\ e_i e_j = -e_j e_i = e_k, \end{cases}$$

where (i, j, k) is a triple on some segment or a circle of Figure 1 put in the order shown by its arrows. The automorphism group G_2 of \mathcal{C} is a subgroup of $SO(7)$, where the metric on \mathcal{C} is given by

$$(x, y) = \Re(x\bar{y}) = \sum_{i=0}^7 x^i y^i, \quad \text{for } x = \sum_{i=0}^7 x^i e_i \text{ and } y = \sum_{i=0}^7 y^i e_i.$$

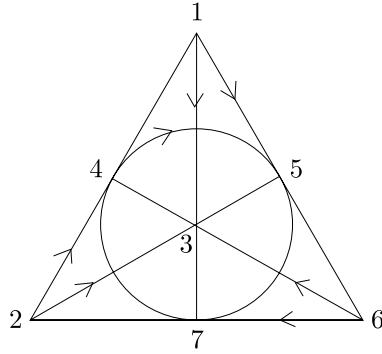


Figure 1

The Lie algebra \mathfrak{g} of G_2 is given as follows (see [OT]). Let E_{ij} be the standard basis of 7×7 matrices with \mathbb{R} -coefficients. Put $G_{ij} = E_{ij} - E_{ji}$, $i, j = 1, \dots, 7$, and put

$$\mathfrak{g}_i = \left\{ \eta_1 G_{i+1i+3} + \eta_2 G_{i+2i+6} + \eta_3 G_{i+4i+5} \mid \eta_j \in \mathbb{R}, \sum_{j=1}^3 \eta_j = 0 \right\},$$

for $1 \leq i \leq 7$. Then \mathfrak{g} is given by

$$(9) \quad \mathfrak{g} = \sum_{i=1}^7 \mathfrak{g}_i,$$

which satisfies $[\mathfrak{g}_i, \mathfrak{g}_i] = 0$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_k$, where (i, j, k) is as before. Note that $[G_{ij}, G_{jk}] = G_{ik}$ for any $1 \leq i, j, k \leq 7$. Note also that (9) is an orthogonal decomposition with respect to the metric $(,)$ on \mathfrak{g} given by

$$(X, Y) = -\frac{1}{2} \text{Tr } XY.$$

For later use, we decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where

$$\mathfrak{k} = \mathfrak{g}_3 + \mathfrak{g}_4 + \mathfrak{g}_6,$$

$$\mathfrak{p} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_5 + \mathfrak{g}_7.$$

Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of \mathfrak{g} , and let τ be the involutive automorphism of $\mathfrak{g}^{\mathbb{C}}$ given by $\tau(X) = \bar{X}$. Then $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$ is the Cartan decomposition. We identify $\sqrt{-1}\mathfrak{g}$ with \mathfrak{g} by $\sqrt{-1}X \mapsto X$. Take a maximal abelian

subspace $\mathfrak{a} = \mathfrak{g}_1 = \{\xi_1 G_{24} + \xi_2 G_{37} + \xi_3 G_{56} \mid \xi \in \mathbb{R}, \sum_{i=1}^3 \xi_i = 0\}$ of \mathfrak{g} , whose dimension, called the *rank of* $(\mathfrak{g}^{\mathbb{C}}, \tau)$, is 2. Let α be a linear form on \mathfrak{a} , and put

$$\begin{aligned}\mathfrak{k}_\alpha &= \{X \in \mathfrak{k} \mid (\text{ad}H)^2(X) = -\alpha(H)^2 X \quad \text{for all } H \in \mathfrak{a}\}, \\ \mathfrak{p}_\alpha &= \{X \in \mathfrak{p} \mid (\text{ad}H)^2(X) = -\alpha(H)^2 X \quad \text{for all } H \in \mathfrak{a}\}.\end{aligned}$$

Note that for $H \in \mathfrak{a}$ and a linear form α on \mathfrak{a} such that $\alpha(H) \neq 0$, $\text{ad}H$ maps \mathfrak{k}_α (resp., \mathfrak{p}_α) isomorphically onto \mathfrak{p}_α (resp., \mathfrak{k}_α) (see (12) below). Selecting a suitable ordering in the dual space of \mathfrak{a} , let Σ_+ be the set of positive roots of \mathfrak{g} with respect to \mathfrak{a} , and let $\Sigma_+^* = \{\alpha \in \Sigma_+, \frac{\alpha}{2} \notin \Sigma_+\}$. We have

$$(10) \quad \begin{aligned}\Sigma_+^* &= \{\alpha_1 = -\xi_2, \alpha_2 = \xi_1 - \xi_2, \alpha_3 = \xi_1, \\ &\quad \alpha_4 = \xi_1 - \xi_3, \alpha_5 = -\xi_3, \alpha_6 = \xi_2 - \xi_3\},\end{aligned}$$

and the root vectors $X_i \in \mathfrak{k}_{\alpha_i}$ and $T_i \in \mathfrak{p}_{\alpha_i}$ are given by

$$(11) \quad \begin{aligned}X_1 &= G_{46} + G_{52} - 2G_{71}, & X_4 &= G_{46} - G_{52} & \in \mathfrak{g}_3 \\ X_2 &= G_{72} - G_{34}, & X_5 &= G_{72} + G_{34} - 2G_{15} & \in \mathfrak{g}_6 \\ X_3 &= G_{57} + G_{63} - 2G_{12}, & X_6 &= G_{57} - G_{63} & \in \mathfrak{g}_4 \\ T_1 &= G_{26} + G_{45} - 2G_{13}, & T_4 &= G_{26} - G_{45} & \in \mathfrak{g}_7 \\ T_2 &= G_{23} + G_{47}, & T_5 &= G_{47} - G_{23} - 2G_{16} & \in \mathfrak{g}_5 \\ T_3 &= G_{35} + G_{67} + 2G_{14}, & T_6 &= -G_{35} + G_{67} & \in \mathfrak{g}_2.\end{aligned}$$

We have immediately

$$(12) \quad \text{ad}H(X_i) = \alpha_i(H)T_i, \quad \text{ad}H(T_i) = -\alpha_i(H)X_i.$$

Note that any two of the above vectors are mutually orthogonal.

Now, let $H = \xi_1 G_{24} + \xi_2 G_{37} + \xi_3 G_{56}$ be a regular element of \mathfrak{a} , and let $H^\perp = (\xi_3 - \xi_2)G_{24} + (\xi_1 - \xi_3)G_{37} + (-\xi_1 + \xi_2)G_{56}$ be an element of \mathfrak{a} orthogonal to H . For a hypersurface $M = \text{Ad } G_2(\tilde{H})$, where $\tilde{H} = H/\|H\|$, by using (12) and $\|H^\perp\| = \sqrt{3}\|H\|$ we can express the second fundamental tensor $A_{\tilde{H}^\perp}$ of M with respect to the unit normal vector $\tilde{H}^\perp = H^\perp/\|H^\perp\|$ at \tilde{H} by (see [TT])

$$A_{\tilde{H}^\perp} X_i = -\tilde{\nabla}_{X_i} \tilde{H}^\perp = -\frac{1}{\alpha_i(\tilde{H})} \frac{d}{dt} \Big|_{t=0} (\text{Ad exp } tT_i) \tilde{H}^\perp$$

$$\begin{aligned}
&= -\frac{1}{\alpha_i(\tilde{H})}[T_i, \tilde{H}^\perp] = -\frac{\alpha_i(\tilde{H}^\perp)}{\alpha_i(\tilde{H})}X_i = -\frac{\alpha_i(H^\perp)}{\sqrt{3}\alpha_i(H)}X_i, \\
A_{\tilde{H}^\perp}T_i &= -\tilde{\nabla}_{T_i}\tilde{H}^\perp = \frac{1}{\alpha_i(\tilde{H})}\frac{d}{dt}\Big|_{t=0}(\text{Ad exp } tX_i)\tilde{H}^\perp \\
&= \frac{1}{\alpha_i(\tilde{H})}[X_i, \tilde{H}^\perp] = -\frac{\alpha_i(\tilde{H}^\perp)}{\alpha_i(\tilde{H})}T_i = -\frac{\alpha_i(H^\perp)}{\sqrt{3}\alpha_i(H)}T_i.
\end{aligned}$$

Thus, the principal curvatures of M are given by

$$\begin{aligned}
(13) \quad \lambda_1 &= -\frac{\xi_1 - \xi_3}{\sqrt{3}\xi_2} = -\frac{1}{\lambda_4}, \\
\lambda_2 &= -\frac{\sqrt{3}\xi_3}{\xi_1 - \xi_2} = -\frac{1}{\lambda_5}, \\
\lambda_3 &= \frac{\xi_2 - \xi_3}{\sqrt{3}\xi_1} = -\frac{1}{\lambda_6},
\end{aligned}$$

and the unit principal vectors corresponding to λ_i are $X_i/\|X_i\|$ and $T_i/\|T_i\|$. Note that by $\lambda_1 > \cdots > \lambda_6$, (13) implies that $\xi_1 > 0 > \xi_2 > \xi_3$, and hence that

$$(14) \quad \alpha_i(H) > 0, \quad 1 \leq i \leq 6$$

follows from (10). Now, putting $e_i = X_i/\|X_i\|$, $e_{\bar{i}} = T_i/\|T_i\|$, we calculate the structure constants $\Lambda_{\alpha\beta}^\gamma$ with respect to this basis of M . As before, using (12), we obtain $X_i = (\|H\|/(\alpha_i(H)))\frac{d}{dt}\Big|_{t=0}\text{Ad}(\exp tT_i)\tilde{H}$. Here we have

$$(15) \quad \nabla_{X_i}X_j = \frac{\|H\|}{\alpha_i(H)}\frac{d}{dt}\Big|_{t=0}\text{Ad}(\exp tT_i)X_j = \frac{\|H\|}{\alpha_i(H)}[T_i, X_j].$$

Similarly, we have

$$(16) \quad \nabla_{X_i}T_j = \frac{\|H\|}{\alpha_i(H)}[T_i, T_j],$$

$$(17) \quad \nabla_{T_i}X_j = -\frac{\|H\|}{\alpha_i(H)}[X_i, X_j],$$

$$(18) \quad \nabla_{T_i}T_j = -\frac{\|H\|}{\alpha_i(H)}[X_i, T_j].$$

Then, noting that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we see that $\Lambda_{ij}^k = \Lambda_{i\bar{j}}^{\bar{k}} = \Lambda_{\bar{i}\bar{j}}^{\bar{k}} = \Lambda_{\bar{i}j}^k = 0$, $1 \leq i, j, k \leq 6$ (be careful for the indices with and without bars).

Moreover, by (11) and (12), we obtain $\Lambda_{\alpha\beta}^\gamma = 0$ if two of the indices—say, (α, β) —satisfy $\alpha \in [1]$ and $\beta \in [4]$, or $\alpha \in [2]$ and $\beta \in [5]$, or $\alpha \in [3]$ and $\beta \in [6]$, where $[i] = \{i, \bar{i}\}$. Thus, the possible nonzero $\Lambda_{\alpha\beta}^\gamma$ are

$$\begin{aligned} \{[\alpha], [\beta], [\gamma]\} &= \{1, 2, 3\}, \{1, 2, 6\}, \{1, 3, 5\}, \{1, 5, 6\}, \\ &\{2, 3, 4\}, \{2, 4, 6\}, \{3, 4, 5\}, \{4, 5, 6\}. \end{aligned}$$

For computation of these, Table 1 shows $[A, B], A, B \in \mathfrak{g}$.

Table 1

	X_2	T_2	X_3	T_3	X_4	T_4	X_5	T_5	X_6	T_6
X_1	$-X_3$	$-T_3$	$2X_5 + 3X_2$	$2T_5 + 3T_2$			$-2X_3 - 3X_6$	$-2T_3 - 3T_6$	X_5	T_5
T_1	T_3	$-X_3$	$-2T_5 + 3T_2$	$2X_5 - 3X_2$			$-2T_3 + 3T_6$	$2X_3 - 3X_6$	T_5	$-X_5$
X_2			$-X_1$	T_1	$-X_6$	$-T_6$			X_4	T_4
T_2			$-T_1$	$-X_1$	T_6	$-X_6$			T_4	$-X_4$
X_3					X_5	T_5	$2X_1 - 3X_4$	$-2T_1 - 3T_4$		
T_3					$-T_5$	X_5	$2T_1 - 3T_4$	$2X_1 + 3X_4$		
X_4							X_3	$-T_3$	$-X_2$	T_2
T_4							T_3	X_3	$-T_2$	$-X_2$
X_5									$-X_1$	T_1
T_5									$-T_1$	$-X_1$

REMARK 3.1. M is a Kähler manifold with complex structure J defined by $JX_i = T_i, JT_i = -X_i$. This is a general theory, but the vanishing of the torsion N and ∇J can be shown directly from Table 1 and (15)–(18).

Here we may assume that $\lambda_1 = 2 + \sqrt{3} = -(\xi_1 - \xi_3)/\sqrt{3}\xi_2$, from which it follows that $\xi_1/\xi_2 = -(2 + \sqrt{3})$. Thus, noting (14), we obtain

$$(19) \quad \begin{aligned} \frac{\|H\|}{\alpha_1(H)} &= \sqrt{3}(\sqrt{3} + 1), & \frac{\|H\|}{\alpha_2(H)} &= 1, & \frac{\|H\|}{\alpha_3(H)} &= \sqrt{3}(\sqrt{3} - 1), \\ \frac{\|H\|}{\alpha_4(H)} &= (\sqrt{3} - 1), & \frac{\|H\|}{\alpha_5(H)} &= \sqrt{3}, & \frac{\|H\|}{\alpha_6(H)} &= (\sqrt{3} + 1). \end{aligned}$$

Now, it follows that

$$(20) \quad \begin{aligned} \begin{pmatrix} \Lambda_{16}^2 & \Lambda_{16}^{\bar{2}} \\ \Lambda_{16}^{\bar{2}} & \Lambda_{16}^2 \end{pmatrix} &= \begin{pmatrix} \Lambda_{16}^2 & \Lambda_{16}^{\bar{2}} \\ \Lambda_{16}^{\bar{2}} & \Lambda_{16}^2 \end{pmatrix} = \begin{pmatrix} \Lambda_{42}^3 & \Lambda_{42}^{\bar{3}} \\ \Lambda_{42}^{\bar{3}} & \Lambda_{42}^3 \end{pmatrix} = \begin{pmatrix} \Lambda_{42}^3 & \Lambda_{42}^{\bar{3}} \\ \Lambda_{42}^{\bar{3}} & \Lambda_{42}^3 \end{pmatrix} = 0, \\ \begin{pmatrix} \Lambda_{16}^5 & \Lambda_{16}^{\bar{5}} \\ \Lambda_{16}^{\bar{5}} & \Lambda_{16}^5 \end{pmatrix} &= -\frac{\sqrt{3}(\sqrt{3} + 1)}{\sqrt{2}}J, & \begin{pmatrix} \Lambda_{16}^5 & \Lambda_{16}^{\bar{5}} \\ \Lambda_{16}^{\bar{5}} & \Lambda_{16}^5 \end{pmatrix} &= -\frac{\sqrt{3}(\sqrt{3} + 1)}{\sqrt{2}}I, \end{aligned}$$

$$\begin{aligned}
(21) \quad & \begin{pmatrix} \Lambda_{26}^4 & \Lambda_{26}^{\bar{4}} \\ \Lambda_{26}^{\bar{4}} & \Lambda_{26}^4 \end{pmatrix} = -\frac{1}{\sqrt{2}}J, & \begin{pmatrix} \Lambda_{26}^4 & \Lambda_{26}^{\bar{4}} \\ \Lambda_{26}^{\bar{4}} & \Lambda_{26}^4 \end{pmatrix} = -\frac{1}{\sqrt{2}}I, \\
& \begin{pmatrix} \Lambda_{51}^3 & \Lambda_{51}^{\bar{3}} \\ \Lambda_{51}^{\bar{3}} & \Lambda_{51}^3 \end{pmatrix} = -\sqrt{2}J, & \begin{pmatrix} \Lambda_{51}^3 & \Lambda_{51}^{\bar{3}} \\ \Lambda_{51}^{\bar{3}} & \Lambda_{51}^3 \end{pmatrix} = -\sqrt{2}I, \\
& \begin{pmatrix} \Lambda_{31}^2 & \Lambda_{31}^{\bar{2}} \\ \Lambda_{31}^{\bar{2}} & \Lambda_{31}^2 \end{pmatrix} = \frac{\sqrt{3}(\sqrt{3}-1)}{\sqrt{2}}J, & \begin{pmatrix} \Lambda_{31}^2 & \Lambda_{31}^{\bar{2}} \\ \Lambda_{31}^{\bar{2}} & \Lambda_{31}^2 \end{pmatrix} = \frac{\sqrt{3}(\sqrt{3}-1)}{\sqrt{2}}I, \\
& \begin{pmatrix} \Lambda_{45}^3 & \Lambda_{45}^{\bar{3}} \\ \Lambda_{45}^{\bar{3}} & \Lambda_{45}^3 \end{pmatrix} = -\frac{\sqrt{3}-1}{\sqrt{2}}J, & \begin{pmatrix} \Lambda_{45}^3 & \Lambda_{45}^{\bar{3}} \\ \Lambda_{45}^{\bar{3}} & \Lambda_{45}^3 \end{pmatrix} = -\frac{\sqrt{3}-1}{\sqrt{2}}I,
\end{aligned}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then by (7) and $1/\sin\theta_a = \pm\sqrt{2}(\sqrt{3}+1)$ for $a = 1, 6$, respectively, we have B_ζ and $B_{\bar{\zeta}}$ of M_+ by Lemma 2.1:

$$\begin{aligned}
(22) \quad & B_\zeta = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3}J \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}}J & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}}J & 0 & 0 & 0 \\ -\sqrt{3}J & 0 & 0 & 0 & 0 \end{pmatrix}, \\
& B_{\bar{\zeta}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3}I \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}}I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}}I & 0 & 0 & 0 \\ \sqrt{3}I & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Similarly, denoting the shape operators of M_- by C_ζ and $C_{\bar{\zeta}}$, $\zeta, \bar{\zeta} \in D_1$, we can express these with respect to $D_2 \oplus \cdots \oplus D_6$ as

$$\begin{aligned}
(23) \quad & C_\zeta = \begin{pmatrix} 0 & J & 0 & 0 & 0 \\ -J & 0 & 0 & -\frac{2}{\sqrt{3}}J & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}}J & 0 & 0 & J \\ 0 & 0 & 0 & -J & 0 \end{pmatrix},
\end{aligned}$$

$$C_{\bar{\zeta}} = \begin{pmatrix} 0 & -I & 0 & 0 & 0 \\ -I & 0 & 0 & \frac{2}{\sqrt{3}}I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}}I & 0 & 0 & -I \\ 0 & 0 & 0 & -I & 0 \end{pmatrix}.$$

In particular, M_+ is not congruent to M_- in S^{13} .

§4. Geometry of G_2 orbits

In [M1, Proposition 2.1], we show that an isoparametric hypersurface N^6 in S^7 with $(g, m) = (6, 1)$ is the inverse image of a real Cartan hypersurface $C_{\mathbb{R}}^3 \cong SO(3)/(\mathbb{Z}_2 + \mathbb{Z}_2)$ under the Hopf fibration $\pi : S^7 \rightarrow S^4$. Since the restriction of the fibration to a proper subset of S^4 is trivial, we have a homeomorphism

$$(24) \quad N^6 \cong C_{\mathbb{R}}^3 \times S^3.$$

Note that $C_{\mathbb{R}}^3$ is a principal orbit of the adjoint action of $SO(3)$ on the space of traceless symmetric matrices $\text{Sym}^0(\mathbb{R}, 3)$. We can express (24) in terms of the decomposition of the tangent bundle of N^6 into two integrable distributions $TN = \mathcal{R} \oplus \mathcal{S}$ given by

$$(25) \quad \begin{aligned} \mathcal{R} &= \text{span}\{e_2, e_4, e_6\}, \\ \mathcal{S} &= \text{span}\{e_1 - \lambda_1 e_4, e_2 + \lambda_2 e_5, e_3 + \lambda_3 e_6\}, \end{aligned}$$

where \mathcal{S} is the direction of the Hopf fiber (see [M1, p. 188, line 6]) and is totally geodesic. On the other hand, \mathcal{R} corresponds to the Lie algebra $\mathfrak{so}(3)$ in $\mathfrak{so}(4)$, since $\Lambda_{24}^j = 0, \Lambda_{26}^j = 0, \Lambda_{46}^j = 0$ hold except for the indices consisting of $\{2, 4, 6\}$. Thus, \mathcal{R} is also integrable. Note that N is an arbitrary principal orbit, and λ_i are given by (4) for some $\theta_1 \in (0, \pi/6)$.

In the case $(g, m) = (6, 2)$, a parallel argument is *not* valid because of the lack of corresponding fibrations. Instead, since $SU(3)$ is a subgroup of G_2 , and its Lie algebra is generated by $\mathfrak{a} \oplus \text{span}\{X_2, T_2, X_4, T_4, X_6, T_6\}$, the subspace

$$(26) \quad \mathcal{R} = D_2 \oplus D_4 \oplus D_6$$

defines an integrable distribution on M . The leaves are Cartan hypersurfaces $C_{\mathbb{C}}^6 \cong SU(3)/T_2$, which are half-dimensional Kähler submanifolds of M^{12} . This defines a Kähler fibration $M \rightarrow S^6 \cong G_2/SU(3)$ with fiber $C_{\mathbb{C}}^6$.

We note that $\pi_1(M) = 1 = \pi_1(M_{\pm})$ in these arguments. On M_+ , the space D_6 collapses, and it is easy to see that $\mathfrak{a} \oplus \text{span}\{X_6, T_6\} = \mathfrak{u}(2)$. Thus, M_+ is diffeomorphic to $G_2/U(2) = \mathbb{Q}^5$, the complex quadric. Moreover, (26) implies that the focal submanifold M_+ has a fibration $M_+ \rightarrow S^6$ with fiber $\mathbb{C}P^2 \cong SU(3)/S(U(2) \times U(1))$, which is tangent to $df_6(D_2 \oplus D_4)$. The total space $M_+ \cong \mathbb{Q}^5$ is diffeomorphic to the twistor space of $S^6 = G_2/SU(3)$ given by Bryant [B1].

Similarly, on M_- , the space D_1 collapses, and $\mathfrak{a} \oplus \text{span}\{X_1, T_1\} = \mathfrak{u}(2)$ shows that $M_- = G_2/U(2) = \mathbb{Q}^5$; however, M_+ and M_- are not congruent as is seen from (22) and (23).^{*} In fact, since D_4 is invariant along L_1 ($\Lambda_{14}^\alpha = 0$), the image of the curvature surface L_4 under the focal map f_1 defines a totally geodesic $S^2 = \mathbb{C}P^1$ fibration on M_- . Here, total geodesicity follows since $df_1(D_4)$ belongs to the kernel of all the shape operators (see (23)). It is easy to see that $\text{span}\{H, X_1, T_1\}$ and $\text{span}\{H^\perp, X_4, T_4\}$ are isomorphic to $\mathfrak{so}(3)$, where

$$H = G_{24} + G_{56} - 2G_{37}, \quad H^\perp = G_{24} - G_{56},$$

and hence that the space $\mathfrak{a} \oplus \text{span}\{X_1, T_1, X_4, T_4\}$ is isomorphic to $\mathfrak{so}(4)$. Therefore, the base manifold of this $\mathbb{C}P^1$ fibration is given by $G_2/SO(4)$, the quaternionic Kähler manifold. This implies that M_- is diffeomorphic to the twistor space of $G_2/SO(4)$ given in [B2].

On the other hand, since $\mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_6 = \text{span}\{X_1, \dots, X_6\}$ generates another $\mathfrak{so}(4)$ (see [M1, p. 183]), we have a half-dimensional submanifold at each point of M given by this $SO(4)$ suborbit N^6 . In fact, the tangent space of N^6 is spanned by

$$T_i = \left. \frac{d}{dt} \right|_{t=0} (\text{Ad exp } tX_i)\tilde{H}, \quad i = 1, \dots, 6.$$

From Remark 3.1, we see that N^6 is a Lagrangian submanifold of M^{12} . At each point of M^{12} , the tangent space of N^6 is expressed as $\{e_{\bar{1}}, e_{\bar{2}}, \dots, e_{\bar{6}}\}$; however, the direction of $e_{\bar{i}}$ can be replaced by a suitable combination of e_i and $e_{\bar{i}}$ in each D_i . In fact, $SO(4)$ is embedded in G_2 in a 2-parameter family, such as

$$(27) \quad \begin{aligned} & \sin \varphi X_1 - \cos \varphi T_1, & \sin \psi X_2 - \cos \psi T_2, \\ & \cos(\varphi - \psi)X_3 - \sin(\varphi - \psi)T_3, & \sin(2\psi - 3\varphi)X_4 - \cos(2\psi - 3\varphi)T_4, \end{aligned}$$

^{*}Here N_+^5 is not congruent to N_-^5 in S^7 (see [M1, Proposition 2.5]).

$$\sin(\psi - 2\varphi)X_5 - \cos(\psi - 2\varphi)T_5, \quad \cos(3\varphi - \psi)X_6 - \sin(3\varphi - \psi)T_6.$$

By using Table 1, we can see that the Lie bracket closes in this space, which generates $\mathfrak{so}(4)$ for any fixed φ and ψ . The tangent space of the corresponding $SO(4)$ orbit is the 6-dimensional subspace of TM spanned by

$$(28) \quad \begin{aligned} &\cos \varphi e_1 + \sin \varphi e_{\bar{1}}, & \cos \psi e_2 + \sin \psi e_{\bar{2}}, \\ &\sin(\varphi - \psi)e_3 + \cos(\varphi - \psi)e_{\bar{3}}, & \cos(2\psi - 3\varphi)e_4 + \sin(2\psi - 3\varphi)e_{\bar{4}}, \\ &\cos(\psi - 2\varphi)e_5 + \sin(\psi - 2\varphi)e_{\bar{5}}, & \sin(3\varphi - \psi)e_6 + \cos(3\varphi - \psi)e_{\bar{6}}. \end{aligned}$$

This reflects the fact that the isotropy subgroup T^2 of $M = G_2/T^2$ at $o = T^2$ acts on T_oM as an isometry. Thus, at $o = T^2 \in M$ (and hence at each point of M), there exists a 2-parameter family of the $SO(4)$ orbits which are Lagrangian.

Note that the distribution \mathcal{R} given in (26) and the tangent space of each $SO(4)$ orbit (e.g., spanned by $e_{\bar{1}}, \dots, e_{\bar{6}}$) are *not* transversal; that is, they do not span TM . Now we have almost shown Theorems 1.1 and 1.2, which we restate in a refined way.

MAIN THEOREM. *On every G_2 orbit M_t , $t \in (-1, 1)$, and M_{\pm} , which sweep out S^{13} , there exists a Kähler fibration:*

- (i) $M_t \cong G_2/T^2 \rightarrow S^6 = G_2/SU(3)$ with fiber $C_{\mathbb{C}}^6 = SU(3)/T^2$ tangent to $D_2 \oplus D_4 \oplus D_6$;
- (ii) $M_+ \cong \mathbb{Q}^5 \rightarrow S^6 = G_2/SU(3)$ with fiber $\mathbb{C}P^2 = SU(3)/S(U(2) \times U(1))$ tangent to $df_6(D_2 \oplus D_4)$, where f_6 is the focal map, and which is diffeomorphic to the twistor fibration of S^6 ;
- (iii) $M_- \cong \mathbb{Q}^5 \rightarrow G_2/SO(4)$ with fiber $\mathbb{C}P^1 = SU(2)/S(U(1) \times U(1))$ tangent to $df_1(D_4)$, where f_1 is the focal map, and which is diffeomorphic to the twistor fibration of the quaternionic Kähler manifold $G_2/SO(4)$.

Note that M_+ is not congruent to M_- in S^{13} .

Moreover, at each point of M_t , there exists a 2-parameter family of Lagrangian submanifolds transferred from an $SO(4)$ suborbit N^6 , which is tangent to $\text{span}\{e_{\bar{i}}, 1 \leq i \leq 6\}$, a set of suitably chosen $e_{\bar{i}} \in D_i$. Here, $C_{\mathbb{C}}^6$ and N^6 are not transversal. Such N^6 collapses into $N_{\pm}^5 \cong \mathbb{R}P^2 \times S^3$ on M_{\pm} , where N_+ is tangent to $\text{span}\{df_6(e_i), 1 \leq i \leq 5\}$ and where N_- is tangent to $\text{span}\{df_1(e_i), 2 \leq i \leq 6\}$. In particular, these are minimal Lagrangian submanifolds on M_{\pm} and on M_0 , where the latter is the minimal principal orbit. However, they never define Lagrangian fibrations on M_t or on M_{\pm} .

Proof. Here N^6 collapses into N_{\pm}^5 as D_6 and D_1 collapse on M_{\pm} , respectively. We denote by N_0 the minimal principal $SO(4)$ orbit lying in M_0 . Because N_0 and N_{\pm} are minimal in some totally geodesic 7-sphere of S^{13} , these are minimal in S^{13} , and hence minimal in M_{\pm} and in M_0 , respectively.

Nonexistence of a Lagrangian fibration follows because the topology of N^6 or N_{\pm}^5 is not that of a torus. \square

Since there are 2-parameter isometric deformations of N_{\pm}^5 in M_{\pm} , and N_0^6 in M_0 , we obtain the following.

COROLLARY 4.1. *The nullity of the Lagrangian minimal submanifold N_{\pm}^5 in M_{\pm} , and N_0^6 in M_0 , respectively, is not less than 2.*

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