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NOTES ON BOUNDEDNESS OF SPECTRAL MULTIPLIERS ON HARDY SPACES ASSOCIATED TO OPERATORS

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Abstract. Let L be a nonnegative self-adjoint operator on $L^2(X)$, where X is a space of homogeneous type. Assume that L generates an analytic semigroup e^{-tL} whose kernel satisfies the standard Gaussian upper bounds. We prove that the spectral multiplier $F(L)$ is bounded on $H^p_L(X)$ for $0 < p \le 1$, the Hardy space associated to operator L, when F is a suitable function.

§1. Introduction

Let (X, d, μ) be a metric measure space endowed with a distance d and a nonnegative Borel doubling measure μ on X. Recall that the measure μ satisfies doubling condition if there exists a constant $C > 0$ such that, for all $x \in X$ and for all $r > 0$,

$$V(x, 2r) \le CV(x, r) < \infty, \tag{1}$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$. In particular, X is a space of homogeneous type. (A more general definition and further studies of these spaces can be found in [CW, chapitre 3].) Note that the doubling property implies the following strong homogeneity property:

$$V(x, \lambda r) \le c\lambda^n V(x, r) \tag{2}$$

for some $c, n > 0$ uniformly for all $\lambda \ge 1$ and $x \in X$. There also exist c and $N, 0 \le N \le n$, such that

$$V(y, r) \le c\Big(1 + \frac{d(x, y)}{r}\Big)^N V(x, r) \tag{3}$$

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uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (3) with $N = n$ is a direct consequence of the triangle inequality of the metric d and the strong homogeneity property. To simplify notation, we will often use B for $B(x_B, r_B)$. Also, given that $\lambda > 0$, we will write λB for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset X$, we set

$$S_0(B) = B \qquad \text{and} \qquad S_j(B) = 2^j B \backslash 2^{j-1} B \quad \text{for } j \in \mathbb{N}.$$

In this paper, we assume that L is a nonnegative self-adjoint operator on $L^2(X)$ that satisfies the following assumptions.

The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ whose kernels $p_t(x, y)$ satisfy the Gaussian upper bound; that is, there exist constants C, $c > 0$ such that, for almost every $x, y \in X$,

$$\text{(G)} \qquad |p_t(x,y)| \le \frac{C}{V(x,\sqrt{t})} \exp\Big(-\frac{d^2(x,y)}{ct}\Big), \quad \forall t > 0.$$

The Gaussian upper bound considered in [DOS] is more general; that is, there exist constants C, $c > 0$ such that, for almost every $x, y \in X$, we have

$$(4) \qquad |p_t(x,y)| \le \frac{C}{V(x,t^{1/m})} \exp\Big(-\frac{d(x,y)^{m/(m-1)}}{ct^{1/(m-1)}}\Big), \quad \forall t > 0.$$

However, in the case where $m \neq 2$, the results concerning the Hardy spaces in [HLMMY] may not hold. Consequently, in this paper we restrict ourselves to considering the case of $m = 2$.

By the spectral theorem, for any bounded Borel function $F : [0, \infty) \to \mathbb{C}$, one can define the operator

$$(5) \qquad F(L) = \int_0^\infty F(\lambda)\, dE(\lambda)$$

which is bounded on $L^2(X)$.

The L^p-*boundedness of spectral multipliers* is a well-known problem which has been studied extensively for elliptic operators in [Ho], for sub-Laplacian on nilpotent groups in [C] and [D], for sub-Laplacian on Lie groups of polynomial growth in [A1], for Schrödinger operator on Euclidean space $\mathbb{R}^n$ in [He], and for sub-Laplacian on Heisenberg groups in [MSt], among other examples. (For further background information on this topic, we refer the reader to [A1], [A2], [B], [C], [DeM], [DOS], and [FS] and the references therein.)

Recently, in [DOS], Duong, Ouhabaz, and Sikora investigated the spectral multiplier theorem in a general setting of abstract operators, which we sketch out briefly here. Let L be a nonnegative self-adjoint operator, and let L generate an analytic semigroup e^{-tL} whose kernel satisfies the standard Gaussian upper bounds (equation (4)). It was proved that if, for $q \in [2,\infty], s > (n/2)$, and for some $\eta \in C_c^\infty(\mathbb{R}_+)$,

$$\sup_{t>0} \|\eta \delta_t F\|_{W_s^q} < \infty, \tag{6}$$

where $\delta_t F(\lambda) = F(t\lambda)$ and $\|F\|_{W_s^q} = \|(I - d^2/dx^2)^{s/2} F\|_{L^q}$, then $F(L)$ is of weak type $(1,1)$, and hence, by interpolation, $F(L)$ is bounded on $L^p(X), 1 < p < \infty$.

Working in the same setting as [DOS], this paper is dedicated to studying the boundedness of $F(L)$ when $0 < p \leq 1$. We show that $F(L)$ is bounded on $H_L^p(X)$ for $0 < p \leq 1$, the Hardy space associated to the operator L. Note that the case when $p = 1$ was investigated in [DP] with stronger assumptions imposed on F and s. More precisely, it was proved in [DP] that if the nonnegative self-adjoint L satisfies (G), then $F(L)$ is bounded on $H_L^1(X)$ if (6) holds for $q = \infty$ and $s > n/2$, or (6) holds for $q = 2$ and $s > n/2 + 1/2$.

The remainder of this article is organized into two sections. In Section 2, we review the definitions and basic properties of Hardy spaces associated to operators in [HLMMY] and [DL]. The main results, Theorem 3.1 and Theorem 3.2, are addressed in Section 3.

§2. Hardy spaces associated to operators

The theory of Hardy spaces associated to nonnegative self-adjoint operators satisfying Davies-Gaffney estimates was developed recently by Hofmann, Lu, Mitrea, Mitrea, and Yan [HLMMY]. Here, we use the definitions and characterizations of Hardy spaces $H_L^p(X)$ from both [HLMMY] and [DL].

2.1. The atomic Hardy spaces $H_L^p(X)$ for $p \leq 1$

Let us describe the notion of a $(p, 2, M)$-*atom*, $0 < p \leq 1$, associated to operators on spaces (X, d, μ). In what follows, assume that

$$M \in \mathbb{N} \qquad \text{and} \qquad M > \frac{n(2-p)}{4p}, \tag{7}$$

where the parameter n is the constant in (2). Let us denote by $\mathcal{D}(T)$ the domain of an operator T.

DEFINITION 2.1.1. A function $a(x) \in L^2(X)$ is called a $(p,2,M)$-atom associated to an operator L if there exist a function $b \in \mathcal{D}(L^M)$ and a ball B of X such that

(i) $a = L^M b$;
(ii) $\operatorname{supp} L^k b \subset B, k = 0,1,\dots,M$;
(iii) $\|(r_B^2 L)^k b\|_{L^2(X)} \leq r_B^{2M} V(B)^{1/2-1/p}$, $k = 0,1,\dots,M$.

In the case $\mu(X) < \infty$, the constant function having value $[\mu(X)]^{-1/p}$ is also considered to be an atom.

DEFINITION 2.1.2. Given $0 < p \leq 1$ and $M > n(2-p)/4p$, the atomic Hardy space $H^p_{L,at,M}(X)$ is defined as follows. We say that $f = \sum \lambda_j a_j$ is an atomic $(p,2,M)$-representation if $\{\lambda_j\}_{j=0}^{\infty} \in l^p$, each a_j is a $(p,2,M)$-atom, and the sum converges in $L^2(X)$. Set

$$\mathbb{H}^p_{L,at,M}(X) = \big\{ f : f \text{ has an atomic } (p,2,M)\text{-representation} \big\},$$

with the norm given by

$$\|f\|_{\mathbb{H}^p_{L,at,M}(X)} = \inf \Big\{ \Big(\sum |\lambda_j|^p \Big)^{1/p} : f = \sum \lambda_j a_j \text{ is an atomic } (p,2,M)\text{-representation} \Big\}.$$

The space $H^p_{L,at,M}(X)$ is then defined as the completion of $\mathbb{H}^p_{L,at,M}(X)$ with respect to the quasi-metric d defined by $d(h,g) = \|h-g\|_{\mathbb{H}^p_{L,at,M}(X)}$ for all $h, g \in \mathbb{H}^p_{L,at,M}(X)$.

In this case, the mapping $h \to \|h\|_{H^p_{L,at,M}(X)}, 0 < p < 1$ is not a norm, and $d(h,g)) = \|h-g\|_{H^p_{L,at,M}(X)}$ is a quasi-metric. For $p = 1$, the mapping $h \to \|h\|_{H^1_{L,at,M}(X)}$ is a norm and $H^1_{L,at,M}(X)$ is complete. In particular, $H^1_{L,at,M}(X)$ is a Banach space and $H^1_{L,at,M}(X) \hookrightarrow L^1$. A basic result concerning these spaces is the following proposition.

PROPOSITION 2.1.3. *If a nonnegative self-adjoint operator L satisfies (G), then for every $0 < p \leq 1$ and for all integers $M \in \mathbb{N}$ with $M > (n(2-p)/4p$, the spaces $H^p_{L,at,M}(X)$ coincide and their norms are equivalent.*

For the proof, we refer to [HLMMY, Theorem 5.1] for $p=1$ and to [DL, Section 3] for $p<1$.

We next describe the notion of a $(p,2,M,\epsilon)$-molecule associated to an operator L.

DEFINITION 2.1.4. Let $0<p\leq 1$, let $0<\epsilon$, and let $M\in\mathbb{N}$. A function $\alpha\in L^2(X)$ is called a $(p,2,M,\epsilon)$-molecule associated to L if there exist a function $b\in D(L^M)$ and a ball B such that

(i) $\alpha = L^M b$;

(ii) for every $k=0,1,\ldots,M$ and $j=0,1,\ldots,$ there holds

$$\|(r_B^2 L)^k b\|_{L^2(S_j(B))}\leq r_B^{2M}2^{-j\epsilon}V(2^jB)^{1/2-1/p}.$$

PROPOSITION 2.1.5. *Suppose that* $0<p\leq 1$ *and that* $M>(n(2-p)/4p)$. *If* α *is a* $(p,2,M,\epsilon)$*-molecule or an* $(p,2,M)$*-atom associated to* L*, then* $\alpha\in H^p_L(X)$. *Moreover,* $\|\alpha\|_{H^p_L(X)}$ *is independent of* M.

For the proof, we refer the reader to [HLMMY] for $p=1$ and to [DL] for $p<1$.

2.2. A characterization of $H^p_{L,at,M}(X)$ in terms of square functions

Define

$$S_h f(x) = \Big(\int_0^\infty \int_{d(x,y)<t} |t^2Le^{-t^2L}f(y)|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\Big)^{1/2},\quad x\in X.$$

The space $H^p_{L,S_h}(X)$ is defined as the completion of

$$\big\{f\in L^2(X): \|S_hf\|_{L^p(X)}<\infty\big\}$$

under the norm given by the L^p-norm of the square function; that is,

$$\|f\|_{H^p_{L,S_h}(X)} = \|S_hf\|_{L^p(X)},\quad 0<p\leq 1.$$

Then the *square function* and *atomic* H^p*-spaces* are equivalent, if the parameter $M>n(2-p)/4p$. In fact, we have the following result.

PROPOSITION 2.2.1. *Suppose that* $0<p\leq 1$ *and that* $M>n(2-p)/4p$. *Then we have* $H^p_{L,at,M} = H^p_{L,S_h}(X)$*, and their norms are equivalent.*

Proof. For the proof, see [DL, Theorem 3.12]. ∎

Consequently, as in Definition 2.2.2, one may write $H^p_{L,at}$ in place of $H^p_{L,at,M}$ when $M > n(2-p)/4p$. Precisely, we have the following definition.

DEFINITION 2.2.2. The Hardy space $H^p_L(X), p \geq 1$, is the space

$$H^p_L(X) := H^p_{L,S_h}(X) := H^p_{L,at}(X) := H^p_{L,at,M}(X), \quad M > \frac{n(2-p)}{4p}.$$

We end this section with the following result, which plays an important role in the remainder of this article.

PROPOSITION 2.2.3. *Let T be a bounded linear operator on $L^2(X)$. If there exists $C_0 > 0$ such that for any $(p,2,M)$-atom a, $0 < p \leq 1$, one has*

$$\|Ta\|_{H^p_L(X)} \leq C_0,$$

then T can be extended to a bounded operator on $H^p_L(X)$; moreover, there exists $\kappa > 0$ so that $\|T\|_{H^p_L(X) \to H^p_L(X)} \leq \kappa C_0$.

The proof is similar to one in [HM, Lemma 4.1], so we omit details here.

§3. Spectral multiplier theorem on $H^p_L(X), 0 < p \leq 1$

Let T be a bounded linear operator on $L^2(X)$. Let the associated kernel to the operator T be denoted by $K_T(x,y)$. By the kernel $K_T(x,y)$, we mean

$$Tf(x) = \int_X K_T(x,y) f(y)\, d\mu(y),$$

where $K_T(x,y)$ is a measurable function and the formula above holds for each continuous function f with compact support and for almost all x not in the support of f.

Our main results are the following two theorems.

THEOREM 3.1. *Let L be a nonnegative self-adjoint operator satisfying (G). Suppose that $s > n(2-p)/2p$, and suppose that, for any $R > 0$ and for all Borel functions F such that* $\operatorname{supp} F \subset [0,R]$,

$$\int_X |K_{F(\sqrt{L})}(x,y)|^2\, d\mu(x) \leq \frac{C}{V(y,R^{-1})} \|\delta_R F\|^2_{L^q} \tag{8}$$

for some $q \in [2,\infty]$. Then for any Borel function F such that $\sup_{t>0} \|\eta \delta_t F\|_{W^q_s} < \infty$, the operator $F(L)$ is bounded on $H^p_L(X)$ for all $0 < p \leq 1$.

Note that (8) always holds for $q=\infty$ (see [DOS]). If (8) holds for some $q<\infty$, then the pointwise spectrum of L is empty. Indeed, for all $p<\infty$ and all $y\in X$, we have

$$0=C\|\delta_R\chi_{\{a\}}\|_{L^q}\le V(y,1/R)^{1/2}\|K_{\chi_{\{a\}}(\sqrt{L})}(\cdot,y)\|_{L^2},$$

so $\chi_{\{a\}}(\sqrt{L})=0$. Hence, for elliptic operators on compact manifolds, (8) cannot be true for any $q<\infty$. To be able to study these operators as well, we introduce some variation of condition (8). Following [CS] and [DOS] for a Borel function F such that $\operatorname{supp} F\subset[-1,2]$, we define the norm $\|F\|_{N,q}$ by the formula

$$\|F\|_{N,q}=\Big(\frac{1}{3N}\sum_{l=1-N}^{2N}\sup_{\lambda\in[\frac{l-1}{N},\frac{l}{N})}|F(\lambda)|^q\Big)^{1/q},$$

where $q\in[1,\infty)$ and $N\in\mathbb{Z}_+$. For $q=\infty$, we put $\|F\|_{N,q}=\|F\|_{L^\infty}$. It is obvious that $\|F\|_{N,q}$ increases monotonically in q. The next theorem is a variation of Theorem 3.1. This variation can be used in case of operators with nonempty pointwise spectrum (see [CS, Theorem 3.6]).

Theorem 3.2. *Assume that $\mu(X)<\infty$. Let L be a nonnegative self-adjoint operator satisfying (G). Suppose that $s>n/2$ and for any $N\in\mathbb{Z}_+$ and all Borel functions F such that* $\operatorname{supp} F\subset[-1,N+1]$,

$$\int_X |K_{F(\sqrt{L})}(x,y)|^2\,d\mu(x)\le\frac{C}{V(y,1/N)}\|\delta_N F\|^2_{N,q}\tag{9}$$

for some $q\in[2,\infty]$. Then for any Borel function F such that $\sup_{t>0}\|\eta\delta_tF\|_{W^q_s}<\infty$, the operator $F(L)$ is bounded on $H^1_L(X)$.

(For further discussion on conditions (8) and (9), we refer the reader to [DOS, pp. 467–480]).

Remark 3.3. In Theorem 3.1, we can extend $F(L)$ to a bounded operator on $H^p_L(X)$ for all $0<p\le 1$, whereas Theorem 3.2 only establishes the boundedness of $F(L)$ on $H^1_L(X)$. This is a reason why in Theorem 3.2 we require $s>n/2$ instead of $s>n(2-p)/2p$ as in Theorem 3.1.

In both Theorems 3.1 and 3.2, the kernel $K_{F(\sqrt{L})}(x,y)$ of $F(\sqrt{L})$ always exists. Indeed, in virtue of the Fourier inversion formula

$$G(L/R^2)e^{-L/R^2}=\frac{1}{2\pi}\int_{\mathbb{R}}\exp\big((i\tau-1)R^{-2}L\big)\widehat{G}(\tau)\,d\tau,$$

and so

$$K_{F(\sqrt{L})}(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{G}(\tau) p_{(i\tau-1)R^{-2}}(x,y)\, d\tau$$

where $G(\lambda) = [\delta_R F](\sqrt{L})e^{\lambda}$. (For details, we refer the reader to [DOS, p. 454].)

As a preamble to the proof of Theorems 3.1 and 3.2, we record a useful auxiliary result, which is taken from [DOS, Lemma 4.3].

LEMMA 3.4. *Let L be a nonnegative self-adjoint operator satisfying (G).*

(a) *If L satisfies (8) for some $q \in [2,\infty]$, $R > 0$ and $s > 0$, then for any $\epsilon > 0$, there exists a constant $C = C(s,\epsilon)$ such that*

$$(10)\quad \int_X |K_{F(\sqrt{L})}(x,y)|^2 \big(1 + Rd(x,y)\big)^s\, d\mu(x) \le \frac{C}{V(y,R^{-1})} \|\delta_R F\|^2_{W^q_{\frac{s}{2}+\epsilon}}$$

for all Borel functions F such that $\operatorname{supp} F \subseteq [R/4, R]$.

(b) *If L satisfies (9) for some $q \in [2,\infty]$ and if $N > 8$ is a natural number, then for any $s > 0$, $\epsilon > 0$, and function $\xi \in C_c^\infty([-1,1])$, there exists a constant $C = C(s,\epsilon,\xi)$ such that*

$$(11)\quad \int_X |K_{F*\xi(\sqrt{L})}(x,y)|^2 \big(1 + Nd(x,y)\big)^s\, d\mu(x) \le \frac{C}{V(y,R^{-1})} \|\delta_N F\|^2_{W^q_{\frac{s}{2}+\epsilon}}$$

for all Borel functions F such that $\operatorname{supp} F \subseteq [N/4, N]$.

Proof of Theorem 3.1. Since condition $\sup_{t>0} \|\eta\delta_t F\|_{W^q_s} < \infty$ is invariant under the change of variable $\lambda \mapsto \sqrt{\lambda}$ and independent on the choice of η, the $H^p_L(X)$-boundedness of $F(L)$ and $F(\sqrt{L})$ is equivalent. Hence, instead of proving the $H^p_L(X)$-boundedness of $F(L)$, we will show that $F(\sqrt{L})$ is bounded on $H^p_L(X)$. Due to Proposition 2.2.3, it suffices to show that there exists $\epsilon > 0$ such that, for any $(p,2,2M)$-atom $a = L^{2M}b$ in H^p_L, the function

$$\widetilde{a} = F(\sqrt{L})a$$

is a multiple of a $(p,2,M,\epsilon)$-molecule for $M > n(2-p)/4p$. □

By standard argument, fix a function $\phi \in C_c^\infty(1/4,1)$ such that

$$\sum_{j\in\mathbb{Z}} \phi(2^{-j}\lambda) = 1 \quad \text{for } \lambda > 0.$$

Set $j_0 = -\log_2 r_B$. Then, for $0 \le k \le M$, one has

$$
\begin{aligned}
(r_B^2 L)^k \widetilde{b} &= r_B^{2k} \sum_{j \ge j_0} \phi(2^{-j}\sqrt{L}) F(\sqrt{L}) L^{k+M} b \\
&\quad + r_B^{2k} \sum_{j < j_0} \phi(2^{-j}\sqrt{L}) L^M F(\sqrt{L}) L^k b \qquad (12)\\
&= r_B^{2k} \sum_{j \ge j_0} \phi(2^{-j}\sqrt{L}) F(\sqrt{L}) b_1 + r_B^{2k} \sum_{j < j_0} \phi(2^{-j}\sqrt{L}) L^M F(\sqrt{L}) b_2,
\end{aligned}
$$

where $\widetilde{b} = L^M b$.

It is easy to see that

$$
\|b_1\|_{L^2} \le r_B^{2M-2k} V(B)^{\frac{1}{2}-\frac{1}{p}} \qquad \text{and} \qquad \|b_2\|_{L^2} \le r_B^{4M-2k} V(B)^{\frac{1}{2}-\frac{1}{p}}.
$$

Setting

$$
F_j(\lambda) = \begin{cases} F(\lambda)\phi(2^{-j}\lambda), & j \ge j_0 \\ F(\lambda)(2^{-j}\lambda)^{2M}\phi(2^{-j}\lambda), & j < j_0, \end{cases}
$$

then we can rewrite (12) as follows

$$
(r_B^2 L)^k \widetilde{b} = r_B^{2k} \sum_{j \ge j_0} F_j(\sqrt{L}) b_1 + r_B^{2k} 2^{2jM} \sum_{j < j_0} F_j(\sqrt{L}) b_2. \qquad (13)
$$

Since (13) converges in $L^2(X)$, we have, for any $k \ge 0$,

$$
\begin{aligned}
\|(r_B^2 L)^k \widetilde{b}\|_{L^2(S_k(B))} &\le r_B^{2k} \sum_{j \ge j_0} \|F_j(\sqrt{L}) b_1\|_{L^2(S_k(B))} \\
&\quad + r_B^{2k} 2^{2jM} \sum_{j < j_0} \|F_j(\sqrt{L}) b_2\|_{L^2(S_k(B))}.
\end{aligned}
$$

First, let us estimate $\|F_j(\sqrt{L}) b_1\|_{L^2(S_k(B))}$ for $j \ge j_0$. Since $\operatorname{supp} F_j \subset [R/4, R]$ with $R = 2^j$, by applying Lemma 3.4 and the Minskowski inequality, we have, for $s > s' > n(2-p)/2p \ge n/2$ and $k \ge 2$,

$$
\begin{aligned}
&\|F_j(\sqrt{L}) b_1\|_{L^2(S_k(B))} \\
&\le \Big\| \int_B K_{F_j(\sqrt{L})}(x,y) b_1(y)\, d\mu(y) \Big\|_{L^2(S_k(B))} \\
&\le \|b_1\|_{L^1} \sup_{y \in B} \Big(\int_{S_k(B)} |K_{F_j(\sqrt{L})}(x,y)|^2\, d\mu(x) \Big)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\le \|b_1\|_{L^2} V(B)^{\frac{1}{2}} \sup_{y\in B}\Big(\int_{S_k(B)} |K_{F_j(\sqrt{L})}(x,y)|^2\, d\mu(x)\Big)^{1/2}\\
&\le r_B^{2M-2k} V(B)^{1-\frac{1}{p}} \big(2^{-(j+k)s'} r_B^{s'}\big)\\
&\qquad \times \sup_{y\in B}\Big(\int_{S_k(B)} |K_{F_j(\sqrt{L})}(x,y)|^2 \big(1+2^j d(x,y)\big)^{2s'}\, d\mu(x)\Big)^{1/2} \qquad (14)\\
&\le C r_B^{2M-2k} V(B)^{1-\frac{1}{p}} (2^{-(j+k)s'} r_B^{s'}) \sup_{y\in B} \frac{1}{\sqrt{V(y,2^{-j})}} \|\delta_{2^j} F_j\|_{W^q_s}\\
&\le C r_B^{2M-2k} V(B)^{1-\frac{1}{p}} (2^{-(j+k)s'} r_B^{s'}) \sup_{y\in B} \frac{1}{\sqrt{V(y,2^{-j})}}.
\end{aligned}
$$

For $j \ge j_0 = -\log_2 r_B$, we have, by (3),

$$
\sup_{y\in B} \frac{1}{V(y,2^{-j})} = \sup_{y\in B} \frac{1}{V(y, r_B 2^{j_0-j})} \le C \sup_{y\in B} \frac{(2^j r_B)^n}{V(y,r_B)} \le C\frac{(2^j r_B)^n}{V(B)}.
$$

This together with (14) yields

$$
\begin{aligned}
\|F_j(\sqrt{L}) b_1\|_{L^2(S_k(B))} &\le C r_B^{2M-2k} V(B)^{1-\frac{1}{p}} 2^{-(j+k)s'} 2^{-s' j_0} \frac{(2^j r_B)^{\frac{n}{2}}}{V(B)^{\frac{1}{2}}}\\
&\le C r_B^{2M-2k} V(2^k B)^{\frac{1}{2}-\frac{1}{p}} 2^{-k(s'-\frac{n(2-p)}{2p})} 2^{(j-j_0)(\frac{n}{2})-s'}.
\end{aligned}
$$

For $k = 0, 1$, it is not difficult to see that

$$
\|F_j(\sqrt{L}) b_1\|_{L^2(S_k(B))} \le \|b_1\|_{L^2(S_k(B))} \le C r_B^{2M-2k} 2^{-k\epsilon} V(2^k B)^{\frac{1}{2}-\frac{1}{p}},
$$

with $\epsilon = s' - n(2-p)/2p$.

Therefore,

$$
r_B^{2k} \sum_{j\ge j_0} \|F_j(\sqrt{L}) b_1\|_{L^2(S_k(B))} \le C 2^{-k\epsilon} r_B^{2M} V(2^k B)^{\frac{1}{2}-\frac{1}{p}}.
$$

Note that for $j \le j_0$,

$$
\sup_{y\in B} \frac{1}{V(y,2^{-j})} = \sup_{y\in B} \le C \frac{1}{V(y, r_B 2^{j_0-j})} \le \sup_{y\in B} C \frac{1}{V(y,r_B)} = \frac{C}{V(B)}.
$$

At this stage, repeating the argument above, we also obtain

$$
r_B^{2k} 2^{2jM} \sum_{j<j_0} \|F_j(\sqrt{L}) b_2\|_{L^2(S_k(B))} \le C 2^{-k\epsilon} r_B^{2M} V(2^k B)^{\frac{1}{2}-\frac{1}{p}}.
$$

Hence, $\widetilde{a} = F(\sqrt{L})a$ is a multiple of a $(p, 2, M, \epsilon)$-molecule. The proof is complete. ∎

Proof of Theorem 3.2. First, we claim that if F supported in $[-1, N+1]$ satisfies (9), then

$$\|F(\sqrt{L})\|^2_{H^1_L \to H^1_L} \le CN^n \|\delta_N F\|_{N,q}. \tag{15}$$

Since $\mu(X) < \infty$, X is bounded. Therefore, there exists $r_0 > 1$ such that $X \subset B(z, r_0)$ for all $z \in X$.

Let $a = L^M b$ be a $(1, 2, M)$-atom associated to some ball B. We will show that $F(\sqrt{L})a = L^M F(\sqrt{L})b$ is a multiple of $(1, 2, M)$-atom associated to the ball $B(z, \gamma)$ for all $z \in X$ and $\gamma = \max\{r_B, r_0\}$. Indeed, by Minskowski inequality, we have, for all $0 \le k \le M$,

$$\begin{aligned}\|L^k F(\sqrt{L})b\|^2_{L^2(B(z,\gamma))} &= \|F(\sqrt{L})(L^k b)\|^2_{L^2(B(z,\gamma))}\\ &= \Big\|\int_X K_{F(\sqrt{L})}(x,y)(L^k b)(y)\, d\mu(y)\Big\|^2_{L^2(X)}\\ &\le \Big(\int_X \|K_{F(\sqrt{L})}(\cdot, y)\|_{L^2} |(L^k b)(y)|\, d\mu(y)\Big)^2.\end{aligned}$$

Since a is a $(1, 2, M)$-atom,

$$\int_X |(L^k b)(y)|\, d\mu(y) \le V(B)^{-1/2} \|L^k b\|_{L^2(B)} \le r_B^{2M-2k}.$$

So, we get

$$\begin{aligned}\|L^k F(\sqrt{L})b\|^2_{L^2(B(z,\gamma))} &\le C \frac{r_B^{4M-4k}}{V(y, 1/N)} \|\delta_N F\|^2_{N,q}\\ &\le C \frac{(r_0 N)^n}{V(y, r_0)} r_B^{4M-4k} \|\delta_N F\|^2_{N,q}\\ &\le \frac{C}{V(z,\gamma)} \gamma^{4M-4k} N^n \|\delta_N F\|^2_{N,q}.\end{aligned}$$

Hence, $F(\sqrt{L})a$ is a multiple of $(1, 2, M)$-atom associated to the ball $B(z, \gamma)$ for any $z \in X$ with a constant $N^{n/2} \|\delta_N F\|_{N,q}$. Therefore, due to Proposition 2.1.5, one has $\|F(\sqrt{L})a\|^2_{H^1_L} \le CN^n \|\delta_N F\|^2_{N,q}$. So, Proposition 2.2.3 tells us that

$$\|F(\sqrt{L})\|^2_{H^1_L \to H^1_L} \le CN^n \|\delta_N F\|^2_{N,q}.$$

Therefore, in order to prove Theorem 3.2, we can assume that $\operatorname{supp} F \subset [1, \infty]$. Let ϕ be the function as in the proof of Theorem 3.1. We set $F^k(\lambda) = \phi(2^{-k}\lambda)F(\lambda)$, and

$$\tilde{F} = \sum_{k=1}^{\infty} F^k * \xi,$$

where ξ is a function defined in (b) of Lemma 3.4.

By repeating the proof of Theorem 3.1 and using (9) in place of (8), we can prove that the $\tilde{F}(\sqrt{L})$ is bounded on $H_L^1(X)$. Hence, it suffices to show that $F(\sqrt{L}) - \tilde{F}(\sqrt{L})$ is bounded on $H_L^1(X)$. To do this, we write

$$F - \tilde{F} = \sum_k H_k, \quad \text{where } H_k = F^k - F^k * \xi.$$

Since $\operatorname{supp} H_k \subset [-1, 2^k + 1]$, due to (15), we have

$$\|H_k(\sqrt{L})\|_{H_L^1 \to H_L^L} \leq C 2^{kn} \|\delta_{2^k} H_k\|_{2^k, q}.$$

Therefore, to complete our proof, we need only to show that $\sum_k 2^{kn} \|\delta_{2^k} H_k\|_{2^k, q}$. To do this, we make the following claim (see [DOS, Proposition 4.6]).

Proposition 3.5. *Suppose that* $\xi \in C_c^\infty$ *is a function such that* $\operatorname{supp} \xi \subset [-1, 1], \xi \geq 0, \widehat{\xi}(0) = 1$ *and* $\widehat{\xi}^{(k)}(0) = 0$ *for all* $1 \leq k \leq [s] + 2$. *If* $\operatorname{supp} G \subset [0, 1]$, *then*

$$\|G - G * \xi_N\|_{N, q} \leq C N^{-s} \|G\|_{W_s^q}$$

for all $s > 1/q$.

In virtue of Proposition 3.5, we have

$$\begin{aligned}
\sum_k 2^{kn} \|\delta_{2^k} H_k\|_{2^k, q} &= \sum_k 2^{kn} \|\delta_{2^k}[F^k] - \xi_{2^k} * \delta_{2^k}[F^k]\|_{2^k, q} \\
&\leq C \sum_k 2^{nk} 2^{-2ks} \|\delta_{2^k}[F^k]\|_{W_s^q}^2 \\
&\leq C \sup_{k>0} \|\delta_{2^k}[F^k]\|_{W_s^q}^2,
\end{aligned}$$

where ξ_{2^k} denotes the function $\xi(2^{-k} \cdot)$.

This completes our proof. $\square$

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