

KUMMER SURFACES ASSOCIATED TO (1, 2)-POLARIZED ABELIAN SURFACES

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Abstract. The aim of this paper is to describe the geometry of the generic Kummer surface associated to a $(1, 2)$ -polarized abelian surface. We show that it is the double cover of a weak del Pezzo surface and that it inherits from the del Pezzo surface an interesting elliptic fibration with twelve singular fibers of type I_2 .

§1. Introduction

The extensive study of Kummer surfaces is explained by their rich geometry and their multiple roles in the theory of $K3$ surfaces and beyond (see [H], [PŠŠ], [B]).

In this paper, all the Kummer surfaces considered are algebraic. Let A be an abelian surface, and consider the involution which maps a to $-a$ for any a in A . This involution has sixteen fixed points, namely, the sixteen 2-torsion points, of A . The quotient surface has sixteen ordinary double points, and its minimal resolution is a smooth algebraic $K3$ surface called the *Kummer surface associated to A* and denoted by $\text{Kum}(A)$. Nikulin gave a clear way to detect Kummer surfaces among all $K3$ surfaces. Indeed, he proved that any $K3$ surface containing sixteen disjoint smooth rational curves is a Kummer surface (see [Ni1]). However, identifying the associated abelian surface can be a difficult problem. In this paper, we will address this problem in the most generic case.

In Section 1, we use Nikulin's criterion to construct new Kummer surfaces from a given Kummer surface. The idea is to take the minimal model of the double cover of the latter surface branched along eight disjoint smooth rational curves C_1, \dots, C_8 , that are even (see Definition 2.1). If the eight curves are chosen properly, we obtain in this way a new Kummer surface together with a rational map between the two surfaces.

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In Section 2 of this paper, we explain this construction in detail, and we show that the abelian surface associated to the new Kummer surface is isogenous to A . In fact, we prove that the rational map is induced by an isogeny of degree 2 on the associated abelian surfaces.

In Section 3, we describe the geometry of a generic Jacobian Kummer surface and explain its classical double plane model. We also recall a theorem of Naruki [Na, Theorem 1] giving explicit generators of the Néron-Severi lattice of a generic Jacobian Kummer surface.

In Section 4, we apply the construction of Section 2 to the generic Jacobian Kummer surface. We obtain in this way fifteen nonisomorphic Kummer surfaces which are associated to $(1, 2)$ -polarized abelian surfaces.

Finally, in Section 5 we show that the Kummer surfaces of Section 4 admit an elliptic fibration with twelve singular fibers of type I_2 . We also prove that these Kummer surfaces are double covers of a weak del Pezzo surface (i.e., the blowup of \mathbb{P}^2 at seven points) and that, for each of our Kummer surfaces, there exists a decomposition of a very degenerate sextic \mathcal{S} into a quartic Q and a conic C for which we have the following theorem.

THEOREM 1.1. *Let \mathcal{S} be a reducible plane sextic, which is the union of six lines all tangent to a conic. Let $\text{Kum}(A)$ be a generic Kummer surface, and let $B \rightarrow A$ be an isogeny of degree 2.*

- (i) *The isogeny of abelian surfaces induces a rational map of degree 2 $\text{Kum}(B) \xrightarrow{\tau} \text{Kum}(A)$ which decomposes as*

$$\begin{array}{ccc} \text{Kum}(B) & \xrightarrow{\varphi} & T \\ \downarrow \tau & & \downarrow \zeta \\ \text{Kum}(A) & \xrightarrow{\phi} & \mathbb{P}^2 \end{array}$$

where ϕ is the canonical resolution of the double cover of \mathbb{P}^2 branched along \mathcal{S} .

- (ii) *There exists a decomposition of \mathcal{S} into a quartic Q and a conic C such that the maps ζ and φ are the canonical resolutions of the double covers branched along Q and $\zeta^*(C)$, respectively.*

§2. Even eight and Kummer surface

We now introduce the notion of an even eight and the double cover construction associated to it. By applying this construction to special even eights of a Kummer surface, we obtain new Kummer surfaces.

DEFINITION 2.1. Let Y be a $K3$ surface. An *even eight* on Y is a set of eight disjoint smooth rational curves C_1, \dots, C_8 for which $C_1 + \dots + C_8 \in 2S_Y$. Here, S_Y denotes the Néron-Severi group of Y .

If C_1, \dots, C_8 is an even eight on a $K3$ surface Y , then there is a double cover $Z \xrightarrow{p} Y$ branched on $C_1 + \dots + C_8$. If E_i denotes the inverse image of C_i , then $p^*(C_i) = 2E_i$ and $E_i^2 = -1$. Hence, we may blow down E_i to the surface X and obtain the diagram

$$\begin{array}{ccc}
 Z & \xrightarrow{\epsilon} & X \\
 p \downarrow & \swarrow & \uparrow \\
 & & Y
 \end{array}$$

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It turns out that the surface X is again a $K3$ surface and that the covering involution $\iota : X \rightarrow X$ is symplectic with eight fixed points (see [Ni1]).

Suppose now that the $K3$ surface Y is a Kummer surface. We want to exhibit natural even eights lying on it. For this purpose, we recall a central lemma of Nikulin.

LEMMA 2.2 ([Ni1, Lemma 3]). *Let Y be a Kummer surface, and let $E_1, \dots, E_{16} \subset Y$ be sixteen smooth disjoint rational curves. Denote by $I = \{1, \dots, 16\}$ the set of indices for the curves E_i and denote $Q = \{M \subset I \mid (1/2)\sum_{i \in M} E_i \in S_Y\}$. Then for every M in Q , we have $\#|M| = 8$ or 16 , and there exists on I a unique 4-dimensional affine geometry structure over \mathbb{F}_2 whose hyperplanes consist of the subsets $M \in Q$ containing eight elements.*

The existence of such a 4-dimensional affine geometry implies that $I \in Q$ or, equivalently, that $\sum_{i=1}^{16} E_i \in 2S_Y$. We can proceed exactly as for an even eight and take the double cover $V \xrightarrow{p} Y$ branched along $E_1 + \dots + E_{16}$. Again, we blow down the preimage of E_i to a surface A , and we obtain the

diagram

$$\begin{array}{ccc}
 V & \xrightarrow{\epsilon} & A \\
 p \downarrow & \swarrow \pi_A & \\
 Y & &
 \end{array}$$

The difference between this diagram and the previous one is that now the surface A is an abelian surface and the map π_A realizes Y as the Kummer surface associated to A . We point out that, by uniqueness, the affine geometry on I corresponds to the one existing on A_2 , the set of 2-torsion points on A (see [Nil]).

It also follows from the lemma that there exist on Y , with $Y \simeq \text{Kum}(A)$, thirty even eights denoted by M_1, \dots, M_{30} , that is, the thirty affine hyperplanes of I .

Let $M \in \{M_1, \dots, M_{30}\}$ be one of these even eights. We can assume that M consists of the curves E_1, \dots, E_8 . The curves E_9, \dots, E_{16} are then orthogonal to M , that is, we have

$$E_i \cdot E_j = 0 \quad \text{if } 1 \leq i \leq 8 \text{ and } 9 \leq j \leq 16.$$

If $X \xrightarrow{\tau} \text{Kum}(A)$ is the double cover associated to M , then the K3 surface X contains again sixteen disjoint smooth rational curves. Indeed, since the curves E_9, \dots, E_{16} do not intersect the branch locus of the double cover $p : Z \rightarrow Y$, they split under p and pull back to sixteen disjoint smooth rational curves on Z . These sixteen curves are then isomorphically mapped by the blowdown $Z \xrightarrow{\epsilon} X$ to sixteen curves on X . It follows that X contains sixteen disjoint smooth rational curves, and hence that it is a Kummer surface.

PROPOSITION 2.3. *Let M be an even eight on a Kummer surface $\text{Kum}(A)$ such as above. Then the K3 surface X associated to M is a Kummer surface. Moreover, there is an abelian surface B associated to X for which we have the commutative diagram*

$$\begin{array}{ccc}
 B & \xrightarrow{p} & A \\
 \pi_B \downarrow & & \pi_A \downarrow \\
 X = \text{Kum}(B) & \xrightarrow{\tau} & \text{Kum}(A)
 \end{array}$$

where $B \xrightarrow{p} A$ is an isogeny of degree 2.

Proof. Since we have already shown that X is a Kummer surface, we only have to prove that B is degree 2 isogenous to A . Write the abelian surface A as the complex torus \mathbb{C}^2/Λ , and let $E_9, \dots, E_{16} \subset \text{Kum}(A)$ be the eight disjoint smooth rational curves orthogonal to M . These curves also form an even eight, and hence they correspond to an affine hyperplane H in A_2 . Up to translation, we can fix the origin of A in H . Let $[v]/2$ be the generator of A_2/H (it defines a sublattice $\Lambda' \subset \Lambda$). Explicitly, we have that $\Lambda' = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \oplus \mathbb{Z}h_3 \oplus \mathbb{Z}2v$, where $H = \langle [h_1]/2, [h_2]/2, [h_3]/2 \rangle \subset A_2$. The canonical inclusion $\Lambda' \hookrightarrow \Lambda$ induces the commutative diagram

$$\begin{array}{ccc}
 \mathbb{C}^2/\Lambda' & \xrightarrow{p} & \mathbb{C}^2/\Lambda \\
 \downarrow \pi' & & \downarrow \pi \\
 \text{Kum}(\mathbb{C}^2/\Lambda') & \xrightarrow{q} & \text{Kum}(\mathbb{C}^2/\Lambda)
 \end{array}$$

where p is an isogeny of degree 2. The covering involution of p is given by the translation by the 2-torsion point $[v]$ in \mathbb{C}^2/Λ' . It induces the symplectic involution on $\text{Kum}(\mathbb{C}^2/\Lambda')$

$$\sigma : \text{Kum}(\mathbb{C}^2/\Lambda') \rightarrow \text{Kum}(\mathbb{C}^2/\Lambda')$$

which has exactly eight fixed points (see [Ni2]), namely, the projection of the sixteen points on \mathbb{C}^2/Λ' satisfying

$$[z] + [v] = -[z] \quad \text{or equivalently } 2[z] = [v].$$

The isogeny p maps the set $\{[z] \in \mathbb{C}^2/\Lambda' \mid 2[z] = [v]\}$ to $A_2 - H$. In other words, the affine hyperplane $A_2 - H$ corresponds to the even eight M in $\text{Kum}(\mathbb{C}^2/\Lambda)$. Hence the resolution of the rational map q is exactly the double cover of $\text{Kum}(A)$ branched along M , and the abelian surface \mathbb{C}^2/Λ' is B . \square

§3. Jacobian Kummer surface

In this section, we briefly expose the classical geometry of a Jacobian Kummer surface and its beautiful 16₆-configuration. We describe its double plane model, and we give explicit generators for its Néron-Severi lattice. This description follows Naruki [Na].

A Kummer surface $\text{Kum}(A)$ is said to be a Jacobian Kummer surface if the surface A is the Jacobian of a curve C of genus 2. Moreover, it is a generic Jacobian Kummer surface if its Picard rank is 17.

Recall that the degree 2 map given by the linear system $|2C|$, $A \xrightarrow{|2C|} \mathbb{P}^3$, factors through the involution $a \mapsto -a$ and hence defines an embedding $A/\{1, i\} \hookrightarrow \mathbb{P}^3$. The image of this map is a quartic $Y_0 \subset \mathbb{P}^3$ with sixteen nodes. Denote by L_0 the class of a hyperplane section of Y_0 . Projecting Y_0 from a node defines a rational map $Y_0 \dashrightarrow \mathbb{P}^2$. We blow up the center of projection

$$\begin{array}{ccc} Y_1 & & \\ \downarrow & \searrow & \\ Y_0 & \dashrightarrow & \mathbb{P}^2 \end{array}$$

and we call $E_1 \subset Y_1$ the *exceptional divisor* and $L_1 \subset Y_1$ the *pullback of a line* in \mathbb{P}^2 . Finally, we resolve the remaining fifteen singularities of Y_1 , and we obtain the Kummer surface $\text{Kum}(A)$ and a map of degree 2 $\text{Kum}(A) \xrightarrow{\phi} \mathbb{P}^2$. The map ϕ is given by the linear system $|L - E_0|$, where L and E_0 are the pullbacks of L_1 and E_1 , respectively.

The branch locus of the map ϕ is a reducible plane sextic \mathcal{S} , which is the union of six lines l_1, \dots, l_6 , all tangent to a conic W .

Let $p_{ij} = l_i \cap l_j \in \mathbb{P}^2$, where $1 \leq i < j \leq 6$. Index the ten $(3, 3)$ -partitions of the set $\{1, 2, \dots, 6\}$ by the pair (i, j) with $2 \leq i < j \leq 6$. Each pair (i, j) defines a plane conic l_{ij} passing through the sextuplet $p_{1i}, p_{1j}, p_{ij}, p_{lm}, p_{ln}, p_{mn}$, where $\{l, m, n\}$ is the complement of $\{1, i, j\}$ in $\{1, 2, \dots, 6\}$ and where $l < m < n$. The map ϕ factors as

$$\text{Kum}(A) \xrightarrow{\tilde{\phi}} \tilde{\mathbb{P}}^2 \xrightarrow{\eta} \mathbb{P}^2,$$

where η is the blowup of \mathbb{P}^2 at p_{ij} and where $\tilde{\phi}$ is the double cover of $\tilde{\mathbb{P}}^2$ branched along the strict transform of the plane sextic \mathcal{S} in $\tilde{\mathbb{P}}^2$. Denote by $E_{ij} \subset \text{Kum}(A)$ the preimage of the exceptional curves of $\tilde{\mathbb{P}}^2$. The ramification of the map $\tilde{\phi}$ consists of the union of six disjoint smooth rational curves $C_0 + C_{12} + C_{13} + C_{14} + C_{15} + C_{16}$. The preimage of the ten plane conics l_{ij} defines ten more smooth disjoint rational curves $C_{ij} \subset \text{Kum}(A)$, with $2 \leq i < j \leq 6$. Finally, note that $\phi(E_0) = W$. The sixteen curves E_0, E_{ij} , $2 \leq i < j \leq 6$ are called the *nodes* of $\text{Kum}(A)$, and the sixteen curves C_0, C_{ij} , $2 \leq i < j \leq 6$ are called the *tropes* of $\text{Kum}(A)$. These two sets of smooth rational curves satisfy a beautiful configuration called the *16₆-configuration*, that is, each node intersects exactly six tropes and vice versa.

It is now possible to fully describe the Néron-Severi lattice $S_{\text{Kum}(A)}$ of a general Jacobian Kummer surface.

THEOREM 3.1 ([Na, Theorem 1]). *Let $\text{Kum}(A)$ be a generic Jacobian Kummer surface. Its Néron-Severi lattice $S_{\text{Kum}(A)}$ is generated by the classes of E_0, E_{ij}, C_0, C_{ij} , and L , with the relations*

- (1) $C_0 = (1/2)(L - E_0 - \sum_{i=2}^6 E_{1i})$;
- (2) $C_{1j} = (1/2)(L - E_0 - E_{1j} - \dots - E_{j-1j} - E_{jj+1} - \dots - E_{j6})$, where $2 \leq j \leq 6$;
- (3) $C_{jk} = (1/2)(L - E_{1j} - E_{1k} - E_{jk} - E_{lm} - E_{ln} - E_{mn})$, where $2 \leq i < j \leq 6$ and where $\{l, m, n\}$ are as described above.

The intersection pairing is given by the following:

- (1) the E_0, E_{ij} are mutually orthogonal;
- (2) $\langle L, L \rangle = 4, \langle L, E_0 \rangle = \langle L, E_{ij} \rangle = 0$;
- (3) $\langle E_0, E_0 \rangle = \langle E_{ij}, E_{ij} \rangle = -2$;
- (4) the C_0, C_{ij} are mutually orthogonal;
- (5) $\langle L, C_0 \rangle = \langle L, C_{ij} \rangle = 2$.

The action on $S_{\text{Kum}(A)}$ of the covering involution α of the map ϕ is given by

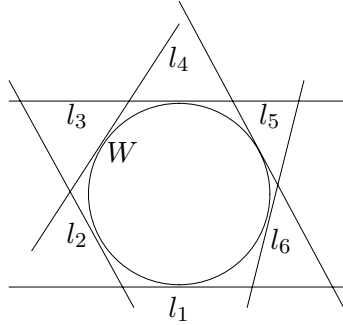
$$\begin{aligned} \alpha(C_0) &= C_0 & \alpha(C_{1j}) &= C_{1j} & 2 \leq j \leq 6 \\ \alpha(E_{ij}) &= E_{ij} & \alpha(L) &= 3L - 4E_0 \\ \alpha(E_0) &= 2L - 3E_0 & \alpha(C_{ij}) &= C_{ij} + L - 2E_0 & 2 \leq i < j \leq 6 \end{aligned}$$

REMARK 3.2. The minimal resolution of the double cover of \mathbb{P}^2 branched along the sextic \mathcal{S} in Figure 1 is a Kummer surface (see [H] for a proof).

§4. (1, 2)-polarized Kummer surfaces

In this section, we apply the construction of Section 2 to a generic Jacobian Kummer surface. We identify all the even eights made out of its nodes and study the associated Kummer surfaces. First, we recall some standard facts about the polarization of abelian varieties.

A polarization on a complex torus \mathbb{C}^g/Λ is the class of an ample line bundle L in the Néron-Severi group. As the latter group is equal (for abelian varieties) to the group of Hermitian forms H on \mathbb{C}^g satisfying $E = \text{Im } H(\Lambda, \Lambda) \subset \mathbb{Z}$, the ample line bundle L corresponds to a positive definite Hermitian form E_L . According to the elementary divisor theorem, there exists a basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ of Λ with respect to which E_L is

Figure 1: The sextic \mathcal{S}

given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \quad \text{with } D = \begin{pmatrix} d_1 & 0 & 0 & \dots \\ 0 & d_2 & 0 & \dots \\ \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & 0 & d_g \end{pmatrix},$$

where $d_i \geq 0$ and $d_i \mid d_{i+1}$ for $i = 1, \dots, g-1$. The vector (d_1, d_2, \dots, d_g) is the type of the line bundle L .

EXAMPLE 4.1 (see [BL]). We have the following.

- (1) If $J(C)$ is the Jacobian of a curve C of genus 2, then the line bundle associated to the divisor C is a polarization of type $(1, 1)$.
- (2) If L is a polarization of type (d_1, \dots, d_g) on a complex torus, then $\chi(L) = d_1 \cdots d_g$.
- (3) If $X_1 \xrightarrow{p} X_2$ is an isogeny of degree 2 of abelian surfaces and if L is a polarization of type $(1, 1)$ on X_2 , then $\chi(p^*(L)) = 2\chi(L) = 2 \cdot 1$. Hence $p^*(L)$ is a polarization of type $(1, 2)$ on X_1 .

PROPOSITION 4.2. *Let $\text{Kum}(A)$ be a generic Jacobian Kummer surface, and let E_0, E_{ij} , $1 \leq i < j \leq 6$ be its sixteen nodes. There exist fifteen even eights made out of its nodes that do not contain E_0 . These even eights are of the form*

$$\Delta_{i,j} = E_{1i} + \cdots + \widehat{E}_{ij} + \cdots + E_{i6} + E_{1j} + \cdots + \widehat{E}_{ij} + \cdots + E_{j6},$$

where $1 \leq i < j \leq 6$ and $E_{11} = 0$.

The Kummer surface $\text{Kum}(B_{ij})$ obtained from the double cover branched along Δ_{ij} is associated to an abelian surface B_{ij} with a (1, 2)-polarization.

Proof. For any couple (i, j) with $1 \leq i < j \leq 6$, consider the divisor $2C_{1i} + 2C_{1j}$, where we set $C_{11} := C_0$. According to the description of the Néron-Severi lattice of a general Jacobian Kummer surface in Section 3, we have the equality

$$2C_{1i} + 2C_{1j} = 2(L - E_0) - (E_{1i} + \cdots + E_{ij} + \cdots + E_{i6} + E_{1j} + \cdots + E_{ij} + \cdots + E_{j6}).$$

Therefore, we have

$$2C_{1i} + 2C_{1j} - 2(L - E_0) + 2E_{ij} = E_{1i} + \cdots + \widehat{E}_{ij} + \cdots + E_{i6} + E_{1j} + \cdots + \widehat{E}_{ij} + \cdots + E_{j6},$$

and consequently

$$E_{1i} + \cdots + \widehat{E}_{ij} + \cdots + E_{i6} + E_{1j} + \cdots + \widehat{E}_{ij} + \cdots + E_{j6}$$

is an even eight not containing E_0 . As there are exactly fifteen choices for i and j , we obtain all of the possible even eights this way.

Let $\text{Kum}(B_{ij})$ be the Kummer surface obtained by taking the double cover branched along such an even eight. By Proposition 2.3, the surface B_{ij} is degree 2 isogenous to A . Since A has a (1, 1)-polarization, it follows from Example 4.1 that B_{ij} has a (1, 2)-polarization. \square

The reason why we only consider the even eights not containing E_0 is because we would obtain the exact same Kummer surface whether we take the double cover branched along an even eight or its complement (see the proof of Proposition 2.3).

For the remainder of this section, we will prove that no two Kummer surfaces $\text{Kum}(B_{ij})$ are isomorphic.

DEFINITION 4.3. The *Nikulin lattice* is an even lattice N of rank 8 generated by $\{c_i\}_{i=1}^8$ and $d = (1/2) \sum_{i=1}^8 c_i$, with the bilinear form $c_i \cdot c_j = -2\delta_{ij}$.

REMARK 4.4. If C_1, \dots, C_8 is an even eight on a $K3$ surface, then the primitive sublattice generated by the class of C_i in the Néron-Severi group of X is a Nikulin lattice.

The following proposition gives a condition on two even eights which give rise to nonisomorphic $K3$ surfaces.

PROPOSITION 4.5. *Let $Y = \text{Kum}(A)$ be a Kummer surface, and let Δ_1 and Δ_2 be two even eights on Y . Denote by X_1 and X_2 the respective double covers of Y . If $N_1, N_2 \subset S_Y$ are the two Nikulin lattices corresponding to Δ_1 and Δ_2 , then*

$$X_1 \simeq X_2 \iff \exists f \in \text{Aut}(Y) \text{ such that } f^*(N_1) = N_2.$$

Proof. We suppose that X_1 is isomorphic to X_2 , and we denote by $X_2 \xrightarrow{g} X_1$ an isomorphism between X_2 and X_1 . Let $X_1 \xrightarrow{i_1} X_1$ and $X_2 \xrightarrow{i_2} X_2$ be the covering involutions with respect to the rational double covers $X_1 \xrightarrow{\tau_1} Y$ and $X_2 \xrightarrow{\tau_2} Y$.

CLAIM. *The following diagram is commutative:*

$$\begin{array}{ccc} H^2(X_1, \mathbb{Z}) & \xrightarrow{g^*} & H^2(X_2, \mathbb{Z}) \\ \downarrow i_1^* & & \downarrow i_2^* \\ H^2(X_1, \mathbb{Z}) & \xrightarrow{g^*} & H^2(X_2, \mathbb{Z}) \end{array}$$

Proof of the claim. Suppose that the above diagram does not commute. Then the surface X_1 would admit two distinct symplectic involutions, namely, i_1 and $g \circ i_2 \circ g^{-1}$. The quotient of X_1 by both of these involutions would be birational to the same Kummer surface Y . However, it follows from [M, Theorem 3.1] that the rational double cover of the Kummer surface $\text{Kum}(A)$ is determined by an embedding $T_X \hookrightarrow T_A$ preserving the Hodge decomposition of T_X and T_A . Since there is a unique embedding of T_X into T_A which preserves the Hodge decomposition, it follows that $i_1 = g^{-1} \circ i_2 \circ g$. \square

Hence $i_2 \circ g = g \circ i_1$, and the isomorphism g descends to an isomorphism on the quotients

$$X_2/i_2 \xrightarrow{g} X_1/i_1.$$

Since this isomorphism maps the eight singular points of X_2/i_2 to the eight singular points of X_1/i_1 , it extends to an automorphism $Y \xrightarrow{f} Y$ for which $f^*(N_1) = N_2$.

Conversely, let $Y \xrightarrow{f} Y$ be an automorphism of Y for which $f^*(N_1) = N_2$. Denote by $Z_i \xrightarrow{p_i} Y$ the double cover of Y branched along the even eight N_i for $i = 1, 2$. Consider the fiber product

$$\begin{array}{ccc} Z_1 \times_Y Y & \xrightarrow{q} & Z_1 \\ \downarrow p & & \downarrow p_1 \\ Y & \xrightarrow{f} & Y \end{array}$$

The map $Z_1 \times_Y Y \xrightarrow{p} Y$ is a double cover of Y branched along the even set N_2 or, equivalently, $Z_1 \times_Y Y = Z_2$. Similarly, by considering the fiber product

$$\begin{array}{ccc} Z_2 \times_Y Y & \xrightarrow{h} & Z_2 \\ \downarrow r & & \downarrow p_2 \\ Y & \xrightarrow{f} & Y \end{array}$$

we see that $Z_2 \times_Y Y = Z_1$. The maps h and $q = h^{-1}$ define an isomorphism between Z_1 and Z_2 which induces the required isomorphism between X_1 and X_2 . \square

Using the same notation as in Proposition 4.2, we prove the following.

PROPOSITION 4.6. *Let Δ_{ij} and $\Delta_{i'j'}$ be two even eights defined as in Proposition 4.2. We have*

$$\text{Kum}(B_{ij}) \simeq \text{Kum}(B_{i'j'}) \Leftrightarrow \{i, j\} = \{i', j'\}.$$

Proof. It is clear that if $\{i, j\} = \{i', j'\}$, then the corresponding Kummer surfaces are equal. Thus we only have to prove the other direction. Without loss of generality, we may assume that $\Delta_{i'j'} = \Delta_{12}$, and we suppose that there exists an automorphism f of $\text{Kum}(A)$ for which $f^*(\Delta_{12}) = \Delta_{ij}$.

CLAIM. *We have*

$$\begin{aligned} & \{f^*(E_{13}), f^*(E_{14}), f^*(E_{15}), f^*(E_{16}), f^*(E_{23}), f^*(E_{24}), f^*(E_{25}), f^*(E_{26})\} \\ &= \{E_{1i}, \dots, \hat{E}_{ij}, \dots, E_{i6}, E_{1j}, \dots, \hat{E}_{ij}, \dots, E_{j6}\} \end{aligned}$$

Proof of the claim. Let N be a Nikulin lattice, and let $D \in N$ be a divisor represented by a smooth rational curve. Note that since D is an effective reduced divisor and N is negative definite, then $D^2 = -2$. It is therefore sufficient to show that the only -2 -classes in N are the c_i , and the claim follows. We write D as $D = \sum_{i=1}^8 \lambda_i c_i + \epsilon d$, where $\lambda_j \in \mathbb{Z}$ and $\epsilon = 0$ or 1 . If $\epsilon = 1$, then the equality

$$D^2 = -2 \sum_{i=1}^8 \lambda_i^2 - 2 \sum_{i=1}^8 \lambda_i - 4 = -2$$

implies that $\sum_{i=1}^8 \lambda_i^2 + \lambda_i = -1$. Since the latter equation has no integer solution, we conclude that $\epsilon = 0$. Hence we have

$$D^2 = -2 \sum_{i=1}^8 \lambda_i^2 = -2$$

or, equivalently, $\sum_{i=1}^8 \lambda_i^2 = 1$. Therefore, there exists a unique λ_k for which $\lambda_k = 1$ and $\lambda_i = 0$ for $i \neq k$. \square

In [K], it is proved that any automorphism of a Jacobian generic Kummer surface induces \pm identity on $D_{S_{\text{Kum}(A)}}$, where $D_{S_{\text{Kum}(A)}}$ is the discriminant group

$$S_{\text{Kum}(A)}^*/S_{\text{Kum}(A)}.$$

We want to apply this fact to the automorphism f . We consider the action of f^* on the following two independent elements of $D_{S_{\text{Kum}(A)}}$:

$$\frac{1}{2}(E_{13} + E_{14} + E_{23} + E_{24}) \quad \text{and} \quad \frac{1}{2}(E_{12} + E_{23} + E_{15} + E_{35}).$$

From the claim, we deduce that

$$f^*(E_{13} + E_{14} + E_{23} + E_{24}) = E_{i_1 i} + E_{i_2 i} + E_{j_1 j} + E_{j_2 j}$$

for some classes $E_{i_1 i}, E_{i_2 i}, E_{j_1 j}, E_{j_2 j} \in \Delta_{ij}$.

From the identity $f_{D_{S_{\text{Kum}(A)}}}^* = \pm \text{id}_{D_{S_{\text{Kum}(A)}}}$, we also deduce that

$$f^*\left(\frac{1}{2}(E_{13} + E_{14} + E_{23} + E_{24})\right) = \pm \frac{1}{2}(E_{13} + E_{14} + E_{23} + E_{24}).$$

Combining these two elements, we find that

$$E_{13} + E_{14} + E_{23} + E_{24} + E_{i_1 i} + E_{i_2 i} + E_{j_1 j} + E_{j_2 j} \in 2S_Y.$$

Since the only even eights containing $E_{13}, E_{14}, E_{23}, E_{24}$ are Δ_{12} and Δ_{34} , we deduce that $\Delta_{ij} = \Delta_{34}$. We proceed similarly for $f^*(E_{12} + E_{23} + E_{13} + E_{35})$, and we find that Δ_{ij} must be equal to Δ_{25} , which yields a contradiction. \square

COROLLARY 4.7. *The fifteen Kummer surfaces $\text{Kum}(B_{ij})$ are not isomorphic.*

§5. Elliptic fibration and weak del Pezzo surface

In this section, we provide an alternate description of the Kummer surfaces $\text{Kum}(B_{ij})$ as the double cover of a weak del Pezzo surface. We relate this construction to the projective double plane model of the generic Jacobian Kummer surface of Section 3. First, we note the existence on $\text{Kum}(B_{ij})$ of an elliptic fibration that will be useful later. For simplicity, we always argue for the Kummer surface $\text{Kum}(B_{12})$.

PROPOSITION 5.1. *Let $\text{Kum}(B_{12})$ be the Kummer surface constructed in Proposition 4.2. The surface $\text{Kum}(B_{12})$ admits a Weierstrass elliptic fibration with exactly twelve singular fibers of type I_2 .*

Proof. Let $\text{Kum}(A) \xrightarrow{\phi} \mathbb{P}^2$ be the double plane model of the generic Jacobian Kummer surface introduced in Section 3. Consider the pencil of lines passing through the point p_{12} in \mathbb{P}^2 . Its preimage in $\text{Kum}(A)$ defines an elliptic fibration given by the divisor class $F = L - E_0 - E_{12}$. The divisors

$$F_1 = E_{15} + E_{16} + 2C_0 + E_{13} + E_{14}$$

and

$$F_2 = E_{25} + E_{26} + 2C_{12} + E_{23} + E_{24}$$

define two fibers of type I_0^* of this fibration. Moreover, the six divisors

$$F_3 = L - E_0 - E_{12} - E_{45} + E_{45},$$

$$F_4 = L - E_0 - E_{12} - E_{46} + E_{46},$$

$$F_5 = L - E_0 - E_{12} - E_{35} + E_{35},$$

$$F_6 = L - E_0 - E_{12} - E_{36} + E_{36},$$

$$F_7 = L - E_0 - E_{12} - E_{34} + E_{34},$$

$$F_8 = L - E_0 - E_{12} - E_{56} + E_{56}$$

define six I_2 fibers. Since the Euler characteristics of the F_i add up to twenty-four (which is equal to the Euler characteristic of a $K3$ surface), we conclude by Shioda's formula [SI, Lemma 1.3] that the F_i are the only singular fibers of the elliptic fibration defined by the linear system $|F|$. Note also that the curves C_{13} , C_{14} , C_{15} , and C_{16} are sections of this fibration.

We now analyze the induced fibration τ^*F on $\text{Kum}(B_{12})$, where $\text{Kum}(B_{12}) \xrightarrow{-\tau} \text{Kum}(A)$ is the rational double cover defined by the even eight Δ_{12} . We remark that the even eight Δ_{12} satisfies

$$\Delta_{12} = F_1 + F_2 - 2(C_0 + C_{12}),$$

which means that the eight components of Δ_{12} are exactly the eight components of the fibers F_1 and F_2 that appear with multiplicity 1. Hence τ^*F_1 and τ^*F_2 are just smooth elliptic curves. However, the six fibers F_3, \dots, F_8 split under the cover and define twelve I_2 fibers of the elliptic fibration on $\text{Kum}(B_{12})$ defined by τ^*F . Again, a computation of Euler characteristics shows that these twelve I_2 fibers are the only singular fibers of the linear system $|\tau^*F|$. Also, the sections C_{13} , C_{14} , C_{15} , and C_{16} of $|F|$ pull back to sections of τ^*F . Hence τ^*F defines a Weierstrass elliptic fibration. \square

We now proceed to the realization of the surface $\text{Kum}(B_{12})$ as a double cover of a weak del Pezzo surface.

THEOREM 5.2. *Let \mathcal{S} be a reducible plane sextic, which is the union of six lines all tangent to a conic (see Figure 1). Let $\text{Kum}(A)$ be a generic Kummer surface, and let $B_{12} \rightarrow A$ be the isogeny of degree 2 defined in Proposition 4.2.*

- (i) *The isogeny of abelian surfaces induces a rational double cover $\text{Kum}(B_{12}) \xrightarrow{-\tau} Y$ which decomposes as*

$$\begin{array}{ccc} \text{Kum}(B_{12}) & \xrightarrow{\varphi} & T \\ \downarrow \tau & & \downarrow \zeta \\ Y & \xrightarrow{\phi} & \mathbb{P}^2 \end{array}$$

where ϕ is the canonical resolution of the double cover of \mathbb{P}^2 branched along \mathcal{S} .

- (ii) *There exists a decomposition of \mathcal{S} into a quartic \mathcal{Q} and conic \mathcal{C} such that the maps ζ and φ are the canonical resolutions of the double covers branched along \mathcal{Q} and $\zeta^*(C)$, respectively.*

Proof. We decompose the sextic \mathcal{S} into the quartic $Q = l_3 + l_4 + l_5 + l_6$ and the conic $C = l_1 + l_2$. Let $T_0 \rightarrow \mathbb{P}^2$ be the double cover of \mathbb{P}^2 ramified over the reducible quartic Q . Its canonical resolution induces the diagram

$$\begin{array}{ccc}
 T & \longrightarrow & T_0 \\
 \downarrow & \searrow \zeta & \downarrow \\
 \tilde{\mathbb{P}}^2 & \longrightarrow & \mathbb{P}^2
 \end{array}$$

where $\tilde{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ is the blowup of \mathbb{P}^2 at the six singular points of Q . The surface T is a nonminimal rational surface containing six disjoint smooth rational curves. Indeed, by Hurwitz’s formula, the canonical divisor of T is given by

$$K_T = \zeta^* \left(K_{\mathbb{P}^2} + \frac{1}{2}(l_3 + l_4 + l_5 + l_6) \right) = -\zeta^*(H),$$

where H is a hyperplane section. Thus, $K_T^2 = 2, H^2 = 2$, and $P_2(T) = 0$. Denote by \tilde{Q} the proper transform of Q in T . Using the additivity of the topological Euler characteristic and the Noether formula, we have that

$$e(T) = e(T - \tilde{Q}) + e(\tilde{Q}) = 10 \Rightarrow \chi(\mathcal{O}_T) = 1 \Rightarrow q(T) = 0.$$

By Castelnuovo’s rationality criterion, T is a rational surface. In fact, we show that T is a weak del Pezzo surface of degree two, that is, the blowup of \mathbb{P}^2 at seven points with nef canonical divisor. Indeed, we successively blow down the preimages in T of the four lines l_3, l_4, l_5 , and l_6 as well as the preimages in T of the three “diagonals” of the complete quadrangle formed by l_3, l_4, l_5, l_6 . The surface obtained after these seven blowdowns is a projective plane.

Consider the following curves of T

$$\zeta^*(C) = \zeta^*(l_1 + l_2) = E_1 + E_2,$$

where E_1 and E_2 are smooth elliptic curves and

$$\zeta^*(W) = W_1 + W_2,$$

where W_1 and W_2 are smooth rational curves with the following intersection properties:

$$\begin{aligned} E_i^2 &= 2, & W_i^2 &= 0, & E_1 \cdot E_2 &= 2, \\ W_1 \cdot W_2 &= 4, & W_i \cdot E_j &= 2 & \text{for } i \neq j \end{aligned}$$

(recall that W is the plane conic tangent to the six lines l_1, \dots, l_6). The linear system $|E_1|$ defines an elliptic fibration on T with six singular fibers of type I_2 . Take the double cover branched along the two fibers $E_1 + E_2 \in 2\text{Pic}(T)$. It induces the canonical resolution commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow & \searrow \varphi & \downarrow \\ \tilde{T} & \longrightarrow & T \end{array}$$

where $\tilde{T} \rightarrow T$ is the blowup of T at the two singular points of $E_1 + E_2$.

CLAIM. X is a Kummer surface.

Proof of the claim. Clearly, $\mathcal{K}_X = \varphi^*(\zeta^*(-H) + (1/2)(E_1 + E_2)) = \mathcal{O}_X$.

(1) The pull back by φ of the six exceptional curves on T defines twelve disjoint smooth rational curves on X .

(2) The two exceptional curves of X give two more rational curves disjoint from (1).

(3) Let $\varphi^*(W_1) = W'_1 + W''_1$ and $\varphi^*(W_2) = W'_2 + W''_2$, and let σ be the lift on X of the covering involution of ζ . Then $\sigma(W'_1) = W'_2$ or $\sigma(W'_1) = W''_2$. Without loss of generality, we can assume that $\sigma(W'_1) = W'_2$, and hence we get the following intersection numbers

$$\begin{aligned} W_i'^2 = W_i''^2 &= -2, & W_i' \cdot W_i'' &= 2 & \text{for } i = 1, 2 \\ W_1' \cdot W_2' = W_1'' \cdot W_2'' &= 4 & \text{and} & & W_1' \cdot W_2'' = W_1'' \cdot W_2' = 0. \end{aligned}$$

One easily checks that W'_1 and W''_2 do not intersect the fourteen curves from (1) and (2).

In particular, the K3 surface X contains sixteen disjoint smooth rational curves. Consequently, X is a Kummer surface. □

Moreover, the surface X contains an elliptic fibration with twelve I_2 fibers. It also admits two nonsymplectic involutions θ and σ , where θ is the cover-

ing involution of the map φ and where σ is the lift of the covering involution of ζ on T encountered earlier. The composition $\iota = \varphi \circ \sigma$ defines a symplectic involution on X whose quotient is a $K3$ surface admitting an elliptic fibration with singular fibers identical to the one defined by F on Y in Proposition 5.1.

In fact, we can now recover sixteen disjoint rational curves on the quotient and conclude that it is our original general Kummer surface Y and that $X \simeq \text{Kum}(B_{12})$. \square

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