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THE RATIONALITY PROBLEM FOR NORM ONE TORI

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To the memory of Professor Masayoshi Nagata

Abstract. We consider the problem of whether the norm one torus defined by a finite separable field extension K/k is stably (or retract) rational over k. This has already been solved for the case where K/k is a Galois extension. In this paper, we solve the problem for the case where K/k is a non-Galois extension such that the Galois group of the Galois closure of K/k is nilpotent or metacyclic.

Introduction

Let K/k be a finite separable field extension, and denote by $R^{(1)}_{K/k}(\mathbb{G}_m)$ the norm one torus defined by K/k, as usual (see, e.g., [V]).

The purpose of this paper is to determine whether the torus $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably (or retract) rational over k. For the case where K/k is Galois, this problem was solved completely in [EM2] and [S]. Hence, we have only to consider this for the case where K/k is non-Galois.

Assume that K/k is non-Galois, and let L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$, and let $H = \mathrm{Gal}(L/K)$. The main results in this paper are the following.

[I] Assume that G is a nilpotent group. Then T is not retract rational over k.

[II] Assume that G is a metacyclic group. Then T is always retract rational over k, and the following conditions are equivalent:

(1) T is stably rational over k;

(2) G is the dihedral group D_n of order $2n$ with n odd or the direct product of the cyclic group C_m of order m and the dihedral group D_n of order $2n$, where m, n are odd, $m, n \geqq 3$, $(m, n) = 1$, and $H \subseteq D_n$ is of order 2.

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[III] Assume that $G = A_n, n \geqq 3$, the alternating group on n letters, and that $H = A_{n-1} \subseteq G$, where H is the stabilizer of one of the letters in G. Then,

(1) T is retract rational over k if and only if n is a prime;

(2) for some $t \geqq 1$, $T^{(t)}$, the product of t copies of T, is stably rational over k if and only if $n = 3, 5$.

For the case of G metacyclic, it is an immediate consequence of [EM2, (1.5)] and [S, (3.14)] that T is retract rational over k. It should be noted that partial results of [I] and [II] have already been given in [CS1].

[I] and [II] are final answers to the problem for the cases of nilpotent groups and metacyclic groups, respectively. [III] can be regarded as an additional remark on the result for symmetric groups in [CS2], [lB], [CK], [LL], and so forth. We will also give another proof of the result for symmetric groups.

§1. Preliminaries

Let G be a finite group. A G-module means a finitely generated left G-module, and a G-module with a $\mathbb{Z}$-basis is said to be a *G-lattice*. A G-lattice M is said to be a *permutation* G-lattice if it has a $\mathbb{Z}$-basis permuted by G, that is, if $M \cong \bigoplus_{1 \leq i \leq m} \mathbb{Z}G/H_i$ for subgroups $H_1, H_2, \ldots, H_m$. M is said to be *invertible* if it is a direct summand of a permutation G-lattice. M is said to be a *quasi-permutation* if there exists an exact sequence of G-lattices

$$0 \to M \to U \to V \to 0,$$

where U and V are permutation lattices. M is said to be *quasi-invertible* if it is a direct summand of a quasi-permutation G-lattice. The dual lattice $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ of a G-lattice M is denoted by M°.

For a subgroup H of G, there exists an exact sequence of G-lattices

$$0 \to I_{G/H} \to \mathbb{Z}G/H \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where ε is the augmentation map and $I_{G/H} = \mathrm{Ker}\, \varepsilon$. The dual lattice $J_{G/H} = (I_{G/H})^\circ$ of $I_{G/H}$ will play a central part in this paper.

When $I_{G/H}$ and $J_{G/H}$ are examined, H can be assumed to contain no normal subgroup of G except $\{1\}$. In fact, let $N \subseteq H$ be a maximal subgroup which is normal in G, set $\overline{G} = G/N$, and set $\overline{H} = H/N$. Then $I_{\overline{G}/\overline{H}} = I_{G/H}$ and $J_{\overline{G}/\overline{H}} = J_{G/H}$, and therefore we may use $\overline{G}$ and $\overline{H}$ instead of G and H, where $\overline{H}$ contains no normal subgroup of $\overline{G}$ except $\{1\}$.

Throughout this paper, a finite group is said to be a *metacyclic* group if all its Sylow subgroups are cyclic.

Let k be a field, let L be a finite Galois extension of k, and let $G = \mathrm{Gal}(L/k)$. Let M be a G-lattice with a $\mathbb{Z}$-basis $\{u_1, u_2, \ldots, u_n\}$. Define the action of G on the rational function field $L(X_1, X_2, \ldots, X_n)$ with variables $X_1, X_2, \ldots, X_n$ over L, as an extension of the action of G over L, as follows. For each $\sigma \in G$,

$$\sigma(X_i) = \prod_{j=1}^{n} X_j^{m_{ij}}, \quad 1 \leqq i \leqq n,$$

when $\sigma u_i = \sum_{j=1}^{n} m_{ij} u_j, m_{ij} \in \mathbb{Z}$, and denote $L(X_1, X_2, \ldots, X_n)$ with this action of G by $L(M)$.

For a given G-lattice M, there exists an algebraic torus T defined over k and split over L such that the character group of T is isomorphic to M as G-lattices, and the invariant subfield $L(M)^G$ of $L(M)$ can be identified with the function field of T.

An extension field F of a basic field k is said to be *rational* over k if it is generated over k by a finite number of elements of F which are algebraically independent over k. F is said to be *stably rational* over k if there exists an extension field of F which is rational over each of k and F. Further, F is said to be *retract rational* over k if there exists an extension field $k(x_1, x_2, \ldots, x_n)$ of F rational over k where $x_1, x_2, \ldots, x_n$ are algebraically independent over k, and if F is the quotient field of a k-subalgebra A of F such that, for some nonzero element s of $k[x_1, x_2, \ldots, x_n]$, we have $A \subseteq k[x_1, x_2, \ldots, x_n][1/s]$ and a k-algebra homomorphism

$$\theta\colon k[x_1, x_2, \ldots, x_n][1/s] \to A$$

whose restriction to A is the identity on A. More generally, F is said to be *unirational* over k if there exists an extension field of F which is rational over k.

It is easy to see that

$$\text{rational} \Longrightarrow \text{stably rational} \Longrightarrow \text{retract rational} \Longrightarrow \text{unirational}.$$

It should be noted that every algebraic torus defined by a separable extension of a field k is unirational over k.

We now have the following.

THEOREM 1.1. *Let L/k be a finite Galois field extension with a group G, and let M be a G-lattice. Then,*

(1) *M is a quasi-permutation G-lattice if and only if $L(M)^G$ is stably rational over k (see, e.g., [EM1, (1.6)]);*
(2) *M is a quasi-invertible G-lattice if and only if $L(M)^G$ is retract rational over k (see [S, (3.14)]).*

Let k be a field, and let K/k be a finite separable extension. Let L/k be the Galois closure of K/k, let $G=\mathrm{Gal}(L/k)$, and let $H=\mathrm{Gal}(L/K)\subseteq G$. The norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ defined by K/k has the lattice $J_{G/H}$ as its character lattice and the field $L(J_{G/H})^G$ as its function field (see [V]). Note that H contains no normal subgroup of G except $\{1\}$, since L/k is the Galois closure of K/k. For the case where K/k is Galois (i.e., $H=\{1\}$), the G-lattices $I_{G/H}$ and $J_{G/H}$ are denoted by I_G and J_G, respectively.

For the case where K/k is Galois, we have the following.

THEOREM 1.2. *Let K/k be a finite Galois field extension with a group G. Then,*

(1) *$R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract rational over k if and only if G is metacyclic (see [EM2, (1.5)], [S, (3.14)]);*
(2) *$R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably rational over k if and only if G is a cyclic group, or a direct product of a cyclic group of order m and a group $\langle\sigma,\tau\mid\sigma^n=\tau^{2^d}=1,\tau\sigma\tau^{-1}=\sigma^{-1}\rangle$, where $d,m\geqq 1$, $n\geqq 3$, m,n odd, and $(m,n)=1$ (see [EM2, (2.3)]).*

Therefore, in this paper, we will consider only the case where K/k is non-Galois, that is, the case where $H\neq\{1\}$.

Let G be a finite group. Let $H_1,H_2,\dots,H_t, t\geqq 2$ be subgroups of G, and let $\varepsilon_i\colon \mathbb{Z}G/H_i\to\mathbb{Z}, 1\leqq i\leqq t$, be the augmentation maps. Then the *multiaugmentation* map

$$\varepsilon=(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_t)\colon \mathbb{Z}G/H_1\oplus\mathbb{Z}G/H_2\oplus\cdots\oplus\mathbb{Z}G/H_t\to\mathbb{Z}$$

is defined by sending $u=(u_i)\in\bigoplus_{i=1}^t\mathbb{Z}G/H_i$ to $\sum_{i=1}^t\varepsilon_i(u_i)\in\mathbb{Z}$.

The following proposition on multiaugmentation maps is simple but very useful.

PROPOSITION 1.3. *Let G be a finite group, and let $H_1,H_2,\dots,H_t$, $t\geqq 2$ be subgroups of G such that $H_{t-1}\supseteq H_t$. Let $\varepsilon_i\colon\mathbb{Z}G/H_i\to\mathbb{Z}, 1\leqq i\leqq t$, be the augmentation maps. Further, let*

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{t-1}, \varepsilon_t) \colon \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} \oplus \mathbb{Z}G/H_t \to \mathbb{Z},$$
$$\varepsilon' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{t-1}) \colon \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} \to \mathbb{Z},$$

be the multiaugmentation maps, set $I = \operatorname{Ker}\varepsilon$, $I' = \operatorname{Ker}\varepsilon'$, *and set* $J = I^\circ$, $J' = (I')^\circ$. *Then* $I \cong I' \oplus \mathbb{Z}G/H_t$ *and* $J \cong J' \oplus \mathbb{Z}G/H_t$.

Proof. Define $\delta_t \colon \mathbb{Z}G/H_t \to \mathbb{Z}G/H_{t-1}$ by $\rho H_t \to \rho H_{t-1}, \rho \in G$, and define

$$\begin{aligned}\delta = (1, 1, \ldots, 1, \delta_t) \colon & \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} \oplus \mathbb{Z}G/H_t \\ & \to \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1}\end{aligned}$$

by sending $(u_1, u_2, \ldots, u_{t-1}, u_t)$ to $(u_1, u_2, \ldots, u_{t-1} + \delta_t(u_t))$. Then δ is a split surjection and $\operatorname{Ker}\delta$ ($\cong \mathbb{Z}G/H_t$) $\subseteq I$. Hence, we can form the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}G/H_t & = & \mathbb{Z}G/H_t & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I & \longrightarrow & \mathbb{Z}G/H_1 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} \oplus \mathbb{Z}G/H_t & \xrightarrow{\varepsilon} & \mathbb{Z} & \to & 0 \\
 & & \downarrow & & \downarrow \delta & & \| & & \\
0 & \longrightarrow & I' & \longrightarrow & \mathbb{Z}G/H_1 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} & \xrightarrow{\varepsilon'} & \mathbb{Z} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then the first column is also split, and so $I \cong I' \oplus \mathbb{Z}G/H_t$ and $J \cong J' \oplus \mathbb{Z}G/H_t$. ∎

Corollary 1.4. *Let* G *be a finite group, and let*

$$\varepsilon^{(t)} \colon [\mathbb{Z}G]^{(t)} \to \mathbb{Z}, \quad t \geqq 2,$$

be the multiaugmentation map of $[\mathbb{Z}G]^{(t)}$, *the direct sum of* t *copies of* $\mathbb{Z}G$, *on* $\mathbb{Z}$ *defined as in Proposition* 1.3 *by augmentation map* : $\mathbb{Z}G \to \mathbb{Z}$. *Let* $I = \operatorname{Ker}\varepsilon^{(t)}$, *and let* $J = I^\circ$. *Then* $I \cong I_G \oplus [\mathbb{Z}G]^{(t-1)}$, *and hence* $J \cong J_G \oplus [\mathbb{Z}G]^{(t-1)}$.

Note that special cases of Proposition 1.3 and Corollary 1.4 have been used in [E] and [CK].

A lattice M over a finite group G is said to be *coflasque* if $H^1(G', M) = 0$ for any subgroup G' of G. Every invertible lattice is coflasque. For any G-lattice M, we can construct an exact sequence

$$0 \to N \to U \to M \to 0,$$

where U is permutation and N is coflasque (see [EM2, (1.1)]). This is said to be a *coflasque resolution* of M.

Proposition 1.5. *Let G be a finite group, and let $0 \to N \to U \to M \to 0$ be an exact sequence of G-lattices with U permutation. Then,*

(1) *M° is a quasi-permutation if and only if N is a quasi-permutation;*

(2) *M° is quasi-invertible if and only if N is quasi-invertible.*

Suppose further that N is coflasque. Then,

(3) *M° is quasi-invertible if and only if N is invertible.*

Proof. For example, see the proof of [EM2, (1.6)]. □

Corollary 1.6. *A lattice over a finite group G is quasi-invertible if and only if it is quasi-invertible over every Sylow subgroup of G.*

Proof. It is well known (see, e.g., [EM2, (1.4)]) that a G-lattice is invertible if and only if it is invertible over every Sylow subgroup of G. Therefore, the assertion follows directly from Proposition 1.5. □

The following proposition is only a slight generalization of Theorem 1.2(1), but this is useful for our problem.

Proposition 1.7. *Let G be a finite group, and let H be a nonnormal Hall subgroup of G. Then $J_{G/H}$ is quasi-invertible over G if and only if all Sylow p-subgroups of G are cyclic for any prime $p \mid [G:H]$.*

Proof. Suppose that there exists a noncyclic Sylow p-subgroup P of G for some prime $p \mid [G:H]$. Then $\mathbb{Z}G/H$ is $\mathbb{Z}P$ free, and therefore $\mathbb{Z}G/H \cong [\mathbb{Z}P]^{(t)}, t \geqq 1$. Hence, by Corollary 1.4, $J_{G/H} \cong J_P \oplus [\mathbb{Z}P]^{(t-1)}$. However, since P is noncyclic, J_P is not quasi-invertible over P. Thus, $J_{G/H}$ is not quasi-invertible over G.

On the other hand, suppose that Sylow p-subgroups of G are cyclic for any prime $p \mid [G:H]$. Let p be a prime divisor of $|G|$, and let P be a Sylow p-subgroup. Assume first that $p \mid [G:H]$. Then, as above, $\mathbb{Z}G/H \cong [\mathbb{Z}P]^{(t)}, t \geqq 1$, as P-lattices, and hence, by Corollary 1.4, $J_{G/H} \cong J_P \oplus [\mathbb{Z}P]^{(t-1)}$. Since P is cyclic by assumption, J_P is quasi-invertible. This shows that $J_{G/H}$ is quasi-invertible over P. Next, assume that $p \mid |H|$. As H is a Hall subgroup, we have $p \nmid [G:H]$, and then the action of P on G/H has a fixed point. Therefore, $J_{G/H}$, as a P-lattice, is a direct summand of $\mathbb{Z}G/H$, which shows also that $J_{G/H}$ is invertible over P. Hence, in both cases, $J_{G/H}$ is quasi-invertible over P. Thus, it follows from Corollary 1.6 that $J_{G/H}$ is quasi-invertible over G. □

§2. Nilpotent groups

In this section, we will prove the following.

THEOREM 2.1. *Let K/k be a finite non-Galois, separable field extension, and let L/k be the Galois closure of K/k. Assume that the Galois group of L/k is nilpotent. Then the norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ defined by K/k is not retract rational over k.*

Let $G = \mathrm{Gal}(L/k)$, and let $H = \mathrm{Gal}(L/K) \subseteq G$. In order to prove Theorem 2.1, it suffices to show by Theorem 1.1(2) that the G-lattice $J = J_{G/H}$ is not quasi-invertible.

We can reduce Theorem 2.1 to the case where G is a p-group for a prime p. In fact, given a nilpotent group G and a nonnormal subgroup $H \subseteq G$, there exists a Sylow p-subgroup P for some $p \mid |G|$ such that $P' = P \cap H$ is nonnormal in P, because nilpotent groups G and H are expressible uniquely as the direct products of their Sylow subgroups. Then we have $\mathbb{Z}G/H \cong [\mathbb{Z}P/P']^{(t)}$ for some $t \geqq 1$ as P-lattices, and so, by Proposition 1.3, $J_{G/H} \cong J_{P/P'} \oplus [\mathbb{Z}P/P']^{(t-1)}$ as P-lattices. Accordingly, it follows that $J_{G/H}$ is not quasi-invertible over G when $J_{P/P'}$ is not quasi-invertible over P.

From now on, we assume that G is a p-group and that $H \subseteq G$ contains no normal subgroup of G except $\{1\}$.

We will prove step by step that $J_{G/H}$ is not quasi-invertible over G.

STEP 1. Case where the center of G is not cyclic.

Proof. Let $Z = Z(G)$ be the center of G. Since H contains no normal subgroup of G except $\{1\}$, we have $H \cap Z = \{1\}$, and so $\mathbb{Z}G/H \cong [\mathbb{Z}Z]^{(t)}$ for some $t \geqq 1$ as Z-lattices. Then, from Corollary 1.4, it follows that $J_{G/H} \cong J_Z \oplus [\mathbb{Z}Z]^{(t-1)}$. Since Z is not cyclic by the assumption, J_Z is not quasi-invertible over Z by Theorem 1.2(1), and so $J_{G/H}$ is not quasi-invertible over G. □

According to Step 1, we may assume from now that the center $Z(G)$ of G is cyclic.

Step 2. Case where p is odd.

Proof. By [Be, (1.4)], there exists a normal subgroup of G as follows:

$$N=\langle\sigma,\tau\mid\sigma^p=\tau^p=1,\sigma\tau=\tau\sigma\rangle.$$

Then, by the above assumption, we may suppose that $N\cap Z(G)=\langle\sigma\rangle$.

Suppose first that $H\cap N=\{1\}$. Then we have $\mathbb{Z}G/H\cong[\mathbb{Z}N]^{(t)}$ for some $t\geqq 1$, as N-lattices, and therefore by Corollary 1.4, $J_{G/H}\cong J_N\oplus[\mathbb{Z}N]^{(t-1)}$, as N-lattices. Since N is not cyclic, J_N is not quasi-invertible over N by Theorem 1.2(1), and hence $J_{G/H}$ is also not quasi-invertible over G.

Next, suppose that $H\cap N\neq\{1\}$. Then we may assume that $H\cap N=\langle\tau\rangle$. As is easily seen, the subgroups $\langle\tau\rangle,\langle\tau\sigma\rangle,\langle\tau\sigma^2\rangle,\dots,\langle\tau\sigma^{p-1}\rangle$ are conjugate under G, because $\langle\tau\rangle$ is not normal in G and $\sigma\in Z(G)$, and so, as N-lattices,

$$\mathbb{Z}G/H\cong[\mathbb{Z}N/\langle\tau\rangle\oplus\mathbb{Z}N/\langle\tau\sigma\rangle\oplus\cdots\oplus\mathbb{Z}N/\langle\tau\sigma^{p-1}\rangle]^{(t)}$$

for some $t\geqq 1$. Let

$$\varepsilon\colon\mathbb{Z}N/\langle\tau\rangle\oplus\mathbb{Z}N/\langle\tau\sigma\rangle\oplus\cdots\oplus\mathbb{Z}N/\langle\tau\sigma^{p-1}\rangle\to\mathbb{Z}$$

be the multiaugmentation map, and set $J=[\operatorname{Ker}\varepsilon]^\circ$. Then it follows from Proposition 1.3 that

$$J_{G/H}\cong J\oplus[\mathbb{Z}N/\langle\tau\rangle\oplus\mathbb{Z}N/\langle\tau\sigma\rangle\oplus\cdots\oplus\mathbb{Z}N/\langle\tau\sigma^{p-1}\rangle]^{(t-1)}.$$

Since J is not quasi-invertible over N by [E, Theorem 2(2)], this implies that $J_{G/H}$ is not quasi-invertible over G. ∎

The following 2-groups are said to be of maximal class (see [Be, p. 26, Definition 2 and (1.7)]):

- the dihedral group
  $D_{2^n}=\langle\sigma,\tau\mid\sigma^{2^n}=\tau^2=1,\tau\sigma\tau^{-1}=\sigma^{-1}\rangle, n\geqq 2,$
- the generalized quaternion group
  $Q_{2^n}=\langle\sigma,\tau\mid\sigma^{2^n}=1,\sigma^{2^{n-1}}=\tau^2,\tau\sigma\tau^{-1}=\sigma^{-1}\rangle, n\geqq 2,$
- the semidihedral group
  $SD_{2^n}=\langle\sigma,\tau\mid\sigma^{2^n}=\tau^2=1,\tau\sigma\tau^{-1}=\sigma^{-1+2^{n-1}}\rangle, n\geqq 3.$

Any subgroup $\neq\{1\}$ of the group Q_{2^n} contains the center $Z(Q_{2^n})=\langle\sigma^{2^{n-1}}\rangle$, and so Q_{2^n} can be omitted from the object of our consideration.

Step 3. Case where $p=2$ and G is of maximal class.

Proof. Assume that $G = D_{2^n}$. Then H is one of the subgroups $\langle\tau\rangle, \langle\tau\sigma\rangle, \ldots, \langle\tau\sigma^{2^{n-1}}\rangle$, and therefore we may assume that $H = \langle\tau\rangle$. Define $N = \langle\sigma^2, \tau\sigma\rangle$. Then N is normal in G and $N \cong D_{2^{n-1}}$ $(n \geqq 3)$ or the elementary abelian group of order 4. Further, we have $\mathbb{Z}G/H \cong \mathbb{Z}N$ as N-lattices, and hence $J_{G/H} = J_N$ is not quasi-invertible over N, again by Theorem 1.2(1). Thus, we conclude that $J_{G/H}$ is not quasi-invertible over G.

Next, assume that $G = SD_{2^n}$. Then H is one of the subgroups $\langle\tau\rangle, \langle\tau\sigma^2\rangle, \ldots, \langle\tau\sigma^{2(2^{n-1}-1)}\rangle$, and therefore we may assume that $H = \langle\tau\rangle$. Note that the subgroups $\langle\tau\sigma\rangle, \langle\tau\sigma^3\rangle, \ldots, \langle\tau\sigma^{2^n-1}\rangle$ of G contain the center $Z(G) = \langle\sigma^{2^{n-2}}\rangle$. Set $N = \langle\sigma^2, \tau\sigma\rangle$. Then N is normal in G and $N \cong Q_{2^{n-1}}$ $(n \geqq 3)$. Further, we have $\mathbb{Z}G/H \cong \mathbb{Z}N$ as N-lattices, and hence, along the same lines as in the dihedral case, we can show that $J_{G/H}$ is not quasi-invertible over G. ∎

Step 4. Case where a 2-group G is not of maximal class and does not have the elementary abelian group of order 8 as its normal subgroup.

Proof. Since G is not of maximal class, there exists an elementary abelian normal subgroup E of order 4 in G by [Be, (1.4)]. The centralizer $C_G(E)$ of E in G is normal in G, and by [Be, (1.8)], we have $E \subsetneq C_G(E)$. Then there is $\rho \in C_G(E) - E$ such that the class $\bar{\rho}$ of ρ in G/E is contained in the center of G/E and is of order 2. Then $N = \langle\rho, E\rangle$ is an abelian, noncyclic normal subgroup of order 8 in G. However, by the assumption, N is not elementary abelian, and therefore it can be expressed as follows:

$$N = \langle\sigma, \tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma\rangle.$$

Since the conjugacy class of σ in G is contained in $\{\sigma, \sigma^3, \sigma\tau, \sigma^3\tau\}$, the conjugacy class of σ^2 in G is $\{\sigma^2\}$, and so we have $\sigma^2 \in Z(G)$. However, by assumption, $Z(G)$ is cyclic. Accordingly, the elements τ and $\sigma^2\tau$ of order 2 in N must be conjugate under G.

Assume first that $H \cap N = \{1\}$. Then $\mathbb{Z}G/H \cong [\mathbb{Z}N]^{(t)}, t \geqq 1$, as N-lattices, and therefore, by Corollary 1.4, $J_{G/H} \cong J_N \oplus [\mathbb{Z}N]^{(t-1)}$, as N-lattices. Since N is not cyclic, we can conclude that $J_{G/H}$ is not quasi-invertible.

Next, assume that $H \cap N \neq \{1\}$. Because $H \cap Z(G) = \{1\}$, $\sigma^2 \notin H$, and so $H \cap N = \{\tau\}$ or $\{\tau\sigma^2\}$. As noted above, τ and $\tau\sigma^2$ are conjugate under G. Hence, we have $\mathbb{Z}G/H \cong [\mathbb{Z}N/\langle\tau\rangle \oplus \mathbb{Z}N/\langle\tau\sigma^2\rangle]^{(s)}, s \geqq 1$, as N-lattices. Let

$$\varepsilon = (\varepsilon_1, \varepsilon_2) \colon U = \mathbb{Z}N/\langle\tau\rangle \oplus \mathbb{Z}N/\langle\tau\sigma^2\rangle \to \mathbb{Z}$$

be the multiaugmentation map, and set $J = [\operatorname{Ker}\varepsilon]^\circ$. Then, by Proposition 1.3, $J_{G/H} \cong J \oplus [\mathbb{Z}N/\langle\tau\rangle \oplus \mathbb{Z}N/\langle\tau\sigma^2\rangle]^{(s-1)}$ as N-lattices. According to Lemma 2.2 given at the end of this section, J is not quasi-invertible over N. Thus, $J_{G/H}$ is not quasi-invertible over G. ∎

STEP 5. Case where the 2-group G has the elementary abelian group E of order 8 as a normal subgroup.

Proof. Let $E = \langle \rho, \sigma, \tau \mid \rho^2 = \sigma^2 = \tau^2 = 1, \rho\sigma = \sigma\rho, \sigma\tau = \tau\sigma, \tau\rho = \rho\tau\rangle$. Since $Z(G)$ is cyclic and $E \cap Z(G) \neq \{1\}$, we may assume that $E \cap Z(G) = \langle\rho\rangle$. Since H contains no normal subgroup of G except $\{1\}$, we have $H \cap Z(G) = \{1\}$, and hence ρ is not contained in any subgroup conjugate to H.

First, suppose that $|H \cap E| = 1$, that is, that $H \cap E = \{1\}$. Then we have $\mathbb{Z}G/H \cong [\mathbb{Z}E]^{(t)}$ for some $t \geqq 1$ as E-lattices, and so the proof is similar to the previous one.

Second, suppose that $|H \cap E| = 2$. Then we may assume that $H \cap E = \langle\sigma\rangle$. Let $E_0 = \langle\rho, \sigma\rangle$. If E_0 is normal in G, then $\{\sigma, \sigma\rho\}$ is a conjugacy class of G. Then the subgroup $E_1 = \langle\rho, \tau\rangle$ does not contain any of σ and $\sigma\rho$. Hence, we have $\mathbb{Z}G/H \cong [\mathbb{Z}E_1]^{(t)}$ for some $t \geqq 1$ as E_1-lattices. On the other hand, if E_0 is not normal in G, then one of the subgroups $\langle\rho, \tau\rangle$ and $\langle\rho, \sigma\tau\rangle$ is normal in G, and we denote it by E_1. Then $E_1 \cap H = \{1\}$, and therefore we have $\mathbb{Z}G/H \cong [\mathbb{Z}E_1]^{(t)}$ for some $t \geqq 1$ as E_1-lattices. Thus, the proof is done in the same way as in the first case.

Finally, suppose that $|H \cap E| = 4$. In this case, we may assume that $H \cap E = \langle\sigma, \tau\rangle$. Now, all the subgroups of order 4 in E are expressible as follows:

$$\langle\sigma,\tau\rangle, \qquad \langle\rho\sigma,\tau\rangle, \qquad \langle\sigma,\rho\tau\rangle, \qquad \langle\rho\sigma,\sigma\tau\rangle,$$
$$\langle\rho,\sigma\rangle, \qquad \langle\rho,\tau\rangle, \qquad \langle\rho,\sigma\tau\rangle.$$

The groups in the second row are not conjugate to those in the first row under G, because $E \cap Z(G) = \langle\rho\rangle$.

We will now show that the groups in the first row of the above list are conjugate under G. Let $E_1 = \langle\rho, \sigma\tau\rangle$, let $E_2 = \langle\rho, \sigma\rangle$, and let $E_3 = \langle\rho, \tau\rangle$. It is easy to see that at least one of E_1, E_2, and E_3 is normal in G, and so we may assume that E_1 is normal in G. Then the centralizer $C_G(\sigma\tau)$ of $\sigma\tau$ in G is a maximal subgroup of G; that is, $[G : C_G(\sigma\tau)] = 2$. Note that E_2 and E_3 are either both normal or both nonnormal in G.

We first consider the case where both E_2 and E_3 are normal in G. Then both $C_G(\sigma)$ and $C_G(\tau)$, the centralizers of σ and τ in G, are maximal in

G. These three maximal subgroups of G are distinct. In fact, if $C_G(\sigma) = C_G(\tau)$, for example, then $C_G(\sigma) = C_G(\tau) = C_G(\sigma\tau)$, since $C_G(\sigma) \cap C_G(\tau) \subseteq C_G(\sigma\tau)$. Setting $C = C_G(\sigma) = C_G(\tau) = C_G(\sigma\tau)$ and taking $\mu \in G - C$, we have $\mu\sigma\mu^{-1} = \rho\sigma$, $\mu\tau\mu^{-1} = \rho\tau$, and $\mu\sigma\tau\mu^{-1} = \rho\sigma\tau$, because E_1, E_2, and E_3 are normal in G. From the equalities $\mu\sigma\mu^{-1} = \rho\sigma$ and $\mu\tau\mu^{-1} = \rho\tau$, it follows that $(\mu\sigma\mu^{-1})(\mu\tau\mu^{-1}) = (\rho\sigma)(\rho\tau) = \sigma\tau$. This contradicts obviously the third equality $\mu\sigma\tau\mu^{-1} = \rho\sigma\tau$. Now, let $\mu \in C_G(\sigma) - C_G(\tau)$ and $\nu \in C_G(\tau) - C_G(\sigma)$. Then we have

$$\begin{aligned} \mu\sigma\mu^{-1} &= \sigma, & \mu\tau\mu^{-1} &= \rho\tau, \\ \nu\sigma\nu^{-1} &= \rho\sigma, & \nu\tau\nu^{-1} &= \tau, \\ (\nu\mu)\sigma(\nu\mu)^{-1} &= \rho\sigma, & (\nu\mu)\tau(\nu\mu)^{-1} &= \rho\tau. \end{aligned}$$

This implies that the groups given in the first row are conjugate under G.

Second, we consider the case where both E_2 and E_3 are nonnormal in G. In this case, the set $\{\sigma, \rho\sigma, \tau, \rho\tau\}$ is a conjugacy class of G, because $[G : C_G(\sigma)] = [G : C_G(\tau)] = 4$. If $C_G(\sigma) = C_G(\tau)$, then $C = C_G(\sigma) = C_G(\tau) \subsetneq C_G(\sigma\tau) \subsetneq G$. Let $\mu \in G - C_G(\sigma\tau)$. Since $\mu \notin C$, we have

$$\mu\sigma\mu^{-1} = \tau, \qquad \mu\tau\mu^{-1} = \rho\sigma$$

or

$$\mu\sigma\mu^{-1} = \rho\tau, \qquad \mu\tau\mu^{-1} = \sigma.$$

Then we have further

$$\mu^2\sigma\mu^{-2} = \rho\sigma, \qquad \mu^2\tau\mu^{-2} = \rho\tau$$

and

$$\mu^3\sigma\mu^{-3} = \rho\tau, \qquad \mu^3\tau\mu^{-3} = \sigma$$

or

$$\mu^3\sigma\mu^{-3} = \tau, \qquad \mu^3\tau\mu^{-3} = \rho\sigma.$$

Therefore, the subgroups given in the first row are conjugate under G. On the other hand, if $C_G(\sigma) \neq C_G(\tau)$, then there exist $\mu \in C_G(\sigma) - C_G(\tau)$ and $\nu \in C_G(\tau) - C_G(\sigma)$. Using these μ, ν, we can show in the same way as in the first case that the four subgroups are conjugate under G. Thus, in both cases, we conclude that the four subgroups are conjugate under G.

Since $H \cap E = \langle \sigma, \tau \rangle$, we have

$$\mathbb{Z}G/H \cong [\mathbb{Z}E/\langle \sigma, \tau \rangle \oplus \mathbb{Z}E/\langle \rho\sigma, \tau \rangle \oplus \mathbb{Z}E/\langle \sigma, \rho\tau \rangle \oplus \mathbb{Z}E/\langle \rho\sigma, \sigma\tau \rangle]^{(t)}$$

for some $t \geqq 1$ as E-lattices. Let

$$\varepsilon\colon \mathbb{Z}E/\langle\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\sigma,\rho\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\sigma\tau\rangle \to \mathbb{Z}$$

be the multiaugmentation map, and set $J = [\operatorname{Ker}\varepsilon]^\circ$. Then it follows from Proposition 1.3 that

$$J_{G/H} \cong J \oplus [\mathbb{Z}E/\langle\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\sigma,\rho\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\sigma\tau\rangle]^{(t-1)}.$$

Since J is not quasi-invertible over E by [E, Theorem 2(1)], this implies that $J_{G/H}$ is not quasi-invertible over G. This completes the proof of this step, and so the proof of Theorem 2.1. □

Finally, we show the following lemma, which has been used in Step 4.

LEMMA 2.2. *Let $G = \langle\sigma,\tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma\rangle$ be the direct product of the cyclic groups $\langle\sigma\rangle$ and $\langle\tau\rangle$. Let*

$$\varepsilon = (\varepsilon_1,\varepsilon_2)\colon U = \mathbb{Z}G/\langle\tau\rangle \oplus \mathbb{Z}G/\langle\tau\sigma^2\rangle \to \mathbb{Z}$$

be the multiaugmentation map, set $I = \operatorname{Ker}\varepsilon$, and set $J = I^\circ$. Then J is not quasi-invertible.

Proof. We construct a concrete coflasque resolution of I. The subgroups of G are as follows:

order 1	$\{1\}$
order 2	$H_0 = \langle\sigma^2\rangle, H_1 = \langle\tau\rangle, H_2 = \langle\tau\sigma^2\rangle$
order 4	$N_0 = \langle\sigma^2,\tau\rangle, N_1 = \langle\sigma\rangle, N_2 = \langle\sigma\tau\rangle$
order 8	G

Under this notation, we have $U = \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2$. Both $\mathbb{Z}G/H_1$ and $\mathbb{Z}G/H_2$ have $\{1,\sigma,\sigma^2,\sigma^3\}$ as representatives of the cosets, so I can be expressed as follows:

$$I = \mathbb{Z}\langle\sigma\rangle(1,-1) + \mathbb{Z}\langle\sigma\rangle(0,\sigma-1) = \mathbb{Z}\langle\sigma\rangle(1,-1) + \mathbb{Z}\langle\sigma\rangle(\sigma-1,0).$$

Here, note that $\tau(1,-1) = (1,-1) - (0,\sigma^2-1)$. Then we have

$$I^{H_1} = \mathbb{Z}\langle\sigma\rangle(1+\sigma^2)(1,-1) + \mathbb{Z}\langle\sigma\rangle(\sigma-1,0),$$

$$I^{H_2} = \mathbb{Z}\langle\sigma\rangle(1+\sigma^2)(1,-1) + \mathbb{Z}\langle\sigma\rangle(0,\sigma-1),$$

$$\begin{aligned} I^{H_0} = I^{N_0} &= \mathbb{Z}\langle\sigma\rangle(1+\sigma^2)(1,-1) + \mathbb{Z}(1+\sigma^2)(\sigma-1,0) \\ &= \mathbb{Z}\langle\sigma\rangle(1+\sigma^2)(1,-1) + \mathbb{Z}(1+\sigma^2)(0,\sigma-1), \end{aligned}$$

$$I^G = I^{N_1} = I^{N_2} = \mathbb{Z}(1+\sigma+\sigma^2+\sigma^3)(1,-1) \cong \mathbb{Z}.$$

Define now the G-homomorphisms

$$\begin{aligned}
\delta_0 &: \mathbb{Z}G \to I && \text{by } \mapsto (1,-1),\\
\delta_1 &: \mathbb{Z}G/H_1 \to I && \text{by } 1 \mapsto (\sigma-1,0),\\
\delta_2 &: \mathbb{Z}G/H_2 \to I && \text{by } 1 \mapsto (0,\sigma-1),\\
\delta_3 &: \mathbb{Z}G/N_0 \to I && \text{by } 1 \mapsto (1+\sigma^2)(1,-1).
\end{aligned}$$

Set $V = \mathbb{Z}G \oplus \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \mathbb{Z}G/N_0$, let

$$\delta = (\delta_0, \delta_1, \delta_2, \delta_3) : V \to I,$$

and let $W = \operatorname{Ker}\delta$. Then it is easy to see that $H^1(H, W) = 0$ for every subgroup H of G. This shows that W is coflasque.

We denote by X^* the completion of a lattice X at 2. Then, $\mathbb{Z}^*G$ is a local ring, and, for any subgroup H of G, $\mathbb{Z}^*G/H$ is indecomposable.

Suppose now that J is quasi-invertible. Then W is invertible by Proposition 1.5(3). Since the Krull-Schmidt theorem holds for permutation lattices over $\mathbb{Z}^*G$, the completion W^* of W at 2 must be a permutation.

From the exact sequences

$$0 \to I \to U \to \mathbb{Z} \to 0,$$
$$0 \to W \to V \to I \to 0,$$

we get the list of the $\mathbb{Z}$-rank for these G-lattices as follows:

H	$\operatorname{rank}_{\mathbb{Z}} U^H$	$\operatorname{rank}_{\mathbb{Z}} I^H$	$\operatorname{rank}_{\mathbb{Z}} V^H$	$\operatorname{rank}_{\mathbb{Z}} W^H$
$\{1\}$	8	7	18	11
H_0	4	3	10	7
H_1	6	5	12	7
H_2	6	5	12	7
N_0	4	3	8	5
N_1	2	1	5	4
N_2	2	1	5	4
G	2	1	4	3

From this list, we can deduce that $W^* \cong \mathbb{Z}^*G \oplus \mathbb{Z}^*G/N_0 \oplus \mathbb{Z}^*$. Now, we have the exact sequence

$$0 \to W^* \to V^* \to I^* \to 0,$$

with $V^* \cong \mathbb{Z}^*G \oplus \mathbb{Z}^*G/H_1 \oplus \mathbb{Z}^*G/H_2 \oplus \mathbb{Z}^*G/N_0$. Setting $V' = \mathbb{Z}^*G/H_1 \oplus \mathbb{Z}^*G/H_2 \oplus \mathbb{Z}^*G/N_0$ and $W' = \mathbb{Z}^*G/N_0 \oplus \mathbb{Z}^*$, and forming the pushout of

$$\begin{array}{ccc} W^* & \longrightarrow & V^* \\ \big\downarrow & & \\ W' & & \end{array}$$

we obtain the exact sequence

$$0 \to W' \to V' \to I^* \to 0,$$

which is obviously a contradiction, because the image of V' in I^* cannot contain the element $(1, -1)$. This concludes that J is not quasi-invertible. □

Remark 2.3. In [CS1, (d3)], it was shown that the torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not rational over k in the case where $\mathrm{Gal}(L/k)$ is the dihedral group D_4 of order 8 and $\mathrm{Gal}(L/K)$ is the subgroup of order 2.

§3. Metacyclic groups

The main result of this section is the following.

Theorem 3.1. *Let K/k be a finite non-Galois, separable field extension, and let L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$, and let $H = \mathrm{Gal}(L/K) \subseteq G$. Assume that G is metacyclic. Then the following conditions are equivalent.*

(1) *The norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ defined by K/k is stably rational over k.*
(2) *G is the dihedral group D_n of order $2n$ with n odd ($n \geqq 3$) or the direct product of the cyclic group C_m of order m and the dihedral group D_n of order $2n$, where m, n are odd, $m, n \geqq 3$, $(m, n) = 1$, and $H \subseteq D_n$ is of order 2.*

Note that Theorem 3.1(2) is equivalent to the following.

(2′) $H = C_2$ is the cyclic group of order 2, and G is isomorphic to a semidirect product $C_r \rtimes H$, $r \geqq 3$ odd, where H acts nontrivially on the cyclic group C_r of order r.

Let G be a nonabelian metacyclic group. Then G is expressible as the semidirect product of the cyclic normal subgroup $N_0 = C_l$ of order l by the cyclic subgroup $H_0 = C_f$ of order f, all Sylow subgroups of which are nonnormal in G, where $l \geqq 3$ odd, $f \geqq 2$, and $(f, l) = 1$. We define

$$i(G) = |\mathrm{Im}(H_0 \to \mathrm{Aut}\, N_0)|.$$

Theorem 3.1 is only a restatement of the following (see [EM2, (2.3) and p. 92, (1′)]).

THEOREM 3.2. *Let G be a nonabelian metacyclic group, and let H be a nonnormal subgroup of G which contains no normal subgroup of G except $\{1\}$. Then the following conditions are equivalent:*
(1) *$i(G) = 2$;*
(2) *$J_{G/H}$ is a quasi-permutation G-lattice;*
(3) *$[J_{G/H}]^{(t)}$ is a quasi-permutation G-lattice for some $t \geqq 1$.*

REMARK 3.3. The partial results of Theorems 3.1 and 3.2 were obtained in [CS1, (R4) and (d1)] and [F, (2.3)]. It is given without proof in [CS1, (d1)] that, for the case of $\mathrm{Gal}(L/k) = D_n$ with n odd, the torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ is rational over k.

Now we will prove Theorem 3.2. In Theorem 3.2 the implication (2) ⇒ (3) is obvious, and so it suffices to prove the implications (1) ⇒ (2) and (3) ⇒ (1).

The proof of (3) ⇒ (1). Assume that $i(G) \geqq 3$.

Case 1. Suppose that $|H| \geqq 3$. Then there exist a subgroup H' of H with $|H'| = 4$ or q an odd prime and a subgroup N' of N_0 with $|N'| = p$ an odd prime such that H' acts faithfully on N' by conjugation. Set $G' = N'H'$, and regard $\mathbb{Z}G/H'$ as a G'-lattice. Then we have $\mathbb{Z}G/H \cong \mathbb{Z}G'/H' \oplus S'$ as G'-lattices, where $S' = 0$ or $S' = \bigoplus_{i=1}^{d} \mathbb{Z}G'/H'_i, d \geqq 1$, for subgroups $H'_i \subseteq H'$, and so, by Proposition 1.3, $J_{G/H} \cong J_{G'/H'} \oplus S'$. Therefore, it suffices to show that $[J_{G'/H'}]^{(t)}$ is not a quasi-permutation over G' for any $t \geqq 1$.

Suppose that $[J_{G'/H'}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$. We have an exact sequence

$$0 \to I_{G'/H'} \to \mathbb{Z}G'/H' \to \mathbb{Z} \to 0.$$

Let σ' be a generator of N'. Then $I_{G'/H'}$ is generated by $\sigma' - 1$. Therefore, defining the map $\phi : \mathbb{Z}G' \to I_{G'/H'}$ by $\phi(1) = \sigma' - 1$ and setting $B' = \mathrm{Ker}\,\phi$, we have an exact sequence

$$0 \to B' \to \mathbb{Z}G' \to I_{G'/H'} \to 0.$$

It is easy to see that B' is coflasque, and so, by Proposition 1.5, B' is invertible. By assumption, we have $[B']^{(t)} \oplus U' \cong V'$ for some permutation G'-lattices U' and V', and so $[B']^{(t)} \oplus U' \cong [[B']^\circ]^{(t)} \oplus U'$. From this it follows that $H^i(G', B') \cong H^i(G', [B']^\circ)$ for any i. Computing the two-dimensional cohomology groups, we obtain

$$H^2(G', B') \cong H^1(G', I'_{G'/H'}) \cong \mathbb{Z}/p\mathbb{Z},$$

$$H^2(G', [B']^\circ) \cong H^3(G', J_{G'/H'}) \cong H^4(G', \mathbb{Z})_p \cong H^4(N', \mathbb{Z})^{H'} \cong [\mathbb{Z}/p\mathbb{Z}]^{H'},$$

where the p-part of a finite abelian group A is denoted by A_p. This implies that H' acts trivially on $H^4(N', \mathbb{Z})$, but, according to [Br, p. 159, Example 6], this is not the case because of $|H'| = 4$ or q.

Case 2. Suppose that $|H| = 2$.

If there exists an odd prime $q \mid i(G)$, then there exist a subgroup H' of H_0 with $|H'| = q$ and a subgroup N' of N_0 with $|N'| = p$ an odd prime such that H' acts nontrivially on N' by conjugation. Setting $G' = N'H'$, and regarding $\mathbb{Z}G/H$ as a G'-lattice, we have $\mathbb{Z}G/H \cong [\mathbb{Z}G']^{(s)}$ for some $s \geqq 1$, as G'-lattices, and therefore, by Corollary 1.4, $J_{G/H} \cong J_{G'} \oplus [\mathbb{Z}G']^{(s-1)}$. According to [EM2, (2.3)], $[J_{G'}]^{(t)}$ is not a quasi-permutation over G' for any $t \geqq 1$. Thus, we conclude that $[J_{G/H}]^{(t)}$ is not a quasi-permutation over G for any $t \geqq 1$.

If $i(G) (\geqq 3)$ is a power of 2, then there exist a subgroup H' of H_0 with $|H'| = 4$ containing H and a subgroup N' of N_0 with $|N'| = p$ an odd prime such that H' acts faithfully on N' by conjugation. Setting $G' = N'H'$, and regarding $\mathbb{Z}G/H$ as a G'-lattice, we have $\mathbb{Z}G/H \cong \mathbb{Z}G'/H \oplus S'$ as G'-lattices, where $S' = 0$ or $S' = \bigoplus_{i=1}^{d} \mathbb{Z}G'/H_i$, $d \geqq 1$, for subgroups $H_i \subseteq H$, and so, by Proposition 1.3, $J_{G/H} \cong J_{G'/H} \oplus S'$. Therefore, it suffices to show that $[J_{G'/H}]^{(t)}$ is not a quasi-permutation over G'. Suppose that $[J_{G'/H}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$. We have an exact sequence

$$0 \to I_{G'/H} \to \mathbb{Z}G'/H \to \mathbb{Z} \to 0.$$

Let $N' = \langle \sigma \rangle$, and let $H' = \langle \tau \rangle$. Then we have $I_{G'/H} = (\sigma - 1, \tau - 1)$. Noticing that $[I_{G'/H}]^{N'} = \mathbb{Z}(\sum_{i=0}^{p-1} \sigma^i)(\tau - 1)$, we can construct the following coflasque resolution of $I_{G'/H}$:

$$0 \to B' \to \mathbb{Z}G' \oplus \bigoplus_{i=1}^{s} \mathbb{Z}G'/H'_i \to I_{G'/H} \to 0,$$

where each H'_i is a subgroup of H'. Then B' is invertible by [EM2, (1.5)]. Since $[J_{G'/H}]^{(t)}$ is a quasi-permutation, $[B']^{(t)}$ is a quasi-permutation by Proposition 1.5, and so we have $[B']^{(t)} \oplus U' \cong V'$ for some permutation G'-lattices U' and V'. From this it follows that $H^i(G', B') \cong H^i(G', [B']^\circ)$ for any i. Computing the cohomology groups $H^2(G', B')$ and $H^2(G', [B']^\circ)$ along the same lines as in Case 1, and considering only the p-parts of the cohomology groups, we finally see that

$$\mathbb{Z}/p\mathbb{Z} \cong H^4(N', \mathbb{Z})^{H'} \cong [\mathbb{Z}/p\mathbb{Z}]^{H'},$$

which is a contradiction. This completes the proof $(3) \Rightarrow (1)$. ∎

The proof of $(1) \Rightarrow (2)$. Assume that $i(G) = 2$.

Under this assumption, the subgroups of H of G as in the theorem are of order 2. Therefore, G and H are expressible as follows:

$$G = \langle \mu, \nu, \tau \mid \mu^m = \nu^n = \tau^2 = 1, \mu\nu = \nu\mu, \mu\tau = \tau\mu, \tau\nu\tau^{-1} = \nu^{-1} \rangle$$

and $H = \langle \tau \rangle$, where m, n are odd, $m \geqq 1$, $n \geqq 3$, and $(m, n) = 1$, that is, that $G = \langle \mu \rangle \times \langle \nu, \tau \rangle$, the direct product of the cyclic group C_m of order m and the dihedral group D_n of order $2n$.

Now we will prove that $J_{G/H}$ is a quasi-permutation, by induction on the number of prime divisors of n. We denote by $\Phi_a(X)$ the ath cyclotomic polynomial and by ζ_a the primitive ath root of unity.

Set $\sigma = \mu\nu$, and set $l = mn$. Let p be a prime divisor of n. Let $n = p^c n'$, $p \nmid n'$, and let $l' = l/p^c$. Further, let $\Psi(X) = \prod_{r \mid l'} \Phi_{p^c r}(X)$ and $\Psi_0(X) = \Psi(X)/\Phi_{p^c}(X)$, and let $\Gamma = \mathbb{Z}G/(\Psi(\sigma))$, $\Gamma_0 = \mathbb{Z}G/(\Psi_0(\sigma))$, and $\Gamma_1 = \mathbb{Z}G/(\Phi_{p^c}(\sigma))$. Then there is an exact sequence of G-lattices

$$0 \to \Gamma_1 \to \Gamma \to \Gamma_0 \to 0.$$

From now on, the tensor products $\otimes$ mean those over $\mathbb{Z}G$ for brevity. As is easily seen, $\Gamma_1 \otimes I_{G/H}$, $\Gamma \otimes I_{G/H}$, and $\Gamma_0 \otimes I_{G/H}$ are torsion free, and hence the following sequence is exact:

$$0 \to \Gamma_1 \otimes I_{G/H} \to \Gamma \otimes I_{G/H} \to \Gamma_0 \otimes I_{G/H} \to 0.$$

From the fact that $\Psi_0(1) = \pm 1$, it follows that $\Gamma_0 \otimes \mathbb{Z} = 0$, and so, tensoring Γ_0 with the exact sequence $0 \to I_{G/H} \to \mathbb{Z}G/H \to \mathbb{Z} \to 0$, we have

$$\Gamma_0 \otimes I_{G/H} \cong \Gamma_0 \otimes \mathbb{Z}G/H.$$

Let $\eta(X) = (X^l - 1)/\Psi_0(X) = (X^{l/p} - 1)\Phi_{p^c}(X)$, and let $\Gamma' = \mathbb{Z}G/(\eta(\sigma))$. Then we have the following exact sequence:

$$0 \to \Gamma' \to \mathbb{Z}G \to \Gamma_0 \to 0.$$

Tensoring $\mathbb{Z}G/H$ with this exact sequence, we obtain the following exact sequence:

$$0 \to \Gamma' \otimes \mathbb{Z}G/H \to \mathbb{Z}G/H \to \Gamma_0 \otimes \mathbb{Z}G/H \to 0.$$

Using these facts, we can form the following pullback diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Gamma' \otimes \mathbb{Z}G/H & = & \Gamma' \otimes \mathbb{Z}G/H & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_1 \otimes I_{G/H} & \longrightarrow & M & \longrightarrow & \mathbb{Z}G/H & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_1 \otimes I_{G/H} & \longrightarrow & \Gamma \otimes I_{G/H} & \longrightarrow & \Gamma_0 \otimes I_{G/H} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now, $\Gamma_1 \cong \mathbb{Z}[\zeta_{p^c}, \tau]$ is the twisted group ring of H over $\mathbb{Z}[\zeta_{p^c}]$, and $\Gamma_1 \otimes I_{G/H} \cong (\zeta_{p^c} - 1) \subseteq \mathbb{Z}[\zeta_{p^c}]$ is an ambiguous ideal of $\mathbb{Z}[\zeta_{p^c}]$. As is easily seen, $\mathbb{Q}(\zeta_{p^c})$ is tamely ramified over $\mathbb{Q}(\zeta_{p^c})^H = \mathbb{Q}(\zeta_{p^c} + \zeta_{p^c}^{-1})$, and then Γ_1 is a nonmaximal, hereditary order in the full matrix algebra $M_2(\mathbb{Q}(\zeta_{p^c} + \zeta_{p^c}^{-1}))$ of degree 2 over $\mathbb{Q}(\zeta_{p^c} + \zeta_{p^c}^{-1})$. Setting $S = \mathbb{Z}[\zeta_{p^c}]$ and $P = (\zeta_{p^c} - 1)$, we have $\Gamma_1 \cong S \oplus P$ as Γ_1-lattices and $(\Gamma_1)^\circ \cong \Gamma_1$, $S^\circ \cong S$. Hence, all of Γ_1, $(\Gamma_1)^\circ$, S°, S, P, P° are Γ_1-projective (see [R], [CR, Section 28]). Since Γ_1 is $\mathbb{Z}H$ free, so is $\Gamma_1 \otimes I_{G/H}$. Therefore, we have

$$\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G/H, \Gamma_1 \otimes I_{G/H}) \cong H^1(H, \Gamma_1 \otimes I_{G/H}) = 0.$$

Accordingly, the second row of the above diagram is split, and so we obtain the exact sequence

$$0 \to \Gamma' \otimes \mathbb{Z}G/H \to [\Gamma_1 \otimes I_{G/H}] \oplus \mathbb{Z}G/H \to \Gamma \otimes I_{G/H} \to 0.$$

We further see that $[\Gamma_1 \otimes I_{G/H}]^\circ (\cong P^\circ)$ is a quasi-permutation $G/\langle \sigma^{p^c} \rangle$-lattice, because both Γ_1 and S are quasi-permutations. On the other hand, there is an exact sequence

$$0 \to \mathbb{Z}G/\langle \sigma^{l/p} \rangle \to \Gamma' \to \Gamma_1 \to 0.$$

Tensoring $\mathbb{Z}G/H$ with this, we obtain an exact sequence

$$0 \to \mathbb{Z}G/\langle \sigma^{l/p} \rangle \otimes \mathbb{Z}G/H \to \Gamma' \otimes \mathbb{Z}G/H \to \Gamma_1 \otimes \mathbb{Z}G/H \to 0.$$

Since $\mathbb{Z}G/\langle \sigma^{l/p} \rangle \otimes \mathbb{Z}G/H$ is a permutation and $\Gamma_1 \otimes \mathbb{Z}G/H \cong \mathbb{Z}[\zeta_{p^c}]$, $[\Gamma' \otimes \mathbb{Z}G/H]^\circ$ is also a quasi-permutation. Hence, setting $U = [\Gamma_1 \otimes I_{G/H}]^\circ \oplus \mathbb{Z}G/H$ and $V = [\Gamma' \otimes \mathbb{Z}G/H]^\circ$, we have an exact sequence

$$\text{(i)} \qquad 0 \to [\Gamma \otimes I_{G/H}]^\circ \to U \to V \to 0,$$

where both U and V are quasi-permutations.

Let $\overline{G} = G/\langle \sigma^{mn'p^{c-1}} \rangle$. Note that $\mathbb{Z}\overline{G} = \mathbb{Z}G/\langle \sigma^{mn'p^{c-1}} \rangle \cong \Psi(\sigma)\mathbb{Z}G$ and that $\Psi(1) = p$. Then we can form the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & I_{G/H} & \longrightarrow & \Gamma \otimes I_{G/H} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}\overline{G}/H & \longrightarrow & \mathbb{Z}G/H & \longrightarrow & \Gamma \otimes \mathbb{Z}G/H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It is easy to see that $Y \cong I_{\overline{G}/H}$, and so we have an exact sequence

$$\text{(ii)} \qquad 0 \to [\Gamma \otimes I_{G/H}]^\circ \to J_{G/H} \to J_{\overline{G}/H} \to 0.$$

Finally, we show that, for any subgroup G' of G,

$$\text{(iii)} \qquad H^0(G', [\Gamma \otimes I_{G/H}]^\circ) = H^0(G', \Gamma \otimes I_{G/H}) = 0.$$

In order to show (iii), we first prove that

$$\text{(iii}'\text{)} \qquad H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0 \quad \text{for any } G' \subseteq G.$$

By definition, $\Psi(X) = \Phi_p(X^{mn'p^{c-1}})$, and so $\Gamma \otimes \mathbb{Z}G/H = \mathbb{Z}[\sigma]/(\Psi(\sigma)) = \mathbb{Z}[\sigma]/(\Phi_p(\sigma^{mn'p^{c-1}})) \cong \mathbb{Z}[\zeta_p] + \mathbb{Z}[\zeta_p]\sigma + \mathbb{Z}[\zeta_p]\sigma^2 + \cdots + \mathbb{Z}[\zeta_p]\sigma^{mn'p^{c-1}(p-1)-1}$, where $\sigma^{mn'p^{c-1}} = \zeta_p$. From this it follows that $[\Gamma \otimes \mathbb{Z}G/H]^{N_0} = 0$, where $N_0 = \langle \sigma^{mn'p^{c-1}} \rangle$.

Assume first that $N_0 \subseteq G'$. Then $[\Gamma \otimes \mathbb{Z}G/H]^{G'} = 0$, so that $H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0$. Next assume that $G' \subseteq N$. From the exact sequence

$$0 \to \mathbb{Z}G/\langle \sigma^{mn'p^{c-1}}, \tau \rangle \to \mathbb{Z}G/H \to \Gamma \otimes \mathbb{Z}G/H \to 0,$$

we obtain the following exact sequence:

$$\to H^0(G', \mathbb{Z}G/H) \to H^0(G', \Gamma \otimes \mathbb{Z}G/H) \to H^1(G', \mathbb{Z}G/\langle \sigma^{mn'p^{c-1}}, \tau \rangle) \to$$

Since $\mathbb{Z}G/H \cong \mathbb{Z}N$ as N-lattices, $H^0(G', \mathbb{Z}G/H) = 0$, and since $\mathbb{Z}G/\langle \sigma^{mn'p^{c-1}}, \tau \rangle$ is a permutation, $H^1(G', \mathbb{Z}G/\langle \sigma^{mn'p^{c-1}}, \tau \rangle) = 0$. Thus, we have $H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0$. Further, assume that $G' = H = \langle \tau \rangle$ (or one of its conjugates). Then, we have $N_{G'}(-(\zeta_p + \zeta_p^2 + \cdots + \zeta_p^{(p-1)/2})u) = u$ for any $u \in [\Gamma \otimes \mathbb{Z}G/H]^{G'}$, which implies that $H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0$. In the other cases, we may assume that $G' = \langle \sigma^{m'n''p^c}, \tau \rangle$, where $m' \mid m$, $n'' \mid n'$ and $m' < m$ or $n'' < n$. Set $N' = \langle \sigma^{m'n''p^c} \rangle$. Then we have $[\Gamma \otimes \mathbb{Z}G/H]^{N'} = (1 + \mu^{m'} + \cdots + (\mu^{m'})^{m/m'-1})(1 + \nu^{n''} + \cdots + (\nu^{n''})^{n'/n''-1})\Gamma \otimes \mathbb{Z}G/H$, and therefore $[\Gamma \otimes \mathbb{Z}G/H]^{G'} = (1 + \mu^{m'} + \cdots + (\mu^{m'})^{m/m'-1})(1 + \nu^{n''} + \cdots + (\nu^{n''})^{n'/n''-1})[\Gamma \otimes \mathbb{Z}G/H]^H = N_{G'}(\Gamma \otimes \mathbb{Z}G/H)$, which implies that $H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0$. This concludes the proof of (iii$'$).

From (iii$'$) and the exact sequence

$$0 \to \Gamma \otimes I_{G/H} \to \Gamma \otimes \mathbb{Z}G/H \to \mathbb{Z}/p\mathbb{Z} \to 0,$$

we obtain an exact sequence

$$H^{-1}(G', \Gamma \otimes \mathbb{Z}G/H) \xrightarrow{\theta} H^{-1}(G', \mathbb{Z}/p\mathbb{Z}) \to H^0(G', \Gamma \otimes I_{G/H}) \to 0$$

for any $G' \subseteq G$. The above map $\theta : H^{-1}(G', \Gamma \otimes \mathbb{Z}G/H) \to H^{-1}(G', \mathbb{Z}/p\mathbb{Z})$ is surjective. In fact, we have $H^{-1}(G', \mathbb{Z}/p\mathbb{Z}) = 0, \mathbb{Z}/p\mathbb{Z}$ when $p \nmid |G'|$, $p \mid |G'|$, respectively. For the case where $p \mid |G'|$, $N_0 = \langle \sigma^{mn'p^{c-1}} \rangle \subseteq G'$ and $N_{N_0}(\Gamma \otimes \mathbb{Z}G/H) = (1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{p-1})\Gamma \otimes \mathbb{Z}G/H = 0$, and therefore $\operatorname{Ker} N_{G'} = \operatorname{Ker} N_{N_0} = \Gamma \otimes \mathbb{Z}G/H$. Thus, θ is surjective; that is, $H^0(G', \Gamma \otimes I_{G/H}) = 0$, which completes the proof of (iii).

By (i), (iii), and [EM2, (2.2)], $[\Gamma \otimes I_{G/H}]^\circ$ is a quasi-permutation. Further, by (ii) and [EM2, (2.2)], $J_{G/H}$ is a quasi-permutation if and only if $J_{\overline{G}/H}$ is so. Note that, for the case where $n = p$, $\overline{G}$ is cyclic of order $2m$, and therefore $J_{\overline{G}/H}$ is a quasi-permutation. Hence, by induction, we can show that $J_{G/H}$ is a quasi-permutation. This completes the proof of the implication (1) $\Rightarrow$ (2). ∎

Remark 3.4. The above proof of (3) $\Rightarrow$ (1) was done in the same way as in [CS1, (R4)]. The proof of (1) $\Rightarrow$ (2) was done by making some modifications on that in [EM2, (2.3)].

§4. Symmetric groups and alternating groups

In this section, we consider the problem for S_n (resp., A_n), the symmetric (resp., alternating) group on n letters. We also assume that the subgroup S_{n-1} (resp., A_{n-1}) of S_n (resp., A_n) is the stabilizer of one of the letters in S_n (resp., A_n).

Let K/k be a non-Galois separable field extension of degree n, and let L/k be the Galois closure of K/k. Let $T_n = R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus defined by K/k.

We give first the following.

Theorem 4.1. *Assume that* $\operatorname{Gal}(L/k) = S_n, n \geqq 2$, *and that* $\operatorname{Gal}(L/K) = S_{n-1}$. *Then,*

(1) T_n *is retract rational over* k *if and only if* n *is a prime;*
(2) T_n *is (stably) rational over* k *if and only if* $n = 2, 3$.

Remark 4.2. The "if" part of Theorem 4.1(1) was first proved in [CS2]. It is well known that, for $n = 2, 3$, T_n is rational over k. The "only if" part of Theorem 4.1(2) was proved in [lB] for the case where n is a prime, and in [CK] for the general case. Note that the "only if" parts of Theorem 4.1(1), (2) were proved implicitly in [LL].

Theorem 4.1 can be restated as follows.

THEOREM 4.3. *Let $S_n, n \geqq 2$ be the symmetric group on n letters. Then we have that*

(1) *$J_{S_n/S_{n-1}}$ is quasi-invertible over S_n if and only if n is a prime;*
(2) *$J_{S_n/S_{n-1}}$ is a quasi-permutation over S_n if and only if $n = 2, 3$.*

Proof. The "if" part of (1) is only a corollary to Proposition 1.7 because S_{n-1} is a Hall subgroup of S_n if n is a prime. Suppose now that n is not a prime.

First assume that there is an odd prime $p \mid n$, and set $m = n/p \geqq 2$. Let P be the elementary abelian p-subgroup of S_n generated by $\rho_1 = (1\ \ 2\ \ \cdots\ \ p)$, $\rho_2 = (p+1\ \ p+2\ \ \cdots\ \ 2p), \ldots,\ \rho_m = ((m-1)p+1\ \ (m-1)p+2\ \ \cdots\ \ mp)$, and set further $P_1 = \langle \rho_2, \rho_3, \ldots, \rho_m \rangle$, $P_2 = \langle \rho_1, \rho_3, \ldots, \rho_m \rangle, \ldots,\ P_m = \langle \rho_1, \rho_2, \ldots, \rho_{m-1} \rangle$. Regarding $\mathbb{Z}S_n/S_{n-1}$ as P-lattices, we have

$$\mathbb{Z}S_n/S_{n-1} \cong \mathbb{Z}P/P_1 \oplus \mathbb{Z}P/P_2 \oplus \cdots \oplus \mathbb{Z}P/P_m,$$

and therefore, by [E, Theorem 2(2)], $J_{S_n/S_{n-1}}$ is not quasi-invertible over P. This implies that $J_{S_n/S_{n-1}}$ is not quasi-invertible over S_n.

Assume next that $n = 2^h, h \geqq 2$. Let P be the subgroup of S_n generated by $(1\ \ 2)(3\ \ 4) \cdot (5\ \ 6)(7\ \ 8) \cdots (2^h-3\ \ 2^h-2)(2^h-1\ \ 2^h)$ and $(1\ \ 3)(2\ \ 4) \cdot (5\ \ 7)(6\ \ 8) \cdots (2^h-3\ \ 2^h-1)(2^h-2\ \ 2^h)$. Then P is an elementary abelian group of order 4, and, as is easily seen, $\mathbb{Z}S_n/S_{n-1} \cong [\mathbb{Z}P]^{(2^{h-2})}$ as P-lattices. Since J_P is not quasi-invertible by Theorem 1.2(1), it follows from Corollary 1.4 that $J_{S_n/S_{n-1}}$ is not quasi-invertible over S_n.

For assertion (2), the "if" part is well known, and so it suffices to prove the "only if" part. However, for n a nonprime, this follows directly from assertion (1). Hence it remains to prove this for $n = p \geqq 5$ a prime. Let $\sigma = (1\ \ 2\ \ \cdots\ \ p)$, and let τ be a $(p-1)$ cycle on the letters $2, 3, \ldots, p$ acting faithfully on $\langle \sigma \rangle$ by conjugation. Set $G' = \langle \sigma, \tau \rangle$, and set $H' = \langle \tau \rangle$. Then we have $\mathbb{Z}S_p/S_{p-1} \cong \mathbb{Z}G'/H'$, and so $J_{S_p/S_{p-1}} \cong J_{G'/H'}$ as G'-lattices. Since $p-1 \geqq 4$, it follows from Theorem 3.2 that $J_{G'/H'}$ is not a quasi-permutation, and so $J_{S_p/S_{p-1}}$ is not a quasi-permutation over S_p. Thus, the proof is complete. ∎

Note that Theorem 4.3(2) can be replaced by the following:

(2′) $[J_{S_n/S_{n-1}}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$ if and only if $n = 2, 3$.

Next, we give the following.

THEOREM 4.4. *Assume that* $\mathrm{Gal}(L/k) = A_n, n \geqq 3$ *and that* $\mathrm{Gal}(L/K) = A_{n-1}$. *Then,*

(1) T_n *is retract rational over* k *if and only if* n *is a prime;*
(2) $[T_n]^{(t)}$ *is stably rational over* k *for some* $t \geqq 1$ *if and only if* $n = 3, 5$.

This can also be reduced to the following.

THEOREM 4.5. *Let* $A_n, n \geqq 3$ *be the alternating group on* n *letters. Then,*
(1) $J_{A_n/A_{n-1}}$ *is quasi-invertible if and only if* n *is a prime;*
(2) $[J_{A_n/A_{n-1}}]^{(t)}$ *is a quasi-permutation for some* $t \geqq 1$ *if and only if* $n = 3, 5$.

Proof. The "if" part of (1) is only a corollary to Proposition 1.7 because A_{n-1} is a Hall subgroup of A_n if n is a prime. In the case where n is not a prime, the assertions can be proved by the same way as in Theorem 4.3. Thus, the proof of (1) is complete. In order to show (2), we may assume that $n = p \geqq 3$ is a prime. The $(p-1)$ cycle τ in the proof of Theorem 4.3(2) is not contained in A_p. Therefore, we must use $G'' = \langle \sigma, \tau^2 \rangle$ and $H'' = \langle \tau^2 \rangle$ instead of G' and H', respectively, in the proof of Theorem 4.3. Then, by Theorem 3.2, we see that $[J_{G''/H''}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$ if and only if $p = 3, 5$. Since $\mathbb{Z}A_p/A_{p-1} \cong \mathbb{Z}G''/H''$ and $J_{A_p/A_{p-1}} \cong J_{G''/H''}$ as G''-lattices, this also shows that $[J_{A_p/A_{p-1}}]^{(t)}$ is not a quasi-permutation over A_p for any $t \geqq 1$ when $p \geqq 7$. On the other hand, J_{A_3/A_2} is a quasi-permutation because A_3 is cyclic of order 3. Further, according to [D, (3.3)], $[J_{A_5/A_4}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$. This completes the proof of (2). ∎

REMARK 4.6. It is an open problem whether J_{A_5/A_4} is a quasi-permutation. This is an interesting problem because we do not know any example of the norm one torus defined by non-Galois separable extension K/k which is stably rational over k except those in Theorem 3.1.

REFERENCES

[Be] Y. Berkovich, *Groups of Prime Power Order, I*, de Gruyter, Berlin, 2008.

[Br] K. S. Brown, *Cohomology of Groups*, Grad. Texts in Math. **87**, Springer, New York, 1982.

[CS1] J.-L. Colliot-Thélène and J.-J. Sansuc, *La R-équivalence sur les tores*, Ann. Sci. Éc. Norm. Supér. (4) **10** (1977), 175–230.

[CS2] ———, *Principal homogeneous spaces under flasque tori: Applications*, J. Algebra **106** (1987), 148–205.

[CK] A. Cortella and B. Kunyavskiĭ, *Rationality problem for generic tori in simple groups*, J. Algebra **225** (2000), 771–793.

[CR] C. W. Curtis and I. Reiner, *Methods of Representation Theory, I*, John Wiley, New York, 1981.

[D] A. Dress, *The permutation class group of a finite group*, J. Pure Appl. Algebra **6** (1975), 1–12.

[E] S. Endo, *On the rationality of algebraic tori of norm type*, J. Algebra **235** (2001), 27–35.

[EM1] S. Endo and T. Miyata, *Invariants of finite abelian groups*, J. Math. Soc. Japan **25** (1973), 7–26.

[EM2] ———, *On a classification of the function fields of algebraic tori*, Nagoya Math. J. **56** (1975), 85–104; *Correction*, Nagoya Math. J. **79** (1980), 187–190.

[F] M. Florence, *Non rationality of some norm one tori*, preprint, 2006.

[lB] L. Le Bruyn, *Generic norm one tori*, Nieuw Arch. Wiskd. (5) **13** (1995), 401–407.

[LL] N. Lemire and M. Lorenz, *On certain lattices associated with generic division algebras*, J. Group Theory **3** (2000), 385–405.

[R] M. Rosen, *Representations of twisted group rings*, Ph.D. dissertation, Princeton University, Princeton, N. J., 1963.

[S] D. J. Saltman, *Retract rational fields and cyclic Galois extensions*, Israel J. Math. **47** (1984), 165–215.

[V] V. E. Voskresenskiĭ, *Algebraic Groups and Their Birational Invariants*, Transl. Math. Monogr. **179**, Amer. Math. Soc., Providence, 1998.

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